Non-Abelian string and particle braiding in topological order: Modular SL(3,$\mathbb{Z}$) representation and (3 + 1)-dimensional twisted gauge theory

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String and particle braiding statistics are examined in a class of topological orders described by discrete gauge theories with a gauge group G and a 4-cocycle twist ω_4 of G's cohomology group $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ in three-dimensional space and one-dimensional time (3 + 1D). We establish the topological spin and the spin-statistics relation for the closed strings and their multistring braiding statistics. The 3 + 1D twisted gauge theory can be characterized by a representation of a modular transformation group, SL(3,$\mathbb{Z}$). We express the SL(3,$\mathbb{Z}$) generators S^{xyz} and T^{xy} in terms of the gauge group G and the 4-cocycle ω_4. As we compactify one of the spatial directions z into a compact circle with a gauge flux b inserted, we can use the generators S^{xy} and T^{xy} of an SL(2,$\mathbb{Z}$) subgroup to study the dimensional reduction of the 3D topological order $\mathcal{C}^{3\text{D}}$ to a direct sum of degenerate states of 2D topological orders $\mathcal{C}_b^{2\text{D}}$ in different flux b sectors: $\mathcal{C}^{3\text{D}} = \oplus_b \mathcal{C}_b^{2\text{D}}$. The 2D topological orders $\mathcal{C}_b^{2\text{D}}$ are described by 2D gauge theories of the group G twisted by the 3-cocycle $\omega_{3(b)}$, dimensionally reduced from the 4-cocycle ω_4. We show that the SL(2,$\mathbb{Z}$) generators, S^{xy} and T^{xy}, fully encode a particular type of three-string braiding statistics with a pattern that is the connected sum of two Hopf links. With certain 4-cocycle twists, we discover that, by threading a third string through two-string unlink into a three-string Hopf-link configuration, Abelian two-string braiding statistics is promoted to non-Abelian three-string braiding statistics.

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I. INTRODUCTION

In the 1986 Dirac Memorial Lectures, Feynman explained the braiding statistics of fermions by demonstrating the plate trick and the belt trick [1]. Feynman showed that the wave function of a quantum system obtains a mysterious (-1) sign by exchanging two fermions, which is associated with the fact that an extra 2π twist or rotation is required to go back to the original state. However, it is known that there is richer physics in deconfined topological phases of 2 + 1D and 3 + 1D spacetime [2]. (Here d + 1D is d-dimensional space and one-dimensional time, while dD is d-dimensional space.) In 2 + 1D spacetime, there are "anyons" with exotic braiding statistics for point particles [3]. In 3 + 1D spacetime, Feynman only had to consider bosonic or fermionic statistics for point particles, without worrying about anyonic statistics. Nonetheless, there are string-like excitations, whose braiding process in 3 + 1D spacetime can enrich the statistics of deconfined topological phases. In this work, we aim to systematically address the string and particle braiding statistics in deconfined gapped phases of 3 + 1D *topological orders*. Namely, we aim to determine what statistical phase the wave function of the whole system gains under the string and particle braiding process.

Since the discovery of 2 + 1D topological orders [4–6] (see Ref. [7] for an overview), we have now gained quite systematic ways to classify and characterize them, by using the induced representations of the mapping class group (MCG) of the $\mathbb{T}^2$ torus (the modular group SL(2,$\mathbb{Z}$) and the gauge/Berry phase structure of ground states [6,8,9]) and the topology-dependent ground-state degeneracy (GSD) [6,10,11], using the unitary fusion categories [12–19] and using simple current algebra [20–23], a pattern of zeros [24–29], and field theories [30–34]. Our better understanding of topologically ordered states also holds the promise of applying their rich quantum phenomena, including fractional statistics [3] and non-Abelian anyons, to topological quantum computation [35].

However, our understanding of 3 + 1 D topological orders is in its infancy and far from systematic. This motivates our work attempting to address question 1.

Q1: *How do we (at least partially) classify and characterize 3D topological orders?*

By *classifying*, we mean counting the number of distinct phases of topological orders and giving them a proper label. By *characterizing*, we mean describing their properties in terms of physical observables. Here our approach to studying dD topological orders is to simply generalize the above 2D approach and to use the GSD on the d torus $\mathbb{T}^d = (S^1)^d$ and the associated representations of the MCG of $\mathbb{T}^d$ (recently proposed in Refs. [19] and [36]):

$$\mathrm{MCG}(\mathbb{T}^d) = \mathrm{SL}(d,\mathbb{Z}). \tag{1}$$

(Refer to Appendix A 4 and references cited therein for a brief review of the computation of 2D topological orders.) For three dimensions, the MCG SL(3,$\mathbb{Z}$) is generated by the modular transformation $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$ [37]:

$$\hat{\mathsf{S}}^{xyz} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{\mathsf{T}}^{xy} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

What are examples of 3D topological orders? One class of them is described by a discrete gauge theory with a finite gauge group G. Another class is described by the *twisted gauge theory* [38], a gauge theory G with a 4-cocycle twist $\omega_4 \in \mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ of G's fourth cohomology group. But the twisted

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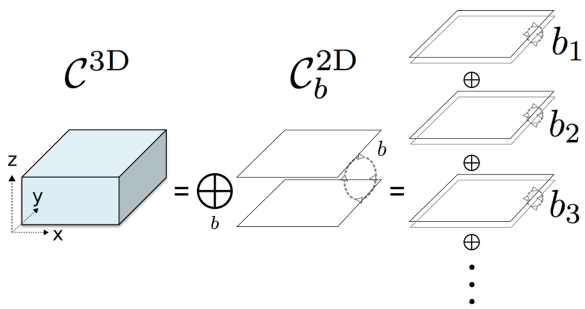


FIG. 1. (Color online) The 3D topological order $\mathcal{C}^{3D}$ can be regarded as the direct sum of 2D topological orders $\mathcal{C}_b^{2D}$ in different sectors b, as $\mathcal{C}^{3D} = \oplus_b \mathcal{C}_b^{2D}$, when we compactify a spatial direction z into a circle. This idea is general and applicable to $\mathcal{C}^{3D}$ without a gauge theory description. However, when $\mathcal{C}^{3D}$ allows a gauge group G description, b stands for a group element (or the conjugacy class for the non-Abelian group) of G. Thus b acts as a gauge flux along the dashed arrow in the compact direction z. Thus, $\mathcal{C}^{3D}$ becomes the direct sum of different $\mathcal{C}_b^{2D}$ values under distinct gauge fluxes b.

gauge theory characterization of 3D topological orders is not one-to-one: different pairs (G,ω_4) can describe the same 3D topological order. In this work, we use $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$ of SL$(3,\mathbb{Z})$ to characterize the topological *twisted discrete gauge theory* with finite gauge group G, which has topology-dependent GSD. The twisted gauge theories describe a large class of 3D gapped quantum liquids in condensed matter. Although we study the SL$(3,\mathbb{Z})$ modular data of the ground-state sectors of gapped phases, these data can capture the gapped excitations such as particles and strings. (This strategy is widely used, especially in two dimensions.) There are two main issues that we focus on. The first is the dimensional reduction from three to two dimensions of SL$(3,\mathbb{Z})$ modular transformation and cocycles to study 3D topological order. The second is the non-Abelian three-string braiding statistics from a twisted discrete gauge theory of an Abelian gauge group.

Dimensional reduction from three to two dimensions for SL$(3,\mathbb{Z})$ *modular* S *and* T *matrices and cocycles*: For the first issue, our general philosophy is as follows. Since 3D topological orders are foreign and unfamiliar to us, we will dimensionally reduce 3D topological orders to several sectors of 2D topological orders in the Hilbert space of ground states (not in the real space; see Fig. 1). Then we will be able to borrow the more familiar 2D topological orders to understand 3D orders.

We compute the matrices S^{xyz} and T^{xy} that generate the SL$(3,\mathbb{Z})$ representation in the quasiparticle- or quasistring-excitation basis of $3+1$D topological order. We find an explicit expression of S^{xyz} and T^{xy}, in terms of the gauge group G and the 4-cocycle ω_4, for both Abelian and non-Abelian gauge groups. (A calculation using a different novel approach, the universal wave-function overlap for the normal untwisted gauge theory, is studied in [39].) We note that SL$(3,\mathbb{Z})$ contains a subgroup SL$(2,\mathbb{Z})$, which is generated by $\hat{\mathsf{S}}^{xy}$ and $\hat{\mathsf{T}}^{xy}$, where

$$\hat{\mathsf{S}}^{xy} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

In the most generic cases of topological orders (potentially without a gauge group description), the matrices S^{xy} and T^{xy} can still be block diagonalized as the sum of several sectors in the quasiexcitation basis, each sector carrying an index of b:

$$\mathsf{S}^{xy} = \oplus_b \mathsf{S}_b^{xy}, \quad \mathsf{T}^{xy} = \oplus_b \mathsf{T}_b^{xy}. \tag{4}$$

The pair $(\mathsf{S}_b^{xy},\mathsf{T}_b^{xy})$, generating an SL$(2,\mathbb{Z})$ representation, describes a 2D topological order $\mathcal{C}_b^{2D}$. This leads to a dimension reduction of the 3D topological order $\mathcal{C}^{3D}$:

$$\mathcal{C}^{3D} = \oplus_b \mathcal{C}_b^{2D}. \tag{5}$$

In the more specific case, when the topological order allows a gauge group G description which we focus on here, we find that the b stands for the gauge flux for group G (that is, b is a group element for an Abelian G, while b is a conjugacy class for a non-Abelian G).

The physical picture of the above dimensional reduction is the following (see Fig. 1): If we compactify one of the 3D spatial directions (say the z direction) into a small circle, the 3D topological order $\mathcal{C}^{3D}$ can be viewed as a direct sum of 2D topological orders $\mathcal{C}_b^{2D}$ with (accidental) degenerate ground states at the lowest energy.

In this work, we focus on a generic finite Abelian gauge group, $G = \prod_i Z_{N_i}$ (isomorphic to products of cyclic groups), with generic cocycle twists from the group cohomology [38]. We examine the $3+1$D twisted gauge theory twisted by 4-cocycle $\omega_4 \in \mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ and reveal that it is a direct sum of $2+1$D twisted gauge theories twisted by a dimensionally reduced 3-cocycle, $\omega_{3(b)} \in \mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$, of G's third cohomology group, namely,

$$\mathcal{C}^{3D}_{G,\omega_4} = \oplus_b \mathcal{C}^{2D}_{G_b,\omega_{3(b)}}. \tag{6}$$

Surprisingly, even for an Abelian group G, we find that such a *twisted Abelian gauge theory* can be dual to a twisted or untwisted *non-Abelian gauge theory*. We study this fact for 3D examples as an extension of the 2D examples in Ref. [40]. By this equivalence, we are equipped with (both untwisted and twisted) non-Abelian gauge theory to study its non-Abelian braiding statistics.

Non-Abelian three-string braiding statistics: We are familiar with the 2D braiding statistics: there is only particle-particle braiding, which yields bosonic, fermionic, or anyonic statistics by braiding a particle around another particle [3]. We find that the 3D topological order introduces both particle-like and string-like excitations. We aim to address question 2:

> **Q2**: *How do we characterize the braiding statistics of strings and particles in $3+1$D topological orders?*

The possible braiding statistics in three dimensions learned in the literature are as follows.

(i) *Particle-particle braiding*, which can only be bosonic or fermionic due to the absence of a nontrivial braid group in three dimensions for point particles.

(ii) *Particle-string braiding*, which is the Aharonov-Bohm effect of $\mathbb{Z}_N$ gauge theory, where a particle such as $\mathbb{Z}_N$ charges braiding around a string (or a vortex line) as $\mathbb{Z}_N$ flux, obtaining an $e^{i\frac{2\pi}{N}}$ phase of statistics [3,41].

(iii) *String-string braiding*, where a closed string (a red loop), shown in Fig. 2(c) excluding the background black

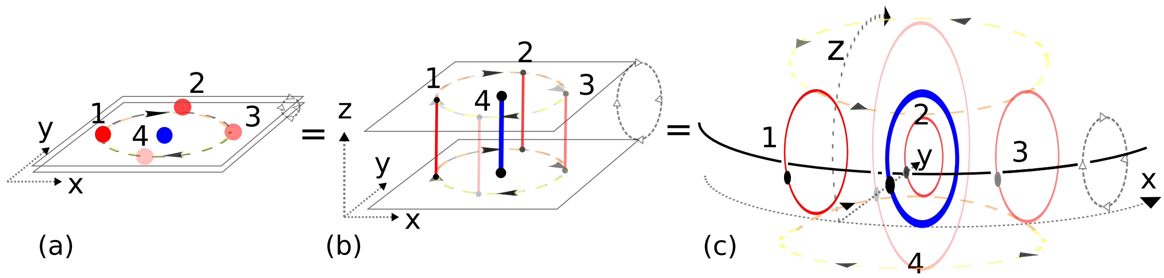


FIG. 2. (Color online) Mutual braiding statistics following the path $1 \to 2 \to 3 \to 4$ along time evolution (see Sec. III C 2). (a) From a 2D viewpoint of dimensional reduced $\mathcal{C}_b^{2\mathrm{D}}$, the 2π braiding of two particles is shown. (b) The compact z direction extends two particles to two closed (red, blue) strings. (c) An equivalent 3D view; the b flux (along the dashed arrow) is regarded as the monodromy caused by the third (black) string. We identify the coordinates x and y and a compact z to see that the full-braiding process is one (red) string going inside the loop of another (blue) string and then going back from the outside. For Abelian topological orders, the mutual braiding process between two excitations (A and B) in (a) yields a statistical Abelian phase $e^{\mathrm{i}\theta_{(\mathrm{A})(\mathrm{B})}} \propto \mathsf{S}^{xy}_{(\mathrm{A})(\mathrm{B})}$ proportional to the 2D S^{xy} matrix. The dimensional-extended equivalent picture (c) implies that the loop braiding yields a phase $e^{\mathrm{i}\theta_{(\mathrm{A})(\mathrm{B}),b}} \propto \mathsf{S}^{xy}_{b\,(\mathrm{A})(\mathrm{B})}$ of Eq. (34) (up to a choice of canonical basis), where b is the flux of the black string. We clarify that in both (b) and (c) our strings may carry both flux and charge. If a string carries only a pure charge, then it is effectively a point particle in three dimensions. If a string carries a pure flux, then it is effectively a loop of a pure string in three dimensions. If a string carries both charge and flux (as a dyon in two dimensions), then it is a loop with string fluxes attached to some charged particles in three dimensions. Therefore the string-string braiding in (c) actually represents several braiding processes—particle-particle, particle-loop, and loop-loop braidings; all processes are threaded with a background (black) string.

string, wraps around a blue loop. The related idea, known as loop-loop braiding, forming the loop braid group, has been proposed mathematically [42]. (See also some earlier studies in Refs. [43] and [44].)

However, we address some extra new braiding statistics among three closed strings:

(iv) *Three-string braiding*, shown in Fig. 2(c), where a closed string (a red loop) wraps around another closed string (a blue loop) but the two loops are both threaded by a third loop (the black string). This braiding configuration was discovered recently in Ref. [45]; Ref. [46] is a related work for a twisted Abelian gauge theory.

The new ingredient of our work on braiding statistics can be summarized as follows: We consider the string and particle braiding of general twisted gauge theories with the most generic finite Abelian gauge group $G = \prod_u Z_{N_u}$, labeled by the data (G,ω_4). We provide a 3D-to-2D reduction approach to realize the three-string braiding statistics in Fig. 2. We first show that the SL$(2,\mathbb{Z})$ representations $(\mathsf{S}^{xy}_b,\mathsf{T}^{xy}_b)$ fully encode the particular type of Abelian three-closed-string statistics shown in Fig. 2. We further find that, for a twisted gauge theory with an Abelian $(Z_N)^4$ group, certain 4-cocycles (called type IV 4-cocycles) will make the twisted theory be a non-Abelian theory. More precisely, *while the two-string braiding statistics of unlinks is Abelian, the three-string braiding statistics of Hopf links, obtained from threading the two strings with a third string, will become non-Abelian.* We also demonstrate that S^{xy}_b encodes this three-string braiding statistics.

Our article is organized as follows. In Sec. II, we address the third question:

> **Q3**: *How do we formulate or construct certain 3 + 1D topological orders in the lattice?*

We outline a lattice formulation of twisted gauge theories in terms of 3D twisted quantum double models, which generalize Kitaev's 2D toric code and quantum double models. Our model is the lattice Hamiltonian formulation of Dijkgraaf-Witten theory [38], and we provide the *spatial lattice* as well as the *spacetime lattice path integral* pictures. In Sec. III, we answer question 4:

> **Q4**: *What are the generic expressions of SL(3,$\mathbb{Z}$) modular data?*

We compute the modular SL$(3,\mathbb{Z})$ representations of S and T matrices, using both the spacetime path integral approach and the representation (Rep) theory approach. In Secs. III C and IV, we address question 5:

> **Q5**: *What is the physical interpretation of SL(3,$\mathbb{Z}$) modular data in three dimensions?*

We use the modular SL$(3,\mathbb{Z})$ data to characterize the braiding statistics of particles and strings. In Sec. V, we discuss the link and knot patterns of string braiding systematically and end with a conclusion. In addition to the text, we organize the following information in Appendix]: (i) group cohomology and cocycles; (ii) projective representation; (iii) some examples of classification of topological orders; and (iv) direct calculations of S and T using cocycle path integrals.

[*Note*: We adopt the name *strings* for the vision of incorporating the excitations from both closed strings (loops) and open strings. Such excitations can have a fusion or braiding process. In this work, however, we focus only on the closed string case. Our notation for a finite cyclic group is either Z_N or $\mathbb{Z}_N$, though they are equivalent mathematically. We use Z_N to denote the gauge group G, the discrete gauge Z_N flux, or the Z_N variables, but $\mathbb{Z}_N$ to denote only the classes of group cohomology or topological order classification. We denote $\gcd(N_i,N_j) \equiv N_{ij}$, $\gcd(N_i,N_j,N_k) \equiv N_{ijk}$ and $\gcd(N_i,N_j,N_k,N_l) \equiv N_{ijkl}$, where gcd stands for the greatest common divisor. We also have $|G|$ as the order of the group, and $\mathbb{R}/\mathbb{Z} = \mathrm{U}(1)$. We may use subindex n for ω_n to indicate an n-cocycle. In principle, we use *types* to count the number of cocycles in cohomology groups. But we use *classes* to count the number of distinct phases in topological orders.

Normally the types outnumber the classes. We use the hat symbol $\hat{\mathsf{S}}$ and $\hat{\mathsf{T}}$ for modular matrices acting on the real space in the x,y,z directions, so $\hat{\mathsf{S}}^{xyz} \cdot (x,y,z) = (z,x,y)$ and $\hat{\mathsf{T}}^{xy} \cdot (x,y,z) = (x+y,y,z)$, while we use the symbols S and T to denote modular matrices in the quasiexcitation basis.]

II. TWISTED GAUGE THEORY AND COCYCLES OF GROUP COHOMOLOGY

In this section, we aim to address question 3:

Q3: *How do we formulate or construct certain $3+1$D topological orders in the lattice?*

We consider $3+1$D twisted discrete gauge theories. Our motivation to study the discrete gauge theory is that it is topological and exhibits Aharonov-Bohm phenomena (see Refs. [3] and [41]). One approach to formulating a discrete gauge theory is the lattice gauge theory [47]. A famous example in both the high-energy and the condensed matter communities is the Z_2 discrete gauge theory in $2+1$ dimensions (also called the Z_2 toric code, Z_2 spin liquids, or Z_2 topological order [48]). Kitaev's toric code and quantum double model [49] provide a simple Hamiltonian,

$$H = -\sum_v A_v - \sum_p B_p, \tag{7}$$

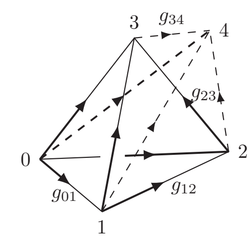
where a space lattice formalism is used, and A_v is the vertex operator acting on vertex v, B_p is the plaquette (or face) term to ensure the zero-flux condition on each plaquette. Both A_v and B_p consist of only Pauli spin operators for the Z_2 model. Such ground states of the Hamiltonian are found to be Z_2 gauge theory with $|G|^2 =$ fourfold topological degeneracy on the $\mathbb{T}^2$ torus. Its generalization to a *twisted Z_2 gauge theory* is the Z_2 double-semions model, captured by the framework of the Levin-Wen string-net model [12,48].

A. Dijkgraaf-Witten topological gauge theory

For a more generic twisted gauge theory, there is indeed another way using the spacetime lattice formalism to construct them by the Dijkgraaf-Witten topological gauge theory [38]. One can formulate the path integral $\mathbf{Z}$ (or partition function) of a $(d+1)$D gauge theory (dD space, 1D time) of a gauge group G as

$$\mathbf{Z} = \sum_{\gamma} e^{iS[\gamma]} = \sum_{\gamma} e^{i2\pi\langle \omega_{d+1}, \gamma(\mathcal{M}_{\text{tri}})\rangle (\text{mod} 2\pi)}$$
$$= \frac{|G|}{|G|^{N_v}} \frac{1}{|G|} \sum_{\{g_{ab}\}} \prod_i (\omega_{d+1}{}^{\epsilon_i}(\{g_{ab}\}))\,|_{v_{c,d} \in T_i}\,, \tag{8}$$

where we sum over all mappings $\gamma: \mathcal{M} \to BG$, from the spacetime manifold $\mathcal{M}$ to BG, the classifying space of G. In the second equality, we triangulate $\mathcal{M}$ to $\mathcal{M}_{\text{tri}}$, with the edge $[v_a v_b]$ connecting the vertex v_a to the vertex v_b. The action $\langle \omega_{d+1}, \gamma(\mathcal{M}_{\text{tri}})\rangle$ evaluates the cocycles ω_{d+1} in the spacetime $(d+1)$-complex $\mathcal{M}_{\text{tri}}$. By the relation between the topological cohomology class of BG and the cohomology group of G: $H^{d+2}(BG,\mathbb{Z}) = \mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})$ [38,50], we can simply regard ω_{d+1} as the $d+1$-cocycles of the cohomology group $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})$ (see more details in Appendix A). The group elements g_{ab} are assigned at the edge $[v_a v_b]$. The $|G|/|G|^{N_v}$ factor is to mod out the redundant gauge equivalence configuration, with the number of vertices N_v. Another extra $|G|^{-1}$ factor mods out the group elements evolving in the time dimension. The cocycle ω_{d+1} is evaluated on all the $d+1$-simplex T_i (namely, a $d+2$-cell) triangulations of the spacetime complex. In the case of our $3+1$ dimensions, we have the 4-cocycle ω_4 evaluated at the 4-simplex (or 5-cell) as



$$= \omega_4{}^{\epsilon}(g_{01}, g_{12}, g_{23}, g_{34}). \tag{9}$$

Here the cocycle ω_4 satisfies the cocycle condition, $\delta\omega_4 = 1$, which ensures that the path integral $\mathbf{Z}$ on the 4-sphere S^4 (the surface of the 5-ball) will be trivial as 1. This is a feature of topological gauge theory. The ϵ is the $\pm$ sign of the orientation of the 4-simplex, which is determined by the sign of the volume determinant of the 4-simplex evaluated by $\epsilon = \text{sgn}(\det(\vec{01},\vec{02},\vec{03},\vec{04}))$.

We utilize Eq. (8) to calculate the path integral amplitude from an initial state configuration $|\Psi_{\text{in}}\rangle$ on the spatial manifold evolving along the time direction to the final state $|\Psi_{\text{out}}\rangle$ (see Fig. 3). In general, the calculation can be done for the MCG on any spatial manifold $\mathcal{M}_{\text{space}}$ as MCG($\mathcal{M}_{\text{space}}$). Here we focus on $\mathcal{M}_{\text{space}} = \mathbb{T}^3$ and MCG($\mathbb{T}^3$) = SL($3,\mathbb{Z}$), as the modular transformation. We first note that $|\Psi_{\text{in}}\rangle = \hat{\mathsf{O}}|\Psi_{\text{B}}\rangle$, such a generic SL($3,\mathbb{Z}$) transformation $\hat{\mathsf{O}}$ under the SL($3,\mathbb{Z}$) representation can be absolutely generated by $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$ of Eq. (2) [37], thus $\hat{\mathsf{O}} = \hat{\mathsf{O}}(\hat{\mathsf{S}}^{xyz},\hat{\mathsf{T}}^{xy})$ as a function of $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$. Calculation of the modular SL($3,\mathbb{Z}$) transformation from $|\Psi_{\text{in}}\rangle$ to $|\Psi_{\text{out}}\rangle = |\Psi_{\text{A}}\rangle$ by filling the 4-cocycles ω_4 into the spacetime-complex triangulation renders the amplitude of

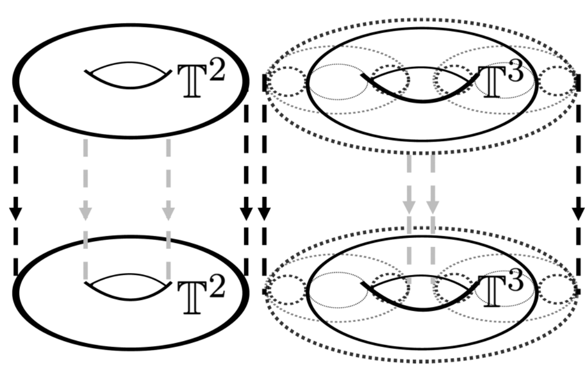


FIG. 3. Illustration for $\mathsf{O}_{(\text{A})(\text{B})} = \langle\Psi_{\text{A}}|\hat{\mathsf{O}}|\Psi_{\text{B}}\rangle$. Evolution from an initial-state configuration $|\Psi_{\text{in}}\rangle$ on the spatial manifold (from the top) along the time direction (dashed line) to the final state $|\Psi_{\text{out}}\rangle$ (at the bottom). For the spatial $\mathbb{T}^d$ torus, the mapping class group MCG($\mathbb{T}^d$) is the modular SL($d,\mathbb{Z}$) transformation. We show schematically the time evolution on the spatial $\mathbb{T}^2$ and $\mathbb{T}^3$. The $\mathbb{T}^3$ is shown as a $\mathbb{T}^2$ attached a S^1 circle on each point.

the matrix element $\mathsf{O}_{(\mathrm{A})(\mathrm{B})}$,

$$\mathsf{O}(\mathsf{S}^{xyz},\mathsf{T}^{xy})_{(\mathrm{A})(\mathrm{B})} = \langle \Psi_{\mathrm{A}} | \hat{\mathsf{O}}(\hat{\mathsf{S}}^{xyz},\hat{\mathsf{T}}^{xy}) | \Psi_{\mathrm{B}} \rangle, \tag{10}$$

where both space and time are discretely triangulated, so this is a spacetime-lattice formalism.

B. Canonical basis and the generalized twisted quantum double model $D^{\omega}(G)$ to the 3D triple basis

So far we have answered question 3 using the *spacetime-lattice path integral*. Our next goal is to construct its Hamiltonian on the space lattice and to find a good basis representing its quasiexcitations, such that we can efficiently read the information of $\mathsf{O}(\mathsf{S}^{xyz},\mathsf{T}^{xy})$ in this *canonical basis*. We outline the twisted quantum double model generalized to three dimensions as the exactly soluble model in the next subsection, where the canonical basis can diagonalize its Hamiltonian.

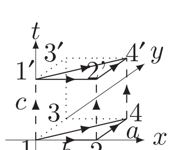
1. Canonical basis

For a gauge theory with the gauge group G, one may naively think that a good basis for the amplitude, Eq. (10), is the group elements $|g_x,g_y,g_z\rangle$, with $g_i \in G$ as the flux labeling three directions of $\mathbb{T}^3$. However, this flux-only label $|g_x,g_y\rangle$ is known to be improper on the $\mathbb{T}^2$ torus already: the canonical basis labeling particles in two dimensions is $|\alpha,a\rangle$, requiring both the charge α (as the representation) and the flux a (the group element or the conjugacy class of G). We propose that the proper way to label excitations for a $3+1$D twisted discrete gauge theory for any finite group G in the canonical basis requires one charge, α, and two fluxes, a and b.

$$|\alpha,a,b\rangle = \frac{1}{\sqrt{|G|}} \sum_{\substack{g_y \in C^a, g_z \in C^b \\ g_x \in Z_{g_y} \cap Z_{g_z}}} \mathrm{Tr}[\widetilde{\rho}_{\alpha}^{g_y,g_z}(g_x)] |g_x,g_y,g_z\rangle, \tag{11}$$

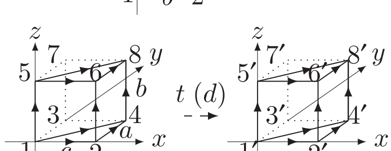
which is the finite-group discrete Fourier transformation on $|g_x,g_y,g_z\rangle$. This is a generalization of the 2D result in Ref. [40] and a very recent 3D Abelian case in Ref. [46]. Here α is the charge of the representation (Rep) label, which is the $\mathsf{C}^{(2)}_{a,b}$ Rep of the centralizers Z_a, Z_b of the conjugacy classes C^a, C^b. (For an Abelian G, the conjugacy class is the group element, and the centralizer is the full G.) $\mathsf{C}^{(2)}_{a,b}$ Rep means an inequivalent unitary irreducible projective representation of G. $\widetilde{\rho}_{\alpha}^{a,b}(c)$ labels this inequivalent unitary irreducible projective $\mathsf{C}^{(2)}_{a,b}$ Rep of G. $\mathsf{C}^{(2)}_{a,b}$ is an induced 2-cocycle, dimensionally reduced from the 4-cocycle ω_4. We illustrate $\mathsf{C}^{(2)}_{a,b}$ in terms of geometric pictures in Eqs. (12) and (13):

$\mathsf{C}_a(b,c)$:



, (12)

$\mathsf{C}^{(2)}_{a,b}(c,d)$:



. (13)

The reduced 2-cocycle $\mathsf{C}_a(b,c)$ is from the 3-cocycle ω_3 in Eq. (12), which triangulates a half of $\mathbb{T}^2$, with a time interval I. The reduced 2-cocycle $\mathsf{C}_a(b,c)$ is from 4-cocycle ω_4 in Eq. (13), which triangulates a half of $\mathbb{T}^3$ with a time interval I. The dashed arrow stands for the time t evolution.

The $\widetilde{\rho}_{\alpha}^{g_y,g_z}(g_x)$ values are determined by the $\mathsf{C}^{(2)}_{a,b}$ projective representation formula:

$$\widetilde{\rho}_{\alpha}^{a,b}(c)\widetilde{\rho}_{\alpha}^{a,b}(d) = \mathsf{C}^{(2)}_{a,b}(c,d)\widetilde{\rho}_{\alpha}^{a,b}(cd). \tag{14}$$

The trace term $\mathrm{Tr}[\widetilde{\rho}_{\alpha}^{g_y,g_z}(g_x)]$ is called *the character* in the math literature. One can view the charge α_x along the x direction, and the flux a,b along the y,z directions. Other details and the calculations of $\mathsf{C}^{(2)}_{a,b}$ Rep, with many examples, are given in Appendix A.

We first recall that, in two dimensions, a reduced 2-cocycle $\mathsf{C}_a(b,c)$ comes from a slant product $i_a\omega(b,c)$ of 3-cocycles [40], which is geometrically equivalent to filling three 3-cocycles in a triangular prism of Eq. (12). This is known to present the *projective representation*, $\widetilde{\rho}_{\alpha}^{a}(b)\widetilde{\rho}_{\alpha}^{a}(c) = \mathsf{C}_a(b,c)\widetilde{\rho}_{\alpha}^{a}(bc)$, because the induced 2-cocycle belongs to the second cohomology group $\mathcal{H}^2(G,\mathbb{R}/\mathbb{Z})$ [40,51–53]. (See its explicit triangulation and a novel use of the projective representation in Sec VI B of Ref. [54].)

Similarly, in three dimensions, a reduced 2-cocycle $\mathsf{C}_a(b,c)$ comes from doing *twice* the slant products of 4-cocycles forming the geometry of Eq. (13) and renders

$$\mathsf{C}^{(2)}_{a,b} = i_b(\mathsf{C}_a(c,d)) = i_b(i_a\omega(c,d)), \tag{15}$$

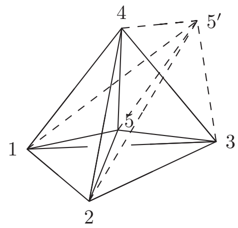
presenting the $\mathsf{C}^{(2)}_{a,b}$ projective representation in Eq. (14), where $\widetilde{\rho}_{\alpha}^{a,b}(c)$: $(Z_a,Z_b) \to \mathrm{GL}\,(Z_a,Z_b)$ can be written as a matrix in the general linear (GL) group. This 3D generalization for the canonical basis in Eq.(11) is not only natural, but also consistent with two dimensions when we turn off the flux along the z direction (e.g., set $b=0$). which reduces the 3D $|\alpha,a,b\rangle$ to $|\alpha,a\rangle$ in the 2D case.

2. Generalizing the 2D twisted quantum double model $D^{\omega}(G)$ to the 3D twisted quantum triple model?

A natural way to combine the Dijkgraaf-Witten theory with Kitaev's quantum double model Hamiltonian approach will enable us to study the Hamiltonian formalism for the twisted gauge theory, which is achieved in Refs. [55] and [53] for $2+1$ dimensions, termed the twisted quantum double model. In two dimensions, the widely used notation $D^{\omega}(G)$ implies the twisted quantum double model with its gauge group G and its cocycle twist ω. It is straightforward to generalize these results to $3+1$ dimensions.

To construct the Hamiltonian on the 3D spatial lattice, we follow Ref. [55] with the form of the twisted quantum double model Hamiltonian of Eq. (7) and put the system on the $\mathbb{T}^3$ torus. However, some modification are adopted for three dimensions: the vertex operator $A_v = |G|^{-1}\sum_{[vv']=g\in G} A_v^g$ acts on the vertices of the lattice by lifting the vertex point

v to point v', living in an extra (fourth) dimension, as Eq. (16),



$$, \qquad (16)$$

and one computes the 4-cocycle filling amplitude as $\mathbf{Z}$ in Eq. (8). To evaluate Eq. (16)'s A_v operator acting on vertex 5, one effectively lifts 5 to 5', and fills 4-cocycles ω into this geometry to compute the amplitude $\mathbf{Z}$ in Eq. (8). For this specific 3D spatial lattice surrounding vertex 5 with one, two, three, and four neighboring vertices, there are four 4-cocycles ω filling in the amplitude of $A_5^{[55']}$.

The plaquette operator $B_p^{(1)}$ still enforces the zero-flux condition on each 2D face (a triangle p) spanned by three edges of a triangle. This will ensure zero flux on each face (along the Wilson loop of a 1-form gauge field). Moreover, zero-flux conditions are required if higher form gauge fluxes are presented. For example, for 2-form field, one adds an additional $B_p^{(2)}$ to ensure zero flux on a 3-simplex (a tetrahedron p). Thus, $\sum_p B_p$ in Eq. (7) becomes $\sum_p B_p^{(1)} + \sum_p B_p^{(2)} + \dots$.

Analogous to Ref. [55], the local operators A_v, B_p of the Hamiltonian have nice commuting properties: $[A_v^g, A_u^h] = 0$ if $v \neq u$, $[A_v^g, B_p] = [B_p, B'_p] = 0$, and also $A_v^{g=[vv']} A_{v'}^h = A_v^{gh}$. Note that A_g defines a ground-state projection operator $\mathsf{P}_v = |G|^{-1} \sum_g A_v^g$ if we consider a $\mathbb{T}^3$ torus triangulated in a cube with only a point v (all eight points are identified). It can be shown that both A_g and P as projection operators project other states to the ground state $|\alpha,a,b\rangle$, and $\mathsf{P}|\alpha,a,b\rangle = |\alpha,a,b\rangle$ and $A_v|\alpha,a,b\rangle \propto |\alpha,a,b\rangle$. Since $[A_v^g, B_p] = 0$, one can simultaneously diagonalize the Hamiltonian, Eq. (7), by this *canonical basis* $|\alpha,a,b\rangle$ as the *ground-state basis*.

A similar 3D model was studied recently in Ref. [46]. There the zero-flux condition is imposed on both the vertex operator and the plaquette operator. Their Hilbert space thus is more constrained than that in Ref. [55] or ours. However, in the ground-state sector, we expect that the physics is the same. It is less clear to us whether the name twisted quantum double model and its notation, $D^\omega(G)$, are still proper usages in three or higher dimensions. With the quantum double basis $|\alpha,a\rangle$ in two dimensions generalized to the triple basis $|\alpha,a,b\rangle$ in three dimensions, we are tempted to call it the *twisted quantum triple model* in three dimensions. It awaits mathematicians and mathematical physicists to explore more details in the future.

C. Cocycle of $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ and its dimensional reduction

To study the twisted gauge theory of a finite Abelian group, we now provide the explicit data on cohomology group and 4-cocycles [56]. Here $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z}) = \mathcal{H}^{d+1}(G,\mathrm{U}(1))$ by $\mathbb{R}/\mathbb{Z} = \mathrm{U}(1)$, as the $(d+1)$th cohomology group of G over G module U(1). Each class in $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})$ corresponds to a distinct $(d+1)$-cocycle. The different 4-cocycles label the distinct topological terms of $3+1$D twisted gauge theories. (However, different topological terms may share the same data for topological orders, such as the same modular data S^{xyz} and T^{xy}. Thus different topological terms may describe the same topological order.) The 4-cocycles ω_4 are 4-cochains but, additionally, satisfy the cocycle condition $\delta\omega = 1$. The 4-cochain is a mapping $\omega_4(a,b,c,d)$: $(G)^4 \to \mathrm{U}(1)$, which inputs $a,b,c,d \in G$ and outputs a U(1) phase. Furthermore, distinct 4-cocycles are not identified by any 4-coboundary $\delta\Omega_3$. (Namely, distinct cocycles ω_4 and ω'_4 do *not* satisfy $\omega_4/\omega'_4 = \delta\Omega_3$, for any 3-cochain Ω_3.) The 4-cochain satisfies the group multiplication rule, $(\omega_4 \cdot \omega'_4)(a,b,c,d) = \omega_4(a,b,c,d) \cdot \omega'_4(a,b,c,d)$ and thus forms a group C^4, the 4-cocycle further forms its subgroup Z^4, and the 4-coboundary further forms the Z^4 subgroup B^4 (since $\delta^2 = 1$). In short, $\mathrm{B}^4 \subset \mathrm{Z}^4 \subset \mathrm{C}^4$. The fourth cohomology group is a kernel Z^4 (the group of 4-cocycles) mod out the image B^4 (the group of 4-coboundaries) relation: $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z}) = \mathrm{Z}^4/\mathrm{B}^4$. We derive the fourth cohomology group of a generic finite Abelian $G = \prod_{i=1}^k Z_{N_i}$ as

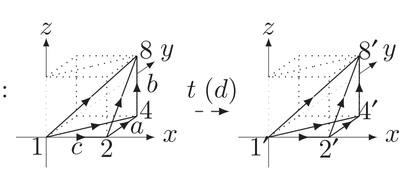
$$\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z}) = \prod_{1 \leqslant i<j<l<m \leqslant k} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{ijl}})^2 \times \mathbb{Z}_{N_{ijlm}}. \quad (17)$$

We construct generic 4-cocycles (not identified by 4-coboundaries) for each type, summarized in Table I.

We call type II first and type II second the 4-cocycles with topological term indices: $p_{\mathrm{II}(ij)}^{(1st)} \in \mathbb{Z}_{N_{ij}}$ and $p_{\mathrm{II}(ij)}^{(2nd)} \in \mathbb{Z}_{N_{ij}}$ of Eq. (17). There are type III first and type III second 4-cocycles for topological term indices: $p_{\mathrm{III}(ijl)}^{(1st)} \in \mathbb{Z}_{N_{ijl}}$ and $p_{\mathrm{III}(ijl)}^{(2nd)} \in \mathbb{Z}_{N_{ijl}}$. There is also a type IV 4-cocycle topological term index: $p_{\mathrm{IV}(ijlm)} \in \mathbb{Z}_{N_{ijlm}}$.

Since we earlier alluded to the relation, Eq. (5), $\mathcal{C}^{3\mathrm{D}} = \oplus_b \mathcal{C}_b^{2\mathrm{D}}$, between 3D topological orders (described by 4-cocycles) as the direct sum of sectors of 2D topological orders (described by 3-cocycles), we wish to see how the dimensionally reduced 3-cocycle from 4-cocycles can hint at the $\mathcal{C}_b^{2\mathrm{D}}$ theory in two dimensions. The slant products $\mathsf{C}_b(a,c,d) \equiv i_b\omega_4(a,c,d)$ are organized in the last column in Table I. The geometric interpretation of the induced 3-cocycle $\mathsf{C}_b(a,c,d) \equiv i_b\omega_4(a,c,d)$ is derived from the 4-cocycle ω_4,

$\mathsf{C}_b(a,c,d)$:



$. \qquad (18)$

The combination of Eq. (18) (with four 4-cocycles filling) times the contribution of Eq. (12) (with three 3-cocycles filling) will produce Eq. (13) with twelve 4-cocycles filling. Luckily, the types II and III ω_4's have a simpler form of $\mathsf{C}_b(a,c,d) = \omega_4(a,b,c,d)/\omega_4(b,a,c,d)$, while the reduced form of type IV ω_4 is more involved [56].

This indeed promisingly suggests the relation in Eq. (6), $\mathcal{C}_{G,\omega_4}^{3\mathrm{D}} = \oplus_b \mathcal{C}_{G,\omega_{3(b)}}^{2\mathrm{D}}$, with $G_b = G$ the original group. If we view b as the gauge flux along the z direction and compactify z into a circle, then a single winding around z acts as a monodromy defect carrying the gauge flux b (group elements or conjugacy classes) [54,57,58]. This implies the geometric picture in Fig. 4.

TABLE I. Cohomology group $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ and 4-cocycles ω_4 for a generic finite Abelian group $G=\prod_{i=1}^{k} Z_{N_i}$. The first column lists the types in $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ of Eq. (17). The second column lists the topological term indices for the $3+1$D twisted gauge theory. (When all indices $p_{...}=0$, it becomes the normal untwisted gauge theory.) The third column lists the explicit 4-cocycle functions $\omega_4(a,b,c,d)$: $(G)^4 \to \mathrm{U}(1)$. Here $a=(a_1,a_2,\dots,a_k)$, with $a\in G$ and $a_i \in Z_{N_i}$. (Same notation for b,c,d.) We define the mod N_j relation by $[c_j+d_j]\equiv c_j+d_j \pmod{N_j}$. The last column lists the induced 3-cocycles from the slant product $\mathsf{C}_b(a,c,d)\equiv i_b\omega_4(a,c,d)$ in terms of types I, II, and III 3-cocycles of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ listed in Table XII.

$\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$	4-cocycle name	4-cocycle form	Induced 3-cocycle $\mathsf{C}_b(a,c,d)$
$\mathbb{Z}_{N_{ij}}$	Type II 1st $p^{(1\text{st})}_{\mathrm{II}(ij)}$	$\omega^{(1\text{st},ij)}_{4,\mathrm{II}}(a,b,c,d)=\exp\big(\frac{2\pi i p^{(1\text{st})}_{\mathrm{II}(ij)}}{(N_{ij}\cdot N_j)}(a_i b_j)(c_j+d_j-[c_j+d_j])\big)$	Types I and II of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ij}}$	Type II 2nd $p^{(2\text{nd})}_{\mathrm{II}(ij)}$	$\omega^{(2\text{nd},ij)}_{4,\mathrm{II}}(a,b,c,d)=\exp\big(\frac{2\pi i p^{(2\text{nd})}_{\mathrm{II}(ij)}}{(N_{ij}\cdot N_i)}(a_j b_i)(c_i+d_i-[c_i+d_i])\big)$	Types I and II of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ijl}}$	Type III 1st $p^{(1\text{st})}_{\mathrm{III}(ijl)}$	$\omega^{(1\text{st},ijl)}_{4,\mathrm{III}}(a,b,c,d)=\exp\big(\frac{2\pi i p^{(1\text{st})}_{\mathrm{III}(ijl)}}{(N_{ij}\cdot N_l)}(a_i b_j)(c_l+d_l-[c_l+d_l])\big)$	Two type IIs of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ijl}}$	Type III 2nd $p^{(2\text{nd})}_{\mathrm{III}(ijl)}$	$\omega^{(2\text{nd},ijl)}_{4,\mathrm{III}}(a,b,c,d)=\exp\big(\frac{2\pi i p^{(2\text{nd})}_{\mathrm{III}(ijl)}}{(N_{li}\cdot N_j)}(a_l b_i)(c_j+d_j-[c_j+d_j])\big)$	Two type IIs of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{ijlm}}$	Type IV $p_{\mathrm{IV}(ijlm)}$	$\omega^{(ijlm)}_{4,\mathrm{IV}}(a,b,c,d)=\exp\big(\frac{2\pi i p_{\mathrm{IV}(ijlm)}}{N_{ijlm}}a_i b_j c_l d_m\big)$	Type III of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$

One can tentatively write the relation

$$\mathcal{C}^{3\mathrm{D}}_{G,\omega_4}=\mathcal{C}^{2\mathrm{D}}_{G,1(\text{untwist})}\oplus_{b\neq 0}\mathcal{C}^{2\mathrm{D}}_{G,\omega_{3(b)}}. \tag{19}$$

There is a zero-flux $b=0$ sector $\mathcal{C}^{2\mathrm{D}}_{G,1(\text{untwist})}$ (with $\omega_3=1$) where the 2D gauge theory with G is untwisted. There are other direct sums of $\mathcal{C}^{2\mathrm{D}}_{G,\omega_{3(b)}}$ with nonzero b flux insertion that have twisted $\omega_{3(b)}$.

However, different cocycles can represent the same topological order with the equivalent modular data. In the next section we examine Eq. (19) more carefully, not in terms of cocycles, but in terms of the modular data, S^{xyz} and T^{xy}.

III. REPRESENTATION FOR S^{xyz} and T^{xy}

The modular transformations $\hat{\mathsf{S}}^{xy}$, $\hat{\mathsf{T}}^{xy}$, and $\hat{\mathsf{S}}^{xyz}$ of Eqs. (2) and (3) act on the 3D real space as

$$\hat{\mathsf{S}}^{xy}\cdot(x,y,z)=(-y,x,z), \tag{20}$$

$$\hat{\mathsf{T}}^{xy}\cdot(x,y,z)=(x+y,y,z), \tag{21}$$

$$\hat{\mathsf{S}}^{xyz}\cdot(x,y,z)=(z,x,y). \tag{22}$$

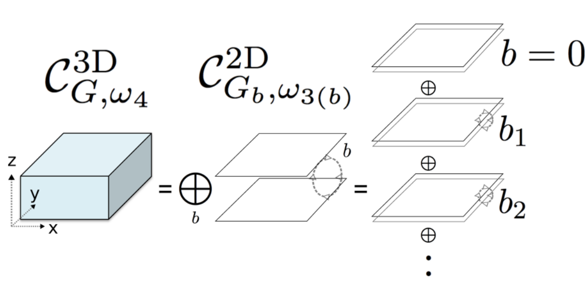
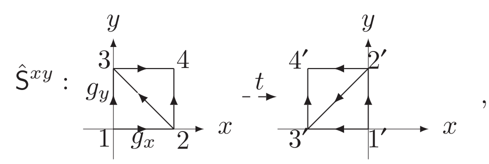
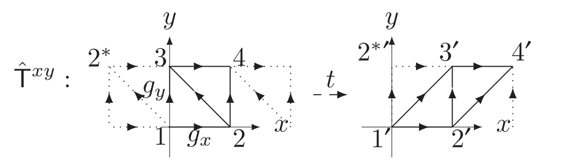


FIG. 4. (Color online) Combining the reasoning in Eq. (18) and Fig. 1, we obtain the physical meaning of dimensional reduction from a $3+1$D twisted gauge theory as a 3D topological order to several sectors of 2D topological orders: $\mathcal{C}^{3\mathrm{D}}_{G,\omega_4}=\oplus_b\mathcal{C}^{2\mathrm{D}}_{G,\omega_{3(b)}}$. Here b stands for the gauge flux (Wilson line operator) of gauge group G. Here ω_3 are dimensionally reduced 3-cocycles from 4-cocycles ω_4. Note that there is a zero-flux $b=0$ sector with $\mathcal{C}^{2\mathrm{D}}_{G,(\text{untwist})}=\mathcal{C}^{2\mathrm{D}}_{G}$.

More explicitly, we present triangulations of them:



(23)



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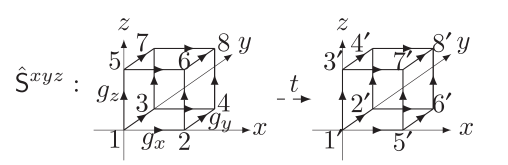

(25)

The modular transformation SL$(2,\mathbb{Z})$ is generated by $\hat{\mathsf{S}}^{xy}$ and $\hat{\mathsf{T}}^{xy}$, while SL$(3,\mathbb{Z})$ is generated by $\hat{\mathsf{S}}^{xyz}$ and $\hat{\mathsf{T}}^{xy}$. The dashed arrow represents the time evolution (as in Fig. 3) from $|\Psi_{\text{in}}\rangle$ to $|\Psi_{\text{out}}\rangle$ under $\hat{\mathsf{S}}^{xy}$, $\hat{\mathsf{T}}^{xy}$, and $\hat{\mathsf{S}}^{xyz}$, respectively. The $\hat{\mathsf{S}}^{xy}$ and $\hat{\mathsf{T}}^{xy}$ transformations on a $\mathbb{T}^3$ torus's x-y plane with the z direction untouched are equivalent to its transformations on a $\mathbb{T}^2$ torus.

Q4: *What are the generic expressions of SL(3,$\mathbb{Z}$) modular data?*

First, in Sec. III A, we apply the *cocycle approach* using the *spacetime path integral* with SL$(3,\mathbb{Z})$ transformation acting along the time evolution to formulate the SL$(3,\mathbb{Z})$ modular data, and then in Sec. III B we use the more powerful *representation* (Rep) *theory* to determine the general expressions of those data in terms of (G,ω_4).

A. Path integral and cocycle approach

The cocycle approach uses the spacetime lattice formalism, where we triangulate the spacetime complex of a 4-

manifold $\mathcal{M} = \mathbb{T}^3 \times I$ (a $\mathbb{T}^3$ torus times a time interval I) of Eqs. (23)–(25) into 4-simplexes. We then apply the path integral $\mathbf{Z}$ in Eq. (8) and the amplitude form in Eq. (10) to obtain

$$\mathsf{T}^{xy}_{(\mathrm{A})(\mathrm{B})} = \langle \Psi_{\mathrm{A}} | \hat{\mathsf{T}}^{xy} | \Psi_{\mathrm{B}} \rangle, \tag{26}$$

$$\mathsf{S}^{xy}_{(\mathrm{A})(\mathrm{B})} = \langle \Psi_{\mathrm{A}} | \hat{\mathsf{S}}^{xy} | \Psi_{\mathrm{B}} \rangle, \tag{27}$$

$$\mathsf{S}^{xyz}_{(\mathrm{A})(\mathrm{B})} = \langle \Psi_{\mathrm{A}} | \hat{\mathsf{S}}^{xyz} | \Psi_{\mathrm{B}} \rangle, \tag{28}$$

$$\mathrm{GSD} = \mathrm{Tr}[\mathsf{P}] = \sum_A \langle \Psi_{\mathrm{A}} | \mathsf{P} | \Psi_{\mathrm{A}} \rangle. \tag{29}$$

Here $|\Psi_{\mathrm{A}}\rangle$ and $|\Psi_{\mathrm{B}}\rangle$ are the ground-state bases on the $\mathbb{T}^d$ torus; for example, they are $|\alpha,a\rangle$ (with α charge and a flux) in $2+1$ dimensions and $|\alpha,a,b\rangle$ (with α charge and a,b fluxes) in $3+1$ dimensions. We also include the data on GSD, where P is the projection operator for ground states discussed in Sec. II B. In the case of dD GSD on $\mathbb{T}^d$ (e.g., 3D GSD on $\mathbb{T}^3$), we simply compute the $\mathbf{Z}$ amplitude filling in $\mathbb{T}^d \times S^1 = \mathbb{T}^{d+1}$. There is no shortcut here except doing explicit calculations [56].

B. Representation theory approach

The cocycle approach in Sec. III A provides nice physical intuition about the modular transformation process. However, the calculation is tedious. There is a powerful approach simply using Rep theory; we present the general formula of $\hat{\mathsf{S}}^{xys}$, $\hat{\mathsf{T}}^{xy}$, $\hat{\mathsf{S}}^{xy}$ data in terms of (G,ω_4) directly. The three steps are outlined as follows: (i) Obtain Eq. (15)'s $\mathsf{C}^{(2)}_{a,b}$ value by doing the slant product twice from the 4-cocycle ω_4 or triangulating Eq. 12. (ii) Derive $\widetilde{\rho}^{\,a,b}_{\alpha}(c)$ of the $\mathsf{C}^{(2)}_{a,b}$ projective Rep in Eq. (14), where $\widetilde{\rho}^{\,a,b}_{\alpha}(c)$ is the GL matrix. (iii) Write the modular data in the canonical basis $|\alpha,a,b\rangle$, $|\beta,c,d\rangle$ of Eq. (11).

After some long computations [56], we find the most general formula S^{xyz} for a group G (both Abelian or non-Abelian) with cocycle twist ω_4:

$$\begin{aligned}\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)} &= \frac{1}{|G|} \langle \alpha_x, a_y, b_z | \sum_w \mathsf{S}^{xyz}_w | \beta_{x'}, c_{y'}, d_{z'} \rangle \\ &= \frac{1}{|G|} \sum_{\substack{g_y \in C^a \cap Z_{g_z} \cap Z_{g_x},\\ g_z \in C^b \cap C^c,\\ g_x \in Z_{g_y} \cap Z_{g_z} \cap C^d}} \mathrm{Tr}\widetilde{\rho}^{\,g_y,g_z}_{\alpha_x}(g_x)^*\, \mathrm{Tr}\widetilde{\rho}^{\,g_z,g_x}_{\beta_y}(g_y) \\ &\quad \times \delta_{g_x,h_{z'}} \delta_{g_y,h_{x'}} \delta_{g_z,h_{y'}}.\end{aligned} \tag{30}$$

Here C^a, C^b, C^c, and C^d are conjugacy classes of the group elements $a,b,c,d \in G$. In the case of a non-Abelian G, we should regard a,b as the conjugacy class C^a, C^b in $|\alpha,a,b\rangle$. Z_g means the centralizer of the conjugacy class of g. For an Abelian G, it simplifies to

$$\begin{aligned}&\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)} \\ &= \frac{1}{|G|} \mathrm{Tr}\widetilde{\rho}^{\,a,b}_{\alpha}(d)^*\, \mathrm{Tr}\widetilde{\rho}^{\,b,d}_{\beta}(a) \delta_{b,c} \equiv \frac{1}{|G|} \mathsf{S}^{\alpha,\beta}_{d,a,b} \delta_{b,c} \\ &= \frac{1}{|G|} \mathrm{Tr}\widetilde{\rho}^{\,a_y,b_z}_{\alpha_x}(d_{z'})^* \mathrm{Tr}\widetilde{\rho}^{\,b_z,d_{z'}}_{\beta_{x'}}(a_y) \delta_{b_z,c_{y'}} \equiv \frac{1}{|G|} \mathsf{S}^{\alpha_x,\beta_y}_{d_x,a_y,b_z} \delta_{b_z,c_{y'}}.\end{aligned}$$

We write $\beta_{x'} = \beta_y$, $d_{z'} = d_x$ due to the coordinate identification under $\hat{\mathsf{S}}^{xyz}$. The assignments of the directions of gauge fluxes (group elements) are clearly expressed in the second line. We may use the first-line expression for simplicity.

We also provide the most general formula of T^{xy} in the $|\alpha,a,b\rangle$ basis:

$$\mathsf{T}^{xy} = \mathsf{T}^{a_y,b_z}_{\alpha_x} = \frac{\mathrm{Tr}\widetilde{\rho}^{\,a_y,b_z}_{\alpha_x}(a_y)}{\dim(\alpha)} \equiv \exp\left(\mathrm{i}\Theta^{a_y,b_z}_{\alpha_x}\right). \tag{31}$$

Here $\dim(\alpha)$ means the dimension of the representation or, equivalently, the rank of the matrix of $\widetilde{\rho}^{\,a,b}_{\alpha_x}(c)$. Since $\mathrm{SL}(2,\mathbb{Z})$ is a subgroup of $\mathrm{SL}(3,\mathbb{Z})$, we can express the $\mathrm{SL}(2,\mathbb{Z})$ value of S^{xy} as the $\mathrm{SL}(3,\mathbb{Z})$ values of S^{xyz} and T^{xy} (an expression for both the real spatial basis and the canonical basis):

$$\mathsf{S}^{xy} = ((\mathsf{T}^{xy})^{-1}\mathsf{S}^{xyz})^3(\mathsf{S}^{xyz}\mathsf{T}^{xy})^2\mathsf{S}^{xyz}(\mathsf{T}^{xy})^{-1}. \tag{32}$$

For an Abelian G, and when $\mathsf{C}^{(2)}_{a,b}(c,d)$ is a 2-coboundary (cohomologically trivial), the dimensionality of Rep is $\dim(\mathrm{Rep}) \equiv \dim(\alpha) = 1$, and the S^{xy} is simplified:

$$\mathsf{S}^{xy}_{(\alpha,a,b)(\beta,c,d)} = \frac{1}{|G|} \frac{\mathrm{tr}\widetilde{\rho}^{\,a,b}_{\alpha}(ac^{-1})^*}{\mathrm{tr}\widetilde{\rho}^{\,a,b}_{\alpha}(a)} \frac{\mathrm{tr}\widetilde{\rho}^{\,c,d}_{\beta}(ac^{-1})}{\mathrm{tr}\widetilde{\rho}^{\,c,d}_{\beta}(c)} \delta_{b,d}. \tag{33}$$

We can verify the above results by first computing the cocycle path integral approach in Sec. III A and substituting from the flux basis to the canonical basis in Eq. (11). We have made several consistent checks, by comparing our $\hat{\mathsf{S}}^{xy}$, $\hat{\mathsf{T}}^{xy}$, and $\hat{\mathsf{S}}^{xyz}$ to (i) the known 2D case for the untwisted theory of a non-Abelian group [40], (ii) the recent 3D case for the untwisted theory of a non-Abelian group [39], and (iii) the recent 3D case for the twisted theory of an Abelian group [46]. And our expression works for all cases: the (un)twisted theory of a (non-)Abelian group. More detailed calculations are provided in Appendix B.

C. Physics of S and T in three dimensions

S^{xy} and T^{xy} in two dimensions are known to have precise physical meanings. At least for Abelian topological orders, there is no ambiguity that S^{xy} in the quasiparticle basis provides the mutual statistics of two particles (winding one around the other as 2π), while T^{xy} in the quasiparticle basis provides the self statistics of two identical particles (winding one around the other as π). Moreover, the intimate spin-statistics relation shows that the statistical phase $e^{\mathrm{i}\Theta}$ gained by interchanging two identical particles is equal to the spin s as $e^{\mathrm{i}2\pi s}$. Figure 5 illustrates the spin-statistics relation [59]. Thus, people also call T^{xy} in two dimensions the *topological spin*. Here we ask question 5:

> **Q5**: *What is the physical interpretation of* $\mathrm{SL}(3,\mathbb{Z})$ *modular data in three dimensions?*

Our approach, again, is by dimensional reduction of Fig. 1, via Eqs. (4) and (5): $\mathsf{S}^{xy} = \oplus_b \mathsf{S}^{xy}_b$, $\mathsf{T}^{xy} = \oplus_b \mathsf{T}^{xy}_b$, and $\mathcal{C}^{3\mathrm{D}} = \oplus_b \mathcal{C}^{2\mathrm{D}}_b$, reducing the 3D physics to the direct sum of 2D topological phases in different flux sectors, so we can retrieve the familiar physics in two dimensions to interpret three dimensions. For our case with a gauge group description, b (subindex of S^{xy}_b, T^{xy}_b, $\mathcal{C}^{2\mathrm{D}}_b$) labels the gauge flux (group element or conjugacy class C^b) winding around the compact z

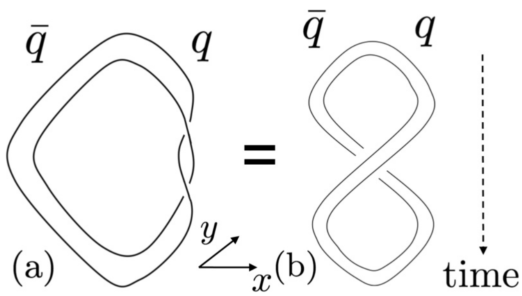

FIG. 5. Both process (a) and process (b) start from the creation of a pair of particle q and antiparticle $\bar{q}$, but the worldlines evolve along time to the bottom differently. Process (a) produces a phase $e^{i2\pi s}$ due to 2π rotation of q, with spin s. Process (b) produces a phase $e^{i\Theta}$ due to the exchange statistics. The homotopic equivalence by deformation implies $e^{i2\pi s} = e^{i\Theta}$.

direction in Fig. 1. This b flux can be viewed as the by-product of a monodromy defect causing a branch cut (a symmetry twist [55,57,58,60]), such that the wave function will gain a phase by winding around the compact z direction. Now we further regard the b flux as a string threading around in the background, so that winding around this background string [e.g., black string threading in Figs. 2(c), 6(c), and 7(c)] obtains the b flux effect if there is a nontrivial winding in the compact direction z. The dashed arrow along the compact z schematically indicates such a b flux effect from the background string threading.

1. T_b^{xy} and topological spin of a closed string

We apply the above idea to interpret T_b^{xy}, shown in Fig. 6. From Eq. (31), we have $\mathsf{T}_b^{xy} = \mathsf{T}_{\alpha_x}^{a_y,b_z} \equiv \exp(i\Theta_{\alpha_x}^{a_y,b_z})$ with a fixed b_z label for a given b_z flux sector. For each b, T_b^{xy} acts as a familiar 2D T matrix, $\mathsf{T}_{\alpha_x}^{a_y}$, which provides the topological spin of a quasiparticle (α,a) with charge α and flux a, in Fig. 6(a).

From the 3D viewpoint, however, this $|\alpha,a\rangle$ particle is actually a closed string compactified along the compact z direction. Thus, in Fig. 6(b), the self-2π rotation of the topological spin of a quasiparticle $|\alpha,a\rangle$ is indeed the self-2π rotation of a framed closed string. (Physically we understand that there the string can be framed with arrows, because the inner texture of the string excitations are allowed in a condensed matter system, due to defects or the finite-size lattice geometry.) Moreover, from the equivalent 3D view in Fig. 6(c), we can view the self-2π rotation of a framed closed string as the *self-2π flipping* of a framed closed string, which flips the string inside-out and then outside-in, back to its original status. This picture works for both the $b = 0$ zero-flux sector and $b \neq 0$ under the background string threading. *We thus propose T_b^{xy} as the topological spin of a framed closed string, threaded by a background string carrying a monodromy b flux.*

2. S_b^{xy} and three-string braiding statistics

Similarly, we apply the same philosophy to do 3D-to-2D reduction for S_b^{xy}, each effective two dimensions threading with a distinct gauge flux b. We can obtain S_b^{xy} from Eq. (32) with $\mathrm{SL}(3,\mathbb{Z})$ modular data. Here we focus on interpreting S_b^{xy} in the Abelian topological order. Writing S_b^{xy} in the canonical basis $|\alpha,a,b\rangle$, $|\beta,c,d\rangle$ of Eq. (11), we find that, true to Abelian topological order,

$$\mathsf{S}_b^{xy} = \mathsf{S}_{(\alpha,a,b)(\beta,c,d)}^{xy} \equiv \frac{1}{|G|}\mathsf{S}_{a,c\ (b)}^{2\mathrm{D}\ \alpha,\beta}\delta_{b,d}. \tag{34}$$

As we predict the generality in Eq. (4), the S_b^{xy} here is diagonalized with the b and d identified (as the z-direction flux created by the background string threading). For a given fixed-b-flux sector, the only free indices are $|\alpha,a\rangle$ and $|\beta,c\rangle$, all collected in $\mathsf{S}_{a,c\ (b)}^{2\mathrm{D}\ \alpha,\beta}$. (Explicit data are be presented in Sec. IV B.) Our interpretation is shown in Fig. 2. From a 2D viewpoint, S_b^{xy} gives the full 2π braiding statistics data for two quasiparticle $|\alpha,a\rangle$ and $|\beta,c\rangle$ excitations in Fig. 2(a). However, from the 3D viewpoint, the two particles are actually two closed strings compactified along the compact z direction. Thus, the full-2π braiding of two particles in Fig. 2(a) becomes that of two closed strings in Fig. 2(b). More explicitly, in the equivalent 3D view in Fig. 2(c), we identify the coordinates x,y,z carefully to see that such a full-braiding process is *one (red) string going inside to the loop of another (blue) string and then going back from the outside.*

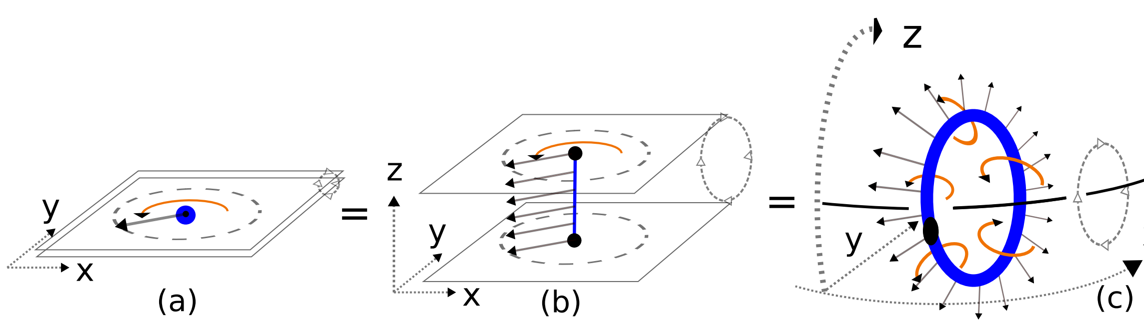

FIG. 6. (Color online) Topological spin of (a) a particle by 2π-self rotation in two dimensions; (b) a framed closed string by 2π-self rotation in three dimensions with a compact z; (c) a closed string [(blue) loop] by 2π-self flipping, threaded by a background (black) string creating monodromy b flux (along the dashed arrow), under a single Hopf link 2_1^2 configuration. All above equivalent pictures describe the physics of topological spin in terms of T_b^{xy}. For Abelian topological orders, the spin of an excitation (say A) in (a) yields an Abelian phase $e^{i\Theta_{(A)}} = \mathsf{T}_{(A)(A)}^{xy}$ proportional to the diagonal of the 2D T^{xy} matrix. The dimensional-extended equivalent picture (c) implies that the loop-flipping yields a phase $e^{i\Theta_{(A),b}} = \mathsf{T}_{b\,(A)(A)}^{xy}$ of Eq. (31) (up to a choice of canonical basis), where b is the flux of the black string.

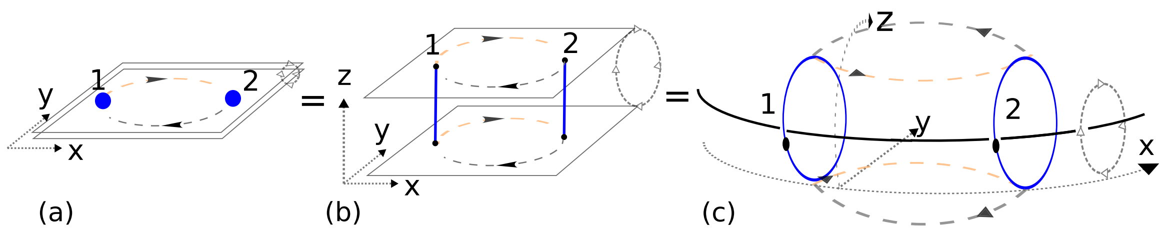

FIG. 7. (Color online) Exchange statistics of (a) two identical particles at positions 1 and 2 by a π winding (half-winding), (b) two identical strings by a π winding in three dimensions with a compact z, and (c) two identical closed strings [(blue) loops] with a π-winding around, both threaded by a background (black) string creating monodromy b flux, under the Hopf links $2_1^2\#2_1^2$ configuration. (a)–(c) describe the equivalent physics in three dimensions with a compact z direction. The physics of exchange statistics of a closed string turns out to be related to the topological spin in Fig. 6, discussed in Sec. III C 3.

The above picture works again for both the $b=0$ zero-flux sector and $b \neq 0$ under background string threading. When $b \neq 0$, the third (black) background string in Fig. 2(c) threads through the two (red and blue) strings. The third (black) string creates the monodromy defect/branch cut in the background and carries b flux along z acting on two (red and blue) strings which have nontrivial winding on the third string. This three-string braiding was first emphasized in a recent paper [45]; here we make further connections to the S_b^{xy} data and illuminate its physics in a 3D-to-2D reduction under b flux sectors.

We have proposed and shown that S_b^{xy} can capture the physics of three-string braiding statistics with two strings threaded by a third background string causing b flux monodromy, where the three strings have the linking configuration as the connected sum of two Hopf links $2_1^2\#2_1^2$.

3. Spin-statistics relation for closed strings

Since a spin-statistics relation for 2D particles is shown in Fig. 5, we may wonder, by using our 3D-to-2D reduction picture, whether a spin-statistics relation for a closed string holds? To answer this question, we should compare the topological spin picture of $\mathsf{T}_b^{xy} = \mathsf{T}_{\alpha_x}^{a_y,b_z} \equiv \exp(\mathrm{i}\Theta_{\alpha_x}^{a_y,b_z})$ to the exchange statistic picture of two closed strings in Fig. 7. Figure 7 essentially takes a half-braiding of the S_b^{xy} process in Fig. 2 and considers doing half-braiding on the same excitations in $|\alpha,a,b\rangle = |\beta,c,d\rangle$. In principle, one can generalize the framed worldline picture of particles in Fig. 5 to the framed worldsheet picture of closed strings. (The framed worldline is like a worldsheet; the framed worldsheet is like a worldvolume.) This interpretation shows that the topological spin in Fig. 6 and the exchange statistics in Fig. 7 carry the same data, namely,

$$\mathsf{T}_b^{xy} = \mathsf{T}_{\alpha_x}^{a_y,b_z} = \left(\mathsf{S}^{\text{2D}\ \alpha_x,\alpha_x}_{a_y,a_y\ (b_z)}\right)^{\frac{1}{2}} \quad \text{or} \quad \left(\mathsf{S}^{\text{2D}\ \alpha_x,\alpha_x}_{a_y,a_y\ (b_z)}\right)^{\frac{1}{2}*}, \tag{35}$$

from the data of Eqs. (31) and (34). The equivalence holds, up to a [complex conjugate (*)] sign caused by the orientation of the rotation and the exchange.

In Sec. IV B, we show, for the twisted gauge theory of Abelian topological orders, that such an interpretation of Eq. (35) is correct and agrees with our data. We term this the *spin-statistics relation for a closed string*.

In this section, we have obtained the explicit formulas of S^{xyz}, T^{xy}, and S^{xy} (Secs. III A and III B) as well as capturing the physical meanings of S_b^{xy} and T_b^{xy} (Sec. III C 3). Before concluding, we note that the full understanding of S^{xyz} seems to be intriguingly related to its 3D nature. It is *not* obvious to us that the use of 3D-to-2D reduction can capture all the physics of S^{xyz}. We comment on this issue again in Sec. V.

IV. SL(3,$\mathbb{Z}$) MODULAR DATA AND MULTISTRING BRAIDING

A. Ground-state degeneracy and particle and string types

We now proceed to study the topology-dependent GSD, modular data S, T of the $3+1$D twisted gauge theory with finite group $G=\prod_i Z_{N_i}$. We comment that the GSD on $\mathbb{T}^2$ of 2D topological order counts the number of quasiparticle excitations, which, from the representation theory, is simply counting the number of charges α and fluxes a forming the quasiparticle basis $|\alpha,a\rangle$ spanning the ground-state Hilbert space. In two dimensions, GSD counts the number of types of quasiparticles (or anyons) as well as the number of bases $|\alpha,a\rangle$. For higher dimensions, GSD on $\mathbb{T}^d$ of dD topological order still counts the number of canonical bases $|\alpha,a,b,\dots\rangle$, however, it may overcount the number of types of particles (with charge), strings (with flux), etc., excitations. From an untwisted Z_N field theory perspective, the fluxed string may be described by a two-form B field, and the charged particle may be described by a one-form A field, with a BF action $\int B dA$. As we can see, fluxes a and b are overcounted.

We suggest that counting the number of types of particles of d dimensions is equivalent to the process in Fig. 8, where we dig a ball B^d with a sphere S^{d-1} around particle q, which resides on S^d. And we connect it through an S^1 tunnel to its antiparticle $\bar{q}$. This process causes creation-annihilation from vacuum and counts how many types of q sectors are equivalent to

$$\text{the number of particle types} = \text{GSD on } S^{d-1}\times I, \tag{36}$$

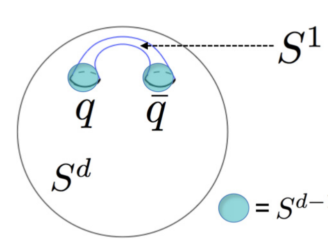

FIG. 8. (Color online) Number of particle types = GSD on $S^{d-1}\times S^1$.

TABLE II. $\mathsf{S}^{xyz}=\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)}\equiv\frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{bc}$ modular data on the 3 + 1D twisted gauge theories with $G=Z_{N_1}\times Z_{N_2}\times Z_{N_3}$. In the last column, $\mathcal{H}^3$ stands for $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$; the induced S^{xy}_b is listed in Table IV.

$\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$	4-cocycle	$\mathsf{S}^{\alpha,\beta}_{d,a,b}$	Induced S^{xy}_b
$\mathbb{Z}_{N_{12}}$	Type II 1st	$\exp\left(\sum_k \frac{2\pi i}{N_k}\,(\beta_k a_k-\alpha_k d_k)\right)\cdot\exp\left(\frac{2\pi i p^{(1st)}_{II(12)}}{(N_{12}\cdot N_2)}(a_1d_2+a_2d_1)b_2-2a_2b_1d_2\right)$	Types I and II of $\mathcal{H}^3$
$\mathbb{Z}_{N_{12}}$	Type II 2nd	$\exp\left(\sum_k \frac{2\pi i}{N_k}\,(\beta_k a_k-\alpha_k d_k)\right)\cdot\exp\left(\frac{2\pi i p^{(2nd)}_{II(12)}}{(N_{12}\cdot N_1)}(a_1d_2+a_2d_1)b_1-2a_1b_2d_1\right)$	Types I and II of $\mathcal{H}^3$
$\mathbb{Z}_{N_{123}}$	Type III 1st	$\exp\left(\sum_k \frac{2\pi i}{N_k}\,(\beta_k a_k-\alpha_k d_k)\right)\cdot\exp\left(\frac{2\pi i p^{(1st)}_{III(123)}}{(N_{12}\cdot N_3)}(a_1b_2-a_2b_1)d_3+(b_2d_1-b_1d_2)a_3\right)$	Two type IIs of $\mathcal{H}^3$
$\mathbb{Z}_{N_{123}}$	Type III 2nd	$\exp\left(\sum_k \frac{2\pi i}{N_k}\,(\beta_k a_k-\alpha_k d_k)\right)\cdot\exp\left(\frac{2\pi i p^{(2nd)}_{III(123)}}{(N_{31}\cdot N_2)}(a_3b_1-a_1b_3)d_2+(b_1d_3-b_3d_1)a_2\right)$	Two type IIs of $\mathcal{H}^3$

with $I\simeq S^1$ for this example. For the spacetime integral, one evaluates Eq. (29) with $\mathcal{M}=S^{d-1}\times S^1\times S^1$.

For counting closed string excitations, one may naively use $\mathbb{T}^2$ to enclose a string, analogously to using $\mathbb{S}^2$ to enclose a particle in three dimensions. Then one may deduce the number of string types = GSD on $\mathbb{T}^2\times S^1\overset{?}{=}\mathbb{T}^3$, and that of spacetime integrals on $\mathbb{T}^4$, as mentioned earlier, which is *incorrect* and overcounting. We suggest

$$\text{the number of string types} = \mathsf{S}^{xy}\text{ and }\mathsf{T}^{xy}\text{'s number of blocks,} \tag{37}$$

whose blocks are labeled b in the form of Eq. (4). We show explicit examples of counting using Eqs. (36) and (37) in Sec. IV B.

B. Abelian examples: 3D twisted $Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ gauge theories with types II and III 4-cocycle twists

We first study the most generic 3 + 1D finite Abelian twisted gauge theories with types II and III 4-cocycle twists in Table I. It is general enough for us to consider $G=Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ with nonvanished gcd N_{ij},N_{ijl}. Types II and III (both the first and the second kinds) twisted gauge theories have GSD $=|G|^3$ on the spatial $\mathbb{T}^3$ torus. As such, the canonical basis $|\alpha,a,b\rangle$ of the ground-state sector labels the charge (α along x) and two fluxes (a and b along y and z); each of the three has $|G|$ kinds. Thus, naturally from the Rep theory viewpoint, we have GSD $=|G|^3$. However, as mentioned in Sec. IV A, $|G|^3$ overcounts the number of strings and particles. By using Eqs. (36) and (37), we find that there are $|G|$ types of particles and $|G|$ types of strings. The canonical basis $|\alpha,a,b\rangle$ (GSD on $\mathbb{T}^3$) counts twice the flux sectors.

In Table II, we list their S^{xyz} values by computing Eq. (30), where we denote $a=(a_1,a_2,a_3,\dots)$, with $a_j\in Z_{N_j}$, and the same notation for other b, c, and d fluxes.

Here we extract the $\mathsf{S}^{\alpha,\beta}_{d,a,b}$ part of S^{xyz}, ignoring the $|G|^{-1}$ factor:

$$\mathsf{S}^{xyz}=\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)}\equiv\frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{b,c}. \tag{38}$$

The S matrix reads $g_{x_k}=d_k$, $g_{y_k}=a_k$ in Eq. (30). In Table III, we show T^{xy}. Here for an Abelian G, where $\mathsf{C}^{(2)}_{a,b}(c,d)$ is a 2-coboundary (cohomologically trivial) and thus dim(Rep) = 1, we compute S^{xy} by Eq. (33) and that reduces to Eq. (34), $\mathsf{S}^{xy}_b=(\mathsf{S}^{xy})_{(\alpha,a,b)(\beta,c,d)}\equiv\frac{1}{|G|}\mathsf{S}^{2\mathrm{D}\,\alpha,\beta}_{a,c\,(b)}\delta_{b,d}$. In Table IV, we list S^{xy} in terms of $\mathsf{S}^{2\mathrm{D}\,\alpha,\beta}_{a,c\,(b)}$ for simplicity.

Several remarks follow. (i) For an untwisted gauge theory (topological term $p_{..}=0$), which is the direct product of Z_N gauge theory or Z_N toric code, its statistics has the form $\exp(\sum_k\frac{2\pi i}{N_k}\,(\beta_k a_k-\alpha_k d_k))$ and $\exp(\sum_k\frac{2\pi i}{N_k}\,\alpha_k\cdot a_k)$. This is described by the BF theory of $\int BdA$ action, withith α,β as the charge of particles (1-form gauge field A) and a,b as the flux of string (2-form gauge field B). This essentially describes the braiding between a pure particle and a pure string.

(ii) Both S^{xy} and T^{xy} have block diagonal forms S^{xy}_b and T^{xy}_b with respect to the b flux (along z), which correctly reflects what Eq. (4) predicts.

(iii) T^{xy} is in the SL(3,$\mathbb{Z}$) canonical basis automatically and fully diagonal, but S^{xy} may not be in the canonical basis for each block of S^{xy}_b, due to its SL(2,$\mathbb{Z}$) nature. We can find the proper basis for each b block by the method of Ref. [61]. Nevertheless, the eigenvalues of S^{xy} in Table IV are still proper and invariant regardless of the basis.

(iv) To characterize the topological orders, we can further compare the 3D S^{xy}_b data to the SL(2,$\mathbb{Z}$) data on the 2D S^{xy} of

TABLE III. T^{xy} modular data of the 3 + 1D twisted gauge theories with $G=Z_{N_1}\times Z_{N_2}\times Z_{N_3}$. We can view this in terms of the index b for blocks of $\mathsf{T}^{xy}_b=\mathsf{T}^{a_y,b_z}_{\alpha_x}$, with the flux b along the compact z direction.

$\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$	4-cocycle	$\mathsf{T}^{a,b}_{\alpha}$	Induced T^{xy}_b
$\mathbb{Z}_{N_{12}}$	Type II 1st	$\exp\left(\sum_k\frac{2\pi i}{N_k}\,\alpha_k\cdot a_k\right)\cdot\exp\left(\frac{2\pi i p^{(1st)}_{II(12)}}{(N_{12}\cdot N_2)}(a_2b_1-a_1b_2)(a_2)\right)$	Types I and II of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{12}}$	Type II 2nd	$\exp\left(\sum_k\frac{2\pi i}{N_k}\,\alpha_k\cdot a_k\right)\cdot\exp\left(\frac{2\pi i p^{(2nd)}_{II(12)}}{(N_{12}\cdot N_1)}(a_1b_2-a_2b_1)(a_1)\right)$	Types I and II of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{123}}$	Type III 1st	$\exp\left(\sum_k\frac{2\pi i}{N_k}\,\alpha_k\cdot a_k\right)\cdot\exp\left(\frac{2\pi i p^{(1st)}_{III(123)}}{(N_{12}\cdot N_3)}(a_2b_1-a_1b_2)(a_3)\right)$	Two type IIs of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$
$\mathbb{Z}_{N_{123}}$	Type III 2nd	$\exp\left(\sum_k\frac{2\pi i}{N_k}\,\alpha_k\cdot a_k\right)\cdot\exp\left(\frac{2\pi i p^{(2nd)}_{III(123)}}{(N_{31}\cdot N_2)}(a_1b_3-a_3b_1)(a_2)\right)$	Two type IIs of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$

TABLE IV. S^{xy} modular data on the $3+1$D twisted gauge theories with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$. There are two more columns [$\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ and induced S_b^{xy}] not shown here, since the data simply duplicate the first and fourth columns in Table II. The basis chosen here is not canonical for excitations, in the sense that particle braiding around a trivial vacuum still obtains a nontrivial statistic phase. Finding the proper canonical basis for each b block of S_b^{xy} can be done by the method in Ref. [61].

ω_4	$\mathsf{S}^{\text{2D}\,\alpha,\beta}_{a,c\,(b)} = \text{tr}\widetilde{\rho}_\alpha^{a,b}(a^2c^{-1})^*\text{tr}\widetilde{\rho}_\beta^{c,b}(ac^{-2})$
II 1st	$\exp\left(\sum_k \frac{2\pi \text{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\right)\cdot\exp\left(\frac{2\pi \text{i} p_{\text{II}(12)}^{(\text{1st})}}{(N_{12}\cdot N_2)} b_1(2a_2c_2-2a_2^2-2c_2^2)+b_2(2a_1a_2+2c_1c_2-a_1c_2-a_2c_1)\right)$
II 2nd	$\exp\left(\sum_k \frac{2\pi \text{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\right)\cdot\exp\left(\frac{2\pi \text{i} p_{\text{II}(12)}^{(\text{2nd})}}{(N_{12}\cdot N_1)} b_2(2a_1c_1-2a_1^2-2c_1^2)+b_1(2a_1a_2+2c_1c_2-a_1c_2-a_2c_1)\right)$
III 1st	$\exp\left(\sum_k \frac{2\pi \text{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\right)\cdot\exp\left(\frac{2\pi \text{i} p_{\text{III}(123)}^{(\text{1st})}}{(N_{12}\cdot N_3)} b_1(a_2c_3+a_3c_2-2a_2a_3-2c_2c_3)+b_2(2a_1a_3+2c_1c_3-a_1c_3-a_3c_1)\right)$
III 2nd	$\exp\left(\sum_k \frac{2\pi \text{i}}{N_k}(\alpha_k(c_k-2a_k)+\beta_k(a_k-2c_k))\right)\cdot\exp\left(\frac{2\pi \text{i} p_{\text{III}(123)}^{(\text{2nd})}}{(N_{31}\cdot N_2)} b_3(a_1c_2+a_2c_1-2a_1a_2-2c_1c_2)+b_1(2a_3a_2+2c_3c_2-a_3c_2-a_2c_3)\right)$

$\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ in Table XII. (See Appendix A for data.) All of the dimensional reductions of these data (S_b^{xy} in Tables II and IV and T_b^{xy} in Table III) agree with the 3-cocycle (induced from the 4-cocycle ω_4) in the final column in Table I. Combining all the data, we conclude that Eq. (19) becomes explicit. For example, type II twists for $G=(Z_2)^2$ as

$$\mathcal{C}^{\text{3D}}_{(Z_2)^2,1} = 4\mathcal{C}^{\text{2D}}_{(Z_2)^2,1}, \tag{39}$$

$$\mathcal{C}^{\text{3D}}_{(Z_2)^2,\omega_{4,\text{II}}} = \mathcal{C}^{\text{2D}}_{(Z_2)^2} \oplus \mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{I}}} \oplus 2\mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{II}}}. \tag{40}$$

Such a type II $\omega_{4,\text{II}}$ can produce a $b=0$ sector of (Z_2 toric code $\otimes$ Z_2 toric code) of two dimensions as $\mathcal{C}^{\text{2D}}_{(Z_2)^2}$, a $b\neq 0$ sector of (Z_2 double-semions $\otimes$ Z_2 toric code) as $\mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{I}}}$, and another $b\neq 0$ sector $\mathcal{C}^{\text{2D}}_{(Z_2)^2,\omega_{3,\text{II}}}$, for example. This procedure can be applied to other types of cocycle twists.

(v) To classify the topological orders, we interpret the decomposition in Eq. (19) as the implication of classification. Let us do the counting of the number of phases in the simplest example of type II, $G=Z_2\times Z_2$ twisted theory. There are four types in $(p_{\text{II}(12)}^{(1st)},p_{\text{II}(12)}^{2nd}) \in \mathcal{H}^4(G,\mathbb{R}/\mathbb{Z}) = (\mathbb{Z}_2)^2$. However, we find that there are only two distinct topological orders of four. One is the trivial $(Z_2)^2$ gauge theory as Eq. (39); the other is the nontrivial type as Eq. (40). There are two ways to see this: (i) from the full S^{xyz}, T^{xy} data and (ii) by viewing the sector of S_b^{xy}, T_b^{xy} under distinct fluxes b, which is from an $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ perspective. We should be aware that, in principle, tagging particles, strings, or gauge groups is not allowed, so one can identify many seemingly different orders by relabeling their excitations. We give more examples of counting 2D and 3D topological orders in Appendix A.

(vi) The spin-statistics relation of closed strings in Eq. (35) is verified to be correct here, while we take the complex conjugate in Eq. (35). This is why we draw the orientations in Figs. 6 and 7 oppositely. Interpreting T^{xy} as the topological spin also holds.

(vii) For all the above data (types II and III), there is a special cyclic relation for S^{xyz} in three dimensions when the charge labels are equal, $\alpha=\beta$ for $\mathsf{S}^{\alpha,\beta}_{a,b,d}$ (e.g., for pure fluxes $\alpha=\beta=0$, namely, for pure strings):

$$\mathsf{S}^{\alpha,\alpha}_{a,b,d}\cdot\mathsf{S}^{\alpha,\alpha}_{b,d,a}\cdot\mathsf{S}^{\alpha,\alpha}_{d,a,b} = 1. \tag{41}$$

However, such a cyclic relation does not hold (even at zero charge) for $\mathsf{S}^{\text{2D}\,\alpha,\beta}_{a,c\,(b)}$, namely, $\mathsf{S}^{\text{2D}\,\alpha,\beta}_{a,c\,(b)}\cdot\mathsf{S}^{\text{2D}\,\alpha,\beta}_{c,b\,(a)}\cdot\mathsf{S}^{\text{2D}\,\alpha,\beta}_{b,a\,(c)} \neq 1$ in general. Some other cyclic relations have been studied recently in Refs. [45] and [46], with which we have not yet made detailed comparisons, but the perspectives may be different. In Ref. [46], their cyclic relation is determined by triple-linking numbers associated with the membrane operators. In Ref. [45], their cyclic relation is related to the loop braiding in Fig. 2, which is relevant instead for $\mathsf{S}^{\text{2D}\,\alpha,\beta}_{a,c\,(b)}$, not our cyclic relation of $\mathsf{S}^{\alpha,\beta}_{a,b,d}$ in three dimensions. We comment more on the difference and the subtlety of S^{xy} and S^{xyz} in Sec. V B.

C. Non-Abelian examples: 3D twisted $(Z_n)^4$ gauge theories with type IV 4-cocycles

We now study a more interesting example, the generic $3+1$D finite Abelian twisted gauge theory with type IV 4-cocycle twists with $p_{ijlm}\neq 0$ in Table I. For generality, our formula also incorporates type IV twists together with the aforementioned types II and III twists. So all 4-cocycle twists are discussed in this subsection. Different from the example of Abelian topological order with Abelian statistics in Sec. IV B, we show that the type IV 4-cocycle $\omega_{4,\text{IV}}$ will cause the gauge theory to become non-Abelian, having non-Abelian statistics even if the original G is Abelian. Our inspiration, rooted in the 2D example of the type III 3-cocycle twist in Table XII, will cause a similar effect, discovered in Ref. [40]. In general, one can consider $G=Z_{N_1}\times Z_{N_2}\times Z_{N_3}\times Z_{N_4}$ with nonvanished gcd N_{1234}; however, we focus on $G=(Z_n)^4$ with $N_{1234}=n$, with n prime for simplicity. From $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_n^{21}$, we have n^{21} types of theories; n^{20} are Abelian gauge theories, and $n^{20}\cdot(n-1)$ types with type IV ω_4 show non-Abelian statistics.

D. Ground-state degeneracy

We compute the GSD of gauge theories with a type IV twist on the spatial $\mathbb{T}^3$ torus, truncated from $=|G|^3=|n^4|^3=n^{12}$ to

$$\text{GSD}_{\mathbb{T}^3,\text{IV}} = (n^8+n^9-n^5)+(n^{10}-n^7-n^6+n^3) \tag{42}$$

$$\equiv \text{GSD}^{\text{Abel}}_{\mathbb{T}^3,\text{IV}} + \text{GSD}^{\text{nAbel}}_{\mathbb{T}^3,\text{IV}}. \tag{43}$$

(We derive the above only for a prime n. The GSD truncation is less severe and is in between $\text{GSD}_{\mathbb{T}^3,\text{IV}}$ and $|G|^3$ for a nonprime n.) As such, the canonical basis $|\alpha,a,b\rangle$ of the ground-state sector on $\mathbb{T}^3$ no longer has $|G|^3$ labels with the $|G|$ number charge and two pairs of $|G| \times |G|$ fluxes as in Sec. IV B. This truncation is due to the nature of non-Abelian physics of type IV $\omega_{4,\text{IV}}$ twisted. We explain our notation in Eq. (43); the (n)Abel indicates the contribution from (non-)Abelian excitations. From the Rep theory viewpoint, we can recover the truncation back to $|G|^3$ by carefully reconstructing the quantum dimension of excitations. We obtain

$$\begin{aligned}|G|^3 &= \left(\text{GSD}^{\text{Abel}}_{\mathbb{T}^3,\text{IV}}\right) + \left(\text{GSD}^{\text{nAbel}}_{\mathbb{T}^3,\text{IV}}\right) \cdot n^2 \\ &= \{n^4 + n^5 - n\} \cdot n^4 \cdot (1)^2 \\ &\quad + \{(n^4)^2 - n^5 - n^4 + n\} \cdot n^2 \cdot (n)^2 \\ &= \left\{\text{Flux}^{\text{Abel}}_{\text{IV}}\right\} \cdot n^4 \cdot (\dim_1)^2 + \left\{\text{Flux}^{\text{nAbel}}_{\text{IV}}\right\} \cdot n^2 \cdot (\dim_n)^2.\end{aligned} \tag{44}$$

$\dim_m$ means that the dimension of Rep as dim(Rep) is m, which is also the *quantum dimension* of excitations. Here we have dimension 1 for Abelian and n for non-Abelian. In summary, we understand the decomposition precisely in terms of each (non-)Abelian contribution as follows:

$$\begin{aligned}\text{Flux sectors} &= |G|^2 = |n^4|^2 \\ &= \text{Flux}^{\text{Abel}}_{\text{IV}} + \text{Flux}^{\text{nAbel}}_{\text{IV}}. \\ \text{GSD}_{\mathbb{T}^3,\text{IV}} &= \text{GSD}^{\text{Abel}}_{\mathbb{T}^3,\text{IV}} + \text{GSD}^{\text{nAbel}}_{\mathbb{T}^3,\text{IV}}. \\ \dim(\text{Rep})^2 &= 1^2, n^2. \\ \text{Numbers of charge Rep} &= n^4, n^2.\end{aligned} \tag{45}$$

Actually, the canonical basis $|\alpha,a,b\rangle$ (GSD on $\mathbb{T}^3$) still works, and the sum of Abelian $\text{Flux}^{\text{Abel}}_{\text{IV}}$ and non-Abelian $\text{Flux}^{\text{nAbel}}_{\text{IV}}$ counts the flux number of a,b as the unaltered $|G|^2$. The charge Rep α is unchanged with a number of $|G| = n^4$ for the Abelian sector with a rank 1 matrix (1-dim linear or projective) representation, however, the charge Rep α is truncated to a smaller number n^2 for the non-Abelian sector also, with a larger, rank n matrix (n-dim projective) representation.

Another view of $\text{GSD}_{\mathbb{T}^3,\text{IV}}$ can be inspired by a generic formula like Eq. (4),

$$\text{GSD}_{\mathcal{M}' \times S^1} = \oplus_b \text{GSD}_{b,\mathcal{M}'} = \sum_b \text{GSD}_{b,\mathcal{M}'}, \tag{46}$$

where we sum over GSD in all b flux sectors, with b flux along S^1. Here we can take $\mathcal{M}' \times S^1 = \mathbb{T}^3$ and $\mathcal{M}' = \mathbb{T}^2$. For the non–type IV (untwisted, types II and III) ω_4 case, we have $|G|$ sectors of b flux and each has $\text{GSD}_{b,\mathbb{T}^2} = |G|^2$. For the type IV ω_4 case $G = (Z_n)^4$ with a prime n, we have

$$\begin{aligned}&\text{GSD}_{\mathbb{T}^3,\text{IV}} \\ &= |G|^2 + (|G| - 1) \cdot |Z_n|^2 \cdot (1 \cdot |Z_n|^3 + (|Z_n|^2 - 1) \cdot n) \\ &= n^8 + (n^4 - 1) \cdot n^2 \cdot (1 \cdot n^3 + (n^3 - 1) \cdot n).\end{aligned} \tag{47}$$

As we expect, the first part is from the zero-flux $b = 0$, which is the normal untwisted 2 + 1D $(Z_n)^4$ gauge theory (toric code) as $\mathcal{C}^{2D}_{(Z_n)^4}$ with $|G|^2 = n^8$ on the 2-torus. The remaining $(|G| - 1)$ copies are inserted with nonzero flux ($b \neq 0$) as $\mathcal{C}^{2D}_{(Z_n)^4,\omega_3}$ with type III 3-cocycle twists in Table XII. In some but not all cases, $\mathcal{C}^{2D}_{(Z_n)^4,\omega_3}$ is $\mathcal{C}^{2D}_{(Z_n)_{\text{untwist}} \times (Z_n)^3_{\text{twist}},\omega_3}$. In either case, the $\text{GSD}_{b,\mathbb{T}^2}$ for $b \neq 0$ has the same decomposition always, equivalent to an untwisted Z_n gauge theory with $\text{GSD}_{\mathbb{T}^2} = n^2$ direct product with

$$\text{GSD}_{\mathbb{T}^2,\omega_{3,\text{III}}} = (1 \cdot n^3 + (n^3 - 1) \cdot n) \tag{48}$$

$$\equiv \text{GSD}^{\text{Abel}}_{\mathbb{T}^2,\omega_{3,\text{III}}} + \text{GSD}^{\text{nAbel}}_{\mathbb{T}^2,\omega_{3,\text{III}}}, \tag{49}$$

from which we generalize the result derived for 2 + 1D type III ω_3 twisted theory with $G = (Z_2)^3$ in Ref. [40] to $G = (Z_n)^3$ of a prime n. One can repeat the counting for 2 + 1 dimensions as in Eqs. (44) and (45); see Appendix A.

To summarize, from the GSD counting, we foresee that there exist non-Abelian strings in 3 + 1D type IV twisted gauge theory, with a quantum dimension n. These non-Abelian strings (fluxes) carry dim(Rep) = n non-Abelian charges. Since charges are sourced by particles, these non-Abelian strings are not pure strings but are attached to non-Abelian particles. (For a projection perspective from three to two dimensions, a non-Abelian string of $\mathcal{C}^{3\text{D}}$ is a non-Abelian dyon with both charge and flux of $\mathcal{C}^{2\text{D}}_b$.)

E. Modular 3D T^{xy}

We compute T^{xy} and S^{xyz} using the formula derived in Sec. III B for type IV ω_4 theory (for generality, we also include the twists by types II and III ω_4). Due to the large GSD and the quantum dimension of a non-Abelian nature, we focus on the simplest example $G = (Z_2)^4$ theory to have the smallest number of data. By $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{21}$, we have 2^{21} types of theories, where 2^{20} types of type IV are endorsed with non-Abelian statistics (while 2^{20} types are Abelian gauge theories of non-type IV, with their T and S data reported in Sec. IV B). For $G = (Z_2)^4$, there are still $\text{GSD}_{\mathbb{T}^3,\text{IV}} = 1576$. Thus both T and S are matrices of rank 1576. T^{xy} has 1576 components along the diagonal.

For $G = (Z_2)^4$, we first define a quantity η_{g_1,g_2,g_3} of convenience from the $\mathsf{C}^{(2)}_{a,b}(c,d)$ in Eq. (15):

$$\eta_{g_1,g_2,g_3} \equiv \begin{cases} 0 & \text{if} \quad \mathsf{C}^{(2)}_{g_1,g_2}(g_3,g_3) = +1; \\ 1 & \text{if} \quad \mathsf{C}^{(2)}_{g_1,g_2}(g_3,g_3) = -1. \end{cases} \tag{50}$$

Below the p_{lm} and p_{lmn} are the shorthand for types II and III (both first and second) topological term labels; $p_{lm} f_{_{lm}}(a,b,c)$ and $p_{lmn} f_{_{lmn}}(a,b,c)$ abbreviate the function forms in the exponents of types II and III ω^4 in Table I. Namely, we regard their 4-cocycle $\omega_4(a,b,c,d)$ as a trivial 2-cocycle $\mathsf{c}_{a,b}(c,d)$ written as $\mathsf{c}_{a,b}(c,d) = \frac{\eta_{a,b}(c)\eta_{a,b}(d)}{\eta_{a,b}(c+d)}$, where $\eta_{a,b}(c)$ is a 1-cochain: $\eta_{a,b}(c) = \exp(\text{i} p_{lm} f_{_{lm}}(a,b,c)) = \exp(\frac{2\pi \text{i}}{N_{lm} N_m} p_{lm} a_l b_m c_m))$ for the type II case. $\eta_{a,b}(c) = \exp(\text{i} p_{lmn} f_{_{lmn}}(a,b,c)) = \exp(\frac{2\pi \text{i}}{N_{lm} N_n} p_{lmn} a_l b_m c_n))$ for the type III case. We derive $\mathsf{T}^{xy} = \mathsf{T}^{a_y,b_z}_{\alpha_x}$ of Eq. (31) in Table V.

F. Modular 3D S^{xyz}

The S^{xyz} matrix has 1576×1576 components. We organize S^{xyz} into four blocks, denoting by nAbel(Abel) (non-Abelian)Abelian with 736 (840) components. Defining

TABLE V. SL(3,$\mathbb{Z}$) modular data $\mathsf{T}^{xy} = \mathsf{T}^{a_y,b_z}_{\alpha_x}$ for the $(Z_2)^4$ theory with type IV ω^4. The formula of T^{xy} is separated into two sets: the first set, with 736 components (from the sector $\mathrm{GSD}^{\mathrm{Abel}}_{\mathbb{T}^3,\mathrm{IV}}$), and another 840 components (from the sector $\mathrm{GSD}^{\mathrm{nAbel}}_{\mathbb{T}^3,\mathrm{IV}}$). $F=(a_i,b_i)$ are fluxes with eight components: $(a_1,a_2,a_3,a_4)\in(Z_2)^4$ and $(b_1,b_2,b_3,b_4)\in(Z_2)^4$. The number of distinct fluxes in $F(j_{\mathrm{Abel}})$ is 46 ($=\mathrm{Flux}^{\mathrm{Abel}}_{\mathrm{IV}}$); the number of distinct fluxes in $F(j_{\mathrm{nAbel}})$ is 210($=\mathrm{Flux}^{\mathrm{nAbel}}_{\mathrm{IV}}$). This table lists *all* 2^{20} kinds of T^{xy} for the non-Abelian theories in $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2^{21}$ (half of 2^{21}). The $((\pm)_a,(\pm)_b)$ pair makes up the numbers of charge Rep $n^2=2^2$ in Eq. (45). Details of the rank 2 matrix Rep are given in Appendix A.

Excitations (α,a,b)	$\mathsf{T}^{a,b}_{\alpha}$
$(\alpha,F(j_{\mathrm{Abel}}))$	$\exp\left(\sum_{k=1}^4 \pi\mathrm{i}\,\alpha_k a_k\right)\to$ e.g., ±1
$(((\pm)_a,(\pm)_b),F(j_{\mathrm{nAbel}}))$	$e^{\mathrm{i}\frac{\pi}{2}(\sum_{l,m,n\in\{1,2,3,4\},\,l<m<n} p_{lm}f_{lm}(a,b,a)+p_{lmn}f_{lmn}(a,b,a))}(\pm)_a(\pm)_b(\mathrm{i})^{\eta_{a,b,a}}\to$ e.g., ±1 or $\pm\mathrm{i}$

$\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)}\equiv\frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{a,b,d}\delta_{b,c}$, we obtain the following.

$$\mathsf{S}^{xyz}=\frac{1}{|G|}\begin{array}{c} \begin{array}{cc}\text{736 components} & \text{840 components}\\ (\beta_1,\beta_2,\beta_3,\beta_4,\,c,d) & ((\pm)_c,(\pm)_d,\,c,d)\end{array}\\ \left(\begin{array}{c|c}\mathsf{S}_{\mathrm{Abel,Abel}} & \mathsf{S}_{\mathrm{Abel,nAbel}}\\ \hline \mathsf{S}_{\mathrm{nAbel,Abel}} & \mathsf{S}_{\mathrm{nAbel,nAbel}}\end{array}\right)\end{array}\begin{array}{l}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\,a,b)\\ ((\pm)_a,(\pm)_b,\,a,b)\end{array} \tag{51}$$

$$\begin{aligned}
\mathsf{S}_{\mathrm{Abel,Abel}}&=1\cdot\exp\left(\sum_k\frac{2\pi\mathrm{i}}{N_k}(-\alpha_k d_k+\beta_k a_k)\right)\cdot\delta_{b,c}=(-1)^{(-\alpha_k d_k+\beta_k a_k)}\cdot\delta_{b,c},\\
\mathsf{S}_{\mathrm{Abel,nAbel}}&=2\cdot(-1)^{(-\alpha_k d_k)}\cdot e^{\mathrm{i}\frac{\pi}{2}(\sum_{l,m,n\in\{1,2,3,4\},\,l<m<n} p_{lm}f_{lm}(b,d,a)+p_{lmn}f_{lmn}(b,d,a))}(\pm1)_b(\pm1)_d\cdot(\mathrm{i})^{\eta_{b,d,a}}\delta_{a\in\{\mathbf{1},b,d,bd\}}\cdot\delta_{b,c},\\
\mathsf{S}_{\mathrm{nAbel,Abel}}&=2\cdot(-1)^{(\beta_k a_k)}\cdot e^{-\mathrm{i}\frac{\pi}{2}(\sum_{l,m,n\in\{1,2,3,4\},\,l<m<n} p_{lm}f_{lm}(a,b,d)+p_{lmn}f_{lmn}(a,b,d))}(\pm1)_a(\pm1)_b\cdot(\mathrm{i})^{\eta_{a,b,d}}\delta_{d\in\{\mathbf{1},a,b,ab\}}\cdot\delta_{b,c},\\
\mathsf{S}_{\mathrm{nAbel,nAbel}}&=4\cdot e^{-\mathrm{i}\frac{\pi}{2}(\sum_{l,m,n\in\{1,2,3,4\},\,l<m<n} p_{lm}f_{lm}(a,b,d)+p_{lmn}f_{lmn}(a,b,d))}\cdot e^{\mathrm{i}\frac{\pi}{2}(\sum_{l,m,n\in\{1,2,3,4\},\,l<m<n} p_{lm}f_{lm}(b,d,a)+p_{lmn}f_{lmn}(b,d,a))}\\
&\quad\cdot(\pm1)_a(\pm1)_b(\pm1)_c(\pm1)_d\cdot(-\mathrm{i})^{\eta_{a,b,d}}\cdot(\mathrm{i})^{\eta_{b,d,a}}\cdot\delta_{a\in\{b,d,bd\}}\cdot\delta_{d\in\{a,b,ab\}}\cdot\delta_{b,c}.
\end{aligned} \tag{52}$$

The $\exp(\sum_k\frac{2\pi\mathrm{i}}{N_k}(-\alpha_k d_k+\beta_k a_k))$ factor in the top-left block shows the pure-particle pure-string braiding of untwisted Z_N gauge theory (no ω_4 dependence). We define $\delta_{a\in\{b,d,bd\}}=1$ if $a\in\{b,d,bd\}$; otherwise, $\delta_{a\in\{b,d,bd\}}=0$. Some other technical details follow: for $G=(Z_2)^4$, the constraint $\delta_{a\in\{b,d,bd\}}\cdot\delta_{d\in\{a,b,ab\}}$ reduces to $\delta_{d\in\{a,ab\}}$. The survival nonzero $\mathsf{S}_{\mathrm{nAbel,nAbel}}$ values occur in only two kinds of forms: either $d=a$ or $d=ab$:

$$\mathsf{S}_{\mathrm{nAbel,nAbel}}=\begin{cases}\mathsf{S}^{\alpha,\beta}_{a,b,a}\delta_{b,c}\delta_{d,a},\\ \mathsf{S}^{\alpha,\beta}_{a,b,ab}\delta_{b,c}\delta_{d,ab}.\end{cases} \tag{53}$$

Some remarks follow.

(i) Regarding dimensional reduction from 3D to 2D sectors with b flux, from the above S^{xyz} and T^{xy}, there is no difficulty deriving S^{xy} from Eq. (32). From all these modular S^{xy}_b and T^{xy}_b data, we find consistency with the dimensional reduction of 3D topological order by comparison with the induced 3-cocycle ω_3 from ω_4. Let us consider a single specific example, given the type IV index, $p_{1234}=1$, and other zero types II and III indices, $p_{..}=p_{...}=0$:

$$\begin{aligned}
\mathcal{C}^{3\mathrm{D}}_{(Z_2)^4,\omega_{4,\mathrm{IV}}}&=\oplus_b\mathcal{C}^{2\mathrm{D}}_b\\
&=\mathcal{C}^{2\mathrm{D}}_{(Z_2)^4}\oplus 10\,\mathcal{C}^{2\mathrm{D}}_{(Z_2)\times(Z_2)^3_{(ijl)},\omega^{(ijl)}_{3,\mathrm{III}}}\oplus 5\mathcal{C}^{2\mathrm{D}}_{(Z_2)^4,\omega_{3,\mathrm{III}}\times\omega_{3,\mathrm{III}}\times\dots}\\
&=\mathcal{C}^{2\mathrm{D}}_{(Z_2)^4}\oplus 10\,\mathcal{C}^{2\mathrm{D}}_{(Z_2)\times(D_4)}\oplus 5\mathcal{C}^{2\mathrm{D}}_{(Z_2)^4,\omega_{3,\mathrm{III}}\times\omega_{3,\mathrm{III}}\times\dots}.
\end{aligned} \tag{54}$$

The $\mathcal{C}^{2\mathrm{D}}_{(Z_2)^4}$, again, is the normal $(Z_2)^4$ gauge theory at $b=0$. The 10 copies of $\mathcal{C}^{2\mathrm{D}}_{(Z_2)\times(D_4)}$ have an untwisted dihedral D_4 gauge theory ($|D_4|=8$) product with the normal (Z_2) gauge theory. The duality to D_4 theory in two dimensions can be expected [40]; see Table VI. [As a by-product of our work, we go beyond Ref. [40] to give the complete classification of all twisted 2D ω_3 of $G=(Z_2)^3$ and their corresponding

TABLE VI. $D^{\omega}(G)$, the twisted quantum double model of G in 2 + 1 dimensions, and their 3-cocycle ω_3 (involving type III) types in $\mathcal{C}^{2\mathrm{D}}_{(Z_2)^3,\omega_3}$. We classify the 64 types of 2D non-Abelian twisted gauge theories into five classes, which agree with Ref. [62]. Each class has distinct non-Abelian statistics. Both dihedral group D_4 and quaternion group Q_8 are non-Abelian groups of order 8, as $|D_4|=|Q_8|=|(Z_2)^3|=8$. $D^{\omega}(G)$ data can be found in Ref. [62]. Details are rereported in Appendix A.

Class	Twisted quantum double $D^{\omega}(G)$	No. of types
$\omega_3[1]$	$D^{\omega_3[1]}(Z_2{}^3)$, $D(D_4)$	7
$\omega_3[3d]$	$D^{\omega_3[3d]}(Z_2{}^3)$, $D^{\gamma^4}(Q_8)$	7
$\omega_3[3i]$	$D^{\omega_3[3i]}(Z_2{}^3)$, $D(Q_8)$, $D^{\alpha_1}(D_4)$, $D^{\alpha_2}(D_4)$	28
$\omega_3[5]$	$D^{\omega_3[5]}(Z_2{}^3)$, $D^{\alpha_1\alpha_2}(D_4)$	21
$\omega_3[7]$	$D^{\omega_3[7]}(Z_2{}^3)$	1

topological orders and twisted quantum double $D^{\omega}(G)$ in Appendix A.] The remaining 5 copies $\mathcal{C}^{2D}_{(Z_2)^4,\omega_{3,\text{III}}\times\omega_{3,\text{III}}\times\ldots}$ must contain the twist on the full group $(Z_2)^4$, not just its subgroup. This peculiar feature suggests the following remark.

(ii) Sometimes there may exist a duality between a twisted Abelian gauge theory and an untwisted non-Abelian gauge theory [40]; one may wonder whether one can find a dual non-Abelian gauge theory for $\mathcal{C}^{3D}_{(Z_2)^4,\omega_{4,\text{IV}}}$. We find, however, that $\mathcal{C}^{3D}_{(Z_2)^4,\omega_{4,\text{IV}}}$ cannot be dual to a normal gauge theory (neither Abelian nor non-Abelian) but must be a twisted (Abelian or non-Abelian) gauge theory. The reason is more involved. Let us first recall the more familiar 2D case. One can consider the $G=(Z_2)^3$ example with $\mathcal{C}^{2D}_{(Z_2)^3,\omega_3}$, with $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})=(\mathbb{Z}_2)^7$. There are 2^6 non-Abelian types with type III ω_3 (the other 2^6 are Abelian without type III ω_3). We find that the 64 non-Abelian types of 3-cocycles ω_3 go to 5 classes, labeled $\omega_3[1]$, $\omega_3[3d]$, $\omega_3[3i]$, $\omega_3[5]$, and $\omega_3[7]$, and their twisted quantum double model $D^{\omega}(G)$ values are listed in Table VI. The number in brackets following ω_3 (first column) is related to the number of pairs of $\pm$i in the T matrix and the d/i stand for the linear dependence (d)/independence (i) of fluxes generating cocycles.

In Table VI, we show that two classes of 3-cocycles for $D^{\omega_3}(Z_2)^3$ of two dimensions can have dual descriptions by gauge theory of a non-Abelian dihedral group D_4 and quaternion group Q_8. However, the other three classes of 3-cocycles for $D^{\omega_3}(Z_2)^3$ do not have a dual (untwisted) non-Abelian gauge theory.

Now let us reconsider 3D $\mathcal{C}^{3D}_{G,\omega_{4,\text{IV}}}$, with $|Z_2|^4=16$. From Ref. [39], we know that $3+1$D D_4 gauge theory undergoes decomposition by its five centralizers. Applying the rule of decomposition to other groups implies that for an untwisted group G in 3D $\mathcal{C}^{3D}_{G}$, we can decompose it into sectors of $\mathcal{C}^{2D}_{G_b,b}$; here G_b becomes the *centralizer* of *conjugacy class* (flux) b: $\mathcal{C}^{3D}_{G}=\oplus_b\mathcal{C}^{2D}_{G_b,b}$. Some useful information is

$$\mathcal{C}^{3D}_{(Z_2)^4}=16\mathcal{C}^{2D}_{(Z_2)^4},\tag{55}$$

$$\mathcal{C}^{3D}_{D_4}=2\mathcal{C}^{2D}_{D_4}\oplus 2\mathcal{C}^{2D}_{(Z_2)^2}\oplus\mathcal{C}^{2D}_{Z_4},\tag{56}$$

$$\mathcal{C}^{3D}_{Z_2\times D_4}=4\mathcal{C}^{2D}_{Z_2\times D_4}\oplus 4\mathcal{C}^{2D}_{(Z_2)^3}\oplus 2\mathcal{C}^{2D}_{Z_2\times Z_4},\tag{57}$$

$$\mathcal{C}^{3D}_{Q_8}=2\mathcal{C}^{2D}_{Q_8}\oplus 3\mathcal{C}^{2D}_{Z_4},\tag{58}$$

$$\mathcal{C}^{3D}_{Z_2\times Q_8}=4\mathcal{C}^{2D}_{Z_2\times Q_8}\oplus 6\mathcal{C}^{2D}_{Z_2\times Z_4},\tag{59}$$

and we find that no such decomposition is possible from $|G|=16$ groups to match Eq. (54)'s. Furthermore, if there exists a non-Abelian G_{nAbel}, to have Eq. (54) those $(Z_2)^4$, $(Z_2)\times(D_4)$ or the twisted $(Z_2)^4$ must be the centralizers of G_{nAbel}. But one of the centralizers (the centralizer of the identity element as a conjugacy class $b=0$) of G_{nAbel} must be G_{nAbel} itself, which was already ruled out from Eqs. (55) and (57). Thus, we prove that $\mathcal{C}^{3D}_{(Z_2)^4,\omega_{4,\text{IV}}}$ is not a normal $3+1$D gauge theory (neither $Z_2\times D_4$, nor Abelian, nor non-Abelian) but must only be a twisted gauge theory.

(iii) We discover that (see Fig. 9) for any twisted gauge theory $\mathcal{C}^{3D}_{G,(\omega_{4,\text{IV}}\cdot\omega_{4,\ldots})}$ with type IV 4-cocycle $\omega_{4,\text{IV}}$ (whose non-Abelian nature is not affected by adding other types II and III $\omega_{4,\ldots}$), by threading a third string through the two-string unlink 0_1^2 into the three-string Hopf links $2_1^2\#2_1^2$ configuration,

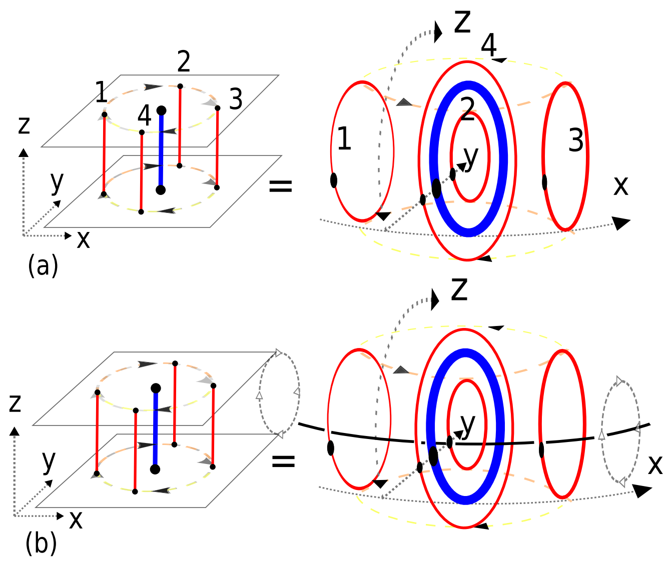

FIG. 9. (Color online) For the $3+1$D type IV $\omega_{4,\text{IV}}$ twisted gauge theory $\mathcal{C}^{3D}_{G,\omega_{4,\text{IV}}}$: (a) the two-string statistics in the unlinked 0_1^2 configuration is Abelian (the $b=0$ sector as $\mathcal{C}^{2D}_{G}$); and (b) the three-string statistics in the two Hopf links $2_1^2\#2_1^2$ configuration is non-Abelian (the $b\neq 0$ sector in $\mathcal{C}^{2D}_{b}=\mathcal{C}^{2D}_{G,\omega_{3,\text{III}}}$). The $b\neq 0$ flux sector creates a monodromy effectively acting as the third (black) string threading the two (red and blue) strings.

Abelian two-string statistics is promoted to non-Abelian three-string statistics. We can see the physics from Eq. (54); the $\mathcal{C}^{2D}_{b}$ is Abelian in the $b=0$ sector but non-Abelian in the $b\neq 0$ sector. The physics in Fig. 9 is then obvious; applying our discussion in Sec. III C about the equivalence between string threading and the $b\neq 0$ monodromy causes a branch cut.

(iv) Regarding the cyclic relation for non-Abelian S^{xyz} in three dimensions, interestingly, for the $(Z_2)^4$ twisted gauge theory with non-Abelian statistics, we find that a similar cyclic relation, Eq. (41), still holds as long as two conditions are satisfied: (a) the charge labels are equivalent, $\alpha=\beta$; and (b) $\delta_{a\in\{b,d,bd\}}\cdot\delta_{d\in\{a,b,ab\}}\cdot\delta_{b\in\{d,a,da\}}=1$. However, Eq. (41) is modified by a factor depending on the dimensionality of Rep α:

$$\mathsf{S}^{\alpha,\alpha}_{a,b,d}\cdot\mathsf{S}^{\alpha,\alpha}_{b,d,a}\cdot\mathsf{S}^{\alpha,\alpha}_{d,a,b}\cdot|\dim(\alpha)|^{-3}=1.\tag{60}$$

This identity should hold for any type IV non-Abelian strings. This is a cyclic relation of a 3D nature, instead of the dimensional-reducing 2D nature of $\mathsf{S}^{2D\,\alpha,\beta}_{a,c\,(b)}$ in Fig. 2.

V. CONCLUSION

A. Knot-and-link configuration

Throughout this paper, we have indicated that the mathematics of knots and links may be helpful in organizing our string-braiding patterns in three dimensions. Here we illustrate them more systematically. We use Alexander-Briggs notation for the knots and links (see Fig. 10).

The knots and links for our string-braiding patterns are organized in Table VII. We recall that, in Sec. III C, the topological spin for a closed string in the $b=0$ flux sector of

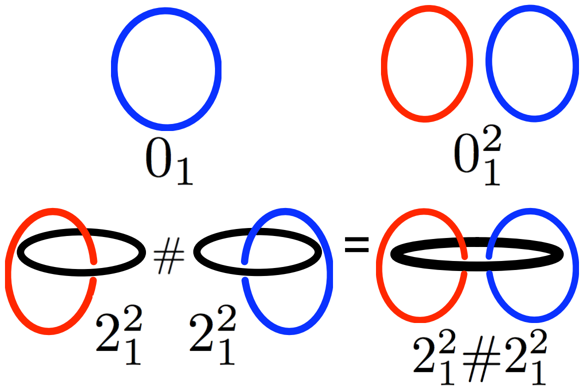

FIG. 10. (Color online) Under Alexander-Briggs notation, an unknot is 0_1, and two unknots can form an unlink 0_1^2. A Hopf link is 2_1^2, and the connected sum of two Hopf links is $2_1^2\#2_1^2$.

$\mathcal{C}_b^{2\text{D}}$ does a self-2π flipping under the 0_1 unknot configuration. Due to our spin-statistics relation of a closed string, we can view the topological spin of the $b=0$ sector as the exchange statistics of two identical strings in the 0_1^2 unlink configuration. On the other hand, for the topological spin in the $b \neq 0$ flux sector, we effectively thread a (black) string through the (blue) unknot, which forms a Hopf link, 2_1^2. Meanwhile, we can view the topological spin of the $b \neq 0$ sector as the exchange statistics of two identical strings threaded by a third (black) string in a connected sum of two Hopf links in the $2_1^2\#2_1^2$ configuration. Furthermore, we can promote two-string Abelian statistics under the 0_1^2 unlink of the $b=0$ sector to three-string Abelian (Sec. IV B) or non-Abelian (in Sec. IV C) statistics under Hopf links $2_1^2\#2_1^2$ of the $b \neq 0$ sector.

Nothing prevents us from considering more generic knot-and-link patterns for three-string or multistring braiding. Our reason is this: From the full modular SL(3,$\mathbb{Z}$) group viewpoint, S^{xyz} is a necessary generator to access the complete data on the SL(3,$\mathbb{Z}$) group. However, we have learned that our 3D-to-2D reduction by Eq. (4) using the SL(2,$\mathbb{Z}$) subgroup data S^{xy} and T^{xy} already encodes all the physics of braidings under the simplest knots and links in Fig. 10. These include

TABLE VII. Various string-braiding patterns in terms of knots and links in Alexander-Briggs notation: the topological spin of a loop and the exchange/braiding statistics of two loops without any background string inserted ($b=0$ sector) or with another background string inserted ($b \neq 0$ sector). Here we effectively view the string-braiding statistics of 3D topological order in terms of 2D sectors: $\mathcal{C}^{3\text{D}} = \oplus_b \mathcal{C}_b^{2\text{D}}$.

$\mathcal{C}_b^{2\text{D}}$	Physics of strings	Knots and links
$b=0$	Topological spin (T)	0_1
	Exchange statistics	0_1^2
	2-string braiding	0_1^2
$b \neq 0$	Topological spin (T)	2_1^2
	Exchange statistics	$2_1^2\#2_1^2$
	3-string braiding	$2_1^2\#2_1^2, \dots$

self-flipping topological spin and exchange/braiding statistics (Secs. III C and IV). This suggests that S^{xyz} contains more than these string-braiding configurations. In addition, there are more generic MCGs, MCG($\mathcal{M}_{\text{space}}$), beyond MCG($\mathbb{T}^3$) = SL(3,$\mathbb{Z}$), which potentially encode more exotic multistring braidings.

Indeed, as noted in Sec. IV, the 3D S matrix essentially contains the information on three fluxes $(d,a,b) = (d_x,a_y,b_z)$ in Eq. (38), $\mathsf{S}^{xyz} = \mathsf{S}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{bc}$. Since strings carry fluxes in three dimensions, this further suggests that we should look for the braiding involving three strings; three-loop braiding was also recently emphasized in Refs. [45] and [46].

The configuration we have studied so far, with three strings, is the Hopf link $2_1^2\#2_1^2$. We propose using more general three-string patterns, such as the link

$$\mathcal{N}_m^3$$

or its connected sum, to study topological states. ($\mathcal{N}_m^3$ is in Alexander-Briggs notation; here 3 means that there are three closed loops, $\mathcal{N}$ is the crossing number, and m is the label for different kinds of $\mathcal{N}^3$ linking.) For example, three-string braiding can include links of 6_1^3, 6_2^3, and 6_3^3 in Fig. 11. The configurations in Fig. 11 are potentially promising for study of the braiding statistics of strings to classify or characterize topological orders.

To examine whether multistring braiding is topologically well defined, we propose a way to check that (such as the braiding processes in Figs. 9 and 11)

> the path that one (red) loop A winds around another (blue) loop B along the time evolution is nontrivial in the complement space of loop B and base (black) loop C. Namely, the path of A needs to be a nontrivial element of the fundamental group

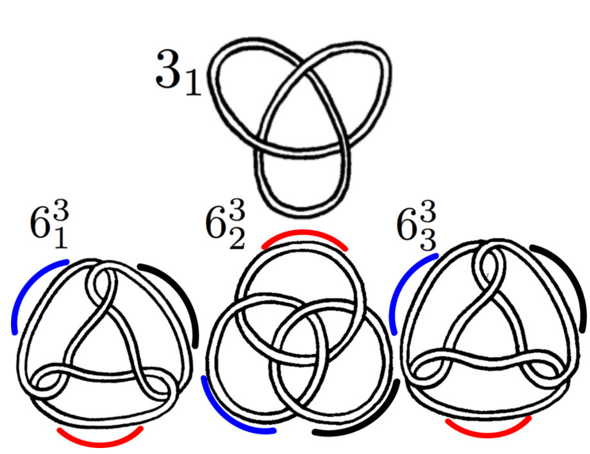

FIG. 11. (Color online) The trefoil knot is 3_1. Some other simplest three-string links (beyond Hopf links $2_1^2\#2_1^2$) are shown: 6_1^3, 6_2^3 (Borromean rings), 6_3^3. From the spin-statistics relation of a closed string discussed in Sec. III C, where the topological spin of certain knot/link configurations (0_1 for the monodromy flux $b=0$ and 2_1^2 for $b \neq 0$) is equivalent to the exchange statistics of certain knot/link configurations (0_1^2 for $b=0$ and $2_1^2\#2_1^2$ for $b \neq 0$) under Eq. (35). Therefore, we may further conjecture that the topological spin of a trefoil knot 3_1 may relate to the braiding statistics of 6_1^3, 6_2^3, 6_3^3.

for the complement space of B and C. Thus the path must be homotopically nontrivial.

Before concluding this subsection, another final remark is in order: In Sec. III C 3, we mention generalizing the framed worldline picture of particles in Fig. 5 to the framed worldsheet picture of closed strings. (*Note*: The framed worldline is like a worldsheet; the framed worldsheet is like a worldvolume.) Thus, it may be interesting to study how incorporating the framing of particles and strings (with worldline/worldsheet/worldvolume) can provide richer physics and textures in the knot-link pattern.

B. Cyclic identity for Abelian and non-Abelian strings

In Secs. IV B and IV C, we discuss cyclic identity for Abelian and non-Abelian strings, particularly for $3+1$D twisted gauge theories. We find Eq. (60), the "cyclic identity of the 3D S^{xyz} matrix of Eq. (38), $\mathsf{S}^{xyz}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{\alpha,\beta}_{d,a,b}\delta_{b,c}$":

$$\mathsf{S}^{\alpha,\alpha}_{a,b,d} \cdot \mathsf{S}^{\alpha,\alpha}_{b,d,a} \cdot \mathsf{S}^{\alpha,\alpha}_{d,a,b} \cdot |\dim(\alpha)|^{-3} = 1. \tag{61}$$

For the Abelian case, the dimension of Rep is simply $\dim(\alpha) = 1$, which reduces to Eq. (41).

On the other hand, we find that there is another cyclic identity, based on the 2D $\mathsf{S}^{xy}_b = \mathsf{S}^{xy}_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|}\mathsf{S}^{2\mathrm{D}\,\alpha,\beta}_{a,c\,(b)}\delta_{b,d}$ matrix of Eq. (34), written in terms of $\mathsf{S}^{2\mathrm{D}\,\alpha,\beta}_{a,c\,(b)}$, at least for Abelian strings of types II and III 4-cocycle twists, namely, the "cyclic identity of the 2D S^{xy} matrix":

$$\mathsf{S}^{2\mathrm{D}\,0,0}_{a_i,c_k\,(b_j)} \cdot \mathsf{S}^{2\mathrm{D}\,0,0}_{c_k,b_j\,(a_i)} \cdot \mathsf{S}^{2\mathrm{D}\,0,0}_{b_j,a_i\,(c_k)} = 1. \tag{62}$$

This Eq. (62) cyclic identity has two additional criteria: (i) Here $\alpha = \beta = 0$ means that all strings must have 0 charges; and (ii) in addition, the $\prod_i Z_{N_i}$ flux labels a_i,b_j,c_k must satisfy $a_i = |a|\hat{e}_i$, $b_j = |b|\hat{e}_j$, $c_k = |c|\hat{e}_k$, as a multiple of a single-unit flux, each only carrying one $\prod_i Z_{N_i}$ flux. Note that $\hat{e}_j \equiv (0,\ldots,0,1,0,\ldots,0)$ is defined to be a unit vector with a nonzero component as the jth component fore Z_{N_j} flux. Equation (62) is true even in the noncanonical basis, such as the case for the b flux sector in Table IV. Thus, whether or not it is in the canonical basis [61] does not affect the identity, Eq. (62), at least for the example of Abelian types II and III 4-cocycles.

This 2D S^{xy}_b cyclic identity in Eq. (62) is indeed the cyclic relation in Ref. [45]. The fact that we associate the 2D S^{xy}_b matrix with the dimensional reduction of string braiding in Fig. 2 shows that the Abelian statistical angle $\theta_{a_i,c_k,(b_j)}$ can be defined, up to a basis [61], as

$$\mathsf{S}^{2\mathrm{D}\,0,0}_{a_i,c_k\,(b_j)} = \exp(\mathrm{i}\theta_{a_i,c_k,(b_j)}). \tag{63}$$

Thus Eq. (62) implies a cyclic relation for Abelian statistical angles:

$$\theta_{a_i,c_k,(b_j)} + \theta_{c_k,b_j,(a_i)} + \theta_{b_j,a_i,(c_k)} = 0 \pmod{2\pi}. \tag{64}$$

In contrast, the 3D cyclic relation works for both Abelian and non-Abelian strings, and it is not restricted to zero charge but only to equal charges, $\alpha = \beta$. More importantly, Eq. (61) allows any flux for each a, b, and c, instead of being limited to a single-unit flux or a multiple of a single-unit flux in Eq. (62).

C. Main results

We have studied string and particle excitations in $3+1$D twisted discrete gauge theories, which belong to a class of topological orders. These 3D theories are gapped topological systems with topology-dependent GSD. The twisted gauge theory contains the data on gauge group G and 4-cocycle twist $\omega_4 \in \mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ of the fourth cohomology group of G. Such data provide many *types* of theories, however, several types of theories belong to the same *class* of a topological order. To better characterize and classify topological orders, we use the MCG on the $\mathbb{T}^3$ torus, as $\mathrm{MCG}(\mathbb{T}^d) = \mathrm{SL}(d,\mathbb{Z})$, to extract the $\mathrm{SL}(3,\mathbb{Z})$ modular data S^{xyz} and T^{xy} in the ground-state sectors, which, however, reveal information on gapped excitations of particles and strings. We have posed five main questions (**Q1–Q5**) and other subquestions throughout this work and have addressed each of them in some depth. We summarize our results and approaches below and make comparisons with some recent work.

1. *Dimensional reduction*

By inserting a gauge flux b into a compactified circle z of 3D topological order $\mathcal{C}^{3\mathrm{D}}$, we can realize $\mathcal{C}^{3\mathrm{D}} = \oplus_b \mathcal{C}^{2\mathrm{D}}_b$, where $\mathcal{C}^{3\mathrm{D}}$ becomes a direct sum of degenerate states of 2D topological orders $\mathcal{C}^{2\mathrm{D}}_b$ in different flux b sectors. We should emphasize that this dimensional reduction is *not* real-space decomposition along the z direction, *but* decomposition in the Hilbert space of ground states [excitation basis such as the canonical basis of Eq. (11)]. We propose that this decomposition in Eq. (5) will work for a generic topological order without a gauge group description. In the most general case, b becomes the certain basis label of the Hilbert space. The recent study in Ref. [39] implements the dimensional reduction idea on the normal gauge theories described by the 3D Kitaev Z_N toric code and 3D quantum double models without cocycle twists using the spatial Hamiltonian approach. In our work, we consider more generic twisted gauge theories with a lattice realization in the twisted 3D quantum double models under the framework of Dijkgraaf-Witten theory [38]. We apply both the spatial Hamiltonian approach and the spacetime path integral approach.

2. *Modular data*

We find explicit formula representations of the $\mathrm{SL}(3,\mathbb{Z})$ modular data S and T using (i) the path integral and cocycle approach and (ii) the representation) theory approach. The Rep theory approach is convenient and perhaps contains more general and simplified expressions. While recent work either focuses on Abelian statistics [45,46] or focuses on normal gauge theories [39], our formula embodies generic non-Abelian twisted gauge theories and thus is the most powerful.

3. *Classification and characterization*

We use the modular data S and T to characterize the braiding statistics of some 2D and 3D topological orders. We can further use the modular data S and T taking into account excitation relabeling to classify (or partially classify) topological orders. Explicit 2D examples are $G = (Z_2)^3$

twisted gauge theories, and 3D examples are $G = (Z_2)^4$ twisted gauge theories. Some of our results are compared with the mathematics literature inAppendix A. Some 2D results are compared to twisted quantum double models $D^{\omega}(G)$.

Our result can also facilitate the study of symmetric protected topological states protected by a global symmetry G_s [51]. By gauging the G_s symmetry of symmetric protected topological states, one can use the induced dynamical gauge theory to study the braiding of excitations and to characterize symmetric protected topological states [45,63–65].

4. Physics of string and particle braiding

We provide the physics meaning of the topological spin and spin-statistics relation for a closed string. We also interpret the three-string braiding statistics first studied in Ref. [45] from a new perspective: a dimensional reduction with b flux monodromy. We find that with the type IV 4-cocycle twist for the twisted gauge theory, by threading a third string through the two-string unlink 0_1^2 into the three-string Hopf link $2_1^2\#2_1^2$ configuration, Abelian two-string statistics is promoted to non-Abelian three-string statistics. In Ref. [39], an effect somewhat the opposite of ours is found: The normal (untwisted) non-Abelian 3D topological order was found with non-Abelian statistics in the $b = 0$ sector, but there may be Abelian statistics in the $b \neq 0$ sector. Incorporating this understanding, we have the more unified picture organized in Table VIII, for the string-braiding statistics of twisted/untwisted Abelian/non-Abelian gauge theories as topological orders. Since string deformation on the lattice can blur the Abelian U(1) phase, our non-Abelian string-braiding statistics provides a better alternative for a robust physical observable than Abelian string-braiding statistics [45,46] to be tested numerically or experimentally in the future. Last but not least, we propose the use of more general patterns, such as $\mathcal{N}_m^3$ (or $\mathcal{N}_m^l\#\ldots$) knots/links of Alexander-Briggs, to study three-string (or multistring) braiding statistics.

TABLE VIII. Braiding statistics, Abelian or non-Abelian, in terms of (G,ω_4), gauge group G, and cocycle twist ω_4 of 3D topological order $\mathcal{C}^{3\mathrm{D}}_{G,\omega_4}$. G_{Abel}, Abelian G; G_{nAbel}, non-Abelian G; stats, statistics. The normal gauge theory has $\omega_4 = 1$ with no cocycle twist. (Non-)Abelian stats: either non-Abelian or pure Abelian statistics. (For example, any $b \neq 0$ sector of an untwisted S_3 gauge theory has pure Abelian statistics, because S_3 centralizers of nonindentity elements are Abelian, but some $b \neq 0$ sectors of untwisted D_4 and Q_8 gauge theories have non-Abelian statistics.) $b = 0$ two-string 0_1^2 braiding is the process in Fig. 9(a); $b \neq 0$ three-string $2_1^2\#2_1^2$ braiding is the process in Fig. 9(b).

Braiding statistics (G,ω_4) of $\mathcal{C}^{3\mathrm{D}}_{G,\omega_4} = \oplus_b \mathcal{C}^{2\mathrm{D}}_b$	$b = 0$ braiding: 2-string 0_1^2	$b \neq 0$ braiding: 3-string $2_1^2\#2_1^2$
$(G_{\mathrm{Abel}},1)$	Abelian stats	Abelian stats
$(G_{\mathrm{Abel}}$, without$\omega_{4,\mathrm{IV}})$	Abelian stats	Abelian stats
$(G_{\mathrm{Abel}}$, with$\omega_{4,\mathrm{IV}})$	Abelian stats	Non-Abelian stats
$(G_{\mathrm{nAbel}},1)$	Non-Abelian stats	(Non-)Abelian stats
$(G_{\mathrm{nAbel}},\ \omega)$	Non-Abelian stats	Non-Abelian stats

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APPENDIX A: GROUP COHOMOLOGY AND COCYCLES

1. Cohomology group

Here we review the cohomology group $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z}) = \mathcal{H}^{d+1}(G,\mathrm{U}(1))$ by $\mathbb{R}/\mathbb{Z} = \mathrm{U}(1)$, as the $(d+1)$th cohomology group of G over the G module U(1). Each class in $\mathcal{H}^{d+1}(G,\mathbb{R}/\mathbb{Z})$ corresponds to a distinct $(d+1)$-cocycle. The n-cocycle is an n-cochain additionally satisfying the n-cocycle conditions $\delta\omega = 1$. The n-cochain is a mapping $\omega(A_1,A_2,\ldots,A_n)$: $G^n \to \mathrm{U}(1)$ [which inputs $A_i \in G$, $i = 1,\ldots,n$, and outputs a U(1) phase]. The n-cochains satisfy the group multiplication rule,

$$(\omega_1 \cdot \omega_2)(A_1,\ldots,A_n) = \omega_1(A_1,\ldots,A_n)\cdot\omega_2(A_1,\ldots,A_n), \tag{A1}$$

and thus form a group. The coboundary operator δ,

$$\begin{aligned}\delta \mathsf{c}(g_1,g_2,\ldots,g_{n+1}) &\equiv \mathsf{c}(g_2,\ldots,g_{n+1})\mathsf{c}(g_1,\ldots,g_n)^{(-1)^{n+1}} \\ &\quad\cdot \prod_{j=1}^{n} \mathsf{c}(g_1,\ldots,\ g_j g_{j+1},\ \ldots,g_{n+1})^{(-1)^j},\end{aligned} \tag{A2}$$

defines the n-cocycle condition $\delta\omega = 1$ (a pentagon relation in two dimensions). We check that the distinct n-cocycles are not equivalent by n-coboundaries. The n-cochain forms a group C^n, the n-cocycle forms a subgroup Z^n of C^n, and the n-coboundary further forms a subgroup B^n of Z^n (since $\delta^2 = 1$). Overall, this shows $\mathrm{B}^n \subset \mathrm{Z}^n \subset \mathrm{C}^n$. The n-cohomology group is exactly the relation of a kernel Z^n (the group of n-cocycles) modding out an image B^n (the group of n-coboundaries):

$$\mathcal{H}^n(G,\mathrm{U}(1)) = \mathrm{Z}^n/\mathrm{B}^n. \tag{A3}$$

To derive the expression of $\mathcal{H}^d(G,\mathrm{U}(1))$ in terms of groups explicitly, we apply some key formulas, as follows.

TABLE IX. Some facts about the cohomology group. For a finite Abelian group G, we have $\mathcal{H}^2(G,Z) = \mathcal{H}^1(G,\mathrm{U}(1)) = G$.

0	$\mathcal{H}^0(G,M) = M$	$\mathcal{H}^0(G,Z) = \mathbb{Z}$	$\mathcal{H}^0(G,\mathrm{U}(1)) = \mathrm{U}(1)$
1	$\mathcal{H}^1(G,M)$	$\mathcal{H}^1(G,Z) = \mathbb{Z}_1$	$\mathcal{H}^1(G,\mathrm{U}(1)) = G$ (1D Rep of group)
2	$\mathcal{H}^2(G,M)$	$\mathcal{H}^2(G,Z) = \mathcal{H}^1(G,\mathrm{U}(1))$	$\mathcal{H}^2(G,\mathrm{U}(1))$ (projective Rep of group)
3	$\mathcal{H}^3(G,M)$	$\mathcal{H}^3(G,Z) = \mathcal{H}^2(G,\mathrm{U}(1))$	
$d \geqslant 2$	$\mathcal{H}^d(G,M)$	$\mathcal{H}^d(G,Z) = \mathcal{H}^{d-1}(G,\mathrm{U}(1))$	

a. Künneth formula

We denote a ring R, $\mathbb{M}$ and $\mathbb{M}'$ are the R modules, and X and X' are chain complexes. The Künneth formula shows the cohomology of chain complex $X \times X'$ in terms of the cohomology of chain complex X and another chain complex, X'. For topological cohomology H^d, we have

$$\begin{aligned} &H^d(X \times X',\mathbb{M} \otimes_R \mathbb{M}') \\ &\quad \simeq \left[\oplus_{k=0}^{d} H^k(X,\mathbb{M}) \otimes_R H^{d-k}(X',\mathbb{M}')\right] \\ &\qquad \oplus \left[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^R(H^k(X,\mathbb{M}),H^{d-k+1}(X',\mathbb{M}'))\right]; \end{aligned} \tag{A4}$$

$$\begin{aligned} &H^d(X \times X',\mathbb{M}) \\ &\quad \simeq \left[\oplus_{k=0}^{d} H^k(X,\mathbb{M}) \otimes_{\mathbb{Z}} H^{d-k}(X',\mathbb{Z})\right] \\ &\qquad \oplus\left[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^k(X,\mathbb{M}),H^{d-k+1}(X',\mathbb{Z}))\right]. \end{aligned} \tag{A5}$$

The above is valid for both topological cohomology H^d and group cohomology $\mathcal{H}^d$ (for G' is a finite group):

$$\begin{aligned} &\mathcal{H}^d(G \times G',\mathbb{M}) \\ &\quad \simeq \left[\oplus_{k=0}^{d} \mathcal{H}^k(G,\mathbb{M}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G',\mathbb{Z})\right] \\ &\qquad \oplus \left[\oplus_{k=0}^{d+1} \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{H}^k(G,\mathbb{M}),\mathcal{H}^{d-k+1}(G',\mathbb{Z}))\right]. \end{aligned} \tag{A6}$$

b. Universal coefficient theorem

The universal coefficient theorem can be derived from the Künneth formula, Eq. (A5), by taking $X = 0$ or Z_1 or a point, thus only $H^0(X',\mathbb{M}) = \mathbb{M}$ survives:

$$H^d(X',\mathbb{M}) \simeq \mathbb{M} \otimes_{\mathbb{Z}} H^d(X',\mathbb{Z}) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{M},H^{d+1}(X',\mathbb{Z})). \tag{A7}$$

Using the universal coefficient theorem, we can rewrite Eq. (A5) as a decomposition below.

c. Decomposition

$$H^d(X \times X',\mathbb{M}) \simeq \oplus_{k=0}^{d} H^k[X,H^{d-k}(X',\mathbb{M})]. \tag{A8}$$

The above is valid for both topological cohomology and group cohomology,

$$\mathcal{H}^d(G \times G',\mathbb{M}) \simeq \oplus_{k=0}^{d} \mathcal{H}^k[G,\mathcal{H}^{d-k}(G',\mathbb{M})], \tag{A9}$$

provided that both G and G' are finite groups.

The expression of the Künneth formula is in terms of the tensor-product operation $\otimes_R$ and the torsion-product operation Tor_1^R of a base ring R, which we write $\boxtimes_R \equiv \mathrm{Tor}_1^R$ in shorthand. Their properties are

$$\begin{aligned} \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{M}' &\simeq \mathbb{M}' \otimes_{\mathbb{Z}} \mathbb{M}, \\ \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{M}, \\ \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{M}/n\mathbb{M}, \\ \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathrm{U}(1) &\simeq \mathrm{U}(1) \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0, \\ \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n &= \mathbb{Z}_{\gcd(m,n)}, \\ (\mathbb{M}' \oplus \mathbb{M}'') \otimes_R \mathbb{M} &= (\mathbb{M}' \otimes_R \mathbb{M}) \oplus (\mathbb{M}'' \otimes_R \mathbb{M}), \\ \mathbb{M} \otimes_R (\mathbb{M}' \oplus \mathbb{M}'') &= (\mathbb{M} \otimes_R \mathbb{M}') \oplus (\mathbb{M} \otimes_R \mathbb{M}'') \end{aligned} \tag{A10}$$

TABLE X. The exponent of the $\mathbb{Z}_{\gcd\otimes_i^m(N_i)}$ class in $\mathcal{H}^d(G,\mathrm{U}(1))$ for $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$. We define the shorthand $\mathbb{Z}_{\gcd(N_i,N_j)} \equiv \mathbb{Z}_{N_{ij}} \equiv \mathbb{Z}_{\gcd\otimes_i^2(N_i)}$, etc., also for other, higher gcd's. Our definition of type m derives from its number m of cyclic gauge groups in the gcd class $\mathbb{Z}_{\gcd\otimes_i^m(N_i)}$. The number of exponents can be systematically obtained by adding all the numbers in the previous column from the top row to the row before the number one wishes to determine. For example, the table shows that we derive $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z}) = \prod_{1\leqslant i<j<l\leqslant n}\mathbb{Z}_{N_i} \times \mathbb{Z}_{N_{ij}} \times \mathbb{Z}_{N_{ijl}}$ and $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z}) = \prod_{1\leqslant i<j<l<m\leqslant n}(\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{ijl}})^2 \times \mathbb{Z}_{N_{ijlm}}$, etc.

	Type I: $\mathbb{Z}_{N_i}$	Type II: $\mathbb{Z}_{N_{ij}}$	Type III: $\mathbb{Z}_{N_{ijl}}$	Type IV: $\mathbb{Z}_{N_{ijlm}}$	Type V: $\mathbb{Z}_{\gcd\otimes_i^5(N^{(i)})}$	Type VI: $\mathbb{Z}_{\gcd\otimes_i^6(N_i)}$	...: $\mathbb{Z}_{\gcd\otimes_i^m(N_i)}$	...: $\mathbb{Z}_{\gcd\otimes_i^{d-1}N_i}$	...: $\mathbb{Z}_{\gcd\otimes_i^d N^{(i)}}$
$\mathcal{H}^1(G,\mathrm{U}(1))$	1								
$\mathcal{H}^2(G,\mathrm{U}(1))$	0	1							
$\mathcal{H}^3(G,\mathrm{U}(1))$	1	1	1						
$\mathcal{H}^4(G,\mathrm{U}(1))$	0	2	2	1					
$\mathcal{H}^5(G,\mathrm{U}(1))$	1	2	4	3	1				
$\mathcal{H}^6(G,\mathrm{U}(1))$	0	3	6	7	4	1			
$\mathcal{H}^d(G,\mathrm{U}(1))$	$\frac{(1-(-1)^d)}{2}$	$\frac{d}{2} - \frac{(1-(-1)^d)}{4}$	...	...	...	...	...	$d-2$	1

TABLE XI. Some derived facts about the cohomology group and its cocycles.

$(d+1)$dim	$\mathcal{H}^{d+1}(G,\mathrm{U}(1))$	Künneth formula in $\mathcal{H}^{d+1}(G,\mathrm{U}(1))$	Path integral form in "fields"
$0+1$D	$\mathbb{Z}_{n_1}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1)]$
$1+1$D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2)]$
$2+1$D	$\mathbb{Z}_{n_1}$	$\mathcal{H}^3(\mathbb{Z}_{n_1},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1dA_1)]$
$2+1$D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1)) \otimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1dA_2)]$
$2+1$D	$\mathbb{Z}_{n_{123}}$	$[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))] \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2A_3)]$
$3+1$D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^3(\mathbb{Z}_{n_2},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2dA_2)]$
$3+1$D	$\mathbb{Z}_{n_{12}}$	$\mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^3(\mathbb{Z}_{n_1},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_2A_1dA_1)]$
$3+1$D	$\mathbb{Z}_{n_{123}}$	$[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1)) \otimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))] \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int (A_1dA_2)A_3)]$
$3+1$D	$\mathbb{Z}_{n_{123}}$	$[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))] \otimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int (A_1A_2)dA_3)]$
$3+1$D	$\mathbb{Z}_{n_{1234}}$	$[[\mathcal{H}^1(\mathbb{Z}_{n_1},\mathrm{U}(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},\mathrm{U}(1))] \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_3},\mathrm{U}(1))] \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_4},\mathrm{U}(1))$	$[\exp(\mathrm{i}k_{..}\int A_1A_2A_3A_4)]$

and

$$\begin{aligned}
\mathrm{Tor}_1^R(\mathbb{M},\mathbb{M}') &\equiv \mathbb{M} \boxtimes_R \mathbb{M}',\\
\mathbb{M} \boxtimes_R \mathbb{M}' &\simeq \mathbb{M}' \boxtimes_R \mathbb{M},\\
\mathbb{Z} \boxtimes_{\mathbb{Z}} \mathbb{M} &= \mathbb{M} \boxtimes_{\mathbb{Z}} \mathbb{Z} = 0,\\
\mathbb{Z}_n \boxtimes_{\mathbb{Z}} \mathbb{M} &= \{m \in \mathbb{M} | nm = 0\},\\
\mathbb{Z}_n \boxtimes_{\mathbb{Z}} \mathrm{U}(1) &= \mathbb{Z}_n, \qquad\qquad \text{(A11)}\\
\mathbb{Z}_m \boxtimes_{\mathbb{Z}} \mathbb{Z}_n &= \mathbb{Z}_{\langle m,n\rangle},\\
\mathrm{Tor}_1^{\mathbb{Z}}(U(1),U(1)) &= 0,\\
\mathbb{M}' \oplus \mathbb{M}'' \boxtimes_R \mathbb{M} &= \mathbb{M}' \boxtimes_R \mathbb{M} \oplus \mathbb{M}'' \boxtimes_R \mathbb{M},\\
\mathbb{M} \boxtimes_R \mathbb{M}' \oplus \mathbb{M}'' &= \mathbb{M} \boxtimes_R \mathbb{M}' \oplus \mathbb{M} \boxtimes_R B.
\end{aligned}$$

For other details, we refer the reader to Ref. [51] and references therein.

We summarize some useful facts in Table IX and some derived results in Table X.

2. Derivation of cocycles

To derive Table X, we find that by carrying out the Künneth formula decomposition carefully for a generic finite Abelian group $G=\prod_i Z_{N_i}$, some corresponding structure becomes transparent. See Table XI.

From the known field theory facts, we know that for $2+1$D twisted gauge theories from $\mathcal{H}^3(G,\mathrm{U}(1)) = \prod_{1\leqslant i<j<l\leqslant m}\mathbb{Z}_{N_i}\times\mathbb{Z}_{N_{ij}}\times\mathbb{Z}_{N_{ijl}}$, the $\mathbb{Z}_{n_i}$ classes are captured by a path integral $\simeq \exp(\mathrm{i}k_{..}\int A_i dA_i)$ up to some normalization factor. (Here we omit the wedge product, denoting $A_i dA_i \equiv A_i \wedge dA_i$. We also schematically denote the quantization factor $k_{..}$; the details of $k_{..}$-level quantizations are given in Ref. [60].) The $\mathbb{Z}_{n_{jl}}$ classes are captured by a path integral $\simeq \exp(\mathrm{i}k_{..}\int A_j dA_l)$, where A is a 1-form gauge field. We deduce that the Künneth formula decomposition in $\mathcal{H}^{d+1}(G,\mathrm{U}(1))$ with the torsion product $\mathrm{Tor}_1^R \equiv \boxtimes_R$ suggests a wedge product $\wedge$ structure in the corresponding field theory, while the tensor product $\otimes_{\mathbb{Z}}$ suggests appending an extra exterior derivative $\wedge d$ structure in the corresponding field theory. For example, $\mathcal{H}^1(\mathbb{Z}_{n_1},U(1)) \boxtimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},U(1)) \to [\exp(\mathrm{i}\int A_1 \wedge A_2)]$, and $\mathcal{H}^1(\mathbb{Z}_{n_1},U(1)) \to [\exp(\mathrm{i}\int A_1)]$, thus $\mathcal{H}^1(\mathbb{Z}_{n_1},U(1)) \otimes_{\mathbb{Z}} \mathcal{H}^1(\mathbb{Z}_{n_2},U(1)) \to [\exp(\mathrm{i}\int A_1 \wedge dA_2)]$. This organization also shows the corresponding form of cocycles for $3+1$ dimensions in Table I and $2+1$ dimensions in Table XII. For example, the relation $A_1 \to a_1$ maps a 1-form field to a gauge flux a_1 (or a group element). The relation $dA_2 \to (b_2 + c_2 - [b_2+c_2])$ maps an exterior derivative to the operation, taking on different edges/vertices in the spacetime complex. We use this fact to determine whether two cocycles are the same forms or whether they are up to coboundaries. We comment that such a path integral is only suggestive so far, not yet being strongly evident enough to formulate a consistent field theoretic path integral. Thus we label them with speculative quotation marks in path integral forms in "fields." The more systematic formulation in terms of field theoretic *partition functions* will be reported in Ref. [60] from the perspective of symmetric protected topological states.

TABLE XII. The cohomology group $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ and 3-cocycle ω_3 for a generic finite Abelian group $G=\prod_{i=1}^n Z_{N_i}$. The first column lists the classes in $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$. The second column lists the topological term indices for the $2+1$D twisted gauge theory. (When all indices $k_{...}=0$, it becomes the normal untwisted gauge theory.) The third column lists explicit 3-cocycle functions $\omega_3(a,b,c)$: $(G)^3 \to \mathrm{U}(1)$. Here $a=(a_1,a_2,\ldots,a_k)$, with $a\in G$ and $a_i \in Z_{N_i}$. The same notation is used for b, c, and d. The last column lists induced 2-cocycles from the slant product $\mathsf{C}_a(b,c)$ using Eq. (A13).

$\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$	3-cocycle name	3-cocycle form	Induced $\mathsf{C}_a(b,c)$
$\mathbb{Z}_{N_i}$	Type I, $k_{\mathrm{I}(i)}$	$\omega_{3,\mathrm{I}}^{(i)}(a,b,c) = \exp\left(\frac{2\pi \mathrm{i}k_i}{N_i^2}\ a_i(b_i+c_i-[b_i+c_i])\right)$	$\exp\left(\frac{2\pi \mathrm{i}k_i}{N_i^2}\ a_i(b_i+c_i-[b_i+c_i])\right)$
$\mathbb{Z}_{N_{ij}}$	Type II, $k_{\mathrm{II}(ij)}$	$\omega_{3,\mathrm{II}}^{(ij)}(a,b,c) = \exp\left(\frac{2\pi \mathrm{i}k_{ij}}{N_iN_j}\ a_i(b_j+c_j-[b_j+c_j])\right)$	$\exp\left(\frac{2\pi \mathrm{i}k_{ij}}{N_iN_j}\ a_i(b_j+c_j-[b_j+c_j])\right)$
$\mathbb{Z}_{N_{ijl}}$	Type III, $k_{\mathrm{III}(ijl)}$	$\omega_{3,\mathrm{III}}^{(ijl)}(a,b,c) = \exp\left(\frac{2\pi \mathrm{i}k_{ijl}}{N_{ijl}}\ a_ib_jc_l\right)$	$\exp\left(\frac{2\pi \mathrm{i}k_{ijl}}{N_{ijl}}\ (a_ib_jc_l - b_ia_jc_l + b_ic_ja_l)\right)$

3. Dimensional reduction from a slant product

In general, for dimensional reduction of cochains, we can use the slant product mapping n-cochain c to $(n-1)$-cochain $i_g\mathsf{c}$:

$$i_g\mathsf{c}(g_1,g_2,\ldots,g_{n-1}) \equiv \mathsf{c}(g,g_1,g_2,\ldots,g_{n-1})^{(-1)^{n-1}} \cdot \prod_{j=1}^{n-1} \mathsf{c}(g_1,\ldots,g_j,(g_1\ldots g_j)^{-1}\cdot g\cdot(g_1\ldots g_j),\ldots,g_{n-1})^{(-1)^{n-1+j}}. \quad \text{(A12)}$$

Here we focus on the Abelian group G. For example, in $2+1$ dimensions, we have 3-cocycle to 2-cocycle:

$$\mathsf{C}_A(B,C) \equiv i_A\omega(B,C) = \frac{\omega(A,B,C)\omega(B,C,A)}{\omega(B,A,C)}. \quad \text{(A13)}$$

In $3+1$ dimensions, we have 4-cocycle to 3-cocycle:

$$\mathsf{C}_A(B,C,D) \equiv i_A\omega(B,C,D) = \frac{\omega(B,A,C,D)\omega(B,C,D,A)}{\omega(A,B,C,D)\omega(B,C,A,D)}. \quad \text{(A14)}$$

In order to study the projective representation (the second cohomology group $\mathcal{H}^2$) from 4-cocycles, we do the slant product again:

$$\mathsf{C}^{(2)}_{AB}(C,D) \equiv i_B\mathsf{C}_A(C,D) = \frac{\mathsf{C}_A(B,C,D)\mathsf{C}_A(C,D,B)}{\mathsf{C}_A(C,B,D)} \quad \text{(A15)}$$

$$= \frac{\omega(B,A,C,D)\omega(B,C,D,A)}{\omega(A,B,C,D)\omega(B,C,A,D)} \cdot \frac{\omega(A,C,B,D)\omega(C,B,A,D)}{\omega(C,A,B,D)\omega(C,B,D,A)} \cdot \frac{\omega(C,A,D,B)\omega(C,D,B,A)}{\omega(A,C,D,B)\omega(C,D,A,B)}. \quad \text{(A16)}$$

4. $2+1$D topological orders of $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$

a. Three-cocycles

Here we organize the known fact about the third cohomology group $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ with $G=\prod_{i=1}^k Z_{N_i}$:

$$\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z}) = \prod_{1\leqslant i<j<l\leqslant m} \mathbb{Z}_{N_i}\times\mathbb{Z}_{N_{ij}}\times\mathbb{Z}_{N_{ijl}}.$$

We study the 2D MCG$(\mathbb{T}^2)=$ SL$(2,\mathbb{Z})$ modular data S and T using the Rep theory approach.

b. Projective Rep and S and T for Abelian topological orders

This subsection simply reviews some known facts for later convenience in discussing new results. Much of the discussion can be absorbed from Refs. [40], [55], [50], and [66]. First, we study the Abelian topological orders from types I and II 3-cocycles ω_3 in Table XII for $2+1$D topological orders. We can determine the C_a projective representation (Rep) and $\widetilde{\rho}^a_\alpha(b)$:

$$\widetilde{\rho}^a_\alpha(b)\widetilde{\rho}^a_\alpha(c) = \mathsf{C}_a(b,c)\widetilde{\rho}^a_\alpha(bc). \quad \text{(A17)}$$

Given that Z_a is the centralizer of $a\in G$, C_a determines the projective Rep of Z_a. Each C_a classifies a class of projective Rep called C_a representations, $\widetilde{\rho}: Z_a\to \mathrm{GL}(Z_a)$. In types I and II ω_3, the irreducible C_A representations $\widetilde{\rho}^g_\alpha$ of Z_g are in one-to-one correspondence with the irreducible linear representations. The linear Rep originating from the normal untwisted $\prod_i Z_{N_i}$ gauge theory/toric code is $\exp(2\pi \mathrm{i}(\sum_i \frac{1}{N_i}\alpha_i h_i))$. It has pure charge (α_i)/pure flux (h_i) coupling formulated by a BF theory in any dimension (a mutual Chern-Simons theory in $2+1$ dimensions). The full C_a representation is

$$\widetilde{\rho}^g_\alpha(h) = \exp\left(2\pi \mathrm{i}\left(\sum_i \frac{1}{N_i}\alpha_i h_i\right)\right)\exp\left(2\pi\mathrm{i}\sum_i \frac{1}{N_i^2}p_i g_i h_i\right) \times \exp\left(2\pi\mathrm{i}\sum_{i,j}\frac{1}{N_iN_j}p_i g_i h_j\right). \quad \text{(A18)}$$

We interpret $(\alpha_1,g_1,\alpha_2,g_2,\alpha_3,g_3)$ and $(\beta_1,h_1,\beta_2,h_2,\beta_3,h_3)$ as the charges α and β and fluxes a and b of particles in a doubled basis, $|\alpha,g\rangle$, $|\beta,h\rangle$. The generic T-matrix formula for modular SL$(2,\mathbb{Z})$ data is [40,55]

$$\mathsf{T}_{(\alpha,A)(\beta,B)} = \mathsf{T}_{(\alpha,A)}\delta_{\alpha,\beta}\delta_{A,B} = \frac{\mathrm{Tr}\widetilde{\rho}^{g^A}_\alpha(g^A)}{\dim(\alpha)}. \quad \text{(A19)}$$

We obtain

$$\mathsf{T}_{(\alpha,A)} = \exp\left(2\pi i\left(\left[\sum_i \frac{1}{N_i}\alpha_i a_i\right] + \sum_{j=1,2,3}\frac{1}{N_j^2}p_j\,(a_j^2) + \sum_{ij=12,23,13}\frac{1}{N_iN_j}p_{ij}\,(a_i a_j)\right)\right), \quad \text{(A20)}$$

where $\mathsf{T}_{(\alpha,A)} = e^{\mathrm{i}\Theta^A_\alpha}$ describes the exchange statistics of two identical particles or the topological spin of the same particle. On the other hand, the generic S-matrix formula in $2+1$ dimensions reads [40,55]

$$\mathsf{S}_{(\alpha,a)(\beta,b)} = \frac{1}{|G|}\sum_{\substack{g\in C^a,h\in C^b\\ gh=hg}} \mathrm{Tr}\widetilde{\rho}^g_\alpha(h)^*\mathrm{Tr}\widetilde{\rho}^h_\beta(g)^*, \quad \text{(A21)}$$

yielding

$$\mathsf{S}_{(\alpha,a)(\beta,b)}(p_j,p_{ij}) = \frac{1}{|G|}\exp\left(-2\pi\mathrm{i}\left(\frac{1}{N_i}\left[\sum_i^2 \alpha_i b_i + \beta_i a_i\right] + 2\sum_{j=1,2,3}\frac{1}{N_i^2}p_j\,(a_j b_j) + \sum_{ij=12,23,13}\frac{1}{N_iN_j}p_{ij}\,(a_ib_j+b_ia_j)\right)\right). \quad \text{(A22)}$$

One can use the K-matrix Chern-Simons theory of an action $\mathbf{S}=\frac{1}{4\pi}\int K_{IJ}a_I\wedge da_J$ to encode the information on $|\alpha,g\rangle$, $|\beta,h\rangle$ into quasiparticle vectors l and l', respectively, and formulate a K with $\mathsf{S}_{l,l'}(p_j,p_{ij}) = \frac{1}{|G|}\exp(-2\pi \mathrm{i} l^T K^{-1} l')$. We

TABLE XIII. Phases of $\mathcal{H}^3((Z_2)^2,\mathbb{R}/\mathbb{Z})=(\mathbb{Z}_2)^3$. There are eight types of 3-cocycles butonly four classes.

Class	$(\mathrm{N}_1,\mathrm{N}_{-1},\mathrm{N}_\mathrm{i},\mathrm{N}_{-\mathrm{i}})$	$(\mathrm{n}_{\pm\mathrm{i}},\mathrm{n}_{\pm1},\mathrm{n}_1)$	No. of types
$\omega_3[0]$	(10,6,0,0)	(0,6,4)	1
$\omega_3[2]$	(8,4,2,2)	(2,4,4)	3
$\omega_3[4]$	(6,2,4,4)	(4,2,4)	3
$\omega_3[6]$	(4,0,6,6)	(6,0,4)	1

can use S and T to study the classifications of classes of topological orders. For example, for $G=(Z_2)^2$ twisted theories, simply using T under basis (particle) relabeling, we find that the diagonal eigenvalues of T can be labeled $(\mathrm{N}_1,\mathrm{N}_{-1},\mathrm{N}_\mathrm{i},\mathrm{N}_{-\mathrm{i}})$, as numbers of eigenvalues for $\mathsf{T}=1,-1,\mathrm{i},-\mathrm{i}$. We show that using the data in Table XIII is enough to match the classes found in Ref. [64]. We denote by $(\mathrm{n}_{\pm\mathrm{i}},\mathrm{n}_{\pm1},\mathrm{n}_1)$ the numbers for (the pair of $\pm\mathrm{i}$, the pair of ±1, the individual 1). Note that $\mathrm{N}_1+\mathrm{N}_{-1}+\mathrm{N}_\mathrm{i}+\mathrm{N}_{-\mathrm{i}}=2\mathrm{n}_{\pm\mathrm{i}}+2\mathrm{n}_{\pm1}+\mathrm{n}_1=\mathrm{GSD}_{\mathbb{T}^2}=|G|^2$. There are eight types of 3-cocycles, but there are only four classes in Table XIII. The number in brackets following ω_3 (first column) indicates the number of $+\mathrm{i}$ (or, equivalently, the number of pairs of $\pm\mathrm{i}$, paired due to the nature of the twisted quantum double model). As another example, for $G=(Z_2)^3$ twisted theories, we find that, in Table XIV, by classifying and identifying the modular S and T data, the 64 Abelian-type 3-cocycles (all with Abelian statistics) in $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ are truncated to only four classes.

TABLE XIV. Phases of $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})=(\mathbb{Z}_2)^7$. Among 128 types of 3-cocycles, there are 64 types of 3-cocycles with Abelian statistics, but there are only four classes.

Class	$(\mathrm{N}_1,\mathrm{N}_{-1},\mathrm{N}_\mathrm{i},\mathrm{N}_{-\mathrm{i}})$	$(\mathrm{n}_{\pm\mathrm{i}},\mathrm{n}_{\pm1},\mathrm{n}_1)$	No. of types
$\omega_3[0]$	(36,28,0,0)	(0,28,8)	1
$\omega_3[8]$	(28,20,8,8)	(8,20,8)	21
$\omega_3[16]$	(20,12,16,16)	(16,12,8)	35
$\omega_3[24]$	(12,4,24,24)	(24,4,8)	7

c. *Projective Rep and* S *and* T *for non-Abelian topological orders*

For $2+1$D $G=(Z_2)^3$ twisted gauge theories of $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})=(Z_2)^7$, with 128 types of theories, we have shown that the 64 types of theories with Abelian statistics (from 64 types of 3-cocycles without type III twist) are truncated to four classes in Table XIV. Here we consider the remaining 64 types of 3-cocycles with type III twist in $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})$. Although the gauge group G is Abelian, the type III cocycle twist promotes the theory to having non-Abelian statistics. Our basic knowledge and formalism are rooted in Ref. [40], where the dual D_4 and Q_8 gauge theories are found for certain type III twists. Here we generalize the results in Ref. [40] to all kinds of 3-cocycle twists.

Our expression is the generalized case where 3-cocycles are based on type III's but can include (or not include) types I and II 3-cocycles. There are 8 Abelian charged particles with zero flux and 14 non-Abelian charged particles (whose projective Rep $\widetilde{\rho}^{a}_{\alpha}(b)$ is 2D, described by a rank 2 matrix) with nonzero fluxes as dyons. For $a,b,c\in G=(Z_2)^3$, we label eight elements in $G=(Z_2)^3$ by (0,0,0), (1,0,0) as (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1). We denote these eight elements $F(0)$, $F(1)$, $F(2)$, $F(3)$, $F(4)$, $F(5)$, $F(6)$, and $F(7)$, respectively. Let us recall that $\widetilde{\rho}^{g_a}_{\alpha}(g_b)$ contains α, meaning the representation as charges; also, g_b means the flux, and g_a indicates, in general, the conjugacy class (i.e., flux) as basis. In short, our notation leads to $\widetilde{\rho}^{g_a}_{\alpha}(g_b)=\widetilde{\rho}^{\text{conjugacyclass(flux)asbasis}}_{\text{representation(charge)}}(\text{flux})$.

(i) $1\cdot 8=8$ particles: $F(0)$; $(\alpha_1,\alpha_2,\alpha_3)$. When the flux is 0, $a=F(0)$ is the conjugacy class $C^{F(0)}$. There are eight linear irreducible representations as charges. These charges can be labeled $(\alpha_1,\alpha_2,\alpha_3)$, with $(\alpha_1,\alpha_2,\alpha_3)\in(Z_2)^3$, $\alpha_1,\alpha_2,\alpha_3\in\{0,1\}$. So we have

$$\widetilde{\rho}^{F(0)}_{F(0),(\alpha_1,\alpha_2,\alpha_3)}(b)=\widetilde{\rho}^{F(0)}_{F(0),(\alpha_1,\alpha_2,\alpha_3)}(b_1,b_2,b_3)=\exp\left(\frac{2\pi i}{m^2}m\left(\sum_{j=1,2,3}\alpha_j b_j\right)\right). \tag{A23}$$

(ii) $7\cdot 2=14$ particles: $F(j)$; $\pm$. The remaining seven kinds of fluxes are $a=F(j)$ for $j=1,\ldots,7$. There are two kinds of representations for each. We can denote these two representations as $+$ or $-$. So together these give 14 more types of particles. In total there are $1\cdot 8+7\cdot 2=22$ quasiparticle excitations as the GSD on the $\mathbb{T}^2$ torus. Generally, the representation is $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(l))$ for some inserting flux $F(l)$. This is a 2D representation. The identity is always assigned $F(0)$; namely, $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(0))=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. We list three more elements: $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1))$, $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2))$, and $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3))$. The remaining $\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(l))$ for $l=4,\ldots,7$ can be determined by Eq. (A17). The representations are adjusted by a 1D projective Rep for type I ω_I and type II ω_{II} 3-cocycles, with topological level quantized coefficients p_1, p_2, p_3 for type I and p_{12}, p_{13}, p_{23} for type II. Under types I and II twists, the type III Rep adjusts to

$$\widetilde{\rho}^{F(j)=a}_{F(j)=a,\pm}(b)\rightarrow\widetilde{\rho}^{F(j)}_{F(j),\pm}(b)e^{i\frac{\pi}{2}(\sum_{\substack{j,l\in\{1,2,3\}\\ j<l}}p_l a_l b_l+p_{ln}a_l b_n)}. \tag{A24}$$

(iii) Two particles: $F(1)$; $\pm j=1$. Here $(a_1,a_2,a_3)=F(1)=(1,0,0)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1))=\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}e^{i\frac{\pi}{2}(p_1a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2))=\begin{pmatrix}0&1\\1&0\end{pmatrix}e^{i\frac{\pi}{2}(p_2a_2+p_{12}a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3))=\begin{pmatrix}1&0\\0&-1\end{pmatrix}e^{i\frac{\pi}{2}(p_3a_3+p_{13}a_1+p_{23}a_2)}.$$

(iv) Two particles: $F(2)$; $\pm j=2$. Here $(a_1,a_2,a_3)=F(2)=(0,1,0)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1))=\begin{pmatrix}1&0\\0&-1\end{pmatrix}e^{i\frac{\pi}{2}(p_1a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2))=\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}e^{i\frac{\pi}{2}(p_2a_2+p_{12}a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3))=\begin{pmatrix}0&1\\1&0\end{pmatrix}e^{i\frac{\pi}{2}(p_3a_3+p_{13}a_1+p_{23}a_2)}.$$

TABLE XV. The modular T^a_α matrix for 2D twisted $(Z_2)^3$ theories with non-Abelian statistics. All 64 non-Abelian theories in $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})$ are listed.

Particle	T^a_α
$((\alpha_1,\alpha_2,\alpha_3),F(0))$	1
$(\pm,F(1)),(\pm,F(2)),(\pm,F(3))$	$\pm \mathrm{i}^{p_1}, \pm \mathrm{i}^{p_2}, \pm \mathrm{i}^{p_3}$
$(\pm,F(4)),(\pm,F(5)),(\pm,F(6))$	$\pm \mathrm{i}^{p_1+p_2+p_{12}}, \pm \mathrm{i}^{p_1+p_3+p_{13}}, \pm \mathrm{i}^{p_2+p_3+p_{23}}$
$(\pm,F(7))$	$\pm \mathrm{i}\cdot \mathrm{i}^{p_1+p_2+p_3+p_{12}+p_{13}+p_{23}}$

(v) Two particles: $F(3)$; $\pm j = 3$. Here $(a_1,a_2,a_3) = F(3) = (0,0,1)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}.$$

(vi) Two particles: $F(4)$; $\pm j = 4$. Here $(a_1,a_2,a_3) = F(4) = (1,1,0)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}.$$

(vii) Two particles: $F(5)$; $\pm j = 5$. Here $(a_1,a_2,a_3) = F(5) = (1,0,1)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}.$$

(viii) Two particles: $F(6)$; $\pm j = 6$. Here $(a_1,a_2,a_3) = F(6) = (0,1,1)$,

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}.$$

(ix) Two particles: $F(7)$; $\pm j = 7$. Here $(a_1,a_2,a_3) = F(7) = (1,1,1)$. (Note, in particular, that for this Rep, our choice $\mp$ differs from that in Ref. [40].)

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(1)) = \mp\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_1 a_1)},\quad \widetilde{\rho}^{F(j)}_{F(j),\pm}(F(2)) = \mp\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} e^{i\frac{\pi}{2}(p_2 a_2 + p_{12} a_1)},$$

$$\widetilde{\rho}^{F(j)}_{F(j),\pm}(F(3)) = \mp\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{i\frac{\pi}{2}(p_3 a_3 + p_{13} a_1 + p_{23} a_2)}.$$

With the above projective Rep $\widetilde{\rho}^a_\alpha(b)$, we can derive the analytic form of the modular data S and T in two dimensions. Here for $G = (Z_2)^3$,

$$\mathsf{T}^A_\alpha = e^{\mathrm{i}\frac{\pi}{2}(\sum_{\substack{l,m\in\{1,2,3,4\} \\ l<m}} p_l {a_l}^2 + p_{lm} a_l a_l)} (\pm)_a (\mathrm{i})^{\eta_{a,a}} \to \mathsf{T}^A_\alpha = \pm 1 \quad \text{or} \quad \pm\,\mathrm{i}; \tag{A25}$$

$$\eta_{g_1,g_2} \equiv \begin{cases} 0 & \text{if} \quad \mathsf{C}_{g_1}(g_2,g_2) = +1, \\ 1 & \text{if} \quad \mathsf{C}_{g_1}(g_2,g_2) = -1. \end{cases} \tag{A26}$$

More explicitly, we compute T^A_α in Table XV.

With the modular $\mathsf{S}^{xy} = \mathsf{S}^{xy}_{(\alpha,a)(\beta,b)}$ matrix [of 64 types of 2D twisted $(Z_2)^3$ theories with non-Abelian statistics],

$$\mathsf{S} = \frac{1}{|G|} \begin{array}{c} \begin{array}{ccc} (\beta_j,0) & (+,b_j) & (-,b_j) \end{array} \\ \left(\begin{array}{c|c|c} 1 & 2(-1)^{b_1\alpha_1+b_2\alpha_2+b_3\alpha_3} & 2(-1)^{b_1\alpha_1+b_2\alpha_2+b_3\alpha_3} \\ \hline 2(-1)^{a_1\beta_1+a_2\beta_2+a_3\beta_3} & \delta_{a,b} 4\cdot(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j + p_{jl} a_j a_l} & -\delta_{a,b} 4(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j + p_{jl} a_j a_l} \\ \hline 2(-1)^{a_1\beta_1+a_2\beta_2+a_3\beta_3} & -\delta_{a,b} 4(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j + p_{jl} a_j a_l} & \delta_{a,b} 4(-1)^{\eta_{a,a}}\cdot(-1)^{\sum_{\substack{j,l=1,2,3\\ j<l}} p_j a_j + p_{jl} a_j a_l} \end{array}\right) \end{array} \begin{array}{c} (\alpha_j,0) \\ (+,a_j) \\ (-,a_j) \end{array} \tag{A27}$$

In Eq. (A27), the factor $(-1)^{\eta_{a,a}}$ is derived from a computation of $(\mathrm{i})^{\eta_{a,b}}\cdot(\mathrm{i})^{\eta_{b,a}}\delta_{a,b} = (-1)^{\eta_{a,a}}\delta_{a,b}$. From Eq. (A26), we note that $\eta_{a,a} = 1$ is nonzero only when $a = (1,1,1) = F(7)$ for the $(Z_2)^3$ flux.

5. Classification of 2 + 1D twisted $(Z_2)^3$ gauge theories, $D^\omega((Z_2)^3)$ and $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})$

The twisted $(Z_2)^3$ gauge theories dual to D_4, Q_8 non-Abelian gauge theories were first discovered in Ref. [40]. Here we present the three other classes which cannot be dual to any non-Abelian gauge theory, but can only be twisted (Abelian or non-Abelian) gauge theories themselves. We again label the diagonal eigenvalues of T as $(\mathrm{N}_1,\mathrm{N}_{-1},\mathrm{N}_{\mathrm{i}},\mathrm{N}_{-\mathrm{i}})$ and their number of eigenvalues as $\mathsf{T} = 1,-1,\mathrm{i},-\mathrm{i}$. We also use shorthand $(\mathrm{n}_{\pm\mathrm{i}},\mathrm{n}_{\pm 1},\mathrm{n}_1)$ instead, which stands for the numbers for (the pair of $\pm\mathrm{i}$, the pair of ± 1, the individual 1) in the diagonal of T. Note that $\mathrm{N}_1+\mathrm{N}_{-1}+\mathrm{N}_{\mathrm{i}}+\mathrm{N}_{-\mathrm{i}} = 2\mathrm{n}_{\pm\mathrm{i}} + 2\mathrm{n}_{\pm 1} + \mathrm{n}_1 = \mathrm{GSD}_{\mathbb{T}^2} = 22$. There are 64 *types* of 3-cocycles corresponding to theories with non-Abelian statistics, but there are only 5 inequivalent *classes* in Table XIII. The number in brackets following ω_3 (first column) indicates the number of $+\mathrm{i}$ (or, equivalently, the number of pairs $\pm\mathrm{i}$, paired due to the nature of the quantum double model).

Although $\omega_3[3d]$ and $\omega_3[3i]$ share the same T-matrix data, they can still be distinguished by the linear dependency of the fluxes which carry three pairs of eigenvalues i. (And, of course, they can be distinguished by the more involved S matrix.) There are 7 types in the $\omega_3[3d]$ class, whose $\pm\mathrm{i}$ are generated by linear-dependent fluxes, and another 28 types in the $\omega_3[3i]$ class, whose $\pm\mathrm{i}$ are generated by linear-independent fluxes. In this notation of linear (in)dependence, we have $\omega_3[1] = \omega_3[1i]$, $\omega_3[5] = \omega_3[5d]$, $\omega_3[7] = \omega_3[7d]$. Such a concept is also used in the mathematics literature in Ref. [62], where the authors study the Frobenius-Schur indicators, Frobenius-Schur exponents, and support of cocycle twist, supp ω, and use these data to classify the twisted quantum double model $D^{\omega}(G)$. Remarkably, we find that using our data is enough to match the classes found in the mathematics literature [62] in the quantum double and module category framework.

These findings, together with Appendix A 4 b, form a complete data set of $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z}) = (\mathbb{Z}_2)^7$, where 128 types of 3-cocycles fall into four distinct classes of Abelian topological orders in Table XIII and five distinct classes of non-Abelian topological orders in Table XVI. In total there are nine distinct classes of topological orders within twisted $(Z_2)^3$ gauge theories. We note that $\omega_3[3i]$, $\omega_3[5]$, and $\omega_3[7]$ can only be twisted gauge theories, not dual to any untwisted non-Abelian gauge theory.

6. $3+1$D topological orders of $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$

This subsection continues the discussion and notations from $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ of $2+1$D to $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ of $3+1$D topological orders. Now we fill in some more information about the data on the projective Rep.

a. Projective Rep and S *and* T *for Abelian topological orders*

The data of $\widetilde{\rho}_{\alpha}^{ab}(c)$ is organized below in Table XVII for $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ of cohomology group $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$. The modular S and T matrices for this Rep are presented in Tables II–IV. In the text, we provide an example of classifying 3D topological orders from $3+1$D $(Z_2)^2$ twisted gauge theories of four types [from $\mathcal{H}^4((Z_2)^2,\mathbb{R}/\mathbb{Z}) = (Z_2)^2$] and find out that the four types are truncated to only two distinct classes of topological orders.

b. Projective Rep and S *and* T *for non-Abelian topological orders*

Below we present the data on twisted gauge theories for those with non-Abelian statistics in $\mathcal{H}^4(G = (Z_2)^4,\mathbb{R}/\mathbb{Z})$ labeled by 4-cocycles ω_4. Among $\mathcal{H}^4((Z_2)^4,\mathbb{R}/\mathbb{Z}) = (Z_2)^{21}$ types of theories, there are 2^{20} types that show non-Abelian statistics. In some cases, we write the formula in terms of a slightly generic $G = (Z_n)^4$, for a prime n.

Analogously to Appendix A 4 c, we recall that the 3D triple basis renders $\widetilde{\rho}_{\alpha}^{g^a,g^b}(g^c) = \widetilde{\rho}_{\text{representation(charge)}}^{\text{conjugacyclass(flux,flux)asbasis}}(\text{flux})$. So we understand that the representation $\widetilde{\rho}(c)$ is constrained by the flux a,b. We consider type IV $\omega_{4,\mathrm{IV}}$ twisted theories, but we include $\omega_{4,\mathrm{IV}}$ further multiplied by type II $\omega_{4,\mathrm{II}}$ and type III $\omega_{4,\mathrm{III}}$ 4-cocycles. Thus, the representation also relates to their topological terms p_{lm} of type II $\omega_{4,\mathrm{II}}$ labeling $(Z_2)^{2\binom{4}{2}} = (Z_2)^{12}$ types of theories and p_{lmn} of type III $\omega_{4,\mathrm{III}}$ labeling $(Z_2)^{2\binom{4}{3}} = (Z_2)^8$ types of theories. In total, all these 4-cocycles multiplied by $\omega_{4,\mathrm{IV}}$ yield the 2^{20} types of theories showing non-Abelian statistics. Under types II and III twists, the type IV Rep is adjusted to

$$
\begin{aligned}
&\widetilde{\rho}_{a,b,(\pm,\pm)}^{a,b}(c)\\
&\quad= \widetilde{\rho}_{F(j_1)=a,F(j_2)=b,(\pm,\pm)}^{F(j_1)=a,F(j_2)=b}(c)\\
&\qquad \cdot e^{i\frac{\pi}{2}(\sum_{\substack{l,m,n\in\{1,2,3,4\}\\ l<m<n}} p_{lm} f_{lm}(a,b,c)+p_{lmn} f_{lmn}(a,b,c))}. \quad \text{(A28)}
\end{aligned}
$$

Note that the trace $\mathrm{Tr}[\widetilde{\rho}_{a,b,(\pm,\pm)}^{a,b}(c)]$ is nonzero only when (i) $c=a$, $c=b$, or $c=ab$, with $\mathrm{Tr}[\widetilde{\rho}_{a,b,(\pm,\pm)}^{a}(c)] \neq 0$, or (ii) $c = F(0)$ zero flux, i.e., $\mathrm{Tr}[\widetilde{\rho}_{a,b,(\pm,\pm)}^{a,b}(F(0))] \neq 0$. Other cases have zero traces. Among the degeneracy sectors on the $\mathbb{T}^3$ torus, we have $\mathrm{GSD}_{\mathbb{T}^3} = (n^8+n^9-n^5)+(n^{10}-n^7-n^6+n^3)$ (ground-state bases in terms of particles and string quasiexcitations), which is 1576 for $n=2$. We can use $|G|^2 = (n^4)^2 = 256$ (doubled) fluxes to do the first labeling. Note that the fluxes form a doubled basis (a,b) in $|\alpha,a,b\rangle$. Among 256 fluxes, there are $n^4+n^5-n = 46$ fluxes carrying Abelian excitations, while the remaining $(n^8-(n^4+n^5-n)) = 210$ are non-Abelian excitations. (*Note*: The bases carry two fluxes and one charge; these bases should *not* be confused with string and particle types.) We may organize the ground-state bases in terms of two kinds, which correspond to Abelian and non-Abelian excitations.

(i) $(n^4+n^5-n)\cdot n^4 = 46\times 16 = 736$ Abelian excitations: $F(j_{ab})$; $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Here $a = F(j_{ab})$ can be zero fluxes or nonzero fluxes by satisfying the following conditions:

$$
\begin{aligned}
&a_1b_2 = a_2b_1, \quad a_1b_3 = a_3b_1, \quad a_1b_4 = a_4b_1,\\
&a_2b_3 = a_3b_2, \quad a_2b_4 = a_4b_2, \quad a_3b_4 = a_4b_3 \pmod N).
\end{aligned} \tag{A29}
$$

There are (n^4+n^5-n) independent solutions for these sets of a, b. The conjugacy class $C^{F(j_{ab})}$ stands for fluxes. There are n^4 representations as charges; these can be labeled $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, with $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in(Z_2)^4$, and $Z_2=\{0,1\}$. We write $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\alpha$. Equation (A28) becomes

$$
\begin{aligned}
\widetilde{\rho}_{F(j_{ab}),(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}^{F(j_{ab})}(c) &= \widetilde{\rho}_{F(0),(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}^{F(0)}(c_1,c_2,c_3,c_4)\\
&= \exp\left(\sum_{k=1}^{4}\frac{2\pi i}{N_k}\alpha_k c_k\right).
\end{aligned} \tag{A30}
$$

For $n=2$, there are $(2^4+2^5-2)=46$ (doubled) fluxes contributing Abelian excitations.

(ii) $(n^8-(n^4+n^5-n))\cdot n^2 = 210\times 4 = 840$ non-Abelian excitations: $F(j_{\text{nAbel}})$; $(\pm,\pm)$. For $n=2$, there are $(n^8-(n^4+n^5-n)) = 210$ (doubled) fluxes contributing non-Abelian excitations. Each of them carries a 2D Rep with two pairs of $(\pm,\pm)$ charge Reps. Thus the number of doubled fluxes multiplied by 4 yields 840 excitations. This is equivalent to counting the $\mathsf{C}_{a,b}^{(2)}(c,d)$ class that they belong to.

TABLE XVI. $D^{\omega}(G)$ is the twisted quantum double of G with a cocycle twist ω of G's cohomology group. Here we consider a 3-cocycle twist ω_3 in $\mathcal{H}^3((Z_2)^3,\mathbb{R}/\mathbb{Z})=(\mathbb{Z}_2)^7$, where ω_3 contains a factor of type III 3-cocycle. We compute the values in the second and the fourth columns and then compare them with the mathematics literature [62] to match for the third column. We find that the 64 types of non-Abelien theories are truncated to five classes.

Class	$(\mathrm{n}_{\pm \mathrm{i}},\mathrm{n}_{\pm 1},\mathrm{n}_1)$	$(\mathrm{N}_1,\mathrm{N}_{-1},\mathrm{N}_{\mathrm{i}},\mathrm{N}_{-\mathrm{i}})$	Twisted quantum double $D^{\omega}(G)$	No. of types
$\omega_3[1]$	(1,6,8)	(14,6,1,1)	$D^{\omega_3[1]}({Z_2}^3)$, $D(D_4)$	7
$\omega_3[3d]$	(3,4,8)	(12,4,3,3)	$D^{\omega_3[3d]}({Z_2}^3)$, $D^{\gamma^4}(Q_8)$	7
$\omega_3[3i]$	(3,4,8)	(12,4,3,3)	$D^{\omega_3[3i]}({Z_2}^3)$, $D(Q_8)$, $D^{\alpha_1}(D_4)$, $D^{\alpha_2}(D_4)$	28
$\omega_3[5]$	(5,2,8)	(10,2,5,5)	$D^{\omega_3[5]}({Z_2}^3)$, $D^{\alpha_1\alpha_2}(D_4)$	21
$\omega_3[7]$	(7,0,8)	(8,0,7,7)	$D^{\omega_3[7]}({Z_2}^3)$	1

There are six $c_l d_m$ terms in type IV 4-cocycles:

$$\begin{aligned}\mathsf{C}^{(2)}_{a,b}(c,d)=\exp\bigg(&\frac{2\pi i p_{\mathrm{IV}(1234)}}{N_{ijlm}}\,(a_4b_3-a_3b_4)c_1d_2\\&+(a_2b_4-a_4b_2)c_1d_3+(a_4b_1-a_1b_4)c_2d_3\\&+(a_3b_2-a_2b_3)c_1d_4+(a_1b_3-a_3b_1)c_2d_4\\&+(a_2b_1-a_1b_2)c_3d_4\bigg).\end{aligned}\tag{A31}$$

Below, each solution is multiplied by 6; due to $\binom{3}{2}\times 2$, three terms, a,b, and ab, can choose 2 as the generator basis for a and b. These terms have $\mathrm{Tr}[\widetilde{\rho}^{a}_{a,b,(\pm,\pm)}(c)]\neq 0$ for $c=0,a,b,ab$. And the permutation of a,b results in an extra multiple of 2. We organize the solutions into the following six *styles*. Each style may contain dimensionally reduced 3-cocycles, as "type III 3-cocycle-like" or "mixed type III 3-cocycles." Here "type III 3-cocycle-like" means that the dimensional reduced 2D theory has an induced 3-cocycle which is a type III 3-cocycle *within* a subgroup $(Z_2)^3$. "Mixed type III 3-cocycle" means that the dimensional reduced 2D theory has an induced 3-cocycle which contains several type III 3-cocycles spanning the full group $(Z_2)^4$. The six styles of solutions are as follows.

Style 1 (type III 3-cocycle-like). $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains one cd term: $\binom{6}{1}\times 6=36$ non-Abelian fluxes.

Style 2 (type III 3-cocycle-like). $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains two cd terms: $(\binom{6}{2}-3)\times 6=72$ non-Abelian fluxes. We have $\binom{6}{2}$ minus 3, because it is impossible to have nonzero coefficient cd terms of $\mathsf{C}^{(2)}_{a,b}(c,d)$ for any of the following terms together: (i) c_3d_4 and c_1d_2 terms, (ii) c_2d_4 and c_1d_3 terms, and (iii) c_2d_3 and c_1d_4 terms.

Style 3 (type III 3-cocycle-like) and style 4 (mixed type III 3-cocycles). $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains three cd terms: $\binom{4}{3}\times 6+\binom{4}{3}\times 6=48$ non-Abelian fluxes.

For style 3 (type III 3-cocycle-like), $\binom{4}{3}\times 6$: $\binom{4}{3}$ of 6 have nonzero coefficients for (i) c_2d_3, c_2d_4, c_3d_4; (ii) c_1d_3, c_1d_4, c_3d_4; (iii) c_1d_2, c_1d_4, c_2d_4; and (iv) c_1d_2, c_1d_3, c_2d_3. Each type has six possible choices for a,b.

For style 4 (mixed type III 3-cocycles), $\binom{4}{3}\times 6$: $\binom{4}{3}$ of 6 have nonzero coefficients for (i) c_1d_2, c_1d_3, c_1d_4; (ii) c_1d_2, c_2d_3, c_2d_4; (iii) c_1d_3, c_2d_3, c_3d_4; and (iv) c_1d_4, c_2d_4, c_4d_4. Each type has six possible choices for a,b.

Style 5 (mixed type III 3-cocycles). $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains four cd terms: $(\binom{6}{4}-\binom{4}{3}\cdot 3)\times 6=3\times 6=18$ non-Abelian fluxes. Among 15 terms (with 4 cd) in $\binom{6}{4}=15$, there are only 3 terms allowed: (i) c_1d_2, c_2d_3, c_1d_4, c_3d_4; (ii) c_1d_3, c_2d_3, c_1d_4, c_2d_4; and (iii) c_1d_2, c_1d_3, c_2d_4, c_3d_4. Terms from $\binom{4}{3}\cdot 3=12$ are not allowed, like c_1d_2, c_1d_3, c_2d_3, c_1d_4 [i.e., choose three elements as $\binom{4}{3}$ and choose one of the three, thus times 3, to pair with the remaining unchosen elements).

Style 6 (mixed type III 3-cocycles). $\mathsf{C}^{(2)}_{a,b}(c,d)$ contains five cd terms: $\binom{6}{5}\times 6=36$ non-Abelian fluxes. Included are (i) c_1d_2, c_1d_3, c_1d_4, c_2d_3, c_2d_4; (ii) c_1d_2, c_1d_3, c_1d_4, c_2d_3, c_3d_4; (iii) c_1d_2, c_1d_3, c_1d_4, c_2d_4, c_3d_4; (iv) c_1d_2, c_1d_3, c_2d_3, c_2d_4, c_3d_4; (v) c_1d_2, c_1d_4, c_2d_3, c_2d_4, c_3d_4; and (vi) c_1d_3, c_1d_4, c_2d_3, c_2d_4, c_3d_4

TABLE XVII. $\widetilde{\rho}^{a,b}_{\alpha}(c)$ for a 3+1D twisted gauge theory with $G=Z_{N_1}\times Z_{N_2}\times Z_{N_3}$ of $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$. We derive $\widetilde{\rho}^{a,b}_{\alpha}(c)$ from the equation introduced in the text, $\widetilde{\rho}^{a,b}_{\alpha}(c)\widetilde{\rho}^{a,b}_{\alpha}(d)=\mathsf{C}^{(2)}_{a,b}(c,d)\widetilde{\rho}^{a,b}_{\alpha}(cd)$, presenting the projective representation, because the induced 2-cocycle belongs to the second cohomology group, $\mathcal{H}^2(G,\mathbb{R}/\mathbb{Z})$. $\widetilde{\rho}^{a,b}_{\alpha}(c)$: $(Z_a,Z_b)\to \mathrm{GL}\,(Z_a,Z_b)$ can be written as a general linear matrix.

$\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$	4-cocycle	$\widetilde{\rho}^{a,b}_{\alpha}(c)$
$\mathbb{Z}_{N_{12}}$	Type II 1st	$\widetilde{\rho}^{(1\mathrm{st})a,b}_{\mathrm{II},\alpha}(c)=\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\alpha_k c_k\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(1\mathrm{st})}_{\mathrm{II}(12)}}{(N_{12}\cdot N_2)}(a_2b_1-a_1b_2)c_2\big)$
$\mathbb{Z}_{N_{12}}$	Type II 2nd	$\widetilde{\rho}^{(2\mathrm{nd})a,b}_{\mathrm{II},\alpha}(c)=\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\alpha_k c_k\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(2\mathrm{nd})}_{\mathrm{II}(12)}}{(N_{12}\cdot N_1)}(a_1b_2-a_2b_1)c_1\big)$
$\mathbb{Z}_{N_{123}}$	Type III 1st	$\widetilde{\rho}^{(1\mathrm{st})a,b}_{\mathrm{III},\alpha}(c)=\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\alpha_k c_k\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(1\mathrm{st})}_{\mathrm{III}(123)}}{(N_{12}\cdot N_3)}(a_2b_1-a_1b_2)c_3\big)$
$\mathbb{Z}_{N_{123}}$	Type III 2nd	$\widetilde{\rho}^{(2\mathrm{nd})a,b}_{\mathrm{III},\alpha}(c)=\exp\big(\sum_k \frac{2\pi \mathrm{i}}{N_k}\alpha_k c_k\big)\cdot\exp\big(\frac{2\pi \mathrm{i} p^{(2\mathrm{nd})}_{\mathrm{III}(123)}}{(N_{31}\cdot N_2)}(a_1b_3-a_3b_1)c_2\big)$

Styles 1–3 are pure type III 3-cocycle ω_3-like, which $\widetilde{\rho}^{a,b}_{a,b,(\pm,\pm)}(c)$ can be deduced from the result $G=(Z_2)^3$ in Appendix A 4 c. Styles 4–6 are mixed type III 3-cocycles in the whole $G=(Z_2)^4$ group, so one needs to assign the Rep $\widetilde{\rho}^{a,b}_{a,b,(\pm,\pm)}(c)$ in a slightly different manner. But it turns out that rank 2 matrices are always sufficient to encode the irreducible projective representation of $\mathsf{C}^{(2)}_{ab}(c,d)$. After finding the $\widetilde{\rho}^{a,b}_{a,b,(\pm,\pm)}(c)$, we analytically derive their 3D non-Abelian S^{xyz} and T^{xy} presented in the text, in Table V, Eq. (51), and Eq. (52).

APPENDIX B: S^{xyz} AND T^{xy} CALCULATION IN TERMS OF THE GAUGE GROUP G AND 4-COCYCLE ω_4

1. Unimodular group and SL($N,\mathbb{Z}$)

In the case of the unimodular group, there are unimodular matrices of rank N forms GL($N,\mathbb{Z}$). S_U and T_U have determinants $\det(\mathsf{S}_\mathsf{U})=-1$ and $\det(\mathsf{T}_\mathsf{U})=1$ for any general N:

$$\mathsf{S}_\mathsf{U}=\begin{pmatrix}0&0&0&\dots&(-1)^N\\1&0&0&\dots&0\\0&1&0&\dots&0\\\vdots&\vdots&\ddots&\dots&\vdots\\0&0&0&\dots&0\end{pmatrix}, \tag{B1}$$

$$\mathsf{T}_\mathsf{U}=\begin{pmatrix}1&1&0&\dots&0\\0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\end{pmatrix}. \tag{B2}$$

Note that $\det(\mathsf{S}_\mathsf{U})=-1$ in order to generate both determinant 1 and determinant -1 matrices.

For the SL($N,\mathbb{Z}$) modular transformation, we denote their generators S and T for a general N with $\det(\mathsf{S})=\det(\mathsf{T})=1$:

$$\mathsf{S}=\begin{pmatrix}0&0&0&\dots&(-1)^{N-1}\\1&0&0&\dots&0\\0&1&0&\dots&0\\\vdots&\vdots&\ddots&\dots&\vdots\\0&0&0&\dots&0\end{pmatrix}, \tag{B3}$$

$$\mathsf{T}=\mathsf{T}_\mathsf{U}. \tag{B4}$$

Here for simplicity, let us denote S^{xyz} as $\mathsf{S}_{3\mathrm{D}}$, S^{xy} as $\mathsf{S}_{2\mathrm{D}}$, and $\mathsf{T}^{xy}=\mathsf{T}_{3\mathrm{D}}=\mathsf{T}_{2\mathrm{D}}$. Recall that SL($3,\mathbb{Z}$) is fully generated by generators $\mathsf{S}_{3\mathrm{D}}$ and $\mathsf{T}_{3\mathrm{D}}$:

$$\mathsf{S}_{3\mathrm{D}}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad \mathsf{T}_{3\mathrm{D}}=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},$$

$$\mathsf{S}_{2\mathrm{D}}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix};$$

$$\mathsf{S}_{2\mathrm{D}}=\left(\mathsf{T}_{3\mathrm{D}}^{-1}\mathsf{S}_{3\mathrm{D}}\right)^3(\mathsf{S}_{3\mathrm{D}}\mathsf{T}_{3\mathrm{D}})^2\mathsf{S}_{3\mathrm{D}}\mathsf{T}_{3\mathrm{D}}^{-1}. \tag{B5}$$

By dimensional reduction (note $\mathsf{T}_{2\mathrm{D}}=\mathsf{T}_{3\mathrm{D}}$), we expect that

$$\mathsf{S}_{2\mathrm{D}}^4=(\mathsf{S}_{2\mathrm{D}}\mathsf{T}_{3\mathrm{D}})^6=1, \tag{B6}$$

$$(\mathsf{S}_{2\mathrm{D}}\mathsf{T}_{3\mathrm{D}})^3=e^{\frac{2\pi \mathrm{i}}{8}c_-}\mathsf{S}_{2\mathrm{D}}^2=e^{\frac{2\pi \mathrm{i}}{8}c_-}C, \tag{B7}$$

where c_- carries the information on central charges. We can write

$$\mathsf{R}\equiv\begin{pmatrix}0&1&0\\-1&1&0\\0&0&1\end{pmatrix}=(\mathsf{T}_{3\mathrm{D}}\mathsf{S}_{3\mathrm{D}})^2\mathsf{T}_{3\mathrm{D}}^{-1}\mathsf{S}_{3\mathrm{D}}^2\mathsf{T}_{3\mathrm{D}}^{-1}\mathsf{S}_{3\mathrm{D}}\mathsf{T}_{3\mathrm{D}}\mathsf{S}_{3\mathrm{D}}. \tag{B8}$$

One can check that

$$\mathsf{S}_{3\mathrm{D}}\mathsf{S}_{3\mathrm{D}}^\dagger=\mathsf{S}_{3\mathrm{D}}^3=\mathsf{R}^6=(\mathsf{S}_{3\mathrm{D}}\mathsf{R})^4=(\mathsf{R}\mathsf{S}_{3\mathrm{D}})^4=1, \tag{B9}$$

$$(\mathsf{S}_{3\mathrm{D}}\mathsf{R}^2)^4=(\mathsf{R}^2\mathsf{S}_{3\mathrm{D}})^4=(\mathsf{S}_{3\mathrm{D}}\mathsf{R}^3)^3=(\mathsf{R}^3\mathsf{S}_{3\mathrm{D}})^3=1, \tag{B10}$$

$$(\mathsf{S}_{3\mathrm{D}}\mathsf{R}^2\mathsf{S}_{3\mathrm{D}})^2\mathsf{R}^2=\mathsf{R}^2(\mathsf{S}_{3\mathrm{D}}\mathsf{R}^2\mathsf{S}_{3\mathrm{D}})^2(\mathrm{mod}3). \tag{B11}$$

Such expressions are known in the mathematic literature; some of them are listed in Ref. [37].

2. Rules for the path integral for the spacetime complex of cocycles

For the branching of a spacetime complex or a simplex, we define that, for any arrow that goes from a small number to a large number, the number ordering is $1<2<3<4<\dots<0'<1'<2'<2^{*\prime}<3'<4'<5'<6'<6^{*\prime}<\cdots$. The time evolves along the fourth direction from the left to the right, or from a smaller number to a larger number. Also, we may write $[01].[12]=[02]$ or, equivalently, $g_{01}.g_{12}=g_{02}$. If $[01]=g$ and $[12]=h$, then $[02]=gh$.

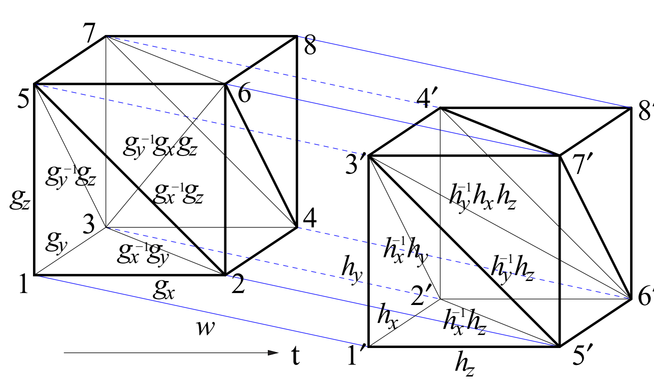


FIG. 12. (Color online) spacetime complex $\mathbb{T}^3\times I$, where $I=[0,1]$ is the time direction. $\mathbb{T}^3\times\{0\}$ and $\mathbb{T}^3\times\{1\}$ are shown. Gray (blue) lines illustrate how the two $\mathbb{T}^3$'s are connected for $t\in(0,1)$. Note that the two $\mathbb{T}^3$'s differ by a rotation S^{xyz}. In other words, when time forms a loop, the two $\mathbb{T}^3$'s are glued together as $1\to1'$, $2\to2'$, $3\to3'$, $4\to4'$, $5\to5'$, $6\to6'$, $7\to7'$, and $8\to8'$.

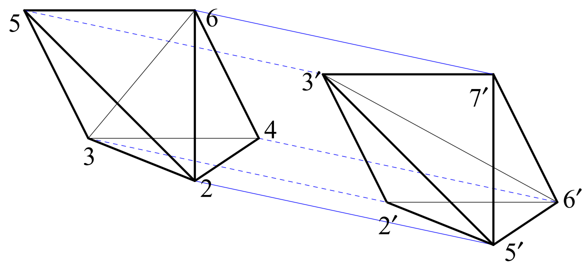


FIG. 13. (Color online) Complex M_1.

3. Explicit expression of S^{xyz} in terms of (G,ω_4)

The S^{xyz} matrix can be computed from the amplitude $A^{xyz}(g_x,g_x,g_z,h_x,h_y,h_z;\text{w})$ of the path integral in spacetime complex $\mathbb{T}^3\times I$ (see Fig. 12). Each $\mathbb{T}^3$ is divided into six tetrahedrons. The amplitude $A^{xyz}(g_x,g_x,g_z,h_x,h_y,h_z;\text{w})$ is the product of the four amplitudes A_i for the four shapes M_i, $i=1,\dots,4$, which are given in Figs. 13–16.

Each shape M_i can be divided into several 4-simplexes. So the amplitude A_i for each M_i is the product of several cocycles on the simplexes. We find that, for M_3,

$$A_3=\frac{\omega_4(g_{12},g_{23},g_{35},g_{51'})\omega_4^{-1}(g_{35},g_{51'},g_{1'2'},g_{2'5'})}{\omega_4(g_{23},g_{35},g_{51'},g_{1'5'})\omega_4(g_{51'},g_{1'2'},g_{2'3'},g_{3'5'})},\quad\text{(B12)}$$

and for M_4,

$$A_4=\frac{\omega_4(g_{67},g_{78},g_{86'},g_{6'7'})\omega_4(g_{84'},g_{4'6'},g_{6'7'},g_{7'8'})}{\omega_4(g_{46},g_{67},g_{78},g_{86'})\omega_4(g_{78},g_{84'},g_{4'6'},g_{6'7'})}.\quad\text{(B13)}$$

To compute the amplitude for M_1, we may view M_1 and a composition of M_1' and M_1'' (see Figs. 17 and 18). The amplitude for M_1' is

$$A_1'=\frac{\omega_4(g_{23},g_{35},g_{56},g_{65'})\omega_4(g_{56},g_{62'},g_{2'3'},g_{3'5'})}{\omega_4^{-1}(g_{35},g_{56},g_{62'},g_{2'5'})\omega_4(g_{62'},g_{2'3'},g_{3'5'},g_{5'7'})}\times\frac{\omega_4^{-1}(g_{34},g_{46},g_{62'},g_{2'5'})\omega_4^{-1}(g_{62'},g_{2'5'},g_{5'6'},g_{6'7'})}{\omega_4(g_{23},g_{34},g_{46},g_{65'})\omega_4^{-1}(g_{46},g_{62'},g_{2'5'},g_{5'6'})}.\quad\text{(B14)}$$

The above eight cocycles come from eight 4-simplexes as illustrated in Fig. 19. The amplitude for M_1'' is

$$A_1''=\omega_4^{-1}(g_{2'3'},g_{3'5'},g_{5'6'},g_{6'7'}),\quad\text{(B15)}$$

and the total amplitude for M_1 is

$$A_1=A_1'A_1''.\quad\text{(B16)}$$

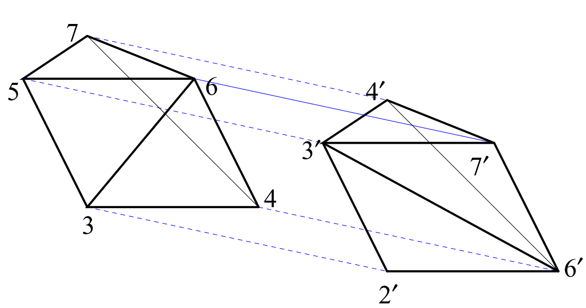


FIG. 14. (Color online) Complex M_2.

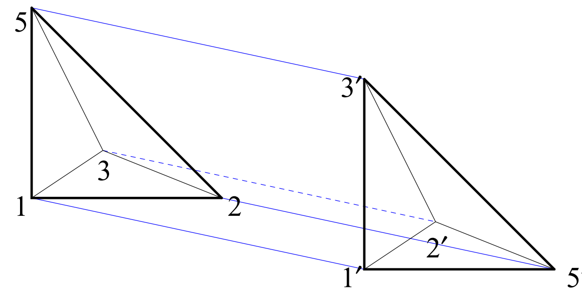


FIG. 15. (Color online) Complex M_3.

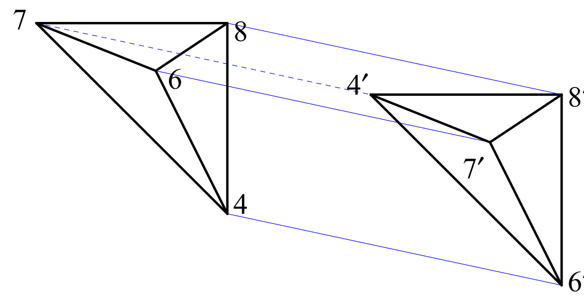


FIG. 16. (Color online) Complex M_4.

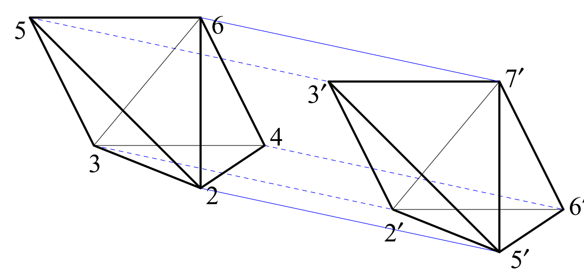


FIG. 17. (Color online) Complex M_1'.

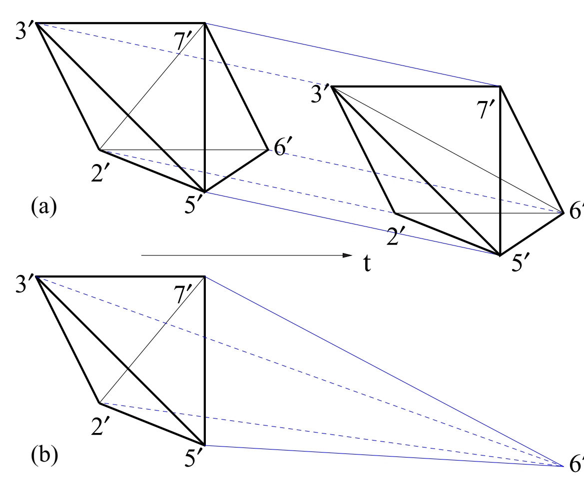


FIG. 18. (Color online) Complex M_1'', which is formed by one 4-simplex. Note that all the vertices in (a) are in the same time slice, but (curved) edge $(2'7')$ is in an earlier time slice and (curved) edge $(3'6')$ is in a later time slice. To realize this using straight edges, we put the vertex $6'$ in a later time slice, and this gives us the 4-simplex in (b).

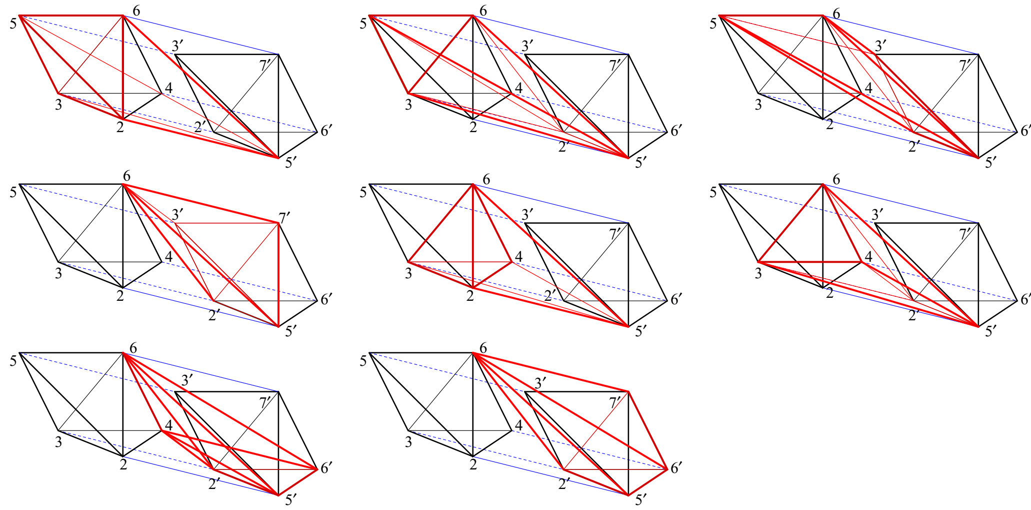


FIG. 19. (Color online) Complex M_1' is formed by eight 4-simplexes.

Similarly, for M_2, we find that

$$A_2 = A_2' A_2'', \tag{B17}$$

where A_2' is the amplitude for M_2' (see Fig. 20),

$$A_2' = \frac{\omega_4(g_{35},g_{56},g_{67},g_{72'})\omega_4(g_{67},g_{72'},g_{2'3'},g_{3'7'})}{\omega_4(g_{56},g_{67},g_{72'},g_{2'3'})\omega_4^{-1}(g_{72'},g_{2'3'},g_{3'4'},g_{4'7'})} \times \frac{\omega_4(g_{46},g_{67},g_{72'},g_{2'6'})\omega_4(g_{72'},g_{2'4'},g_{4'6'},g_{6'7'})}{\omega_4(g_{34},g_{46},g_{67},g_{72'})\omega_4(g_{67},g_{72'},g_{2'6'},g_{6'7'})} \tag{B18}$$

and A_2' is the amplitude for M_2'' (see Fig. 21),

$$A_2'' = \omega_4(g_{2'3'},g_{3'4'},g_{4'6'},g_{6'7'}). \tag{B19}$$

Here g_{ij} is the group element on edge (ij). We have

$$\begin{aligned} &g_{12} = g_{34} = g_{56} = g_{78} = g_x,\\ &g_{13} = g_{24} = g_{57} = g_{68} = g_y,\\ &g_{15} = g_{26} = g_{37} = g_{48} = g_z,\\ &g_{23} = g_{67} = g_x^{-1}g_y, \quad g_{35} = g_{46} = g_y^{-1}g_z,\\ &g_{25} = g_{47} = g_x^{-1}g_z, \quad g_{36} = g_y^{-1}g_xg_z; \end{aligned} \tag{B20}$$

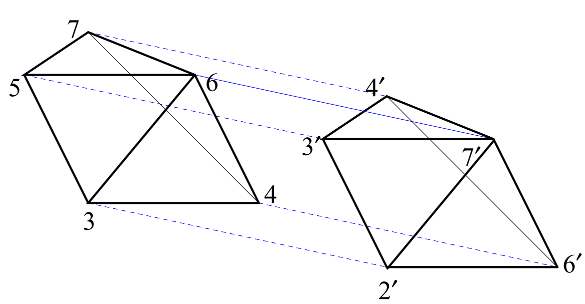


FIG. 20. (Color online) Complex M_2', which is formed by eight 4-simplexes.

$$\begin{aligned} &h_{12} = h_{34} = h_{56} = h_{78} = h_x,\\ &h_{13} = h_{24} = h_{57} = h_{68} = h_y,\\ &h_{15} = h_{26} = h_{37} = h_{48} = h_z,\\ &h_{23} = h_{67} = h_x^{-1}h_y, \quad h_{35} = h_{46} = h_y^{-1}h_z,\\ &h_{25} = h_{47} = h_x^{-1}h_z, \quad h_{36} = h_y^{-1}h_xh_z; \end{aligned} \tag{B21}$$

$$\begin{aligned} &g_{51'} = g_z^{-1}\mathrm{w}, \quad g_{62'} = g_z^{-1}g_x^{-1}g_y\mathrm{w}, \quad g_{84'} = \mathrm{w}h_z^{-1},\\ &g_{65'} = g_{72'} = g_{86'} = \mathrm{w}h_y^{-1}. \end{aligned} \tag{B22}$$

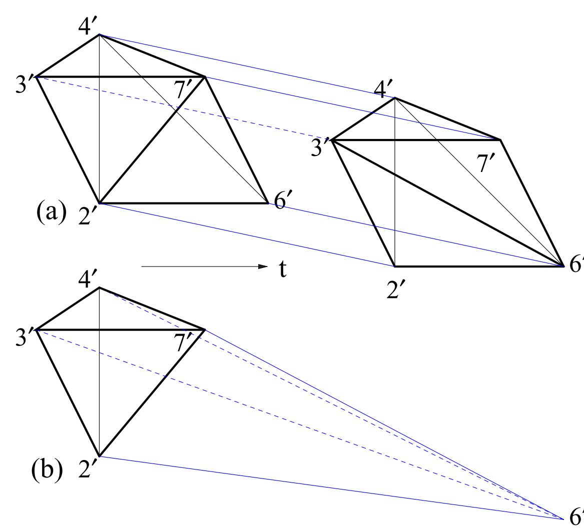


FIG. 21. (Color online) Complex M_2'', which is formed by one 4-simplex. Note that all the vertices in (a) are in the same time slice, but (curved) edge $(2'7')$ is in an earlier time slice and (curved) edge $(3'6')$ is in a later time slice. To realize this using straight edges, we put vertex $6'$ in a later time slice, and this gives us the 4-simplex in (b).

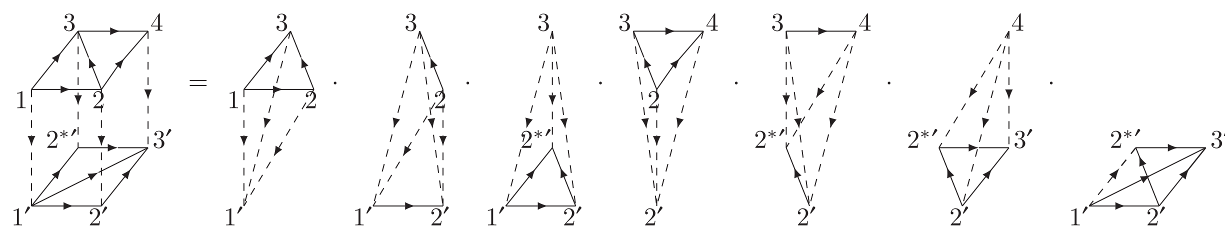
Also, if the following conditions are not satisfied, the amplitude $A^{xyz}(g_x,g_x,g_z,h_x,h_y,h_z;\mathrm{w})$ will be 0:

$$\begin{aligned} g_x \mathrm{w} &= \mathrm{w} h_z, \quad g_y \mathrm{w} = \mathrm{w} h_x, \quad g_z \mathrm{w} = \mathrm{w} h_y, \\ g_x g_y &= g_y g_x, \quad g_y g_z = g_z g_y, \quad g_z g_x = g_x g_z, \\ h_x h_y &= h_y h_x, \quad h_y h_z = h_z h_y, \quad h_z h_x = h_x h_z. \end{aligned} \tag{B23}$$

Note that the above has g_x,g_y,g_z commute due to the identification on a $\mathbb{T}^3$ torus.

4. Explicit expression of T^{xy} in terms of (G,ω_4)

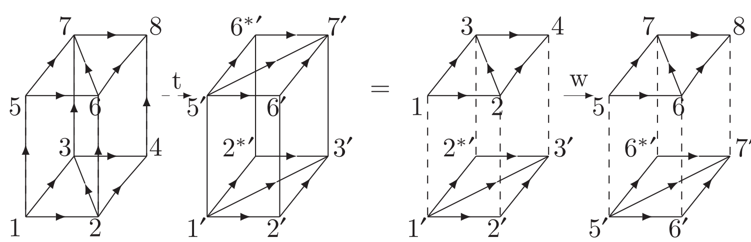
Similarly to S^{xyz}, we can triangulate T^{xy} on $\mathbb{T}^3 \times I$. It is easier to start with a T^{xy} on $\mathbb{T}^2 \times I$ for two dimensions, which we denote $\mathsf{T}_{\mathrm{2D}}(\mathrm{w})$ and triangulate in the following $3! + 1 = 7$ tetrahedra (3-simplex). Here we have the vertex ordering for the arrows: $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 1' < 2' < 2^{*\prime} < 3' < 5' < 6' < 6^{*\prime} < 7'$.

$\mathsf{T}_{\mathrm{2D}}(\mathrm{w}) =$ 

(B24)

The last extra piece is required to change the branching structure of the 3-simplex due to T^{xy} transformation.

For $\mathsf{T}_{\mathrm{3D}}(\mathrm{w})$, we simply have seven pieces of slant products. Each slant product contains four 4-simplexes. So in total there are 28 pieces of 4-cocycles in $\mathsf{T}_{\mathrm{3D}}(\mathrm{w})$.

$\mathsf{T}_{\mathrm{3D}}(\mathrm{w}) =$  $= (\mathsf{T}_1)(\mathsf{T}_2)(\mathsf{T}_3)(\mathsf{T}_4)(\mathsf{T}_5)(\mathsf{T}_6)(\mathsf{T}_7).$

(B25)

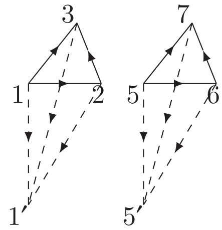
The constraints given by $\mathsf{T}(\mathrm{w})$ are

$$\mathrm{w}^{-1} g_x \mathrm{w} = h_x, \tag{B26}$$

$$\mathrm{w}^{-1} g_x g_y \mathrm{w} = h_y, \tag{B27}$$

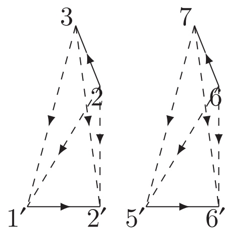
$$\mathrm{w}^{-1} g_z \mathrm{w} = h_z. \tag{B28}$$

Below we explicitly write seven T_i values, where we omit the "w" arrow and do not draw it, which shall connect from the left 3-simplex to the right 3-simplex.

$(\mathsf{T}_1) =$ 

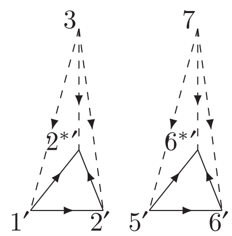
$$= \omega_4([12],[23],[35],[51']) \cdot \omega_4([23],[35],[56],[61']) \cdot \omega_4([35],[56],[67],[71']) \cdot \omega_4^{-1}([56],[67],[71'],[1'5']). \tag{B29}$$

$(\mathsf{T}_2) =$



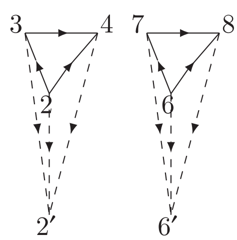
$$= \omega_4^{-1}([23],[36],[61'],[1'2']) \cdot \omega_4([36],[67],[71'],[1'2']) \cdot \omega_4^{-1}([67],[71'],[1'2'],[2'5']) \cdot \omega_4([67],[72'],[2'5'],[5'6']). \tag{B30}$$

$(\mathsf{T}_3) =$



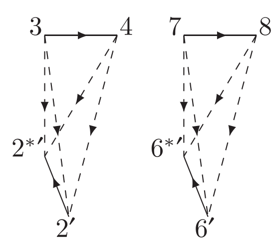
$$= \omega_4([37],[71'],[1'2'],[2'2^{*\prime}]) \cdot \omega_4^{-1}([71'],[1'2'],[2'2^{*\prime}],[2^{*\prime}5']) \cdot \omega_4^{-1}([72'],[2'2^{*\prime}],[2^{*\prime}5'],[5'6']) \cdot \omega_4^{-1}([72^{*\prime}],[2^{*\prime}5'],[5'6'],[6'6^{*\prime}]). \tag{B31}$$

$(\mathsf{T}_4) =$



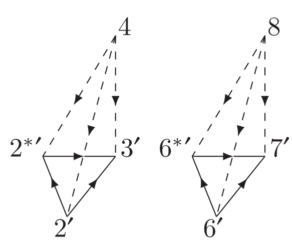
$$= \omega_4^{-1}([23],[34],[46],[62']) \cdot \omega_4^{-1}([34],[46],[67],[72']) \cdot \omega_4^{-1}([46],[67],[78],[82']) \cdot \omega_4([67],[78],[82'],[2'6']). \tag{B32}$$

$(\mathsf{T}_5) =$



$$= \omega_4([34],[47],[72'],[2'2^{*\prime}]) \cdot \omega_4^{-1}([47],[78],[82'],[2'2^{*\prime}]) \cdot \omega_4([78],[82'],[2'2^{*\prime}],[2^{*\prime}6']) \cdot \omega_4^{-1}([78],[82^{*\prime}],[2^{*\prime}6'],[6'6^{*\prime}]). \tag{B33}$$

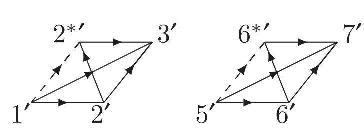
$(\mathsf{T}_6) =$



$$= \omega_4^{-1}([48],[82'],[2'2^{*\prime}],[2^{*\prime}3']) \cdot \omega_4([82'],[2'2^{*\prime}],[2^{*\prime}3'],[3'6']) \cdot \omega_4([82^{*\prime}],[2^{*\prime}3'],[3'6'],[6'6^{*\prime}]) \cdot \omega_4([83'],[3'6'],[6'6^{*\prime}],[6^{*\prime}7']). \tag{B34}$$

For the tricky T_7, we shift $1'$ to a new later time slice, $1''$, and shift $5'$ to a new later time slice, $5''$:

$(\mathsf{T}_7) =$



$$= \omega_4^{-1}([1'2'],[2'2^{*\prime}],[2^{*\prime}3'],[3'5']) \cdot \omega_4([2'2^{*\prime}],[2^{*\prime}3'],[3'5'],[5'6']) \cdot \omega_4^{-1}([2^{*\prime}3'],[3'5'],[5'6'],[6'6^{*\prime}]) \cdot \omega_4([3'5'],[5'6'],[6'6^{*\prime}],[6^{*\prime}7']). \tag{B35}$$

One can also define the projection operator on $\mathbb{T}^3$ as

$$\mathsf{P}_{3\mathrm{D}}(\mathrm{w}) = (\mathsf{T}_1)(\mathsf{T}_2)(\mathsf{T}_3)(\mathsf{T}_4)(\mathsf{T}_5)(\mathsf{T}_6). \tag{B36}$$

Once we have obtained the path integral of 4-cocycles, we can change the flux basis to the canonical basis and follow the procedure outlined in the Appendix of Ref. [55] to derive the Rep theory formula given in Sec. III B. One additional remark: An easier way to check the consistency of formulas for S and T is to use the rules in Appendix B 1 and to apply the discrete Fourier transformation of a finite group, such as

$$\frac{1}{|G|}\sum_{b,d,\beta} \mathrm{tr}\widetilde{\rho}_{\beta}^{b,d}(a)\ \mathrm{tr}\widetilde{\rho}_{\beta}^{b,d}(e)^* = \delta_{a,e}, \tag{B37}$$

$$\frac{1}{|G|}\sum_{a,b,d} \mathrm{tr}\widetilde{\rho}_{\alpha}^{a,b}(d)^*\ \mathrm{tr}\widetilde{\rho}_{\gamma}^{a,b}(d) = \delta_{\alpha,\gamma}. \tag{B38}$$

Using the properties of $\mathsf{C}^{(2)}_{a,b}(c,d)$ and the canonical basis $|\alpha,a,b\rangle$, we can justify that our formulas satisfy the rules (up to some projective representation's complex phases). See also Ref. [67] for the derivation.

Note Added in Proof. At the "Symmetry in Topological Phases" workshop at Princeton University, we became aware that the authors of Ref. [45] were working on the braiding statistics of 3 + 1D gapped phases; their studies intersect some of ours, but also further inspire our work. During the long process of preparing our manuscript, two works appeared (Refs. [45] and [46]) dealing with the Abelian braiding statistics of twisted gauge theories, as well as a preprint (Ref. [68]) considering the surface topological order of symmetric protected topological states with loop braiding statistics.

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