

Nonlinear analysis of magnetization dynamics excited by spin Hall effect

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We investigate the possibility of exciting self-oscillation in a perpendicular ferromagnet by the spin Hall effect on the basis of a nonlinear analysis of the Landau-Lifshitz-Gilbert (LLG) equation. In the self-oscillation state, the energy supplied by the spin torque during a precession on a constant energy curve should equal the dissipation due to damping. Also, the current to balance the spin torque and the damping torque in the self-oscillation state should be larger than the critical current to destabilize the initial state. We find that these conditions in the spin Hall system are not satisfied by deriving analytical solutions of the energy supplied by the spin transfer effect and the dissipation due to the damping from the nonlinear LLG equation. This indicates that the self-oscillation of a perpendicular ferromagnet cannot be excited solely by the spin Hall torque.

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I. INTRODUCTION

Nonlinear dynamics such as fast switching and self-oscillation (limit cycle) has been a fascinating topic in physics [1,2]. Magnetization dynamics excited by the spin transfer effect [3,4] in a nanostructured ferromagnet [5–12] provide fundamentally important examples of such nonlinear dynamics. The magnetization switching was first observed in Co/Cu metallic multilayer in 2000 [5]. Three years later, self-oscillation was reported in a similar system [6]. In these early experiments on the spin transfer effect, linear analysis was used to estimate, for example, the critical current destabilizing the magnetization in equilibrium [13,14]. However, recently it became clear that nonlinear analysis is necessary to quantitatively analyze the magnetization dynamics [2,15–26]. For example, current density to excite self-oscillation can be evaluated by solving a nonlinear vector equation called the Landau-Lifshitz-Gilbert (LLG) equation [23,24].

Originally the spin transfer effect was studied by applying an electric current directly to a ferromagnetic multilayer. Recently, however, an alternative method employing the spin Hall effect has been used to observe the spin transfer effect [27–40]. The spin-orbit interaction in a nonmagnetic heavy metal scatters the spin-up and spin-down electrons to the opposite directions, producing a pure spin current flowing in the direction perpendicular to an applied current. The pure spin current excites the spin torque, called spin Hall torque, on a magnetization in a ferromagnet attached to a nonmagnet. The direction of the spin Hall torque is geometrically determined [27], and its magnitude shows a different angular dependence than the spin torque in the ferromagnetic multilayer [3]. Therefore, it is fundamentally unclear whether the physical phenomena observed in the multilayer [5–12] can be reproduced in the spin Hall system, and thus, new physical analysis is necessary. The magnetization switching of both in-plane magnetized and perpendicularly magnetized ferromagnets by spin Hall torque was recently reported [28–31,36,37]. Accordingly, it might be reasonable to expect reports on self-oscillation by spin Hall torque. However, whereas self-oscillation has been observed in the in-plane magnetized system [32], it has not been reported yet in the perpendicularly magnetized system.

The purpose of this paper is to investigate the possibility of exciting self-oscillation by spin Hall torque based on a nonlinear analysis of the LLG equation. We argue that two physical conditions should be satisfied to excite self-oscillation. The first condition is that the energy that the spin torque supplies during a precession on a constant energy curve should equal the dissipation due to damping. The second condition is that the current to balance the spin torque and the damping torque in the self-oscillation state should be larger than the critical current to destabilize the initial state. This is because the magnetization initially stays at the minimum energy state, whereas the self-oscillation corresponds to a higher energy state. We derive exact solutions of the energy supplied by the spin transfer effect and the dissipation due to damping in the spin Hall system by solving the nonlinear LLG equation, and find that these conditions are not satisfied. Thus, the self-oscillation of a perpendicular ferromagnet cannot be excited solely by the spin Hall torque.

The paper is organized as follows. The physical conditions to excite a self-oscillation is summarized in Sec. II. These conditions are applied to the spin Hall system in Sec. III. Section IV is devoted to the conclusions.

II. PHYSICAL CONDITIONS TO EXCITE SELF-OSCILLATION

Let us first summarize the physical conditions necessary to excite self-oscillation. The magnetization dynamics are described by the LLG equation

$$\frac{d\mathbf{m}}{dt} = -\gamma\mathbf{m} \times \mathbf{H} - \gamma H_s \mathbf{m} \times (\mathbf{p} \times \mathbf{m}) + \alpha \mathbf{m} \times \frac{d\mathbf{m}}{dt}, \quad (1)$$

where \mathbf{m} and \mathbf{p} are the unit vectors pointing in the directions of the magnetization and the spin polarization of the spin current, respectively. The gyromagnetic ratio and the Gilbert damping constant are denoted as γ and α , respectively. The magnetic field \mathbf{H} relates to the energy density of the ferromagnet E via $\mathbf{H} = -\partial E / \partial (M\mathbf{m})$, where M is the saturation magnetization. The strength of the spin torque H_s is proportional to the current density j . Since the LLG equation conserves the norm of the magnetization, the magnetization dynamics can be described as a trajectory on a unit sphere. The energy density E shows constant energy curves on this sphere. For example, when the

system has uniaxial anisotropy, the constant energy curves are latitude lines. The self-oscillation is a steady precession state on a constant energy curve excited by the field torque, the first term on the right-hand side of Eq. (1). This means that the second and third terms of Eq. (1), averaged over the constant energy curve, cancel each other. In other words, the energy supplied by the spin transfer effect during the precession on the constant energy curve equals the dissipation due to the damping. This condition can be expressed as [2,24]

$$\oint dt \frac{dE}{dt} = \mathcal{W}_s + \mathcal{W}_\alpha = 0, \quad (2)$$

where the energy supplied by the spin transfer effect and the dissipation due to the damping during the precession on the constant energy curve of E are given by [2,15–26]

$$\mathcal{W}_s(E) = \gamma M \oint dt H_s [\mathbf{p} \cdot \mathbf{H} - (\mathbf{m} \cdot \mathbf{p})(\mathbf{m} \cdot \mathbf{H})], \quad (3)$$

$$\mathcal{W}_\alpha(E) = -\alpha \gamma M \oint dt [\mathbf{H}^2 - (\mathbf{m} \cdot \mathbf{H})^2]. \quad (4)$$

The time integral is over a precession period on a constant energy curve. We emphasize that Eqs. (3) and (4) are functions of the energy density E . We denote the minimum and maximum values of E as E_{\min} and E_{\max} , respectively. When the energy density also has saddle points E_{saddle} , E_{\max} in the following discussion can be replaced by E_{saddle} . To excite the self-oscillation, there should be a certain value of the electric current density that satisfies Eq. (2) for $E_{\min} < E < E_{\max}$ in a set of real numbers. Therefore, Eq. (2) can be rewritten as

$$\exists j \in \mathbb{R}, \quad \mathcal{W}_s + \mathcal{W}_\alpha = 0. \quad (5)$$

We denote the current satisfying the first condition, Eq. (2), or equivalently Eq. (5), as $j(E)$.

Another condition necessary to excite self-oscillation relates to the fact that the magnetization initially stays at the minimum energy state. To excite any kind of magnetization dynamics, the spin torque should destabilize the initial state, which means that a current density larger than the critical current density $j_c = j(E_{\min})$ should be injected. Then, the condition

$$j(E) > j(E_{\min}) \quad (6)$$

should be satisfied to excite the self-oscillation. If this condition is not satisfied, the magnetization directly moves to a constant energy curve including the saddle point without showing a stable steady precession, and stops dynamics because the spin torque does not balance the damping torque for $E_{\min} < E < E_{\text{saddle}}$. An example of such dynamics is shown below; see Fig. 3. We emphasize that Eqs. (5) and (6) are applicable to any kind of physical system showing a self-oscillation.

III. SPIN HALL SYSTEM

Let us apply the above discussions to the spin Hall system schematically shown in Fig. 1(a), where the electric current flows in the nonmagnet along the x direction, whereas the ferromagnet is attached along the z direction. The spin polarization of the spin current is geometrically determined

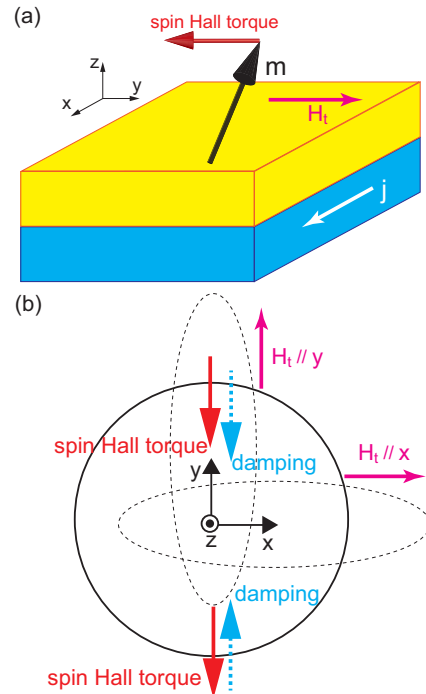


FIG. 1. (Color online) (a) Schematic view of system. The current density j flows in the nonmagnet along the x axis, exciting the spin Hall torque pointing in the y direction on the magnetization \mathbf{m} in the ferromagnet. The applied magnetic field is denoted as H_t . (b) Schematic view of the precession trajectory of the magnetization on the constant energy curve. The solid circle is the trajectory in the absence of the magnetic field or in the presence of the field along the z axis, whereas the dashed elliptical lines are those in the presence of the field in the x and y axes. The solid and dotted arrows represent the directions of the spin Hall torque and the damping torque, respectively.

as $\mathbf{p} = \mathbf{e}_y$. In the spin Hall system, the spin torque strength H_s is given by

$$H_s = \frac{\hbar \vartheta j}{2eMd}, \quad (7)$$

where ϑ and d are the spin Hall angle and the thickness of the ferromagnet, respectively. The magnetic field \mathbf{H} consists of the applied field H_t and the perpendicular anisotropy field $H_K m_z \mathbf{e}_z$. We can assume that $H_t > 0$ without losing generality because the sign of H_t only affects the sign of $j(E)$ derived below. Since we are interested in a perpendicular ferromagnet, we assume that $H_K > H_t > 0$. Figure 1(b) schematically shows the precession trajectory of the magnetization on a constant energy curve, where the directions of the spin Hall torque and the damping torque are represented by the solid and dotted arrows, respectively. The spin Hall torque is parallel to the damping torque for $m_y > 0$, whereas it is antiparallel to the damping torque for $m_y < 0$. This means that the spin Hall torque dissipates energy from the ferromagnet when $m_y > 0$, and supplies the energy to the ferromagnet when $m_y < 0$. Then, due to the symmetry of the trajectory, the net energy supplied by the spin Hall torque \mathcal{W}_s is zero when the applied magnetic field points to the x or z direction. This means that Eq. (2) cannot be satisfied, and thus, self-oscillation cannot be

excited in the spin Hall system in the absence of the applied magnetic field, or in the presence of the field pointing in the x or z direction. Therefore, in the following we focus on the applied magnetic field pointing in the y direction. The magnetic field and the energy density are given by

$$\mathbf{H} = H_t \mathbf{e}_y + H_K m_z \mathbf{e}_z, \quad (8)$$

$$E = -MH_t m_y - \frac{MH_K}{2} m_z^2. \quad (9)$$

The minimum energy of Eq. (9) is

$$E_{\min} = -\frac{MH_K}{2} \left[1 + \left(\frac{H_t}{H_K} \right)^2 \right], \quad (10)$$

which corresponds to a point $\mathbf{m}^{\text{stable}} = (0, H_t/H_K, \sqrt{1 - (H_t/H_K)^2})$. On the other hand, Eq. (9) has a saddle point at $\mathbf{m}^{\text{saddle}} = (0, 1, 0)$, corresponding to the energy density

$$E_{\text{saddle}} = -MH_t. \quad (11)$$

Since the magnetization initially stays at the minimum energy state, and the magnetization dynamics stops when \mathbf{m} reaches the saddle point $\mathbf{m}^{\text{saddle}}$, we consider the energy region of $E_{\min} < E < E_{\text{saddle}}$. To calculate Eqs. (3) and (4), it is necessary to solve a nonlinear equation $d\mathbf{m}/dt = -\gamma \mathbf{m} \times \mathbf{H}$, which determines the precession trajectory of \mathbf{m} on the constant energy curve. Since the constant energy curve of Eq. (9) is symmetric with respect to the yz plane, it is sufficient for the calculation of Eqs. (3) and (4) to derive the solutions of \mathbf{m} for half of the trajectory in the region of $m_x > 0$, which are exactly given by

$$m_x(E) = (r_2 - r_3) \text{sn}(u, k) \text{cn}(u, k), \quad (12)$$

$$m_y(E) = r_3 + (r_2 - r_3) \text{sn}^2(u, k), \quad (13)$$

$$m_z(E) = \sqrt{1 - r_3^2 - (r_2^2 - r_3^2) \text{sn}^2(u, k)}, \quad (14)$$

where $u = \gamma \sqrt{H_t H_K / 2} \sqrt{r_1 - r_3} t$, and r_ℓ are given by

$$r_1(E) = -\frac{E}{MH_t}, \quad (15)$$

$$r_2(E) = \frac{H_t}{H_K} + \sqrt{1 + \left(\frac{H_t}{H_K} \right)^2 + \frac{2E}{MH_K}}, \quad (16)$$

$$r_3(E) = \frac{H_t}{H_K} - \sqrt{1 + \left(\frac{H_t}{H_K} \right)^2 + \frac{2E}{MH_K}}. \quad (17)$$

The modulus of Jacobi elliptic functions, $\text{sn}(u, k)$ and $\text{cn}(u, k)$, is

$$k = \sqrt{\frac{r_2 - r_3}{r_1 - r_3}}. \quad (18)$$

The derivations of Eqs. (12), (13), and (14) are shown in Appendix A. The precession period is

$$\tau(E) = \frac{2\mathbf{K}(k)}{\gamma \sqrt{H_t H_K / 2} \sqrt{r_1 - r_3}}, \quad (19)$$

where $\mathbf{K}(k)$ is the first kind of complete elliptic integral. The work done by spin torque and the dissipation due to damping, \mathcal{W}_s and \mathcal{W}_α , are obtained by substituting Eqs. (12), (13), and (14) into Eqs. (3) and (4), integrating over $[0, \tau/2]$, and multiplying a numerical factor 2 because Eqs. (12), (13), and (14) are the solution of the precession trajectory for a half-period. Then, \mathcal{W}_s and \mathcal{W}_α for $E_{\min} < E < E_{\text{saddle}}$ are exactly given by

$$\mathcal{W}_s = \frac{8MH_s \sqrt{r_1 - r_3}}{3H_t \sqrt{H_K} (2H_t)} \mathcal{H}_s, \quad (20)$$

$$\mathcal{W}_\alpha = -\frac{4\alpha M \sqrt{r_1 - r_3}}{3\sqrt{H_K} (2H_t)} \mathcal{H}_\alpha, \quad (21)$$

where \mathcal{H}_s and \mathcal{H}_α are given by

$$\mathcal{H}_s = H_t \left(\frac{1 - r_1^2}{r_1 - r_3} \right) \mathbf{K}(k) - \left(\frac{E}{M} + \frac{H_t^2}{H_K} \right) \mathbf{E}(k), \quad (22)$$

$$\mathcal{H}_\alpha = H_t \left(\frac{1 - r_1^2}{r_1 - r_3} \right) \mathbf{K}(k) + \left(\frac{5E}{M} + 3H_K + \frac{2H_t^2}{H_K} \right) \mathbf{E}(k). \quad (23)$$

Here $\mathbf{E}(k)$ is the second kind of complete elliptic integral. The derivations of Eqs. (20) and (21) are shown in Appendix B. The current $j(E)$ for $E_{\min} < E < E_{\text{saddle}}$ is given by

$$j(E) = \frac{2\alpha e M d}{\hbar \vartheta} \frac{H_t \mathcal{H}_\alpha}{2\mathcal{H}_s}. \quad (24)$$

The currents for $E \rightarrow E_{\min}$ and $E \rightarrow E_{\text{saddle}}$ are [41]

$$j(E_{\min}) = \frac{2\alpha e M d}{\hbar \vartheta} \frac{H_K}{H_t / H_K} \left[1 - \frac{1}{2} \left(\frac{H_t}{H_K} \right)^2 \right], \quad (25)$$

$$j(E_{\text{saddle}}) = \frac{2\alpha e M d}{\hbar \vartheta} \left(\frac{3H_K - 2H_t}{2} \right). \quad (26)$$

Equation (24) is the current density satisfying Eq. (2), or equivalently Eq. (5). Then, let us investigate whether Eq. (24) satisfies Eq. (6). It is mathematically difficult to calculate the derivative of Eq. (24) with respect to E for an arbitrary value of E , although we can confirm that $j(E_{\min}) > j(E_{\text{saddle}})$ for $H_t < H_K$. We note that a parameter determining whether Eq. (6) is satisfied is only H_t/H_K because the other parameters, such as α and M , are just common prefactors for any $j(E)$. As shown in Fig. 2, $j(E)$ is a monotonically decreasing function of E for a wide range of H_t/H_K , i.e., Eq. (6) is not satisfied. This result indicates that the magnetization stays in the equilibrium state when $j < j_c = j(E_{\min})$, whereas it moves to the constant energy curve of E_{saddle} without showing stable self-oscillation when $j > j_c$ because the spin Hall torque does not balance the damping torque on any constant energy curve between E_{\min} and E_{saddle} . The magnetization finally stops its dynamics at $\pm \mathbf{m}^{\text{saddle}}$ because all torques become zero at these points. Figure 3 shows a typical example of such dynamics, in which the time evolution of each component is shown. Therefore, self-oscillation solely by the spin Hall torque cannot be excited in the perpendicular ferromagnet. This is a possible reason why the self-oscillation has not been reported yet.

Recently many kinds of other torques pointing in different directions or having different angular dependencies,

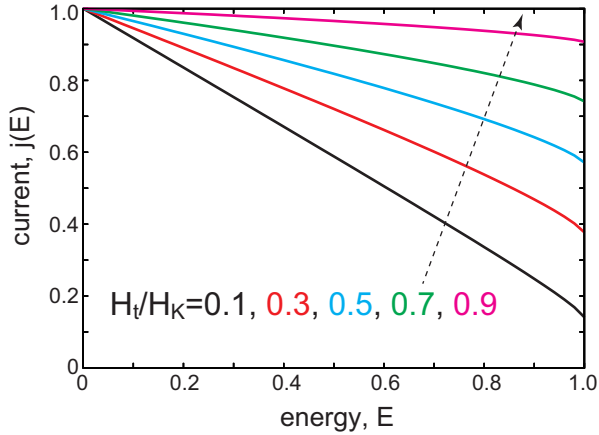


FIG. 2. (Color online) The dependence of the current $j(E)$, Eq. (24), for several values of H_t/H_K on the energy density E . For simplicity, the horizontal and vertical axes are normalized as $j(E)/j_c$ and $E/(E_{\text{saddle}} - E_{\text{min}}) - [E_{\text{min}}/(E_{\text{saddle}} - E_{\text{min}})]$ to make $j(E_{\text{min}}) = 1$, $E_{\text{min}} = 0$, and $E_{\text{saddle}} = 1$.

such as fieldlike and Rashba torques, have been proposed [28,29,36,37,40,43–45]. These effects might change the above conclusions. Adding of an in-plane anisotropy [21,22], tilting the perpendicular anisotropy [40], or using higher order anisotropy might be another candidate. Spin pumping is also an interesting phenomenon because it modifies the Gilbert damping constant [46–49]. It was shown in Refs. [48,50] that the enhancement of the Gilbert damping constant in a ferromagnetic/nonmagnetic/ferromagnetic trilayer system depends on the relative angle of the magnetization. This means that the Gilbert damping constant has an angular dependence. In such a case, it might be possible to satisfy Eqs. (5) and (6) by attaching another ferromagnet to the spin Hall system and by choosing an appropriate alignment of the magnetizations. The above formulas also apply to these studies. In Appendix C we briefly discuss a technical difficulty to include the effect of the fieldlike torque or Rashba torque.

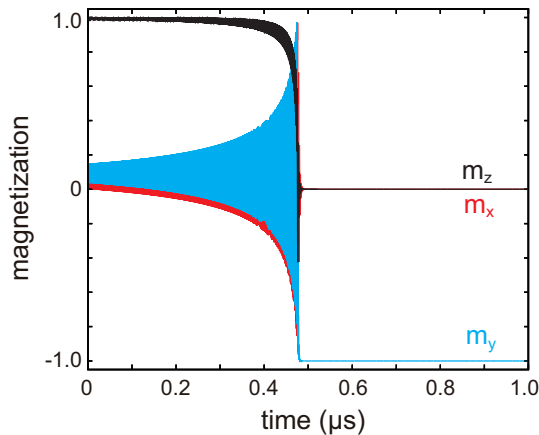


FIG. 3. (Color online) Typical magnetization dynamics excited by the spin Hall effect. The parameter values are taken from experiments [36–38,42] as $M = 1500$ emu/cc, $H_K = 540$ Oe, $\alpha = 0.005$, $\gamma = 1.764 \times 10^7$ rad/(Oe s), $d = 1$ nm, $\vartheta = 0.1$, and $H_t = 50$ Oe. The current magnitude is 14×10^6 A/cm², while the critical current, Eq. (25), is 13×10^6 A/cm².

IV. CONCLUSION

In conclusion, we developed a method for the nonlinear analysis of the LLG equation in the spin Hall system with a perpendicular ferromagnet. We summarized physical conditions to excite self-oscillation by the spin transfer effect. The first condition, Eq. (2), or equivalently Eq. (5), implies that the energy supplied by the spin torque during a precession on a constant energy curve should equal the dissipation due to damping. The second condition, Eq. (6), implies that the current to balance the spin torque and the damping torque in the self-oscillation state should be larger than the critical current to destabilize the initial state. By solving the nonlinear LLG equation, we derived exact solutions of the energy supplied by the spin transfer effect and the dissipation due to damping, and showed that these conditions are not satisfied. These results indicate that self-oscillation cannot be excited solely by the spin Hall torque.

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APPENDIX A: PRECESSION TRAJECTORY ON A CONSTANT ENERGY CURVE

Here we show the derivation of Eqs. (12), (13), and (14). The precession trajectory on a constant energy curve is determined by $d\mathbf{m}/dt = -\gamma\mathbf{m} \times \mathbf{H}$. The y component of this equation is $dm_y/dt = \gamma H_K m_x m_z$. Thus we find

$$\int dt = \frac{1}{\gamma H_K} \int \frac{dm_y}{m_x m_z}. \quad (\text{A1})$$

As mentioned in Sec. III, since the constant energy curve of Eq. (9) is symmetric with respect to the yz plane, it is sufficient to derive the solutions of \mathbf{m} for half of the trajectory in the region of $m_x > 0$. Using E and m_y , m_x , and m_z are expressed as

$$m_x = \sqrt{1 - m_y^2 + \frac{2E}{MH_K} + \frac{2H_t}{H_K} m_y}, \quad (\text{A2})$$

$$m_z = \sqrt{-\frac{2E}{MH_K} - \frac{2H_t}{H_K} m_y}. \quad (\text{A3})$$

The initial state of m_y is chosen as $m_y(0) = r_3$, where r_3 is given by Eq. (17). Then, m_y at a certain time t is determined from Eq. (A1) as

$$\begin{aligned} & \gamma \sqrt{2H_t H_K} \int_0^t dt \\ &= \int_{r_3}^{m_y} \frac{dm'_y}{\sqrt{(m'_y - r_1)(m'_y - r_2)(m'_y - r_3)}}. \end{aligned} \quad (\text{A4})$$

We introduce a new parameter s as $m_y = r_3 + (r_2 - r_3)s^2$. Then we find

$$\gamma \sqrt{\frac{H_t H_K}{2}} \sqrt{r_1 - r_3} t = \int_0^s \frac{ds'}{\sqrt{(1 - s'^2)(1 - k^2 s'^2)}}, \quad (\text{A5})$$

where the modulus k is given by Eq. (18). The solution of s is $s = \text{sn}(u, k)$. Therefore, m_y is given by Eq. (13). Equations (12) and (14) are obtained by substituting Eq. (13) into Eqs. (A2) and (A3).

We note that Eqs. (12), (13), and (14) are periodic functions with the period given by Eq. (19). On the other hand, when $E = E_{\text{saddle}}$, the magnetization stops its dynamics finally at the saddle point $\mathbf{m} = (0, 1, 0)$. The solution of the constant energy curve of E_{saddle} with the initial condition $m_y(0) = r_3$ can be obtained by similar calculations, and are given by

$$m_x = 2 \left(1 - \frac{H_t}{H_K} \right) \frac{\tanh(vt)}{\cosh(vt)}, \quad (\text{A6})$$

$$m_y = -1 + \frac{2H_t}{H_K} + 2 \left(1 - \frac{H_t}{H_K} \right) \tanh^2(vt), \quad (\text{A7})$$

$$m_z = 2 \sqrt{\frac{H_t}{H_K} \left(1 - \frac{H_t}{H_K} \right)} \frac{1}{\cosh(vt)}, \quad (\text{A8})$$

where $v = \gamma \sqrt{H_t(H_K - H_t)}$.

APPENDIX B: DERIVATION OF EQS. (20) AND (21)

Using Eqs. (12), (13), and (14), the explicit form of Eq. (3) for the spin Hall system is given by $\mathcal{W}_s = \gamma M H_s \int dt w_s$, where w_s is given by

$$\begin{aligned} w_s = & (H_t - H_K r_3) (1 - r_3^2) \\ & + \{-2H_t r_3 + H_K [r_3(r_2 + r_3) - (1 - r_3^2)]\} \\ & \times (r_2 - r_3) \text{sn}^2(u, k) \\ & + \{-H_t + H_K(r_2 + r_3)\} (r_2 - r_3)^2 \text{sn}^4(u, k). \end{aligned} \quad (\text{B1})$$

Similarly, Eq. (21) for the spin Hall system is given by $\mathcal{W}_\alpha = -\alpha \gamma M \int dt w_\alpha$, where w_α is given by

$$\begin{aligned} w_\alpha = & (1 - r_3^2)(H_t - H_K r_3)^2 \\ & - [2H_t^2 r_3 - H_K^2(r_2 + r_3)(1 - 2r_3^2) \\ & + 2H_t H_K(1 - r_2 r_3 - 2r_3^2)] \\ & \times (r_2 - r_3) \text{sn}^2(u, k) \\ & - [H_t - H_K(r_2 + r_3)]^2 (r_2 - r_3)^2 \text{sn}^4(u, k). \end{aligned} \quad (\text{B2})$$

Then \mathcal{W}_s and \mathcal{W}_α are obtained by integrating over $[0, \tau/2]$, and multiplying a numerical factor 2. The following integral formulas are useful:

$$\int^u du' \text{sn}^2(u', k) = \frac{u - \text{E}[\text{am}(u, k), k]}{k^2}, \quad (\text{B3})$$

$$\begin{aligned} \int^u du' \text{sn}^4(u', k) = & \frac{\text{sn}(u, k) \text{cn}(u, k) \text{dn}(u, k)}{3k^2} \\ & + \frac{2 + k^2}{3k^4} u \\ & - \frac{2(1 + k^2)}{3k^4} \text{E}[\text{am}(u, k), k], \end{aligned} \quad (\text{B4})$$

where $\text{E}(u, k)$, $\text{am}(u, k)$, and $\text{dn}(u, k)$ are the second kind of incomplete elliptic integral, Jacobi amplitude function, and Jacobi elliptic function, respectively.

APPENDIX C: THE EFFECT OF THE FIELDLIKE TORQUE OR RASHBA TORQUE

The direction of the fieldlike torque or the Rashba torque is given by $\mathbf{m} \times \mathbf{p}$, where \mathbf{p} is the direction of the spin polarization. This means that the effects of these torques can be regarded as a normalization of the field torque $\mathbf{m} \times \mathbf{H}$. Then the energy density E and the magnetic field \mathbf{H} in the calculations of \mathcal{W}_s and \mathcal{W}_α should be replaced with an effective energy density \mathcal{E} and an effective field \mathcal{B} given by

$$\mathcal{E} = E - \beta M H_s \mathbf{m} \cdot \mathbf{p}, \quad (\text{C1})$$

$$\mathcal{B} = \mathbf{H} + \beta H_s \mathbf{p}, \quad (\text{C2})$$

where a dimensionless parameter β characterizes the ratio of the fieldlike torque or Rashba torque to the spin Hall torque. We neglect higher order terms of the torque [44] for simplicity, because these do not change the main discussion here. In principle, $j(E)$ satisfying Eq. (5) can be obtained by a similar calculation shown in Sec. III. However, for example, the right-hand-side of Eq. (24) now depends on the current through \mathcal{E} and \mathbf{H} . Thus, Eq. (24) should be solved self-consistently with respect to the current j , which is technically difficult.

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