Reply to "Comment on 'Reconnection of quantized vortex filaments and the Kolmogorov spectrum' "

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In the preceding Comment [1], Hanninen raised a criticism against the results obtained in my paper "Reconnection of quantized vortex filaments and the Kolmogorov spectrum" [2]. Let me state several counterarguments to his remarks.

I would like to start with the last sentence in the abstract of the Comment, "Therefore, we find the suggestion misleading that the Kolmogorov spectrum in superfluids arises from the reconnection of vortices." In fact, the main (although not unique) suggestion of my work was somewhat different. Indeed, it was that the spectrum $E(k)$ close to the Kolmogorov dependence $E(k) \propto k^{-5/3}$, which was observed in a number of numerical simulations on the dynamics of quantized vortex filaments and which was held in the interval of the order of one decade for around $k \approx 2\pi/\delta$, may appear from reconnecting lines. And, arising out of this, the main claim was that the numerical works (cited as [21–26] in [2]), reporting on the Kolmogorov-type spectrum, cannot be considered as rigorous and definite evidence of the quasiclassic behavior of quantum turbulence. I want to stress once again that due to the extremely great importance of the underlying physical problem (interpretation of the classical turbulence in terms of quantized vortex lines), all arguments and evidence should be carefully scrutinized, and any result which offers dependence close to $k^{-5/3}$ should be alarming. This clarification, first of all, shifts the accents, and secondly, changes the requests to my results; that is, if there are any variants of generating the dependence $E(k) \propto k^{-5/3}$, different from the vortex bundles structure, they can affect our vision of the quasiclassic behavior of quantum turbulence (see discussion in [3]).

The second issue, which has been touched on in the Comment by Hanninen, concerns the self-energy of vortex filament and its contribution into the energy spectrum. Calculation of the spectrum in [2] was implemented with the use of equation

$$E(k) = \frac{\rho_s \kappa^2}{(2\pi)^2} \oint \oint \mathbf{s}'(\xi_1)\mathbf{s}'(\xi_2) d\xi_1 d\xi_2 \frac{\sin(k|\mathbf{s}(\xi_1) - \mathbf{s}(\xi_2)|)}{k|\mathbf{s}(\xi_1) - \mathbf{s}(\xi_2)|}, \tag{1}$$

where only the interaction terms have been considered, i.e., ξ_1 and ξ_2 belong to different lines participating in the reconnection process. Hanninen noticed that the majority of the kinetic energy is contained in the self-energy term, which appeared from integration along the same line. This contribution has a characteristic spectrum of $1/k$, and exceeds the suggested Kolmogorov-type spectrum $E(k) \propto k^{-5/3}$, arising from the interacting term. In particular, in Ref. [3], Hanninen wrote: "The self-energy term at the contact (reconnection) point does not vanish, rather it is the interaction energy which exactly cancels the self-terms, but this occurs only at the contact point and does not change the general conclusions presented here." Let me comment on this conclusion. Of course, the full annihilation occurs in the contact point. However, the behavior of touching curves (see Fig. 1 in [2]) is such that neighboring points remain very close to each other. The shape of contacting lines is crucial for the form of the spectrum, as was proved in the section "Analytical consideration" of paper [2]. It is easy to illustrate the said above in the 2D case. In this case the spectrum can be written as: $E(\mathbf{k}) = \rho_s \mathbf{v_k v_{-k}}/2 = \rho_s \omega_\mathbf{k} \omega_{-\mathbf{k}}/2k^2$ (this follows, e.g., from the observation that in k space the vorticity ω obeys $\omega_\mathbf{k} = \mathbf{k} \times \mathbf{v_k}$, and $\mathbf{k} \cdot \mathbf{v_k} = 0$ due to incompressibility). Putting a vortex-antivortex pair in points $\mathbf{a} = (a,0)$ and $-\mathbf{a} = (-a,0)$ on the xy plane, and using that $\omega_\mathbf{k} = (\kappa/2\pi) \int [\delta(\mathbf{r} - \mathbf{a}) - \delta(\mathbf{r} + \mathbf{a})] \exp(-i\mathbf{k} \cdot \mathbf{r}) d^2r = (\kappa/2\pi)[\exp(-i\mathbf{k} \cdot \mathbf{a}) - \exp(i\mathbf{k} \cdot \mathbf{a})]$, we get for the energy spectrum $E(\mathbf{k}) = (\rho_s \kappa^2/2\pi^2 k^2) \sin^2(\mathbf{k} \cdot \mathbf{a})$. The one-dimensional spectral density $E(k)$ in k space is obtained by integrating over the azimuthal angle

$$E(k) = \frac{\rho_s \kappa^2}{2\pi k}[1 - J_0(2ka)]. \tag{2}$$

It is seen that when the vortex-antivortex pair annihilates ($a = 0$), the full energy vanishes, $E = 0$. However, the effect of screening remains also for small a, since for the small argument $2ka$, the expansion for the Bessel function is $J_0(2ka) \approx 1 - (ka)^2$. Thus, we see that for small separation, the effect of the self-energy disappears, and one-dimensional spectral density $E(k)$ (2) is entirely determined by the intervortex distance. This consideration allows one to assert that for collapsing lines, the main contribution to the energy spectrum comes from interacting terms.

In the case of many (N) vortices with circulation $\kappa_i = \pm\kappa$, placed in points r_i, r_j, formula (2) is generalized to the following form (see, e.g., [4], and references therein):

$$E(k) = \frac{\rho_s}{2\pi k} \sum_{i=1,j=1}^{N} \kappa_i \kappa_j J_0(k|\mathbf{r}_i - \mathbf{r}_j|). \tag{3}$$

Following paper [5] we introduce N-vortex distribution density for a system of N vortices

$$f(t, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(N)}) = \left\langle \prod_{m=1}^{N} \delta(\mathbf{x}^m(t) - \mathbf{r}^m) \right\rangle, \tag{4}$$

where $x^m(t)$ is the trajectory of the m vortex. In the case of the neutral configuration of vortices, consisting of an

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equal number of vortices ($N/2$), which have opposite signs ("vortex plasma"), and in the case of uniform distribution ($f = \text{const}$), the averaged spectrum $\langle E(k)\rangle \to 0$ [just due to the antisymmetry of expression (3) over indices i, j]. Of course, because of the strong interaction between vortices the fully uniform distribution is not possible. Due to the involved dynamics the some structure (vortex clusters in the 2D case, bundles of collapsing lines in the 3D case) can appear, which determines the properties of 2D turbulence and generates nontrivial energy spectra. But as for the contribution from self-energy, the "... cancellation of the far-field velocity profiles for length scales exceeding the largest intervortex separation in any neutral configuration of vortices." is very probable (see [4]). Resuming, we can assert, that screening and averaging diminish the contribution of self-energy from remote lines into spectrum $\langle E(k)\rangle$.

In the three-dimensional case, the similar consideration is more involved; however, we suppose that this line of reasoning is also valid, i.e., the main contribution into spectral density $\langle E(k)\rangle$ appears from the interaction of collapsing filaments, and the shape of lines near the contact point is of crucial importance for the final result.

In my eyes, the misunderstanding appears from the straightforward use of formula (1). This formula, although convenient for the evaluation of spectra, includes in the latent form the preliminary averaging both over an ensemble and angle (see, e.g., [6,7]). For this reason it does not take into account both the screening and the averaging effects. Therefore, one has to treat this formula with precaution, otherwise, indeed, the wholespectrum would be $1/k$. The possible confirmation of that would be the fact that the power-law spectrum close to the $k^{-5/3}$, arising from reconnection lines, had been discussed by other authors, who accomplished calculations with the use of a quite different technique (see, e.g., [8–11].

Another issue, touched on in the Comment, is related to the role of the time averaging over the interval, when vortices approach each other to reconnect. This problem is tightly related to the second motivation of my paper, namely, to the role of hydrodynamic collapse in the formation of turbulent spectra. That is an important topic, which is intensively being discussed in the nonlinear physics society. The striking example of such type of spectra is the Phillips spectrum for water-wind waves, created by white caps—wedges of water surface. Despite the fact that the formation of cups requires the finite time (just as the reconnection of vortex filaments), as was shown in experiments (see, e.g., [12]) the Phillips spectrum is really observed for some intervals of wave numbers. The reason for this behavior could be the following: the formation of coherent structures usually occurs in a self-similar manner. This means that at any given time at the appropriate scales the formation of the required gradients takes place. At larger times, the formation of the steeper gradients on smaller scales occurs. Thus, each collapse (here reconnection) leads to the formation of power-law behavior for all the characteristic times of the collapse formation, involving the greater and greater values of k. Of course, the area uninvolved in the collapses continues to evolve in a regular manner, perhaps on a weakly turbulent scenario. In our case this scenario corresponds to those parts of the vortex lines uninvolved in the collapses, evolving according to their (extremely complicated) dynamics, but it concerns large scales (small $k \ll 2\pi/\delta$) spectra. Thus, the two processes must coexist, and there is some crossover, but the according problem is very hard and is not resolved yet.

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