

# Admittance of a long diffusive SNS junction

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The dynamical properties of hybrid normal metal/superconductor structures have recently come into research focus both experimentally and theoretically. Recent experimental studies of the coherent admittance  $Y(\omega)$  of SNS rings as a function of the phase difference  $\phi_0$  are still not fully understood. Here we concentrate on the linear response regime, calculating  $Y(\omega)$  by solving the Usadel equations, linearized in an electric field. Partially reproducing previously known results, we find qualitatively different behavior in the collisionless regime of  $\tau_{\text{in}}^{-1} \ll \omega \lesssim E_{\text{Th}}$  and high temperature  $T \gg E_{\text{Th}}$  and low temperature  $T \lesssim E_{\text{Th}}$  near the minigap closing  $\phi_0 \sim \pi$ . We find that the dissipative part  $\text{Re}Y(\omega)$  peaks when the minigap closes (at a phase difference of  $\pi$ ) even at high temperatures, when the equilibrium supercurrent is fully suppressed.

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## I. INTRODUCTION

The superconducting proximity effect on the transport properties of normal metal/superconductor structures has been thoroughly studied both theoretically and experimentally [1]. Most of the studies were concentrated on properties of these systems in equilibrium [2,3] or stationary nonequilibrium situations [4,5]. Recently, one of the most basic quantities, characterizing dynamical properties of such structures [the admittance  $Y(\omega) = I(\omega)/V(\omega)$ ], acquired more attention [6–12]. It characterizes the current response  $I(t) = \int (d\omega) I(\omega) e^{-i\omega t}$  to an ac voltage  $V(t) = \int (d\omega) V(\omega) e^{-i\omega t}$  in the linear response regime.

The problem of calculating the current in the tunneling (SIS) junction has been solved long ago for arbitrary time-dependent voltage  $V(t)$  [14]. The phase dynamics of such a junction, coupled to the electromagnetic environment, can usually be described by the RSJC model [15]. The same problem for a superconductor–normal metal–superconductor (SNS) junction is much more complicated, since the ac dynamics of the phase interferes here with the dynamics of the electrons in the normal metal. This introduces two distinct energy scales into the problem: inverse diffusion time over the junction of the length  $L$ ,  $E_{\text{Th}} \equiv \tau_D^{-1} = \frac{\hbar D}{L^2}$  with  $D$  for diffusion constant, and inelastic scattering rate  $\tau_{\text{in}}^{-1}$ . Additionally, multiple Andreev reflections are very important in such junctions, producing highly nontrivial energy distribution of the electrons in the wire [16–18], but they are not essential in the regime of the small voltage which we concentrate on.

As follows, superconducting proximity effect causes  $Y$  to be different from  $Y_N = 1/R_N$ , the admittance of the wire in the normal state. Due to the Josephson relation  $\dot{\phi} = 2eV/\hbar$ , the admittance can be related to the linear susceptibility of the junction with respect to the oscillating superconducting phase difference. In the geometry of an SNS ring, the phase difference  $\phi = -2\pi \frac{\Phi}{\Phi_0}$  (where  $\Phi_0 = h/2e$  is the flux quantum) is controlled by magnetic flux  $\Phi$  penetrating this ring. The corresponding response function  $\chi(\omega) = \frac{\delta I}{\delta \Phi}$  can be directly

measured, and is related to  $Y(\omega)$  as follows:

$$\chi(\omega) = i\omega Y(\omega). \quad (1)$$

In practice, the measurement of  $\chi$  can be conducted by exposing the SNS ring to a weak magnetic field  $B(t) = B_0 + B_{\text{osc}} e^{-i\omega t}$ . While  $B_0$  fixes the stationary part of the superconducting phase difference along the normal wire  $\phi_0 = -2\pi \frac{\Phi_0}{\Phi_0}$ ,  $B_{\text{osc}}$  generates an e.m.f.  $\mathcal{E}(t) = -\frac{d\Phi_{\text{osc}}}{dt}$ , generating an ac electric current.

In the static limit ( $\omega \rightarrow 0$ ),  $\mathcal{E}(t)$  is absent and the equilibrium response function  $\chi(0)$  is recovered:

$$\chi(\omega = 0) = -\frac{2\pi}{\Phi_0} \partial_{\phi} I_S(\phi_0), \quad (2)$$

where  $I_S(\phi_0)$  stands for the current-phase relation of the junction in equilibrium. At finite frequency, the effect of  $\mathcal{E}(t)$  is to both modify the nondissipative response and to generate dissipation in the normal wire.

It can be expected that there should exist a limit, in which the admittance  $Y$  of the junction equals the admittance of two SN junctions connected in series. In this *incoherent* limit,  $Y$  is  $\phi_0$  independent. As we will demonstrate shortly, it is achieved only when frequency is large  $\omega \gg E_{\text{Th}}$ . Peculiarly enough, largeness of temperature alone [ $T \gg E_{\text{Th}}$ ] is not enough to destroy the coherence: at moderate frequency  $\omega \lesssim E_{\text{Th}}$  the admittance has noticeable  $\phi_0$ -dependent contribution even at large temperature. This contribution will be the focus of our discussion. Another interesting point is that by measuring the response of the wire to  $\mathcal{E}$  as a function of frequency and dc phase difference  $\phi_0$  [11,12], one infers the dynamical properties of the Andreev levels in the junction through their effect on the conductive properties of the normal wire. This effect is thus very sensitive to inelastic processes in the wire and can be used as a specific probe.

From the theoretical side, the first study of the coherent contribution to the impedance of an SNS bridge was performed by Lempitskii [19]. He considered the long junction limit ( $E_{\text{Th}} \ll \Delta$ ) at high temperature ( $T \gg E_{\text{Th}}$ ), biased in the *adiabatic* regime ( $\hbar\omega \ll E_{\text{Th}}$ ), and obtained the following

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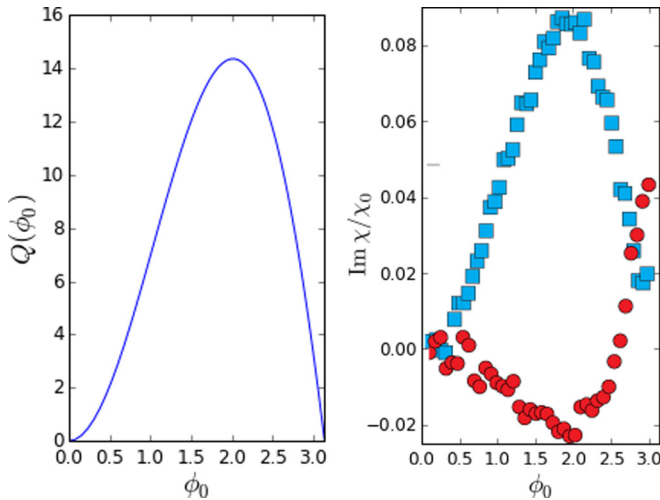


FIG. 1. (Color online) Left: Function  $Q(\phi_0)$  entering Lempitskii's prediction for the coherent part of the dissipation. Right: Dissipative response at  $T \approx 17E_{\text{Th}}$ , data from Ref. [13]. Blue curve: Hydrodynamic regime,  $\hbar\omega = 0.4E_{\text{Th}}$ , red curve: collisionless regime,  $\hbar\omega = 2E_{\text{Th}}$  (arbitrary shifted in the  $c$  axis).

result:

$$Y(\omega) = Y_N \frac{E_{\text{Th}}}{T} \frac{E_{\text{Th}}/\hbar}{\tau_{\text{in}}^{-1} - i\omega} Q(\phi_0), \quad (3)$$

with  $\tau_{\text{in}}$  staying for the inelastic relaxation time and universal function  $Q(\phi_0)$  evaluated numerically. This function has recently been recalculated [9] with better precision, see Fig. 1 for the result.

Nontrivial phase dependence of admittance [Eq. (3)] on phase difference, persisting to large temperature, results from supercurrent-enhanced nonequilibrium population of Andreev levels in the wire. The most fascinating result is that this nonequilibrium population causes the coherent part of  $Y$  to decay (at given  $\tau_{\text{in}}$ ) slowly, as  $E_{\text{Th}}/T$ , at  $T \gg E_{\text{Th}}$ , whereas the equilibrium supercurrent decays exponentially, as  $\propto \exp(-L/L_T)$  with thermal length  $L_T = \sqrt{\hbar D/k_B T}$ . This nonequilibrium enhancement of the superconducting correlations recalls the well known effect of the microwaves enhancement of superconductivity, the phenomena, known as the Dayem-Wyatt effect [20–22], which is observed in microbridges, thin films, and stripes [23–27]. Similar effects exist in the hybrid structures [28,29], but their physics is enriched by the existence of two different time scales: time of diffusion along the normal part  $\tau_D$  and inelastic scattering rate  $\tau_{\text{in}}$ , as was clearly demonstrated recently [30]. In our work we concentrate on how this rich physics shows itself in the linear response function  $Y$ .

Since Lempitskii's work, there has not much theoretical activity on the coherent contribution to  $Y$  with notable exceptions provided by [6,7,31,32]. However, the recent experiments motivated a series of theoretical studies [8–10]. In particular, extensive numerical work [8,9], supported by qualitative analytical treatment, was devoted to study  $Y$  in a wide range of temperatures and frequencies.

Detailed comparison of the existent theoretical predictions to the experimental results was performed in Ref. [13]. It was found that the nondissipative response  $\text{Im}Y$  of the junction can

be well understood on the basis of Lempitskii's theory for all moderate frequencies:  $\hbar\omega \lesssim E_{\text{Th}}$  (in that experiment,  $E_{\text{Th}} = 71$  mK, corresponding to the frequency  $f_{\text{Th}} = 1.5$  GHz). Interestingly, experimental results demonstrate that it is possible to follow the response function while it crosses over from a hydrodynamic ( $\omega\tau_{\text{in}} \ll 1$ ) to collisionless ( $\omega\tau_{\text{in}} \gg 1$ ) regime, and extract inelastic scattering rate  $\tau_{\text{in}}^{-1}$  as a function of temperature. At the highest temperature studied,  $T \approx 1.2$  K, it was found that  $\tau_{\text{in}} \approx 2.5\tau_D$ . Interestingly, the scattering rate, found in this experiment, demonstrates unusual temperature dependence  $\tau_{\text{in}}^{-1} \propto T^2$ . We are not aware of any physical mechanism which can lead to such a dependence in a normal gold wire and believe that this power law is specific to the wire in the conditions of the proximity effect. We expect that it is related to the strong modification of the electronic spectrum in the wire by the superconducting contacts, which should influence electron-electron scattering processes—the effect which certainly deserves future studies.

Experimental results for hydrodynamic and collisionless regimes are presented in Fig. 1. In this figure, phase susceptibility is measured in natural units of  $\chi_0 = Y_N E_{\text{Th}}/\hbar$ . Theoretical prediction, obtained from Eqs. (1) and (3), gives

$$\chi/\chi_0 = \frac{E_{\text{Th}}}{T} \frac{i\omega}{\tau_{\text{in}}^{-1} - i\omega} Q(\phi_0). \quad (4)$$

Figure 1 illustrates one of the most important experimental observations: while at low frequency  $\text{Im}\chi^{\text{expt}}$  fits well with Lempitskii's result [see Fig. 1 for  $Q(\phi_0)$ ], at higher frequency ( $\omega\tau_{\text{in}} > 1$ ) the dissipative response has a very different shape as a function of dc phase bias  $\phi_0$ . Recall that Eq. (4) results from the adiabatic calculation, which assumes  $\omega \ll E_{\text{Th}}$ . One may hope that the full numerical calculation (the one not relying on the expansion in  $\omega/E_{\text{Th}}$ ) in this regime can describe the experimental results. Such a calculation, performed in Ref. [9], implies that the peak in  $\text{Im}\chi^{\text{expt}}$  at minigap closing  $\phi_0 = \pi$  should be absent at  $T \gg E_{\text{Th}}$ , in clear contradiction with experiment.

This contradiction motivated a linear response analysis on the basis of BdG equations [10]. The results of the latter study seem to indicate qualitatively the presence of a maximum at  $\phi_0 = \pi$ . From the theoretical side, it is clear that Lempitskii's prediction concerns deviation from equilibrium of the Andreev pairs only, but quasiparticle excitations in the normal region can also be relevant, especially at not too low a frequency. While it is clear that dissipation due to quasiparticles, excited by electric field, should be sharply peaked at  $\phi_0 = \pi$  at low temperatures  $T \ll E_{\text{Th}}$ , the fate of this peak at high temperature is not obvious *a priori*. As we mentioned above, it was predicted in Ref. [9] that at high temperature this peak should disappear. Our goal is to reconsider this problem and to resolve the apparent contradiction between numerical results and experimental data in this regime.

We start with a simple observation: For the dissipative response the condition for validity of the adiabatic approximation is more stringent than for the nondissipative one. Since the adiabatic contribution to  $\text{Im}\chi$  decreases for  $\omega \gtrsim \tau_{\text{in}}^{-1}$ , the nonadiabatic (proportional to  $\frac{\hbar\omega}{E_{\text{Th}}}$ ) correction to Eq. (3) becomes essential already at  $\hbar\omega \sim \sqrt{\hbar E_{\text{Th}}/\tau_{\text{in}}} \ll E_{\text{Th}}$ . As we will show, the terms of the order of  $\hbar\omega/E_{\text{Th}}$  result from the charge imbalance (induced by the ac electric field) and lead to the enhancement of dissipation at  $\phi_0 \approx \pm\pi$ .

Our approach is based on the Usadel equation, expanded to first order in the electric field, without assuming smallness of the proximity effect (in particular, we take into account all non-perturbative effects, such as the minigap). Although it is impossible to get a response function  $Y(\omega, \phi_0)$  in closed form even in the simplest limiting cases, we go as far as possible analytically, resorting to numerical calculation only at the latest stage, which makes our calculation more controllable than fully numerical solution of the time-dependent Usadel equation.

## II. USADEL EQUATION AND LINEAR RESPONSE

### A. General equations

In what follows we make several additional simplifying assumptions: (i) we treat the system as quasi-one-dimensional, (ii) we treat electron-electron interaction in the wire in the relaxation time approximation, neglecting possible energy and position dependence of the relaxation time as well as its modification by the proximity effect, and (iii) we assume that  $\Delta/E_{\text{Th}} \gg 1$ . We measure the energy in units of  $E_{\text{Th}}$  and length in units of  $L$ . Our starting point is the Usadel equation ( $e = -|e|$ ) in the presence of an electric field. Due to gauge invariance, we are allowed to use scalar potential  $\varphi$  instead of vector potential  $\mathbf{A}$  to define an electric field in our quasi-one-dimensional normal wire:  $E = -\nabla\varphi$ . Then the Usadel equation acquires the form

$$\partial_x(\check{g} \cdot \partial_x \check{g}) + i[\epsilon \hat{\tau}^3, \check{g}] + ie[\varphi, \check{g}] = I_S[\check{g}]. \quad (5)$$

In this equation  $\check{g}(x, t, t')$  is an isotropic part of the quasi-classical Keldysh Green function, which is a matrix in the Nambu-Gorkov space. In terms of this function, the electric current can be expressed as follows ( $S$  stands for the normal wire's cross-section area):

$$I = \frac{\pi \sigma_N S}{4e} \text{tr}[\hat{\tau}^3 \check{j}^K(t, t)], \quad (6)$$

where  $\check{j} = \check{g} \cdot \nabla \check{g}$ .

Neglecting spatial gradients in the superconducting reservoirs, we write for the Green function there:

$$\check{g}_S = \check{S}_\phi \cdot \check{g}_{\text{eq}} \cdot \check{S}_\phi^+, \quad (7)$$

with

$$\check{S}(t, t') = \delta(t - t') e^{i\hat{\tau}_3 e \int^t \varphi(\tau) d\tau}. \quad (8)$$

Here  $\check{g}_{\text{eq}}$  is the equilibrium BCS Green function.

Usadel equation (5) includes spectral and kinetic equations which may be obtained with the use of conventional parametrization  $\hat{g}^K = \hat{g}^R \cdot \hat{H} - \hat{H} \cdot \hat{g}^A$ , where  $\hat{H}$  is a diagonal matrix of distribution functions in the Nambu-Gorkov space. In equilibrium, distribution function equals  $\hat{H} = h(\epsilon) \hat{\tau}_0$ , with  $h(\epsilon) = \tanh \frac{\epsilon}{2T}$ . For retarded Green function the following parametrization is appropriate:

$$\hat{g}_{\text{eq}}^R(\epsilon, x) = \begin{pmatrix} G & F \\ \bar{F} & -G \end{pmatrix}, \quad (9)$$

where  $G = \cosh \theta$ ,  $F = \sinh \theta e^{i\phi}$ , and  $\bar{F} = -\sinh \theta e^{-i\phi}$ . In this parametrization, the spectral angle  $\theta$  satisfies

$$\partial_x^2 \theta + (2i\epsilon - \tau_{\text{in}}^{-1}) \sinh \theta + J^2 \frac{\cosh \theta}{\sinh^3 \theta} = 0, \quad (10)$$

where  $J \equiv J(\epsilon) = i \sinh^2 \theta \partial_x \phi$  is the spectral supercurrent, which is an integral of motion:  $\partial_x J = 0$  and we employed the relaxation time approximation. The boundary conditions for  $\phi$  and  $\theta$  are fixed by the BCS functions.

It is not feasible to write down the solutions of Eq. (10) in a closed analytical form. However, the properties of the solutions are well known and numerical approaches to it are well developed. In order to obtain the solutions, we use a publicly available solver, developed by Virtanen and Heikkilä and described in Ref. [33]. It provides the Green function in the Riccati parametrization, which is related to the trigonometric parametrization by means of the equations presented in Appendix A.

Once the unperturbed solution is found, the effects of the weak electric field can be discussed. In the presence of the oscillating electric potential  $\varphi$ , the Green function becomes time dependent:  $\check{g} = \check{g}_{\text{eq}} + \delta \check{g}$ . The effect of the electric field is twofold. First, it imposes time dependence on the phases of the order parameters in the superconducting contacts, see Eq. (7). This modifies the spectrum of the energy levels in the junction through corrections to retarded and advanced Green functions. Second, it induces interlevel transitions with energy transfer  $\omega$ , changing the populations of these levels through corrections to the distribution function. Contributions of these two types of corrections to the electric current behave very differently at high temperatures: the former decay exponentially  $\propto \exp(-L/L_T)$ , while the latter decreases as a power law by increasing the temperature.

### B. Kinetic corrections

Let us start with a discussion of the correction to the generalized distribution function  $\delta \hat{H}$ . It can be chosen diagonal in the particle-hole space:

$$\delta \hat{H}(\epsilon, \epsilon', x) = [h_L(\epsilon, x) \hat{\tau}_0 + h_T(\epsilon, x) \hat{\tau}_3] \delta(\epsilon - \epsilon' - \omega).$$

In the contacts, the transversal distribution function  $h_T$  is driven out of equilibrium by the time-dependent voltage:

$$h_T(\epsilon, x = 0, 1) = h_{T,0}(\epsilon, x = 0, 1). \quad (11)$$

The function  $h_T(\epsilon, x)$  describes the charge imbalance that is induced in the N region due to an oscillating electric field.

The longitudinal distribution function  $h_L(\epsilon, x)$  describes all deviations from the equilibrium Fermi distribution function  $h(\epsilon)$ , which are related by nonequilibrium in energy distribution, but without any charge imbalance.  $h_L(\epsilon, x)$  remains unperturbed within the linear response regime strictly at the boundaries with both superconductors:

$$h_L(\epsilon, x = 0, 1) = 0, \quad (12)$$

however it varies sharply within a short distance near these boundaries, as will be discussed below.

In the wire  $h_{L,T}$  are governed by conservation laws of energy and charge currents:

$$\partial_x j_L + N(i\omega - \tau_{\text{in}}^{-1}) h_L = 0, \quad (13)$$

$$\partial_x j_T + N(i\omega - \tau_{\text{in}}^{-1}) [h_T - h_{T0}] = 0, \quad (14)$$

where relaxation time approximation is employed. Here

$$h_{T,0}(\epsilon, x) = e\varphi(x) \frac{h(\epsilon - \omega) - h(\epsilon)}{\omega},$$

and  $h(\epsilon) = \tanh(\epsilon/2T)$  is an equilibrium Fermi distribution function. Spatial distribution of the electric potential  $\varphi(x)$  has to be found from the Poisson equation:

$$\Delta\varphi = -\rho, \quad (15)$$

taking into account the fact that the oscillating voltage drop along the wire  $V e^{-i\omega t} = \varphi(0) - \varphi(1)$  is fixed by the applied ac phase modulation. In general, this gives a complicated coupled system of Eqs. (5) and (15) which can be solved iteratively. In general we find that for all frequencies of interest, the corrections to

$$\varphi(x) = \varphi(0) - V x e^{-i\omega t}, \quad (16)$$

which result from the charge redistribution in the wire, do not lead to noticeable modification of the coherent part of the admittance and we neglect them in what follows.

The energy current in Eq. (13) reads

$$j_L = D_L \partial_x h_L - T \partial_x h_T + j h_T, \quad (17)$$

and the charge current is equal to

$$j_T = D_T \partial_x h_T + T \partial_x h_L + j h_L. \quad (18)$$

The transport coefficients, which enter the definitions of the currents  $j_{L,T}$ , have the following physical meaning:  $D_{L,T}$  are diffusion coefficients for energy and charge excitations,  $T$  is responsible for conversion of charge current to energy current and vice versa, while  $N$  plays the role of the DOS of electron excitations. Finally,  $j$  is determined by the spectral supercurrent  $J$ , see Eq. (B4). These quantities are modified compared to their equilibrium values as a result of the time dependence of the electric field, see Appendix B for explicit expressions for them in terms of the unperturbed  $\theta$  and  $\phi$ .

### C. Spectral corrections

Let us now turn to the corrections to the spectral functions  $\delta\hat{g}^{R(A)}$  [we will omit superscripts  $(R, A)$  below, since it cannot lead to any confusion]. Naively, each of these two matrices in the particle/hole space has four components:

$$\delta\hat{g}(\epsilon, \epsilon', x) = \begin{pmatrix} u_{pp}(\epsilon, x) & u_{ph}(\epsilon, x) \\ u_{hp}(\epsilon, x) & u_{hh}(\epsilon, x) \end{pmatrix} \delta(\epsilon - \epsilon' - \omega), \quad (19)$$

but the normalization condition  $\delta(\hat{g} \cdot \hat{g}) = \hat{g} \cdot \delta\hat{g} + \delta\hat{g} \cdot \hat{g} = 0$  allows us to express diagonal components in terms of the off-diagonal ones:

$$\begin{pmatrix} u_{pp}(\epsilon) \\ u_{hh}(\epsilon) \end{pmatrix} = \frac{\hat{M}}{1 - \text{th}^2 \frac{\theta}{2} \text{th}^2 \frac{\theta_-}{2}} \begin{pmatrix} u_{ph}(\epsilon) \\ u_{hp}(\epsilon) \end{pmatrix} \quad (20)$$

with matrix  $M$  given by

$$\hat{M} = \begin{pmatrix} e^{i\phi} \text{th} \frac{\theta}{2} (1 - \text{th}^2 \frac{\theta}{2}) & -e^{i\phi} \text{th} \frac{\theta}{2} (1 - \text{th}^2 \frac{\theta_-}{2}) \\ -e^{-i\phi} \text{th} \frac{\theta}{2} (1 - \text{th}^2 \frac{\theta_-}{2}) & e^{i\phi} \text{th} \frac{\theta_-}{2} (1 - \text{th}^2 \frac{\theta}{2}) \end{pmatrix} \quad (21)$$

and notation  $f_-(\epsilon) = f(\epsilon - \omega)$  is used. Parametrization (20) reduces the number of independent components in  $\delta\hat{g}$  to two:

$u_{ph}, u_{hp}$ . In the contacts, these functions are driven by the time-dependent voltage:

$$u_{ph}(\epsilon, x = 0, 1) = u_{ph,0}(\epsilon, x = 0, 1), \quad (22)$$

$$u_{ph,0}(\epsilon, x) = \frac{e^{i\phi} \sinh \theta + e^{i\phi_-} \sinh \theta_-}{\omega} e\varphi(x). \quad (23)$$

Similar equations valid for  $u_{hp}$  can be obtained from Eqs. (22) and (23) by the replacement  $\phi \rightarrow -\phi$ . In the wire, functions  $u_{ph}(x)$  and  $u_{hp}(x)$  are determined by the conservation laws of the spectral currents, which take the following form:

$$\partial_x j_{ph} + (2i\epsilon - i\omega - \tau_{\text{in}}^{-1}) u_{ph} + i e\varphi [e^{i\phi} \sinh \theta - e^{i\phi_-} \sinh \theta_-] = 0. \quad (24)$$

A similar equation is valid for  $u_{hp}$  and  $j_{hp}$  with substitution  $\phi \rightarrow -\phi$ . The spectral currents read

$$\begin{pmatrix} j_{ph} \\ j_{hp} \end{pmatrix} = \hat{D}_S \begin{pmatrix} \partial_x u_{ph} \\ \partial_x u_{hp} \end{pmatrix} + \hat{J}_S \begin{pmatrix} u_{ph} \\ u_{hp} \end{pmatrix}, \quad (25)$$

where

$$\hat{D}_S = \begin{pmatrix} D_S & e^{i(\phi+\phi_-)} \bar{D}_S \\ e^{-i(\phi+\phi_-)} \bar{D}_S & D_S \end{pmatrix} \quad (26)$$

and

$$\hat{J}_S = \begin{pmatrix} J_S + \frac{J+J_-}{\cosh \theta + \cosh \theta_-} & e^{i(\phi+\phi_-)} \bar{J}_S \\ e^{-i(\phi+\phi_-)} \bar{J}_S & J_S - \frac{J+J_-}{\cosh \theta + \cosh \theta_-} \end{pmatrix}. \quad (27)$$

The spectral transport coefficients  $D_S, \bar{D}_S$  and  $J_S, \bar{J}_S$  which enter these expressions are provided in Appendix B.

## III. RESULTS

We start our presentation from the exemplary results for the distribution functions, which are shown in Figs. 2 and 3. In all the figures, for inelastic rate we have assumed  $\hbar\tau_{\text{in}}^{-1} = T/g$  for definiteness, assuming the standard model for electron-electron inelastic scattering rate [34] (the value of dimensionless film conductance  $g$  is given in the figure captions).

Below we discuss the results for the admittance  $Y/Y_N$ . For comparison with experiment, keep in mind that susceptibility

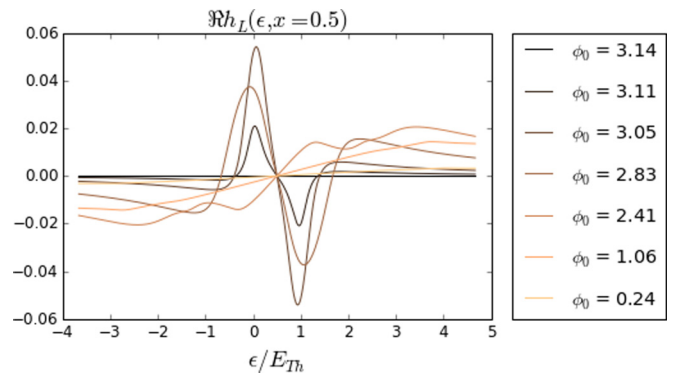


FIG. 2. (Color online) Variation of the longitudinal distribution function at  $T = 15E_{\text{Th}}$ ,  $\hbar\omega = E_{\text{Th}}$  with varying the phase difference  $g = 40$ .

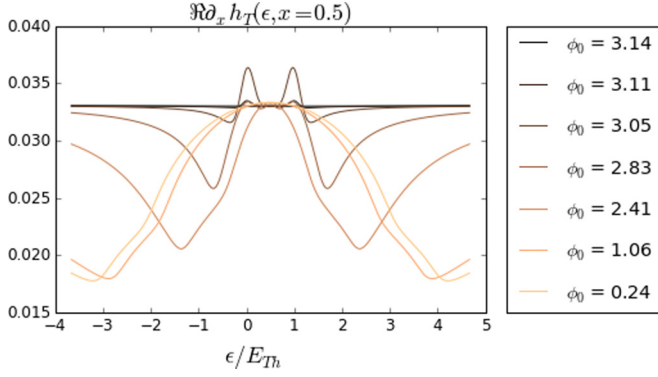


FIG. 3. (Color online) Variation of the transversal distribution function at  $T = 15E_{Th}$ ,  $\hbar\omega = E_{Th}$  with varying the phase difference  $g = 40$ .

to the oscillating phase reads  $\chi/\chi_0 = i \frac{\hbar\omega}{E_{Th}} Y/Y_N$ . For the SNS junction of the experiment, mentioned in the Introduction,  $\chi_0 \approx 35 \mu A/\Phi_0$ .

#### A. High temperature

Let us discuss the variation of the dissipative part of the admittance with frequency at high temperature, see Figs. 4 and 5. As expected, at low frequency, Lempitskii's result is reproduced, see the curve corresponding to  $\omega = 0.1E_{Th}$ . With growth of the frequency, the shape of the curve drastically changes and the dissipative part of  $Y$  acquires a peak at  $\phi_0 = \pi$ , which becomes more prominent with growth of frequency and can be clearly seen up to the largest temperature of  $T = 15E_{Th}$ .

The same kind of evolution of  $\text{Re}Y(\phi_0)/Y_N$  is shown for different temperatures at fixed frequency  $\hbar\omega = E_{Th}$  in Fig. 6. Note the strong peak near the phase equal to  $\pi$  at  $T = 5E_{Th}$ .

In order to understand this result, recall how Eq. (3) was derived. First, we note that at  $T \gg E_{Th}$  the contribution of the spectral corrections  $\delta\hat{g}$  to the electric current can be neglected

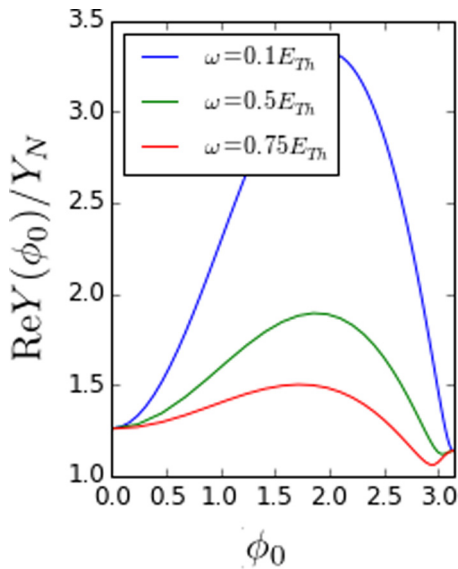


FIG. 4. (Color online) Variation of  $\text{Re}Y(\phi_0)/Y_N$  with frequency at  $T = 15E_{Th}$ ,  $g = 40$ , low frequencies.

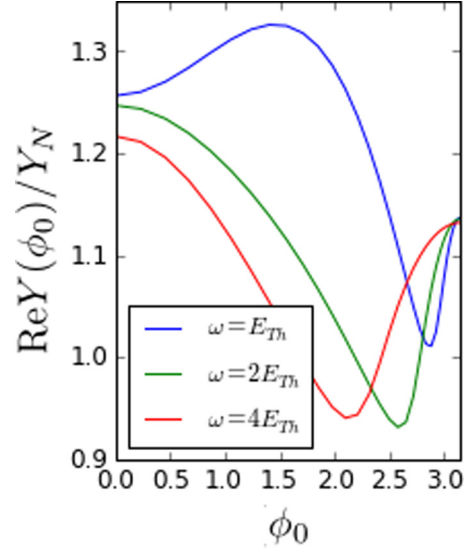


FIG. 5. (Color online) Variation of  $\text{Re}Y(\phi_0)/Y_N$  with frequency at  $T = 15E_{Th}$ ,  $g = 40$ , high frequencies.

and only corrections to the distribution function ( $h_{L,T}$ ) are important. At finite voltage, the charge excitations, described by  $h_T$ , which enter the wire from the superconductor and get converted into energy excitations there, see the last term in Eq. (17). In the limit of  $\hbar\omega/\Delta \ll 1$  the energy excitations [described by  $h_L(\epsilon, x)$ ] are locked in between the superconducting contacts, since the corresponding density of states vanishes at the superconductors. Because of that, a relatively large and almost spatially independent nonequilibrium correction to the longitudinal distribution function  $h_L^{(\text{Lemp.})}(\epsilon) = h_L(\epsilon) - h(\epsilon)$  in the wire is established:

$$h_L^{(\text{Lemp.})} \approx \frac{j}{\langle N \rangle} \frac{eV}{i\omega + \tau_{\text{in}}^{-1}} h'(\epsilon), \quad (28)$$

which contributes to electric current as  $I \propto \int j h_L^{(\text{Lemp.})} d\epsilon$ , leading to Eq. (3). Note that this equation seems to be

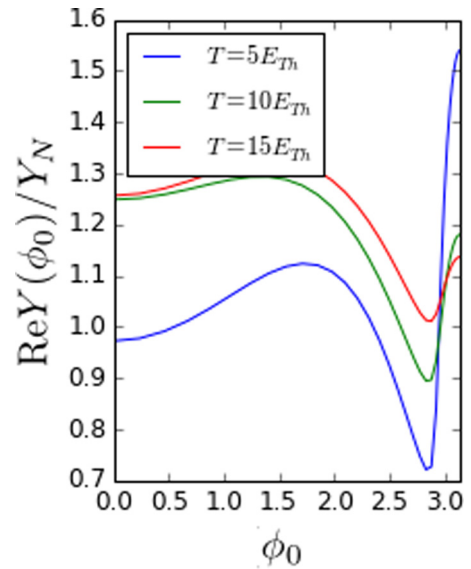


FIG. 6. (Color online) Variation of  $\text{Re}Y(\phi_0)/Y_N$  with temperature at  $\hbar\omega = E_{Th}$ ,  $g = 40$ .

inconsistent with the boundary condition Eq. (12). In fact, in the limit  $E_{\text{Th}} \ll \Delta$  true distribution function differs from  $h_L^{(\text{Lemp.})}$  in the closest vicinity of the boundary, where it exhibits large spatial gradient and sharply varies from  $h_L = 0$  in the superconductor to  $h_L = h_L^{(\text{Lemp.})}$  in the wire. As a consequence, the limit of  $\Delta \rightarrow \infty$  is singular:  $h_L$  has a jump at  $x = 0$ . Expanding the KE in the vicinity of the contact  $x = 0$  we find that Eq. (12) is replaced by an effective boundary condition:

$$\partial_x h_L(\epsilon)|_{x=0} = \frac{\omega}{\zeta(\epsilon)} h_L(\epsilon)|_{x=0}, \quad (29)$$

where  $\zeta(\epsilon) = \partial_x \theta_-^A|_{x=0} - \partial_x \theta_-^R|_{x=0}$ .

It is important that at low frequency  $h_L$  is limited only by inelastic processes: in the limit of  $\tau_{\text{in}} \rightarrow \infty$  one has  $h_L \propto \frac{V}{T} \frac{E_{\text{Th}}}{\hbar\omega}$ . This is why at lowest frequencies the correction to  $h_L(\epsilon)$  leads to the whole effect dominated by Lempitskii's contribution. The properties of the transversal distribution function  $h_T$  are quite different. It describes charge excitations which are free to leave the wire via Andreev reflection, so that corrections to  $h_T$  are relatively small at the lowest frequencies:  $h_T(\epsilon) \propto \frac{V}{T}$ . However, it is clear that at  $\hbar\omega \sim E_{\text{Th}}$  charged excitations described by  $h_T(\epsilon)$  can provide an important contribution to electric current, comparable to that due to excitations of Andreev pairs [described by  $h_L(\epsilon)$ ]. If one is interested in the dissipative part of  $Y$ , the corresponding condition is even more stringent, since the real part of  $h_L$  starts to decay already at  $\hbar\omega \sim \hbar\tau_{\text{in}}^{-1} \ll E_{\text{Th}}$ .

### B. Low temperature

At low temperature, the dissipation is noticeable only in the vicinity of the minigap closing, see Fig. 7. These results are very natural. Indeed, at  $T = 0$  dissipation is nonvanishing only as long as frequency is large enough compared to the minigap  $E_g$ , in particular, at  $\omega = 0$  one has  $\text{Re}Y \propto \delta(\phi_0 - \pi)$ . This peak broadens at finite temperature:  $\delta\phi \propto \hbar\omega, T$ . In addition, at finite  $\omega$  it acquires an additional structure: observe a kink

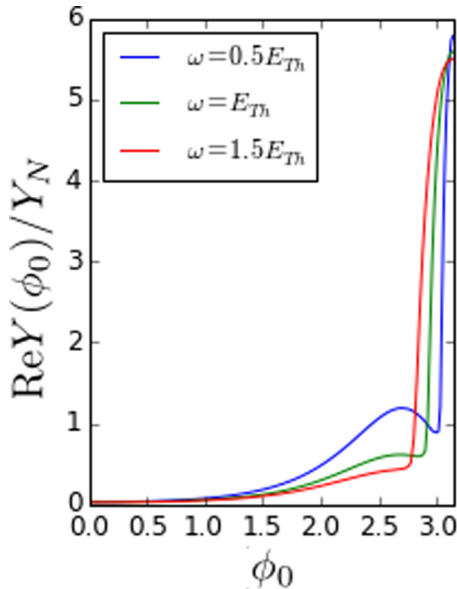


FIG. 7. (Color online) Variation of  $\text{Re}Y(\phi_0)/Y_N$  with frequency at  $T = E_{\text{Th}}$ ,  $g = 40$ .

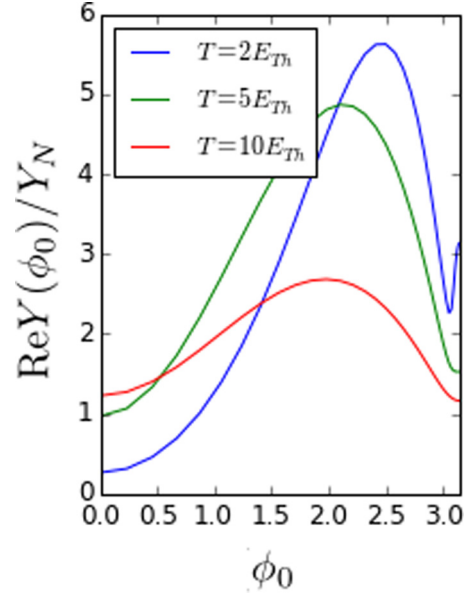


FIG. 8. (Color online) Variation of  $\text{Re}Y(\phi_0)/Y_N$  with temperature at  $\omega = 0.3E_{\text{Th}}$ ,  $g = 20$ .

of the dissipation as  $\phi_0$  departs from  $\pi$ . The position of this kink is determined by the condition  $2E_g(\phi_k) = \omega$ . Indeed, for  $\phi_k \sim \pi$ , one has [35]  $E_g(\phi) \approx \frac{\pi^3}{4} E_{\text{Th}} |1 - \phi/\pi|$ , which gives for  $\hbar\omega = 0.5E_{\text{Th}}$ :  $\phi_k \approx 3.04$ . It can also be followed how this kink shifts with growth of the frequency. At larger temperature it becomes smoothed away, see for example the evolution of the curve in Fig. 8 from  $T = 2E_{\text{Th}}$  to  $T = 5E_{\text{Th}}$ .

### C. Low frequency

Another interesting crossover in the shape of  $\text{Re}Y(\phi_0)/Y_N$  is seen at low frequencies upon variation of the temperature. It is illustrated in Fig. 8. At moderately low temperature  $T = 2E_{\text{Th}}$  a strong peak of dissipation is found at the phase difference  $\phi_0 \approx 0.75\pi$ ; with temperature increase, this peak becomes more rounded and shifts further away from  $\pi$ , so that curve becomes more and more similar to Lempitskii's function  $Q_0(\phi)$ .

## IV. CONCLUSIONS

We have developed a fully microscopic approach to the calculation of a nonstationary ac linear response function of a SNS junction under the dc phase bias, that is valid at arbitrary relations between temperature  $T$ , Thouless energy  $E_{\text{Th}}$ , and frequency  $\omega$ . We assumed an energy gap in the S terminals  $\Delta$  to be much larger than all these energy scales and took into account inelastic relaxation rate  $\hbar/\tau_{\text{in}} \leq E_{\text{Th}}$ . The shape of the dissipative response  $\text{Re}Y(\phi_0)/Y_N$  is shown to be very sensitive to the relations between  $T$ ,  $E_{\text{Th}}$ ,  $\hbar\omega$ , and  $\hbar/\tau_{\text{in}}$ . Explicit results for the function  $\text{Re}Y(\phi_0)/Y_N$  can be found for any choice of the above parameters using the published codes. In particular, we have shown that an accurate solution reproduces many of the qualitative features of the experimental results [13], including peak at the phase difference equal to  $\pi$  at high frequencies and high temperature; we interpret this peak as the result of

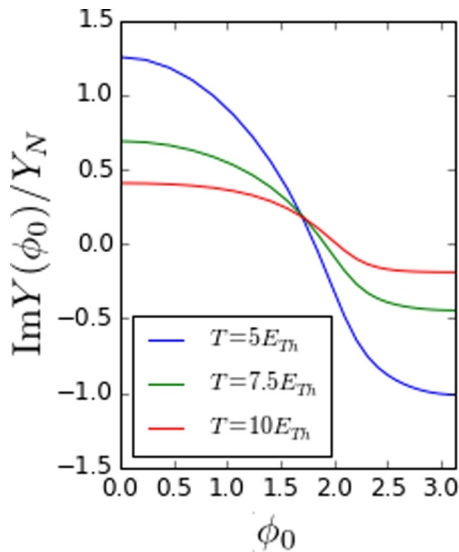


FIG. 9. (Color online) Variation of  $\text{Im}Y(\phi_0)/Y_N$  with temperature at  $\omega = 5E_{Th}$ ,  $g = 20$ .

charge imbalance induced by a high-frequency electric field. Still some quantitative disagreement exists: The experimental value of dissipation at  $\phi_0 = \pi$  is higher (at the same values of  $T$  and  $\omega$ ) than our computations provide. A possible source of this disagreement may be related to nonzero resistance of SN interfaces which we did not take into account in the present calculations, since we assume interfaces to be perfectly transmitting. It is a straightforward task to include nonzero interface resistance into the calculational scheme developed.

In our discussion, we did not touch the issue of the nondissipative part of  $Y(\phi_0)$ , which, at moderate frequency, seems to be reasonably well described in Lempitskii's approximation. What lies outside this approximation is an interesting feature at high frequency at  $\phi_0 \sim \pi$ , which is observed in the experiment [13], as Fig. 6.21 of this reference shows. In our model we obtain the flattening of  $\text{Im}Y$  at  $\phi_0 \sim \pi$  at  $\omega \gtrsim E_{Th}$ . For example, see Fig. 9 for the results at  $\omega = 5E_{Th}$ , which at high temperature are rather close to the experiment. However, we do not see a qualitative change of behavior by lowering the temperature and do not get the large drop at  $\phi_0 = \pi$  which is observed in experiment. The nature of this drop is a very interesting problem for a future study. Another interesting problem is to include a more realistic description of electron-electron interaction into the linear response calculation. It can be as interesting and as important due to the specific spectral properties of the electrons, confined between superconducting reservoirs and the great sensitivity of the admittance to inelastic processes in the experimentally relevant regime of frequency and temperature.

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#### APPENDIX A: RICATTI PARAMETRIZATION OF THE GF

For numerical solution of the unperturbed Usadel equation, it is more convenient to use Ricatti parametrization:

$$\hat{G}_0^R = \frac{1}{1-ab} \begin{pmatrix} 1+ab & 2a \\ -2b & -1-ab \end{pmatrix}. \quad (\text{A1})$$

In this parametrization, the spectral Usadel equation reads

$$\begin{aligned} Da'' + 2i\epsilon a &= -\frac{2Dba'^2}{1-ab}, \\ Db'' + 2i\epsilon b &= -\frac{2Dab'^2}{1-ab}. \end{aligned} \quad (\text{A2})$$

In the main text we hold to trigonometric parametrization, see Eq. (9), which makes the formulas more compact. The relationship between the two parametrizations is as follows:

$$a = e^{i\phi} \tanh \frac{\theta}{2}, \quad b = e^{-i\phi} \tanh \frac{\theta}{2}. \quad (\text{A3})$$

#### APPENDIX B: TRANSPORT COEFFICIENTS AT FINITE FREQUENCY

Here we present expressions for transport coefficients at nonzero frequency, which enter Eqs. (13) and (14). Energy/charge diffusion coefficients read

$$\begin{aligned} D_{L,T} &= 1 - \cosh \theta^R \cosh \theta_-^A \\ &\quad \pm \cos(\phi_-^A - \phi^R) \sinh \theta^R \sinh \theta_-^A, \end{aligned} \quad (\text{B1})$$

anomalous transport coefficient:

$$T = -i \sin(\phi_-^A - \phi^R) \sinh \theta^R \sinh \theta_-^A, \quad (\text{B2})$$

and density of states:

$$N = \cosh \theta^R - \cosh \theta_-^A. \quad (\text{B3})$$

Finally, the spectral supercurrent reads

$$j = J^R - J_-^A. \quad (\text{B4})$$

The frequency of oscillations enters these expressions by the energy shifts, which are shown by the following notation:  $f_{\pm}(\epsilon) = f(\epsilon \pm \omega)$ . The spectral transport coefficients, which enter Eq. (24), read

$$D_S = \frac{1 + \cosh \theta \cosh \theta_-}{\cosh \theta + \cosh \theta_-}, \quad (\text{B5})$$

$$\bar{D}_S = \frac{\sinh \theta \sinh \theta_-}{\cosh \theta + \cosh \theta_-}, \quad (\text{B6})$$

and

$$J_S = (1 + \cosh \theta \cosh \theta_-) \partial_x \frac{1}{\cosh \theta + \cosh \theta_-}, \quad (\text{B7})$$

$$\bar{J}_S = \sinh \theta \sinh \theta_- \partial_x \frac{1}{\cosh \theta + \cosh \theta_-}. \quad (\text{B8})$$

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