Critical point scaling of Ising spin glasses in a magnetic field

Joonhyun Yeo[1,*] and M. A. Moore[2,†]

[1]*Division of Quantum Phases and Devices, School of Physics, Konkuk University, Seoul 143-701, Korea*
[2]*School of Physics and Astronomy, University of Manchester, Manchester M13 9PL, United Kingdom*

Critical point scaling in a field H applies for the limits $t \to 0$ (where $t = T/T_c - 1$) and $H \to 0$ but with the ratio $R = t/H^{2/\Delta}$ finite. Δ is a critical exponent of the zero-field transition. We study the replicon correlation length ξ and from it the crossover scaling function $f(R)$ defined via $1/(\xi H^{4/(d+2-\eta)}) \sim f(R)$. We have calculated analytically $f(R)$ for the mean-field limit of the Sherrington-Kirkpatrick model. In dimension $d = 3$, we have determined the exponents and the critical scaling function $f(R)$ within two versions of the Migdal-Kadanoff (MK) renormalization group procedure. One of the MK versions gives results for $f(R)$ in $d = 3$ in reasonable agreement with those of the Monte Carlo simulations at the values of R for which they can be compared. If there were a de Almeida-Thouless (AT) line for $d \leqslant 6$, it would appear as a zero of the function $f(R)$ at some negative value of R, but there is no evidence for such behavior. This is consistent with the arguments that there should be no AT line for $d \leqslant 6$, which we review.

I. INTRODUCTION

Despite decades of research, the nature of the ordered state of spin glasses remains controversial. There are two main competing pictures. The first is based on the broken replica symmetry (RSB) ideas of Parisi and co-workers [1], which are indeed exact in the limit of infinite dimensions—the Sherrington-Kirkpatrick (SK) model [2]. The other picture is the droplet model [3–5], which is based on the properties of excitations (droplets of reversed spins) in the ordered phase.

A key discriminator between the two pictures is the existence or not of the de Almeida-Thouless (AT) line [6]. According to the RSB picture in an applied field H there is a line in the H-T plane, $h_{AT}(T)$, at which there is a transition from the high-temperature replica symmetric state to a low-temperature broken replica symmetry state. On the other hand, in the droplet picture there is no such transition line. The application of the field H removes the phase transition completely just as does a uniform field applied to an Ising ferromagnet. There is no AT line in the droplet picture as the low-temperature phase in zero field is replica symmetric. It has been argued [7,8] that $d = 6$ is the lower critical dimension for the RSB state and that there is no AT line for $d \leqslant 6$. The absence of RSB for $d < 6$ has been rigorously established for a (unphysical) choice of the distribution function of the spin couplings J_{ij} [9].

The question of the existence or not of an AT line in three dimensions can be investigated experimentally [10]. The chief problem are those of achieving equilibration. There are also extensive Monte Carlo simulations on the properties of spin glasses in a field [11–15]. For simulations the chief problem is that of finite-size effects: these can give rise an appearance of RSB even if it is absent in the thermodynamic limit [16,17].

In this paper, we shall study critical point scaling. By this we mean the scaling behavior in the limit of small fields close to the transition temperature T_c. It is useful to

introduce the reduced $t = T/T_c - 1$ as the measure of the temperature difference from T_c. The crossover from the zero field regime to the field dominated regime is measured by the ratio $R = t/H^{2/\Delta}$ [18]. The exponent Δ is the gap exponent and is such that $\Delta = \beta + \gamma = \nu(d + 2 - \eta)/2$. Above the upper critical dimension, which is six for Ising spin glasses, $\beta = \gamma = 1$, $\Delta = 2$, and $\nu = 1/2$. (The zero-field correlation length varies as $1/t^\nu$.) The decay of the zero-field bond-averaged spin-spin correlation function, $\overline{\langle S_i S_j \rangle^2}$ with distance at T_c is as $1/r^{d-2+\eta}$; $\eta = 0$ for $d > 6$. There is the usual failing of hyperscaling for $d > 6$. For $d < 6$, all the exponents take on nontrivial values and have been extensively studied [19] in three dimensions and investigated via epsilon expansions below six dimensions [20,21]. In the presence of a field, there are many correlation lengths and the longest of these is the replicon length scale ξ [6]. Right at T_c, this grows as $\xi \sim 1/H^{4/(d+2-\eta)}$ and for $d > 6$ this reduces to $\xi \sim 1/H^{1/2}$. When t is nonzero, ξ becomes

$$\frac{1}{\xi H^{4/(d+2-\eta)}} \sim f\left(\frac{t}{H^{2/\Delta}}\right) = f(R). \tag{1}$$

The focus of this paper is on the form of the function $f(R)$. We shall refer to it as the crossover function as it describes how the correlation length at $T = T_c$ (i.e., $t = 0$) is modified when t is nonzero. It is important to keep in mind that Eq. (1) only strictly applies in the limits $t \to 0$, $H \to 0$, with the ratio R fixed. For applications one always has t and H finite and one needs to allow for corrections to scaling. We can determine $f(R)$ analytically in the mean-field limit, i.e., for the SK model and this is done in Sec. II. For three dimensions we shall calculate it with two different versions of the Migdal-Kadanoff RG procedure in Sec. V and compare them with the results of the Monte Carlo simulations of Ref. [15].

Since there is an AT line for $d > 6$ and in the SK model we need to understand why there is no trace of it in the crossover function $f(R)$. The answer lies in the scaling form of the AT line, i.e., the behavior of h_{AT} as $|t| \to 0$. When $d > 8$ $h_{AT}^2 \sim |t|^3$, but for $8 > d > 6$, $h_{AT}^2 \sim |t|^{d/2-1}$ [6,18,22]. Then for all $d > 6$, R goes to $-\infty$ at the AT line in the limit $|t| \to 0$.

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*jhyeo@konkuk.ac.kr
†m.a.moore@manchester.ac.uk

However, for $d < 6$, if there is an AT line, it would occur at a finite value of $-R$ as $h_{AT}^2 \sim |t|^\Delta$ [18]. Because ξ diverges at the AT line, $f(R)$ has to have a zero if there is an AT line. However, we shall see no evidence for such a zero in the work for $d = 3$ reported in Sec. V. The absence of an AT line for $d = 3$ is consistent with the argument of Ref. [7] that the AT line only exists when $d > 6$. This argument is sketched in Sec. III; it is because for $8 > d > 6$, $h_{AT}^2 \sim (d-6)^4 |t|^{d/2-1}$, *when* $|t| \to 0$, which indicates that in the scaling limit, the AT line is going away as d approaches 6. The comments of Ref. [23] on this argument will be reviewed.

II. THE CROSSOVER FUNCTION $f(R)$ IN THE SK LIMIT

The Hamiltonian for the SK Ising spin glass in a field [2] is given by

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j - H \sum_i S_i, \tag{2}$$

where the Ising spins S_i take the value ± 1 and $\langle ij \rangle$ means that the sum is over all pairs i and j. The couplings J_{ij} are chosen independently from a Gaussian distribution with zero mean and a standard deviation (width) $J/N^{1/2}$. We shall calculate for this Hamiltonian the scaling function $f(R)$. This is easy to do as the calculation is done in the region where there is replica symmetry. There the Edwards-Anderson order parameter $q = 1/N \sum_i \langle S_i \rangle^2$ is obtained by solving the equation

$$q = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \tanh^2(\beta J \sqrt{q} x + \beta H). \tag{3}$$

It is convenient to introduce the notation $Q = \beta^2 J^2 q + \beta^2 H^2$, and $t = T/T_c - 1$, where for the SK model $T_c = J$. Then to order t^2 and $h^2 (= H^2/T_c^2)$, one can obtain by expanding the argument of tanh in Eq. (3) the following approximation to the equation of state:

$$H^2/T_c^2 \equiv h^2 = 2tQ + 2Q^2. \tag{4}$$

To determine the crossover function $f(R)$, we need to determine the replicon correlation length. This is the length scale associated with the decay of the (bond-averaged) replicon correlation function $G_R(ij)$,

$$G_R(ij) = \overline{(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^2}. \tag{5}$$

For the replica symmetric state, de Almeida and Thouless [6] calculated the eigenvalues of the Hessian associated with the replica symmetric solution. The smallest eigenvalue was that in the replicon sector and is given by

$$\lambda_R = 1 - \beta^2 J^2 \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \operatorname{sech}^4(\beta J \sqrt{q} x + \beta H). \tag{6}$$

To order t and h, this reduces to $\lambda_R = 2t + 2Q$. Eliminating Q using Eq. (4) gives

$$\lambda_R \approx t + \sqrt{t^2 + 2h^2}. \tag{7}$$

We next use the Ornstein-Zernike approximation to determine the replicon correlation length by setting $1/\xi^2 \sim \lambda_R$. Then we get on using the notation $R = t/h$, which is what

$R = t/h^{2/\Delta}$ becomes for $\Delta = 2$,

$$\frac{1}{\xi h^{1/2}} = f(R) \sim (R + \sqrt{2 + R^2})^{1/2}. \tag{8}$$

Note that the power of H in Eq. (1), $4/(d + 2 - \eta)$, can be re-written as $2\nu/\Delta$, which reduces to $1/2$ for the SK model where $\nu = 1/2$, and $\Delta = 2$. Equation (8) is the mean-field approximation for $f(R)$. Exactly the same result for the crossover function $f(R)$ is obtained for a Gaussian distribution of fields whose variance is H^2, so there are universality features associated with it. In Sec. V, the form of $f(R)$ is studied in three dimensions, and it is found to be at least qualitatively similar to that of the mean-field limit in Eq. (8).

Notice that when t is negative, $f(R)$ decreases to zero as $R \to -\infty$.

$$f(R) \sim 1/|R|^x, \qquad R \to -\infty, \tag{9}$$

where in the SK limit the exponent x takes the value $\frac{1}{2}$. We would expect the same value for x to hold from infinite dimension down to the upper critical dimension, $d = 6$.

III. THE AT LINE FOR $d > 6$

Since we are essentially studying the replicon correlation length ξ, one might have expected to see in the scaling function $f(R)$ behavior associated with the AT line. The AT line is the line in the H-T plane at which ξ goes to infinity. Near T_c, the AT line for the SK model is given by $h_{AT}^2 \sim |t|^3$ [6], so on the AT line $R = t/H^{2/\Delta} = t/h_{AT}$. This diverges to $-\infty$ in the critical scaling limit $|t| \to 0$: the AT line is thus outside the critical scaling limit for the SK model, which explains its absence from $f(R)$.

We next review how the AT line evolves with dimension d. The SK form for the AT line as $|t| \to 0$ is expected to hold down to $d = 8$, and changes for $8 > d > 6$, to $h_{AT}^2 \sim |t|^{d/2-1}$ [18,22]. Provided $d > 6$, R at the AT line is outside the critical scaling limit. However, as $d \to 6^+$, the value of R on the AT line is diverging to infinity less and less strongly (as $1/|t|^{\frac{d-6}{4}}$). A change must occur at $d = 6$. *If there were an AT line for $d \leqslant 6$, it would be in the critical scaling region and occur at a finite value of R.* In Sec. V, we shall study the computer simulations of the Janus group [15] and the MK approximation for $f(R)$ in three dimensions for evidence for such a zero at a finite value of R and find none. The more detailed calculation of Ref. [7] provide a clue as to what happens: the AT transition does not occur for $d \leqslant 6$.

The more detailed calculations started from the replicated Ginzburg-Landau-Wilson free-energy functional for the Ising spin glass which, written in terms of the replica order parameter field $Q_{\alpha\beta}$, is

$$F[\{Q_{\alpha\beta}\}] = \int d^d x \left[\frac{1}{2} r \sum_{\alpha<\beta} Q_{\alpha\beta}^2 + \frac{1}{2} \sum_{\alpha<\beta} (\nabla Q_{\alpha\beta})^2 \right.$$
$$+ \frac{w}{6} \sum_{\alpha<\beta<\gamma} Q_{\alpha\beta} Q_{\beta\gamma} Q_{\gamma\alpha}$$
$$\left. - h^2 \sum_{\alpha<\beta} Q_{\alpha\beta} + O(Q^4) \right]. \tag{10}$$

The coefficient r is essentially a measure of the distance from T_c, i.e., it is basically t. When $d < 8$, the Q^4 terms are irrelevant. Conventional RG methods were applied [20], but the calculation was done above the upper critical dimension, $d_u = 6$. It is useful to define $\epsilon = d - 6$. Then for small ϵ [7,23],

$$h_{AT}^2 \sim \frac{w|r|^{d/2-1}}{[(2w^2/\epsilon)(1 - |r|^{\epsilon/2}) + 1]^{5d/6-1}}. \qquad (11)$$

In Moore and Bray [7], it was argued that Eq. (11) implied that near six dimensions in the limit when $|r| \to 0$,

$$h_{AT}^2 \sim \left(\frac{\epsilon}{2w^2}\right)^4 w|r|^{d/2-1}. \qquad (12)$$

This result strongly suggests that the AT line will disappear as $d \to 6$ in the critical scaling limit, $|r| \to 0$. At fixed $|r|$ in the limit of $\epsilon \to 0$, Eq. (11) gives

$$h_{AT}^2 \sim \left(\frac{1}{w^2 \ln|r| + 1}\right)^4 w|r|^{d/2-1}, \qquad (13)$$

which agrees with the expression for the AT line in six dimensions given by Parisi and Temesvári [23]. However, Eq. (11) is not valid for this limit.

To see that, it is useful to note that the general form of the AT line (at least for $6 < d < 8$) is

$$h_{AT}^2 = \frac{|r|^2}{w} g(w^2|r|^{\epsilon/2}). \qquad (14)$$

At small values of $y = w^2|r|^{\epsilon/2}$, one can construct the perturbative expansion for $g(y)$ [22]. The renormalization group calculation that leads to Eq. (11) is effectively just a resummation of the perturbative calculation and will only be useful and valid at small values of y, which means only for the critical scaling limit $|r| \to 0$ at fixed ϵ.

However, we would agree with the authors of Ref. [23] that it would be good to have an argument that at any fixed value of $|t|$, h_{AT} went to zero in the limit $d \to 6$, rather than just for the scaling limit $|t| \to 0$. This is a nonperturbative task, but perhaps not impossible. In Ref. [8], a $1/m$ expansion of the value of the AT field, H_{AT}, at zero temperature was undertaken, for an m-spin component spin glass ($m = 1$ is the Ising spin glass, $m = 3$ is the Heisenberg spin glass). It was found that the first term in the $1/m$ expansion went to zero around $d = 6$.

IV. DROPLET SCALING AND $f(R)$ FOR $d \leqslant 6$

In Sec. V, we find that there is no evidence that there is an AT line in $d = 3$. In other words, the crossover function $f(R)$ has no finite zero on the negative axis. In the limit when $R \to -\infty$, it decays towards zero, as $1/|R|^x$. In this section, we show that this behavior for the crossover function is predicted by droplet scaling and determine how the exponent x depends on the critical exponents and θ.

According to the droplet picture the length scale ξ is determined by an Imry-Ma argument [3,4] where the energy cost of flipping a region of linear extent ξ, $\Upsilon\xi^\theta$, is balanced against the magnetic field energy which could be gained, $\sqrt{q}H\xi^{d/2}$. The interface energy term Υ scales as $|t|^{\nu\theta}$ and q scales as $|t|^\beta \sim |t|^{\nu(d-2+\eta)/2}$ at small $|t|$. Thus, according to

the droplet picture,

$$\frac{1}{\xi} \sim \left(\frac{\sqrt{q}}{\Upsilon}H\right)^{\frac{1}{d/2-\theta}} \sim \left[\frac{H}{|t|^{\nu[\theta-(d-2+\eta)/4]}}\right]^{\frac{1}{d/2-\theta}}. \qquad (15)$$

The Imry-Ma argument for ξ is a scaling argument itself, which should hold for all $T < T_c$ in the limit $H \to 0$. As a consequence, it should coincide with the $R \to -\infty$ limit for the critical scaling function, at least for $d \leqslant 6$. This enables us to relate the exponent x for the decay of $f(R)$ as $1/|R|^x$ to the other exponents. Multiplying both sides of Eq. (15) by $1/H^{\frac{4}{d+2-\eta}}$ gives

$$\frac{1}{\xi H^{4/(d+2-\eta)}} = \left[\frac{1}{|t|^{\nu[\theta-(d-2+\eta)/4]}}\right]^{\frac{1}{d/2-\theta}} H^{(\frac{1}{d/2-\theta} - \frac{4}{d+2-\eta})}. \qquad (16)$$

This will approach $\frac{1}{|R|^x}$ as $R \to -\infty$. Note that

$$\frac{1}{|R|^x} = \left[\frac{H^{\frac{4}{\nu(d+2-\eta)}}}{|t|}\right]^x. \qquad (17)$$

The exponents of both $|t|$ and H in Eqs. (17) and (16) must both agree. This happens if

$$x = \frac{\nu}{d/2-\theta}\left(\theta - \frac{d-2+\eta}{4}\right). \qquad (18)$$

We shall give numerical estimates of the value of x for $d = 3$ in Sec. V. For $d > 6$, $x = \frac{1}{2}$. We do not expect that Eq. (18) should necessarily give $x = \frac{1}{2}$ as $d \to 6^-$. This is because for $d > 6$, x is associated with the Gaussian perturbative fixed point, while for $d \leqslant 6$, it is associated with the zero-temperature fixed point of droplet scaling (see Sec. VI for a further discussion of fixed points).

V. THE CROSSOVER FUNCTION $f(R)$ IN THREE DIMENSIONS

In this section, we shall study two variants (called scheme I and scheme II) of the Migdal-Kadanoff RG procedure to first get the critical exponent η and then the crossover function $f(R)$. We find the Janus result for η lies between the values of scheme I and that of scheme II. The MK results for $f(R)$ have been obtained over a much wider range of R than those of the Janus group [14,15,19]. For scheme II, there is reasonable agreement on $f(R)$ with that of the Janus group at the R values at which they they can be directly compared. Scheme I is less satisfactory; it also leads to a value for $\eta > 0$ and a value for $x < 0$, both of which seem rather unlikely. There is no good evidence of a zero of $f(R)$ at a finite value of R in either scheme, which means they provide no evidence for an AT line in three dimensions.

The Edwards-Anderson Hamiltonian for the Ising spin glass in a field is given by

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j - H \sum_i S_i, \qquad (19)$$

where the Ising spins S_i take the value ± 1 and $\langle ij \rangle$ means that the sum is over all nearest neighbor pairs i and j. The couplings J_{ij} are chosen independently from a Gaussian distribution with zero mean and a standard deviation (width) J. The temperature
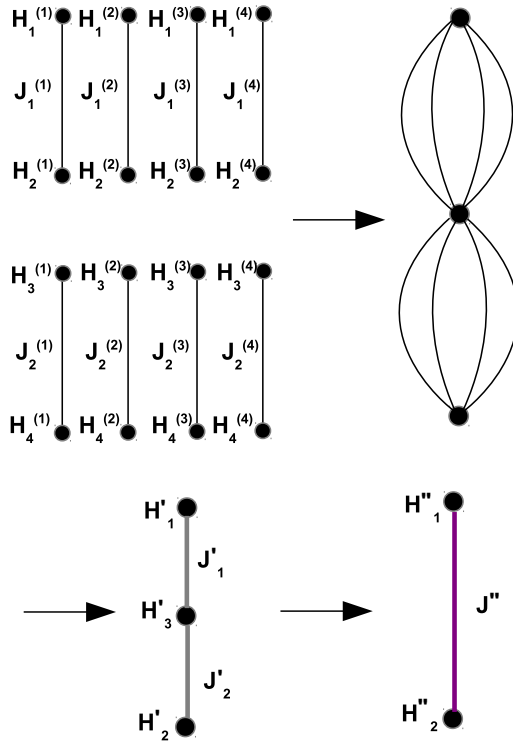
FIG. 1. (Color online) Bond-moving scheme in the Migdal-Kadanoff approximation. In the first step two sets of four bonds are moved together. The renormalized couplings and fields are assigned in the second step according to the given scheme. Finally, a trace is taken over the middle spin.

T enters into the problem through $\exp(-\mathcal{H}/T)$ and the flows of the couplings J_{ij}/T and the fields H/T are studied in the Migdal-Kadanoff RG.

We use the approximate bond moving schemes. For a three-dimensional cubic lattice, $2^{d-1} = 4$ (when $d = 3$) bonds are put together to form a new bond. The coupling constant of this new bond is just given by the sum of the coupling constants of the four bonds. Using the notations in Fig. 1, we have

$$J_1' = \sum_{i=1}^{4} J_1^{(i)}, \quad J_2' = \sum_{i=1}^{4} J_2^{(i)}. \quad (20)$$

As for the external field, however, there is a certain freedom in how the on-site field variables are moved in the bond-moving scheme. In this paper, we employ two different schemes for moving the fields. The first one, which we call scheme I, is due to Ref. [24]. In this scheme, when three bonds are moved to combine with a bond, the fields on the three bonds are moved to the site that is to be traced over. This procedure prevents the fields from increasing indefinitely, and allows us to work with the uniform applied field. In terms of the notations in Fig. 1, we have

$$H_1' = H_1^{(1)}, \quad H_2' = H_4^{(1)}, \quad (21)$$

and

$$H_3' = \sum_{i=2}^{4} H_1^{(i)} + \sum_{i=1}^{4} H_2^{(i)} + \sum_{i=1}^{4} H_3^{(i)} + \sum_{i=2}^{4} H_4^{(i)}. \quad (22)$$

In the other scheme, scheme II, the fields stay with the bonds when moved. This kind of field moving method was used in Ref. [25] for the random field Ising model. From Fig. 1, we assign in this case

$$H_1' = \sum_{i=1}^{4} H_1^{(i)}, \quad H_2' = \sum_{i=1}^{4} H_4^{(i)}, \quad (23)$$

and

$$H_3' = \sum_{i=1}^{4} H_2^{(i)} + \sum_{i=1}^{4} H_3^{(i)}. \quad (24)$$

Since the field grows indefinitely as the iteration continues, the uniform external field is not appropriate in this case. Instead, we use a random external field of zero mean and the standard deviation H. In both schemes, once the renormalized fields, H_n' are determined, a trace is performed over the spins at the site connecting the two new bonds. The decimation procedure can be continued n times for a system of size $L = 2^n$.

We perform the MK RG numerically for given temperature T and given uniform field H (scheme I) or given field width H (scheme II). All these quantities are measured in units of J. In the numerical calculations, we prepare 10^6 bonds. On each bond, the couplings are chosen independently from the Gaussian distribution with zero mean and width $1/T$. On each end of the given bond, we either assign $H/(2dT)$ for scheme I or choose the field from the Gaussian distribution of zero mean and width $H/(2dT)$. The factor of $1/(2d)$ is used to account for the coordination number in the d-dimensional cubic lattice. We then randomly select two sets of 4 bonds out of 10^6 to form two new bonds, and follow the procedure described in Fig. 1 to obtain renormalized couplings and fields. This procedure is continued until we get 10^6 new bonds, which completes the first iteration. As we iterate the same procedure, we can obtain the flow of the couplings and fields as a function of iteration number n.

At each step of the iteration, we measure the standard deviations of the couplings, $J(n)$. The RG flow of these widths of the couplings at fixed external field is shown in Fig. 2. For a finite field, the coupling strength always flows to zero in all cases we studied indicating the absence of a phase transition. In general, the decay of the coupling strength is slower at low temperatures. The decaying part of $J(n)$ can be described as $\exp[-L/\xi(T,H)]$ with $L = 2^n$, where the $\xi(T,H)$ is interpreted as the correlation length at temperature T and field H. Note that the decay in the scheme II is slower than that in scheme I, so the correlation length in scheme II is generally larger than that in scheme I.

In Fig. 3, the RG flow of the couplings is shown at fixed temperature. The results at zero field are also included in these figures. We can see that the coupling strength flows to infinity if $T < T_c$ for $H = 0$. We estimate the zero-field transition temperature as $T_c = 1.77$. Note that the presence of the external field makes the correlation length decrease.

In scheme I, we start from the uniform external field. As the iteration continues, however, the field becomes no longer uniform but shows a random distribution. The mean value of the fields is not changed, but the standard deviation σ_H keeps increasing until it saturates after the coupling drops to a small
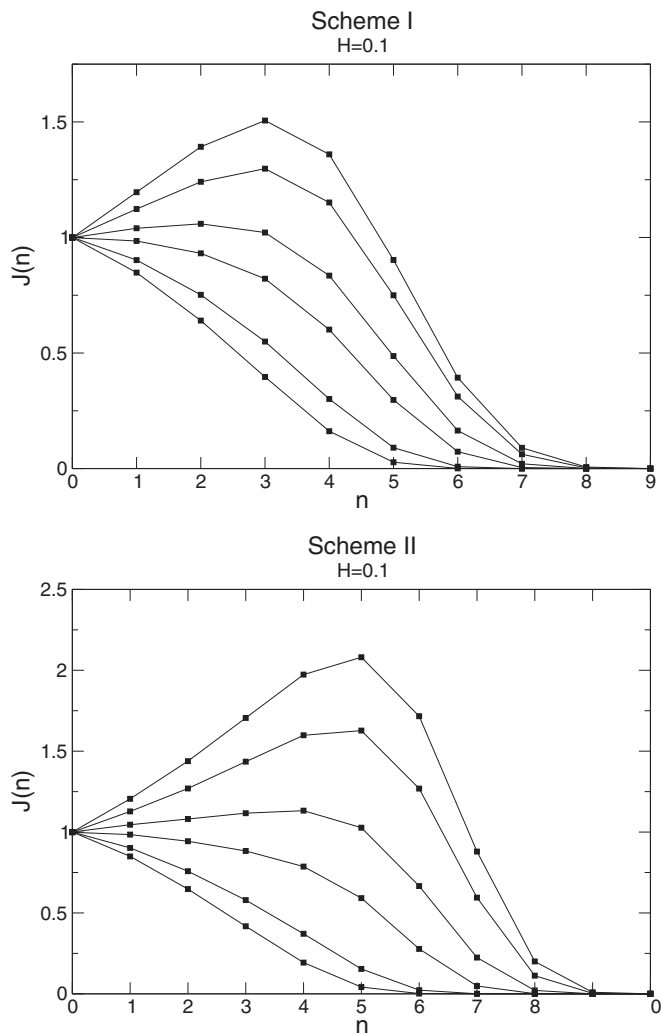
FIG. 2. The coupling strength $J(n)$ at iteration step n at fixed external field $H = 0.1$ in scheme I (top) and scheme II (bottom). The temperatures are $T = 0.1, 1.0, 1.6, 2.0, 2.6,$ and 3.0 from top to bottom.
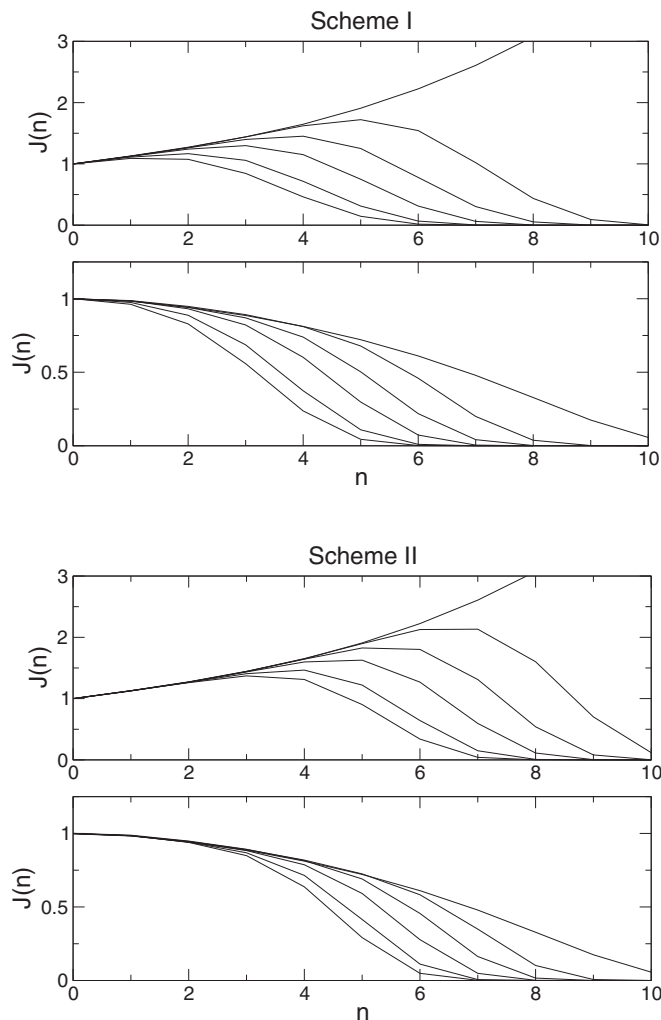

FIG. 3. The flow of the coupling $J(n)$ at fixed temperature and varying fields in schemes I and II. The upper panel for each scheme is at $T = 1.0$ and the lower one is for $T = 2.0$. From top to bottom, $H = 0, 0.02, 0.05, 0.1, 0.2,$ and 0.3.

value as can be seen in Fig. 4. In scheme II, the standard deviation σ_H also increases from the initial value $\sigma_H(0) = H$ for the random external field as shown in the lower figure of Fig. 4. But it does not saturate in this case, but increases at a slower rate. In both cases, the initial increase of σ_H is well described by $\sigma_H \sim L^{d/2}$ as $T \to 0$.

Finally, we use our results for the critical exponents and ξ to determine the crossover function $f(R)$. The results are displayed in Fig. 5 for scheme I and in Fig. 6 for scheme II.

Also on the plots are the results of the Janus collaboration [15]. In neither case is there good agreement between the MK results and the Janus simulation. However, the apparent discrepancy is just an artifact due to using different definitions of ξ. We extracted ξ by studying the decay of the couplings J with iteration number. The Janus ξ was obtained from a particular moment of the replicon correlation function G_R. A different choice of moment would alter their estimate of ξ. However, one expects the different definitions of ξ to have essentially the same t and H dependence, i.e., they differ

mostly by a multiplicative constant. We chose that constant to make $\xi_{\text{Janus}}(t = 0, H = 0.2) = \xi_{\text{MK}}(t = 0, H = 0.2)$. For scheme I, that constant is 3.79, while for scheme II it is 7.13. We then multiplied the Janus data by that factor and the resulting plots are as shown in the lower panels of Figs. 5 and 6. The Janus data are now in reasonable agreement with the MK results over the ranges of R for which there is overlapping scaling data at least for scheme II. However, it is only for the MK results that we have data over a large enough range to see the decay of $f(R)$ as $1/|R|^x$ at large values of $-R$. There is no evidence that $f(R)$ has a zero at any finite negative value of R, which implies that there is no AT line. We end this section by summarizing the values found or used for the various exponents.

Critical point scaling exponents. (1) The exponent η was evaluated from Fig. 7. From $\xi(T_c, H) \sim H^{-4/(d+2-\eta)}$, we have $\eta \sim 0.18$ for scheme I and $\eta \sim -0.56$ for scheme II. The Janus collaboration [19] estimate is $\eta \sim = -0.39$. (2) The MK value for $1/\nu = 0.356$ [26] for both schemes I and II. The Janus estimate [19] is $1/\nu = 0.39$.
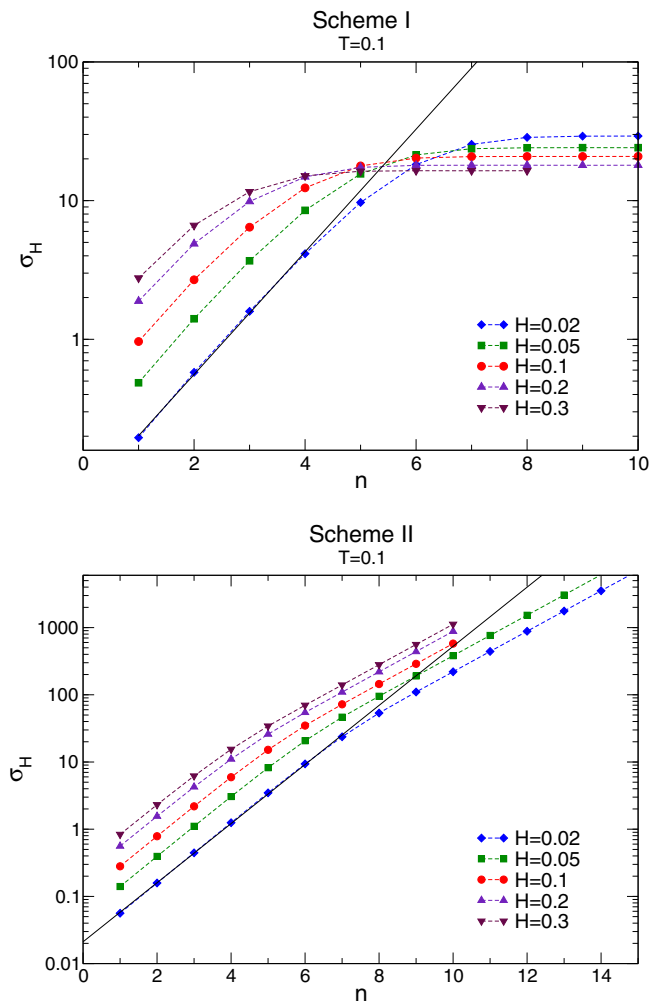
Scheme I
T=0.1

Scheme II
T=0.1

FIG. 4. (Color online) The standard deviation of the fields as a function of iteration in scheme I (top) for fixed temperature $T = 0.1$ and varying initial uniform field H. The solid line in the upper figure is the fit $\sigma_H \sim L^{1.47}$. The lower figure is for scheme II also for fixed temperature $T = 0.1$ and varying standard deviation H of the initial random field. The solid line is a fit to $\sigma_H \sim L^{1.46}$.

Droplet scaling exponents. (1) We found by studying the growth of $J(L) \sim L^\theta$ at zero field and at very low temperature that $\theta = 0.26$. The value obtained for $d = 3$ in Ref. [27] was $\theta = 0.24$. (2) Using Eq. (18) in $d = 3$ for MK scheme I, $x = -0.079$, for scheme II, $x = 0.34$. For the Janus exponents, $x = 0.18$.

VI. DISCUSSION

In Figs. 5 and 6, the MK results include data points right down to $T = 0$, which are well outside the critical regime. Some of the points from low T have departures from the universal curve, which are visible if the large $|R|$ region is put on an expanded scale, as in Fig. 8. The reason for this is that the critical scaling forms $\Upsilon \sim |t|^{\nu\theta}$ and $q \sim |t|^\beta$ do not hold accurately right down to $T = 0$. However, despite this, the critical scaling collapse of the data is surprisingly good.

Critical scaling strictly applies only for the limit $t \to 0$, $H \to 0$, with the ratio $R = t/H^{2/\Delta}$ fixed. One always hopes
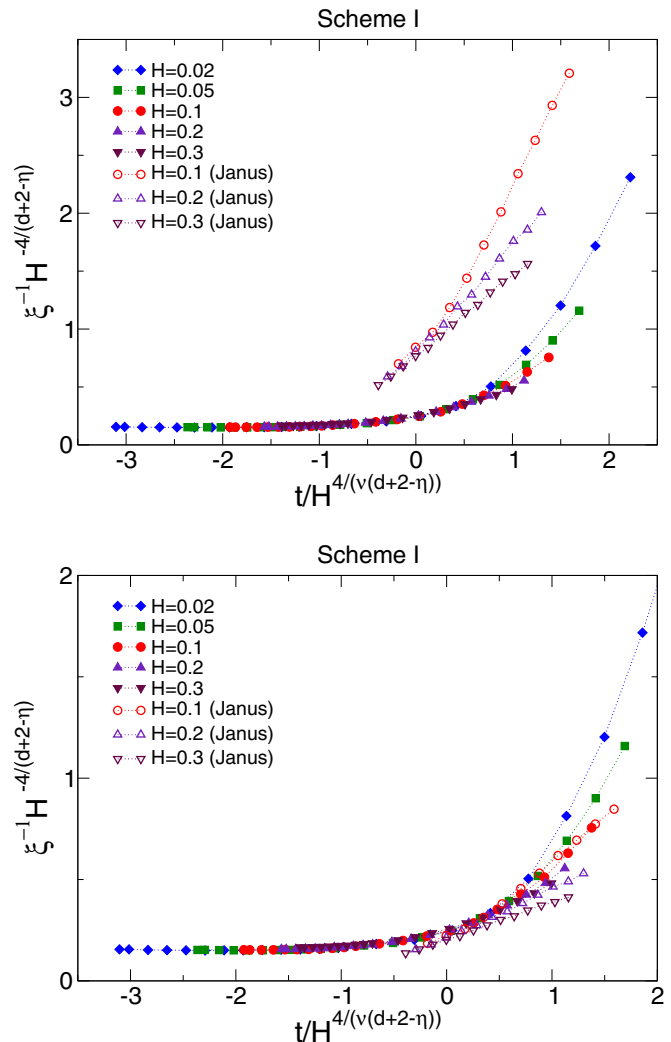

Scheme I

Scheme I

FIG. 5. (Color online) The upper panel is $\xi^{-1} H^{\frac{4}{d+2-\eta}} \equiv f(R)$ vs $t/H^{4/(\nu(d+2-\eta))} = R$ for MK scheme I and Janus data [15]. The lower panel is the same but with the Janus data for ξ rescaled by the factor 3.79, as described in the text.

though that there is a sizable critical region where the scaling forms are good approximations. We have seen that when $T < T_c$ the critical region extends to a good approximation down to $T = 0$. However, the figures show the critical region is much smaller when $T > T_c$. Many of the MK and Janus data points are off the scaling line, indicating that they were calculated from values of t and H, which were too large to be in the scaling region.

Another surprise is that when $d < 6$, the critical scaling function $f(R)$ involves the exponent θ, which is an exponent of the zero-temperature fixed point. The RG flows near the AT line were first investigated by Bray and Roberts [28] in a perturbative treatment appropriate for $d \to 6^-$ and no stable fixed point was found. It has been suggested from time to time that such runaway flows perhaps indicate that the behavior is controlled by a zero temperature fixed point. Within the MK approximation it is possible to follow the flows of the fields H and the couplings J under the RG iterations. If one starts out at a temperature below T_c, when the fields H are very
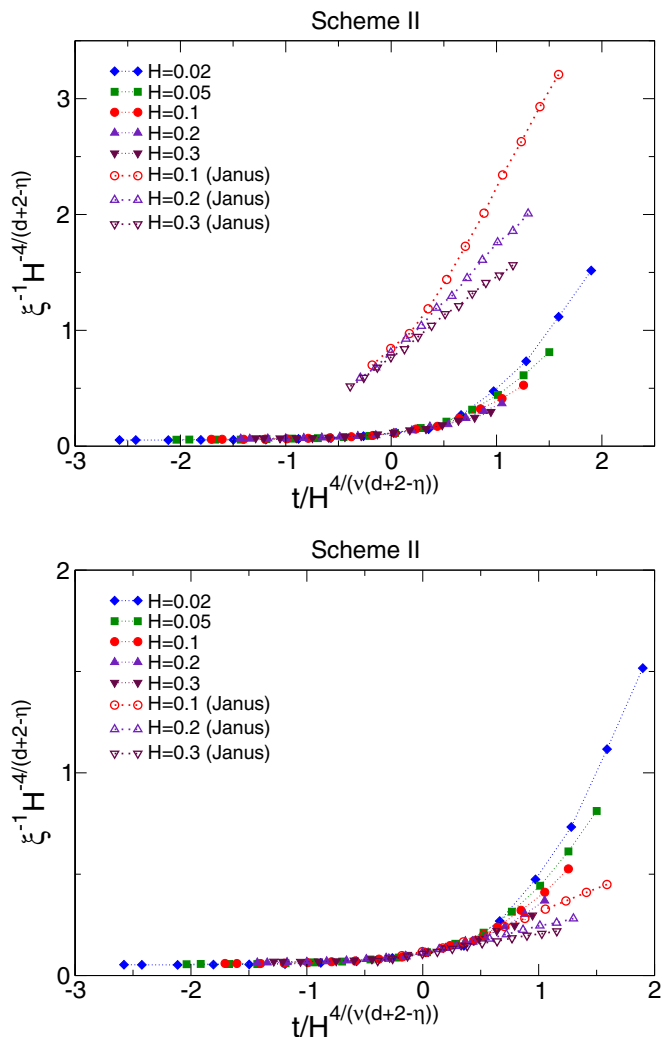
FIG. 6. (Color online) The upper panel is $\xi^{-1} H^{\frac{4}{d+2-\eta}} \equiv f(R)$ vs $t/H^{4/(\nu(d+2-\eta))} = R$ for MK scheme II and Janus data [15]. The lower panel is the same but with the Janus data for ξ rescaled by the factor 7.13, as described in the text.
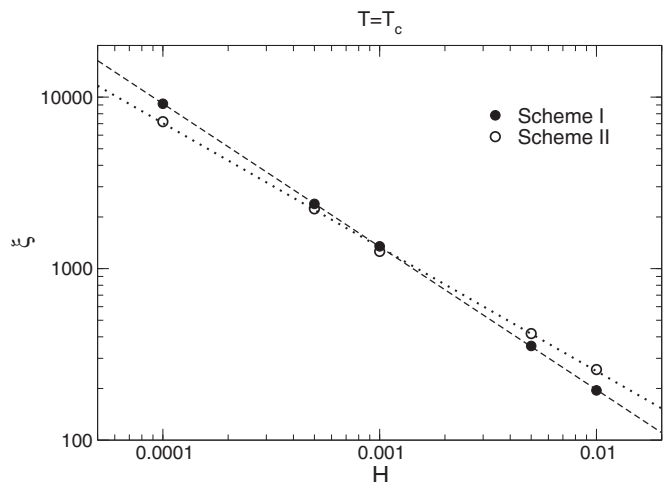

FIG. 8. (Color online) The log-log plot of $\xi^{-1} H^{\frac{4}{d+2-\eta}} \equiv f(R)$ vs $-t/H^{4/(\nu(d+2-\eta))} = -R$ for MK scheme I (upper panel) and scheme II (lower panel) to illustrate the power-law behavior $f(R) \sim 1/|R|^x$, which is predicted to set in at large values of $-R$. The straight line is the expected slope if x is predicted by Eq. (18).

small, the flows take one very close to the zero temperature fixed point associated with $H = 0$ before they finally run off to large H/J values. Thus it may be that the RG runaway flows found by Bray and Roberts are related to the effects produced by a zero-temperature fixed point. A role for a zero-temperature fixed point has also been suggested recently by Angelini and Biroli [29] in a very unconventional scenario for the high-dimensional behavior of spin glasses in a field.

FIG. 7. The correlation length $\xi(T_c, H)$ at zero-field transition temperature T_c for small external field H. The dashed lines are fits, $\xi \sim H^{-0.83}$ for scheme I and $\xi \sim H^{-0.72}$ for scheme II.
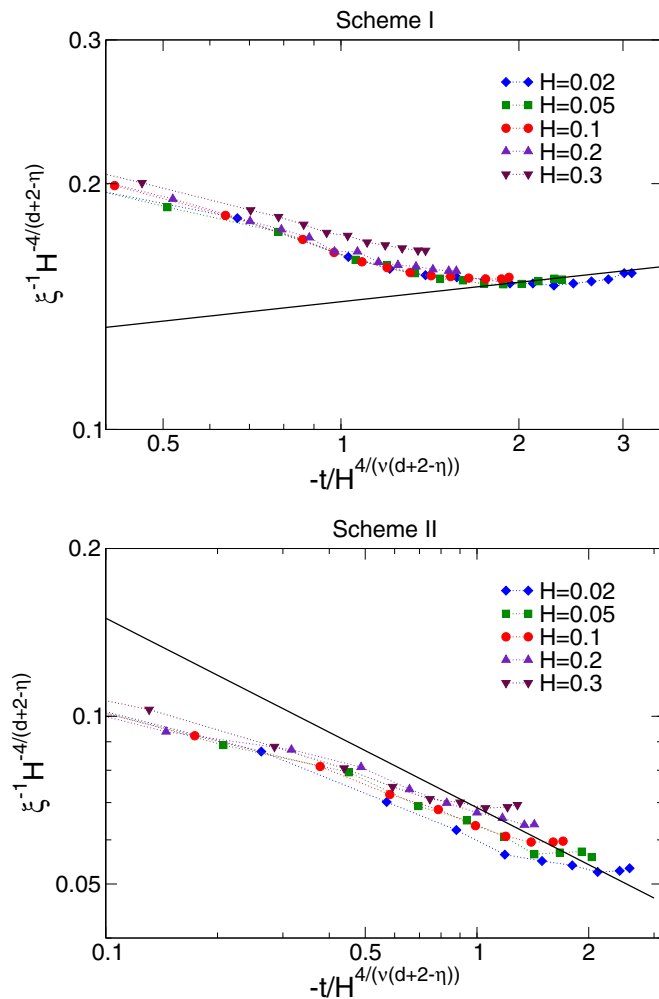
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