# Interacting tunneling model for two-level systems in amorphous materials and its predictions for their dephasing and noise in superconducting microresonators

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We use a tunneling model for two-level systems in insulators that takes into account the interaction between them and a slow power-law dependence of their density of states. We show that the predictions of this model are in a perfect agreement with the recent studies of the noise in high quality superconducting resonators. The predictions also agree with the temperature dependence of the TLS dephasing rates observed in phase qubits. Both observations are impossible to explain in the framework of the standard tunneling model of TLS. We discuss the origin of the universal dimensionless parameter that controls the interaction between TLS in glasses and show that it is consistent with the assumptions of the model.

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## I. INTRODUCTION

Thin-film high quality superconducting resonators are important for a number of applications, ranging from quantum computation to submillimeter and far-infrared astronomy [1]. The performance of these devices has improved dramatically over the past decades and resonator quality factors above $10^6$ are now routinely fabricated using single-layer superconductors deposited on high quality low-loss crystalline substrate. Achieving resonators with high quality factors requires minimization of all potential sources of dissipation and noise.

The major source of dissipation and noise in the resonators is two-level systems (TLSs) located in the amorphous dielectrics. The presence and importance of TLSs was proven by the measurements of the resonators' frequency shifts as a function of the temperature [2]. These experiments have shown unambiguously that even in the devices that do not use a deposited dielectric and consist only of a patterned superconducting film on high quality crystalline substrate, a thin, TLS-hosting layer is present on the surface of the device. In particular, TLSs in the thin amorphous surface layer of the microresonators are responsible for the noise in the resonator frequency that is the subject of this paper. This noise has been carefully characterized in the last few years. The early works reported unusual behavior of the noise spectral density, $S \sim f^{-1/2}$ [2–6], but all recent works [7–10] agree on a more conventional $S \sim 1/f$ spectrum. The noise spectrum also shows a square-root dependence on the applied power $S \sim P^{-1/2}$. Furthermore, the recent work by Burnett *et al.* [8] shows that the dependence on the applied power is also temperature dependent. The most striking feature of the frequency noise is its spectrum temperature dependence: $S \sim T^{-\beta}$ with $\beta = 1.2$–$1.73$ [4,7,8] which is at odds with the expectation that any kind of noise should disappear as $T \to 0$.

On the theoretical side, the observations cannot be explained by the conventional phenomenological model of TLSs known as the standard tunneling model (STM) [11,12]. This model was very successful in explaining the anomalous bulk properties of amorphous glasses at low temperature. However, its predictions for the frequency noise are in a strong disagreement with the data. This problem was noted by the works [4,13] which observed that the data can be fitted by a single empirical equation that describes the noise dependence on microwave power and temperature. This equation however cannot be derived in the framework of the STM.

In this paper we propose a modification of the STM that is capable of explaining all the features of the noise. The model develops on our previous ideas [14]; it differs from other models in that it assumes relatively large interaction between TLSs. The unusual properties of the frequency noise spectra are mostly associated with this large interaction. For a better fit to the data, the model also assumes a slightly nonuniform density of states of TLSs at low energies which might be a consequence of the large interaction. In the bulk of the paper we show that the model explains all features of the noise spectral density, namely, the frequency dependence of the spectrum $S \sim f^{-1}$, the temperature dependence $S \sim T^{-\beta}$, and the applied power dependence $S \sim P^{-1/2}$ as well as the saturation of the noise with the power at the temperature-dependent level. In addition, the model and STM gives the same shifts in the resonant frequency as a function of temperature that were originally interpreted as the indication for the presence of TLSs [2,15]. Thus, the predictions of the model agree well with the results of most experiments on resonators [16–21]. Furthermore, the model provides the explanation for the recent spectroscopy data on the temperature dependence of the dephasing rate of TLSs located in the Josephson-junction barriers of the phase qubits [22].

The paper is organized as follows. Section II gives standard (Sec. II A) and generalized (Sec. II B) tunneling models and discusses the effects of TLS interactions (Sec. II C). The detailed calculations of the frequency noise spectrum are given in Sec. III while Sec. IV summarizes the assumptions and approximations we use in our calculations. Section V compares the predictions of the model for the noise power spectrum with that of the STM and with the experimental data. Finally, Sec. VI gives conclusions and discusses the possible origin of the larger interaction assumed by the model.

 

## II. MODEL

### A. Standard tunneling model and its predictions

The existence of TLSs in amorphous materials was conjectured four decades ago [11,12] in order to explain the anomalous bulk properties of these materials at low temperatures, i.e., the temperature dependence of the specific heat and the thermal conductivity. The phenomenological model describing the TLS is known as the STM for its simplicity and wide application. It assumes the existence of localized excitations with very low energy $E$ that are visualized as excitations in double-well potentials that happen to be nearly symmetric. It is generally believed that the existence of the double-well potentials is due to the disorder, so that local rearrangement of atoms might switch the system between adjacent local energy minima. For a given $T$, double-well potentials with $E \sim k_B T$ dominate the thermodynamic properties. In the double-well potentials a transition between the two minima is due to the quantum tunneling. Therefore, they are referred to as tunneling systems which are characterized by an asymmetry $\Delta$ and a tunneling matrix element $\Delta_0$. The unperturbed Hamiltonian $H_{\text{TLS}}$ of each independent tunneling system is

$$H_{\text{TLS}} = \frac{\Delta}{2}\sigma^z + \frac{\Delta_0}{2}\sigma^x. \tag{1}$$

Here $\sigma^a$, $a = x,y,z$ are Pauli matrices. In the rotated basis, the Hamiltonian is simply $H = ES^z$, where $E = \sqrt{\Delta^2 + \Delta_0^2}$ is the TLS energy splitting and $S^z = \frac{1}{2}(\cos\theta\sigma^z + \sin\theta\sigma^x)$ with $\tan\theta = \Delta_0/\Delta$. The STM assumes that the energy distribution of $\Delta$ is flat while $\Delta_0$ is exponential in the barrier width and thus has an exponentially wide distribution, so that the probability density of TLS is given by

$$P(\Delta,\Delta_0) = \frac{\bar{P}_0}{\Delta_0}\Theta(\Delta_0 - \Delta_{0,\text{min}}), \qquad \frac{\Delta_{0,\text{min}}}{k_B} \simeq 10^{-7}\ \text{K}. \tag{2}$$

The form of $P(\Delta,\Delta_0)$ implies that the distribution of the energy splitting $P(E)$ is uniform. Experimentally it turns out that for most glasses $\bar{P}_0$ is in the range (0.5–3) $\times 10^{20}$ eV$^{-1}$ cm$^{-3}$ [23].

In the insulating materials TLSs are coupled to the environment by the interaction with phonons and photons that can excite or relax the TLS eigenstates. The phonon interaction Hamiltonian reads

$$H_{\text{TLS-ph}} = \gamma\sigma_z\epsilon, \tag{3}$$

where $\epsilon$ is the strain field and $\gamma \sim 1$ eV is the typical coupling constant. Because of this coupling the TLS acquires a relaxation rate $\Gamma_1^{\text{ph}}$ and a dephasing rate $\Gamma_2^{\text{ph}}$. The golden rule formula gives the relaxation rate [24]:

$$\Gamma_1^{\text{ph}} = \frac{\gamma^2}{2\pi\zeta\hbar^4 v^5}\Delta_0^2 E \coth[E/2k_B T], \tag{4}$$

where $\zeta$ the density of the glass and $v$ is the sound velocity. The dephasing rate is due to decay: $\Gamma_2^{\text{ph}} = \frac{1}{2}\Gamma_1^{\text{ph}}$. At low temperature, assuming that $\Delta_0/E$ has little or no $E$ dependence, one concludes that $\Gamma_1 \sim E^3$.

Because in this work we are considering TLSs that are located in a very thin layer of material on the surface of metals, we also briefly review STM predictions for the relaxation of TLSs in metals [25]. In these materials, TLSs also interact with the conduction electrons. The interacting Hamiltonian reads

$$H_{\text{TLS-el}} = \sigma_z \sum_{kk'\eta} V_{kk'} c_{k\eta}^{\dagger} c_{k'\eta},$$

where $V_{kk'}$ describes the scattering potential and $c_{k\eta}^{\dagger}(c_{k\eta})$ creates (annihilates) a fermion of wave vector $k$ and spin $\eta$. The coupling between TLSs and electrons is described quite generally by a parameter $\mathcal{K}$, which for a weak $s$-wave potential ($V_{kk'} = V$) is $\mathcal{K} = \frac{1}{2}(\nu_F V)^2$, where $\nu_F$ is the electron density of states at the Fermi level. It is known that in metallic glasses $\mathcal{K}$ must be less than 1/2. This typically strong interaction leads to a short relaxation time for the TLS:

$$\Gamma_1^{\text{el}} = \pi\mathcal{K}E \coth[E/2k_B T]. \tag{5}$$

At low temperatures, $\Gamma_1^{\text{el}} \sim E$.

TLSs can also interact between themselves due to the exchange of virtual phonons, photons, or electronic excitations. In all cases the interaction falls off as $1/r^3$:

$$H_2^{\text{int}} = \frac{1}{2}\sum_{i,j} U_{ij}\sigma_i^z\sigma_j^z, \qquad U_{ij} = \frac{u_{ij}}{r_{ij}^3}. \tag{6}$$

In the case of photon exchange, the interaction is essentially the instantaneous dipole-dipole one [26]:

$$u_{ij} = \sum_i \sum_{i\neq j} \frac{\vec{d}_i\cdot\vec{d}_j - 3(\hat{r}_{ij}\cdot\vec{d}_i)(\hat{r}_{ij}\cdot\vec{d}_j)}{4\pi\varepsilon}. \tag{7}$$

In the case of phonon exchange, calculations using elasticity theory [27] showed that the retardation can be ignored if the distance between TLSs is less than the wavelength of the phonon, i.e., $r_{ij} < (vE)^{-1}$. This condition is satisfied for characteristic distances and energies of relevant TLSs. By neglecting the retardation, the interaction is [28]

$$u_{ij} = \sum_i \sum_{i\neq j} \frac{\gamma_i\gamma_j}{\zeta v^2}. \tag{8}$$

The interaction scale is set by $U_0 \approx d^2/\varepsilon$ ($U_0 \approx \gamma^2/\zeta v^2$) for electric (elastic) interactions. Comparing the interaction between TLS at a typical distance $r^3 \sim 1/\bar{P}_0$ with the distance between the levels, one concludes that the effects of the interaction are controlled by the dimensionless parameter $\chi = \bar{P}_0 U_0$. The crucial assumption of the STM is that this parameter is very small, $\chi \ll 1$, so that the effect of the interaction on TLS can be mostly ignored. In particular, one expects that the TLS density of states remains constant at low energies, $\rho(E) = \bar{P}_0$. Ultrasound attenuation experiments that measure the product $\bar{P}_0 U_0$ show that $\chi$ is indeed small in bulk amorphous insulators and has almost universal value $\chi \approx 10^{-3}$–$10^{-2}$. In metals, the interaction between TLSs is similar to Ruderman-Kittel-Kasuya-Yosida interaction between spins, so that $U_0 = E_F/k_F^3$, where $E_F$ is the Fermi energy and $k_F$ is the Fermi wave vector. In metallic glasses $U_0 \sim 10^5$ K Å$^3$; as a result the constant $\chi$ has the same order of magnitude as the phonon mediated interaction. To summarize, in the framework of the STM the interactions of different origins add together to form an effective interaction $U_0/r_{ij}^3$ that is characterized by the constant $U_0 \sim 10^5\,\mathrm{K\,\AA^3}$. This conclusion relies on the assumption that the TLS sizes are much smaller than the distance between them, that allows one to estimate $r^3 \sim 1/\bar{P}_0$.

The small value of the dimensionless parameter $\chi \ll 1$ implies that the relaxation of the TLS induced by their mutual interaction is negligible. Indeed, two interacting TLSs ($i$ and $j$) exchange energy if the resonant condition $|E_i - E_j| < U_0$ is satisfied. By computing the number $N_0$ of TLSs that form a resonant pair with a given one, we get $N_0 \approx \chi \ln(\frac{L}{a})$, where $L$ is the size of the system and $a$ is the minimum distance between two TLSs [29]. Because the number of resonant neighbors $N_0 \ll 1$ for any reasonable sample size $L$, the STM assumes that different TLSs are independent and their relaxation rate $\Gamma_1$ is dominated by phonons.

### B. Generalized tunneling model

In this work we show that in order to explain the data we need to do two modifications to the standard tunneling model. We shall refer to this model as the generalized tunneling model (GTM) and the two modifications are the following:

(i) The interaction between TLSs is not neglected. In fact, we show that the latter has a significant effect on the TLS relaxation at sufficiently low temperatures even if $\chi \ll 1$.

(ii) We allow a nonflat probability density of the asymmetry energy $\Delta$:

$$P(\Delta_0,\Delta) = p(\Delta_0)\begin{cases}(1+\mu)\left(\frac{\Delta}{\Delta_{\max}}\right)^{\mu} & \text{if } 0 \leqslant \Delta \leqslant \Delta_{\max};\\ 0 & \text{otherwise,}\end{cases} \tag{9}$$

where

$$p(\Delta_0) = \begin{cases}\Delta_0^{-1} & \text{if } \Delta_{\min} \leqslant \Delta_0 \leqslant \Delta_{\max};\\ 0 & \text{otherwise.}\end{cases} \tag{10}$$

Here $\mu < 1$ is a small positive parameter whose value will be discussed in detail below.

We notice that the second assumption might be in fact the consequence of the first. Indeed, a strong interaction between discrete degrees of freedom always decreases the density of states at low energies, $\rho(E) = \rho_0(E/E_{\max})^{\mu}$. For Coulomb interaction this effect results in a very large suppression of the density of states and the formation of Efros-Shklovkii pseudogap [30]. Dipole-dipole interaction is marginal and it would result in logarithmic corrections of the density of states for pointlike TLSs which might be difficult to distinguish from a power law with $\mu \approx 0.3$ [31]. Because larger than expected interaction implies that the assumption of pointlike defects is probably wrong, we do not attempt to derive the probability distribution (9) in some microscopic picture but take it as an assumption.

It is worthwhile to mention here that the suppression of the density of states at low energies was reported previously by a number of experimental works. Historically, first the specific-heat measurements performed in the 1980's indicated that at low temperatures ($T \leqslant 1$ K) the density of states is $\rho(E) \sim E^{\mu}$ with $\mu \approx 0.2$–$0.3$ [32]. Another indirect evidence comes from the old fluorescence experiments [33] that showed homogeneous line broadening with anomalously large magnitude and unusual temperature dependence $\sim T^{1.3}$ in glasses. It was argued [34] that this low-temperature anomaly is due to TLSs. However, to fit the data one needs to assume a nonconstant density of states $\rho(E) = \rho_0(E/E_{\max})^{\mu}$, with $\mu \approx 0.3$. More recently, experiments by Skacel *et al.* [35] directly probed the TLS density of states in thin $a$-SiO films by measuring losses in superconducting lumped element resonators and reported that $\rho(E) \propto E^{0.28}$ in agreement with previous measurements in glasses.

The importance of the interaction between TLSs was conjectured by Yu and Leggett in 1988 [36] who argued that the apparent universality of the dimensionless parameter $\chi$ can be only understood as a consequence of the many-body interactions. In this picture each TLS is a complicated many-body excitation formed by many local degrees of freedom. However, despite the effort of many workers [37–41] the consistent first-principles theory of TLSs is not available. Experimentally, the first evidence for interactions between TLSs were found in thin $a$-$SiO_{2+x}$ layers, where it was shown that dipole-dipole interactions between TLSs play a key role up to 100 mK [42]. Very recently experiments performed on superconducting microresonators showed that the electromagnetic response of thin oxide layers is not described by STMs. In particular, the observed weak (logarithmic) power dependence of the loss is in a striking contrast with the square-root prediction of STMs but agrees perfectly well with the interacting picture [14].

### C. Main predictions of the generalized tunneling model

Non-negligible interactions between TLSs provides a mechanism for their dephasing and relaxation that might dominate at low temperatures when relaxation caused by phonons becomes very inefficient. In this section we compute the broadening of TLS levels that is due to their mutual interaction. Then we explain why this width is crucial for the low-frequency noise of the high quality resonators.

It is convenient to divide the TLSs into coherent (quantum) and fluctuators (classical) TLSs. Coherent TLSs are characterized by a small phonon induced decoherence rate, $\Gamma_2^{\mathrm{ph}} < E$, while fluctuators have $\Gamma_2^{\mathrm{ph}} \geqslant E$. Among coherent TLSs we distinguish high-, $E \gg k_B T$, and low-, $E \lesssim k_B T$, energy TLSs. The noise in high quality resonators is generated by the TLSs that have energies close to the resonator frequency $\nu_0$. We shall assume that the frequency of the resonator is high, $\nu_0 \gg k_B T$, so that the TLSs responsible for the noise are high-energy coherent TLSs. Their properties are affected by the environment that consists of slow fluctuators and thermally activated coherent TLSs with energies $E \lesssim k_B T$.

The linewidth of an individual high-frequency TLS is due to the combined effect of the surrounding thermally excited TLSs that change their state emitting and absorbing phonons. We begin by evaluating the effect of a single thermally excited TLS and then sum over many of them.

In the rotated basis the Hamiltonian of the high-frequency TLS (denoted by subscript 0) interacting with a thermally excited one (denoted by subscript $T$) at distance $r$ is given by

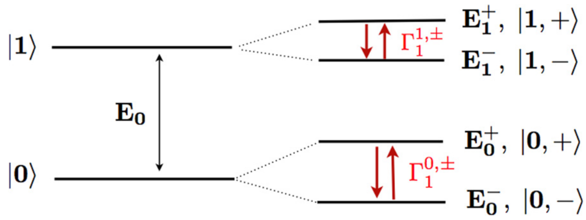


FIG. 1. (Color online) Schematics of the energy levels of the Hamiltonian $H_{\rm int}$. The solid arrows indicate phonon induced relaxation.

$H = E_0 S_0^z + E S_T^z + H_{\rm ph} + H_{\rm int}$ with

$$H_{\rm int} = 4U(r) S_0^z \left( \frac{\Delta}{E} S_T^z + \frac{\Delta_0}{E} S_T^x \right), \qquad (11)$$

where $U(r) = U_0 r^{-3}$ is the interaction energy. We denote the two states of the high-frequency TLS as $|0\rangle$ and $|1\rangle$ ($S_0^z|0\rangle = -1/2\,|0\rangle$). In the Hamiltonian (11) we neglected the terms proportional to $S_0^x$ that lead to decay of the excited state. These terms are irrelevant for TLSs with very different energies, $E_0 \gg E$.

Hamiltonians of the type (11) have been studied extensively in the context of the anomalous homogeneous optical linewidths in glasses ([34] and references therein). We now outline the main assumptions and results of these studies. Due to the interaction the high-frequency and the thermally activated TLSs form a four-level quantum system (see Fig. 1) which can be diagonalized by rotating the basis of thermally activated TLSs:

$$|n,-\rangle = \frac{1}{\sqrt{2}}[\sqrt{1+\eta_n}|n,0\rangle - \sqrt{1-\eta_n}|n,1\rangle],$$
$$|n,+\rangle = \frac{1}{\sqrt{2}}[\sqrt{1-\eta_n}|n,0\rangle + \sqrt{1+\eta_n}|n,1\rangle], \qquad (12)$$

$$H_{\rm int} = \sum_{n=0,1} \sum_{k=-,+} E_n^k |n,k\rangle\langle n,k| \qquad (13)$$

with eigenvalues

$$E_0^{\mp} = -\frac{E_0}{2} \mp \sqrt{\left(\frac{E}{2}\right)^2 + U(r)\Delta + U(r)^2},$$
$$E_1^{\mp} = +\frac{E_0}{2} \mp \sqrt{\left(\frac{E}{2}\right)^2 - U(r)\Delta + U(r)^2}, \qquad (14)$$

where

$$\eta_n = \frac{E + (-1)^n 2U(r)(\Delta/E)}{\sqrt{E^2 + (-1)^n 4U(r)\Delta + 4U(r)^2}}.$$

The width of the sublevels of the four-level system can be found by evaluating the matrix elements describing the phonon emission or absorption between the states (12). A typical thermally excited TLS is characterized by $\Delta_0 \ll \Delta \simeq E$, so these matrix elements are very close to the ones of noninteracting TLSs with the same energy,

$$\Gamma_1^{0,+} + \Gamma_1^{0,-} \simeq \Gamma_1^{1,+} + \Gamma_1^{1,-} \simeq \Gamma_1^{\rm ph}(E).$$

It is convenient to define the effective decoherence rate as the sum of the widths of the sublevels weighted with their probabilities:

$$\Gamma_{\rm eff} = \frac{1}{2} \sum_{k=\pm} p_k \left( \Gamma_1^{0,k} + \Gamma_1^{1,k} \right) \simeq \Gamma_1^{\rm ph}(E). \qquad (15)$$

In the limit of significant interaction energy, $U(r) > \Gamma_{\rm eff}$, the width of the high-frequency TLS level, $\Gamma_2$, coincides with the effective rate (15). This can be also seen by arguing that transition between sublevels changes the energy of the fast TLS by $U(r)$. After such transition its wave function acquires the phase $\delta\phi = U(r)t$ and thus leads to dephasing after a time $1/U(r)$. For large $U(r)$ this time is short compared to the time between transitions, so the dephasing rate is given by $\Gamma_{\rm eff}$. Note that small values of $\Gamma_{\rm eff} \ll T$ and the fast dependence of $U(r) \sim 1/r^3$ imply that a typical thermally activated TLS with $U(r) > \Gamma_{\rm eff}$ has $U(r) \ll T$.

In the opposite limit of very small $U(r) < \Gamma_{\rm eff}$ the phonon process does not affect the high-frequency TLS immediately. After the thermally excited TLS changes its state, the energy of the fast TLS changes by $U(r)$, so the phase $U(r)t$ that it acquires is much smaller than unity at a time when the TLS flips again. As a result the effect of phonon processes averages out.

Both limits can be treated analytically for thermally excited TLSs with $\Delta_0 \ll \Delta$, in which one can neglect the rotation of the basis (12) induced by phonon processes. In this case the fluctuations of the TLS energy are given by

$$\langle \delta E(t) \delta E(0) \rangle = U(r)^2 \cosh^{-2}(E/2T) \exp\left(-\Gamma_1^{\rm ph} t\right).$$

They result in the dephasing of the high-frequency TLS,

$$\langle S_0^+(t) S_0^-(0) \rangle \sim \left\langle \exp\left[ -i \int_0^t dt_1 \delta E(t_1) \right] \right\rangle.$$

In the limit $\Gamma_1^{\rm ph} t \gg 1$ the energy $\delta E(t)$ experiences many fluctuations and the average can be evaluated in the Gaussian approximation,

$$\langle S_0^+(t) S_0^-(0) \rangle \sim \exp\left( -\frac{u^2}{\Gamma_1^{\rm ph}} t \right),$$

where $u = U(r)\cosh^{-1}(E/2T)$. In this approximation the level width is $\Gamma_2 = u^2/\Gamma_1^{\rm ph}$. The assumption $\Gamma_1^{\rm ph} t \gg 1$ is valid provided that $\Gamma_2 \ll \Gamma_1^{\rm ph}$ which is correct for $u \ll \Gamma_1^{\rm ph}$.

To summarize, the level width of the high-frequency TLS is given by

$$\Gamma_2(u) = \begin{cases} \Gamma_1^{\rm ph} & \text{if } u \gg \Gamma_1^{\rm ph}, \\ \frac{u^2}{\Gamma_1^{\rm ph}} & \text{if } u \ll \Gamma_1^{\rm ph}. \end{cases} \qquad (16)$$

The full level width of the fast TLS is given by the sum over thermally activated TLSs in its environment:

$$\Gamma_2 = \sum_k \Gamma_2(u_k),$$

which should be averaged over positions [that control $u(r)$], energies, and relaxation rates of the thermally excited TLS. These averages can be performed independently. Because

$u \sim 1/r^3$ the average of (16) over positions is dominated by $u(r) \sim \Gamma_1^{ph}$. Estimating the integral over $r$ we get

$$\Gamma_2 = c \int d\Gamma_1 dE\, P(E,\Gamma_1) U_0 \cosh^{-1}(E/2T), \tag{17}$$

where $c \sim 1$ and $P(E,\Gamma_1)$ is the probability density of the TLS characterized by energy $E$ and relaxation rate $\Gamma_1$. Combining the probability distribution (9) and expression for the relaxation rate (4) we get

$$P(E,\Gamma_1) = P_0 \frac{E^\mu}{2E_{max}^\mu \Gamma_1} \tag{18}$$

for $\Gamma_1 < \Gamma_1^{max}$ where $\Gamma_1^{max} = \Gamma_1(\Delta_0 \sim E)$ is the maximum rate possible for the TLS with energy $E$. Performing the average in (17) with the distribution (18) we get

$$\Gamma_2 = c\chi \ln\left(\frac{\Gamma_1^{max}}{\Gamma_1^{min}}\right) \frac{T^{1+\mu}}{E_{max}^\mu}, \tag{19}$$

where $c \sim 1$ and $\Gamma_1^{min}$ is the minimal relaxation rate, $\ln(\Gamma^{max}/\Gamma^{min}) = 2\ln(E/\Delta_0^{min})$. The largest value of $\Gamma^{max}$ associated with the thermally excited TLS is of the order of $10^7$–$10^8$ s$^{-1}$ for $E \sim 11$–$12$ GHz [43] and correspondingly $10^4$–$10^5$ s$^{-1}$ for $T \sim 50$ mK. There is no information available on the precise value of the minimal rate $\Gamma^{min}$ for thermally activated TLSs in glasses, but the electrical noise data show that $1/f$ noise generated by these TLSs extends to very low frequencies $f \lesssim 1$ mHz beyond which the dependence changes. This implies that $\Gamma^{min} \lesssim 10^{-3}$ s$^{-1}$, so the value of $\ln(\Gamma^{max}/\Gamma^{min}) \approx 20$.

A large $\ln(\Gamma^{max}/\Gamma^{min})$ factor appears only for TLSs that are distributed uniformly through a three-dimensional volume so that the integral over the volume produces factor $U_0$ for any $\Gamma_1$ in (17). This factor is expected to be much smaller for surface insulators. In the case of amorphous two-dimensional layers of thickness $d$ with three-dimensional interaction $[U(r) \sim 1/r^3]$ between the TLSs the logarithmic contribution comes from $\Gamma_1 > U_0/d^3$, which provides the lower cutoff of the logarithmic divergence $\Gamma_1^{min} \to U_0/d^3$. In real materials, however, the interaction between TLS might have a two-dimensional character at intermediate scales, $d < r < d_{eff}$, which cuts off the logarithmic divergence at smaller $\Gamma_1^{min} \to U_0/d_{eff}^3$. For the estimates below we shall assume that $\ln(\Gamma^{max}/\Gamma^{min}) \gtrsim 1$ in surface oxides formed in superconducting microresonators.

In a typical low-temperature experiment the dephasing rate $\Gamma_2$ given by (19) dominates over decoherence rate $\Gamma_2^{ph} \sim \Gamma_1^{ph}$ due to phonons. In fact, for $E \sim T$ we estimate the phonon mediated relaxation rate given in (4),

$$\Gamma_1^{ph} \approx \frac{U_0}{a^3}\left(\frac{E}{\omega_D}\right)^3, \tag{20}$$

where $a \sim 0.3$ nm is the atomic distance and $\omega_D = (c_s/a)(6\pi^2)^{1/3} \sim 10^3$ K is the Debye frequency. Estimating the interaction one gets $U/a^3 \approx 300$ K [44]. A typical high-frequency TLS probed by superconducting resonators or phase qubit experiments has energy $E \sim 5$–$10$ GHz; for these TLSs the relaxation rate due to phonon is $\Gamma_1^{ph} \sim \Gamma_2^{ph} \sim 10^2$–$10^3$ s$^{-1}$. At $T \sim 100$ mK, the dephasing rate given by (19) is much larger: $\Gamma_2 \sim 10^6$ s$^{-1}$, assuming that $\mu \approx 0.3$, $E_{max} \approx 100$ K, and $\chi \approx 10^{-3}$. Note that the STM assumption of $\mu \approx 0$ would make this rate even larger by a factor of $\sim 10$.

In contrast to the dephasing rate, the relaxation of high-frequency TLSs due to interaction with others is small. The relaxation rate is proportional to the square of the interaction, which falls off as $1/r^6$. It is thus dominated by the closest TLS which is in resonance with the given one. Because the level width of the TLS is given by $\Gamma_2$, the resonant condition implies that the typical distance between resonant TLSs is $r^3 \sim 1/[\Gamma_2 \rho(E)]$, and the interaction between them is $U_0 \Gamma_2 \rho(E)$. Applying Fermi's "golden rule" we estimate that the relaxation rate due to this interaction is

$$\Gamma_1^{TLS} \approx [U_0 \rho(E)]^2 \Gamma_2 = \chi^2 \left(\frac{E}{E_{max}}\right)^{2\mu} \Gamma_2. \tag{21}$$

The relaxation rate (21) is much smaller than $\Gamma_2$ because it contains two extra factors of $\chi$ which, in contrast to $\Gamma_2$, are not compensated by large logs. Estimating it we get $\Gamma_1^{TLS} \sim 10^{-2}$–$10^0$ s$^{-1}$, which is much smaller than the phonon relaxation rate. We conclude that the phonon relaxation mechanism dominates, i.e., $\Gamma_1 \approx \Gamma_1^{ph}$.

This dephasing rate (19) is in a perfect agreement with the direct experimental observations [22] that used phase qubits to study individual TLSs with energies $E \sim 6$–$8$ GHz. This work observed the temperature dependence $\Gamma_2 \propto T^{1.24}$ and absolute values $\Gamma_2 \sim 10^6$ s$^{-1}$ at $T \sim 50$ mK.

The discussion above does not differentiate between coherent and incoherent thermally excited TLSs. The small fluctuations of the energy of the high-frequency TLS created by coherent and incoherent TLS far away from the fast TLS are indistinguishable. The crucial assumption in the derivation of the level width $\Gamma_2$ (19) was the Gaussian nature of the effective energy fluctuations $\delta E$ which is the sum of the effects produced by many fluctuators. This assumption is confirmed by the large factor $\ln(\Gamma_1^{max}/\Gamma_1^{min})$ that appeared in (19).

The effect of the slow fluctuators requires a separate analysis for those fluctuators that are located so close to the high-frequency TLS that they shift its energy by an amount larger than the width $\Gamma_2$. As mentioned above, the presence of slow fluctuators is revealed by the omnipresent $1/f$ charge and critical current noise that extends to the lowest frequencies [45]. Some of these fluctuators interact strongly with the fast TLS: $U(r) > \Gamma_2$ for $r < R_0$, where $R_0^3 = U_0/\Gamma_2$. These fluctuators create highly non-Gaussian noise that cannot be regarded as a contribution to $\Gamma_2$. Qualitatively, the slow strong fluctuators result in the chaotic motion of individual TLS levels around their average positions as shown in Fig. 2 where we sketch the effect of different fluctuators on high-frequency TLSs. Strongly coupled fluctuators (a) are located within the sphere of radius $R_0$ and brings TLSs in and out of resonance with the external probe. The fluctuator (b) is weakly coupled and contributes to the level width. The fluctuator (c), although strong enough to be non-Gaussian, is not sufficiently strong to bring the TLSs in resonance with the external probe. The chaotic motion of TLS energy level due to the strong fluctuators causes the noise in the external probe, such as resonator frequency. We discuss this noise in the following section.

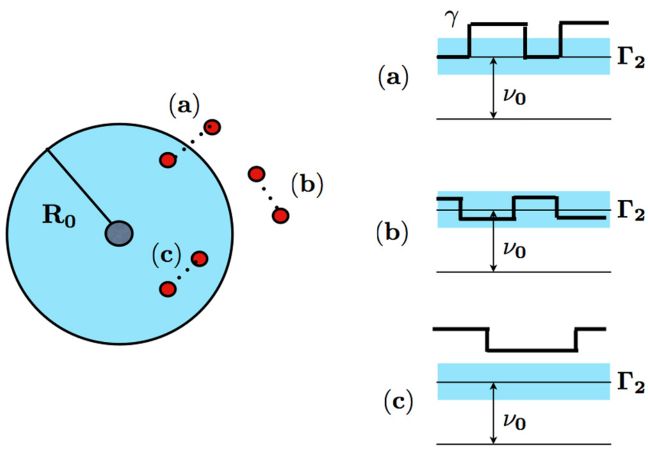


FIG. 2. (Color online) High-frequency TLS (dark small circle) and fluctuators that are coupled to it. The strongly coupled fluctuator (a) brings the TLS in and out of resonance with the external probe. This translates into a large noise measured by the probe. The weakly coupled fluctuator (b) only contributes to the linewidth of the high-frequency TLS. The strongly coupled fluctuator (c) is not strong enough to bring the fast TLS in resonance, so its effect is not observable.

## III. EFFECT OF SLOW FLUCTUATORS ON THE RESONATOR NOISE

The frequency noise in the microresonator is ultimately due to the switching of classical fluctuators that are strongly coupled to TLSs that are in resonance with the resonator electromagnetic mode. The coupling is strong in the sense that the resulting energy drift of the resonant TLS is larger than the broadening of its level, $\Gamma_2$, i.e., $U(r) > \Gamma_2$. The condition $U(r) > \Gamma_2(T)$ is satisfied for all fluctuators in the sphere of radius $R_0$ around the resonant TLS. Because the width $\Gamma_2(T)$ decreases at low temperatures, the volume of the sphere of radius $R_0$ grows at low temperatures. This compensates the decrease in the density of thermally activated fluctuators. The effect of each TLS on the dielectric constant and thereby on the resonator frequency is proportional to $1/\Gamma_2$. Thus, as the temperature goes down, the noise increases: a conclusion that seems to contradict the intuition. We illustrate the mechanism of the resonator noise in Fig. 3. The motion of levels in and out of resonance does not affect the average dielectric constant of the material because the average number of TLSs in resonance with the external frequency remains the same. Thus, one expects that in contrast to temperature-dependent noise, neither internal loss nor average frequency shift of the resonators show anomalous temperature dependence.

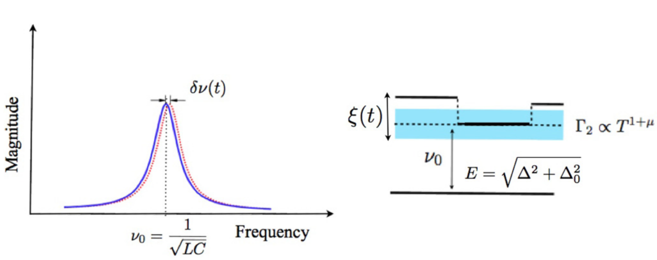


FIG. 3. (Color online) Schematics of the frequency noise generation in microresonators. The noise is due to fluctuators that are strongly coupled to resonant TLS and can induce energy drifts for the resonant TLS larger than the broadening width $\Gamma_2$ by bringing the resonant TLS in and out of resonance with the resonator.

The classical fluctuators responsible for the effects discussed in this section might be slow TLSs that are characterized by small $\Delta_0$ and $\Gamma_1$ or have a different nature. The main results of the following discussion do not depend on the assumption that classical fluctuators have the same nature as TLSs, but when estimating the magnitude of the effect we shall assume that they have similar densities.

We now provide the detailed computation that confirms this qualitative conclusion and provides quantitative estimates of the noise. The interaction between TLSs and electrical field, $\vec{\mathcal{E}}(t) = \vec{\mathcal{E}} \cos \nu_0 t$, in the resonator is due to its dipole moment $\vec{d}_0$:

$$H^{\rm int}_{\rm field} = \vec{d}_0 \cdot \vec{\mathcal{E}}(t)\sigma^z. \tag{22}$$

The dynamics of the coherent TLS can be described by the Bloch equations [46], which coincide with the equation for the TLS density-matrix evolution. These equations includes the phenomenological description of the decay and decoherence process with rates $\Gamma_1$ and $\Gamma_2$. The effect of the classical fluctuators is described by an additional time-dependent contribution to the effective "magnetic" field acting on the pseudospin representing the TLS: $\vec{B}(t) = \vec{B}' + \vec{B}''(t)$, where $\vec{B}' = [0,0,E - \xi(t)]$ and $\vec{B}''(t) = 2(\sin\theta,0,\cos\theta)\vec{d}_0 \cdot \vec{\mathcal{E}}(t)$. The ac electric field $\vec{\mathcal{E}}(t) = \vec{\mathcal{E}} \cos \nu_0 t$ is a small perturbation, so one can linearize the Bloch equations by keeping terms of the first order in the applied electric field. We look for solutions of the Bloch equations of the form $\vec{S}(t) = \vec{S}^0(t) + \vec{S}^1(t)$, where $S^0$ is the solution in the absence of electric field and $S^1 \propto \vec{\mathcal{E}}(t)$. The linearized equations become

$$\begin{aligned} \frac{dS^0_z(t)}{dt} &= \mathcal{I}m S^+(t)\Omega \cos \nu_0 t - \Gamma^{\rm ph}_1\left[S^0_z(t) - m\right],\\ i\frac{dS^+(t)}{dt} &= [E + \xi(t) - i\Gamma_2]\, S^+(t) + \Omega S^0_z(t) \cos \nu_0 t. \end{aligned} \tag{23}$$

Here we have introduced the raising operator $S^+ = S^1_x + iS^1_y$, $\Omega = 2\sin\theta \vec{d}_0 \cdot \vec{\mathcal{E}}$ is the Rabi frequency, and $m = \tanh(E/2k_B T)/2$. The presence of fluctuators (weakly and strongly coupled to the TLS) is accounted by the energy drift $\xi(t)$.

The physical quantity that we need to get from the solution of (23) is the average polarization $\mathbf{P}_{\nu_0}(t)$ produced by the resonant TLS:

$$\mathbf{P}_{\nu_0}(t) = \tfrac{1}{2}\langle \vec{d}_0 \sin\theta \langle S^+(t)\rangle_f \rangle = \varepsilon\chi(\nu_0,t)\vec{\mathcal{E}}, \tag{24}$$

where $\langle\cdot\rangle_f$ denotes the average over the distribution of the strongly coupled fluctuators responsible for the energy drift and the average $\langle\cdot\rangle$ is taken over the distribution of all the coherent TLSs and their dipole moments. The coefficient $\chi(\nu_0,t)$ gives the permittivity which is responsible for the variation of the complex resonance frequency [47]:

$$\frac{\delta f^*}{f^*} = -\int_{V_h} \frac{\chi(\nu_0,t)|\vec{\mathcal{E}}|^2 dV}{2\int_V |\vec{\mathcal{E}}|^2 dV}, \tag{25}$$

where $V_h$ is the TLS host material volume and $V$ is the resonator volume. The real part of (25) gives the relative frequency shift

$$\frac{\delta\nu(t)}{\nu_0} = -\frac{\int_{V_h} \mathrm{Re}[\mathbf{P}_{\nu_0}(t)]\cdot\vec{\mathcal{E}}dV}{2\varepsilon\int_V |\vec{\mathcal{E}}|^2 dV} \tag{26}$$

while the imaginary part is responsible for the internal quality factor $Q$:

$$\left(\frac{1}{Q}-\frac{1}{Q_0}\right) = \frac{\int_{V_h} \mathrm{Im}[\mathbf{P}_{\nu_0}(t)]\cdot\vec{\mathcal{E}}dV}{2\varepsilon\int_V |\vec{\mathcal{E}}|^2 dV}. \tag{27}$$

The frequency noise spectrum measured in the microresonator is defined as

$$\frac{S_{\delta\nu}}{\nu_0^2} = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau\int_0^\tau \frac{\langle\delta\nu(t_1)\delta\nu(t_2)\rangle}{\nu_0^2}e^{i\omega(t_1-t_2)}dt_1dt_2. \tag{28}$$

Notice that both the frequency shifts and the noise are related to the real part of the susceptibility.

Our goal is to get the physical quantities (26)–(28) from the solution of the Bloch equation (23). The assumption that relevant fluctuators are slow implies that we can solve Eq. (23) in the stationary approximation:

$$\mathrm{Re}S^+(t) = \frac{\Omega m\,[\nu_0 - E - \xi(t)]}{[\nu_0 - E - \xi(t)]^2 + \Gamma_2^2 + \Omega^2\Gamma_2\left(\Gamma_1^{\mathrm{ph}}\right)^{-1}}. \tag{29}$$

In order to calculate the average polarization $P_{\nu_0}(t)$ given by (24) we need to average (29) first over the distribution of fluctuators and then over the distribution of coherent TLSs.

Generally, the energy drift caused by fluctuators can be written as $\xi(t) = \sum_k^{\mathcal{N}_f} u_k n_k(t)$, where $\mathcal{N}_f$ is the number of coupled fluctuators, $u_k = U_0/r_k^{-3}$ denotes the interaction strength of the $k$th fluctuator coupled to the resonant TLS and $n_k(t) = \pm 1$ is a random telegraph signal with associated switching rate $\gamma_k$. Effectively each fluctuator produces a random telegraph signal with the following properties:

(i) $n_k(t) = \pm 1$ with probabilities $p(n_k(0) = \pm 1) = 1/2$;

(ii) the number $N_*$ of zero crossings in the interval $(0,t)$ is described by a Poisson process with probabilities:

$$p(N_* = \text{even number}) = e^{-\gamma_k t}\cosh\gamma_k t,$$
$$p(N_* = \text{odd number}) = e^{-\gamma_k t}\sinh\gamma_k t.$$

We now show that weakly coupled fluctuators do not contribute either to the frequency noise or the frequency shifts because their contribution to the real part of the response is equivalent to a mere additional broadening for the resonant TLS. The solution (29) implies that in this computation we can neglect the time dependence of $\xi(t)$, which we emphasize by writing its argument as the subscript $\xi(t) = \xi_t$. In order to average over weakly coupled fluctuators the real part of the response

$$\langle \mathrm{Re}S^+(t)\rangle_f = \int \mathrm{Re}S^+(t)P_{\mathcal{N}_f}(\xi_t)d\xi_t \tag{30}$$

we need to compute the distribution $P_{\mathcal{N}_f}(\xi_t)$ defined by

$$P_{\mathcal{N}_f}(\xi_t) = \prod_{k=1}^{\mathcal{N}_f}\left[\int dz_k P(z_k)\right]\delta\left(\sum_{k'=1}^{\mathcal{N}_f} z_{k'} - \xi_t\right), \tag{31}$$

where $z_k = u_k n_k$ and $P(z_k)$ is the distribution of the $k$th Random Telegraph Signal (RTS). The constraint imposed by the $\delta$ function can be simplified by finding first the Fourier transform, $G_{\mathcal{N}_f}(\lambda) = \int P_{\mathcal{N}_f}(\xi)\exp[i\lambda\xi]\,d\xi$:

$$G_{\mathcal{N}_f}(\lambda) = \left[\int_{-\infty}^{\infty} e^{i\lambda z_k(t)}P(z_k)dz_k\right]^{\mathcal{N}_f} = \left[\frac{1}{V_h}\int dr_k^3\cos\left(\frac{U_0\lambda}{r_k^3}\right)\right]^{\mathcal{N}_f}, \tag{32}$$

where $V_h$ is the fluctuators host material volume. Integrating (32) we find

$$G_{\mathcal{N}_f}(\lambda) = \exp\left\{\frac{\mathcal{N}_f}{V_h}\int dr_k^3\left[\cos\left(\frac{U_0\lambda}{r_k^3}\right)-1\right]\right\} = \exp[-\Gamma_f|\lambda|], \tag{33}$$

where $\Gamma_f = C\rho_{0f}U_0$, $\rho_{0f} \approx \rho_0 T^{1+\mu}/E_{\max}^{\mu}$ is the density of thermally activated fluctuators and $C = \frac{4\pi}{3}\int_0^\infty dy(1-\cos\frac{1}{y}) \approx 6.57$ is a constant. By performing the inverse Fourier transform of Eq. (33) we get the distribution

$$P(\xi(t)) = \int_{-\infty}^{+\infty} d\lambda e^{-i\lambda\xi(t)-\Gamma_f|\lambda|} = \sqrt{\frac{2}{\pi}}\frac{\Gamma_f}{\Gamma_f^2+\xi(t)^2} \tag{34}$$

that is Lorentzian. By substituting (29) and (34) into (30) we get the response:

$$\langle \mathrm{Re}S^+(t)\rangle_f = \frac{\sqrt{2\pi}\,\Omega m\,(\nu_0 - E)}{(\nu_0 - E)^2 + \left(\sqrt{\Gamma_2^2 + \Omega^2\Gamma_2\left(\Gamma_1^{\mathrm{ph}}\right)^{-1}} + \Gamma_f\right)^2}$$

that shows the additional contribution $\Gamma_f$ to the dephasing width. Unlike $\Gamma_2$ (19) this contribution does not contain a large logarithmic factor, so $\Gamma_f \ll \Gamma_2$ for bulk materials. As explained in Sec. II C the logarithmic factor might become of the order of unity for surface dielectrics, so in this case $\Gamma_f \lesssim \Gamma_2$.

We now discuss the effect of strongly coupled fluctuators on the real part of the susceptibility. Estimating the number of strongly coupled fluctuators by $\mathcal{N}_f = \frac{4\pi}{3}\frac{U_0}{\Gamma_2}\rho_0$ we get $\mathcal{N}_f \sim 1$ for two-dimensional surface dielectrics and $\mathcal{N}_f \sim 10^{-1}$ for three-dimensional materials characterized by a large value of $\ln(\Gamma_1^{\max}/\Gamma_1^{\min})$. The same estimate can be obtained directly from the experimental values $\rho_0 \approx 10^{20}\,\mathrm{cm}^{-3}\,\mathrm{eV}^{-1}$, $\Gamma_2 \approx 2\times 10^{-4}$ K and $U_0 \approx 10\,\mathrm{K\,nm}^3$. A strongly coupled fluctuator brings the resonant TLS in and out of resonance inducing a dynamical change of the susceptibility that is described by a random telegraph signal:

$$w_k^{\mathrm{res}} = \lim_{u_k\to 0}\frac{\Omega m\,[\nu_0 - E - u_k]}{[\nu_0 - E - u_k]^2 + \Gamma_2^2 + \Omega^2\Gamma_2\left(\Gamma_1^{\mathrm{ph}}\right)^{-1}},$$
$$w_k^{\mathrm{off}} = 0 \tag{35}$$

with the switching rate $\gamma_k$ of the strongly coupled fluctuator. As a result, it contributes to the average susceptibility as $w_k^{\mathrm{res}}(\tanh E/2T + 1)/2$. By substituting (35) into (26) we estimate the induced frequency shift of the resonator:

$$\frac{\delta\nu}{\nu_0} = \frac{1}{3}\langle\vec{d}_0^2\rangle\frac{\int_{V_h} v(\nu_0,\vec{\mathcal{E}},T)|\vec{\mathcal{E}}|^2 dV}{2\int_V \epsilon|\vec{\mathcal{E}}|^2 dV}, \tag{36}$$

where

$$v(\nu_0,\vec{\mathcal{E}},T) = \int_0^{E_{\max}} dE\,\frac{P(E)\tanh\left(\frac{E}{2T}\right)(E-\nu_0)}{(E-\nu_0)^2+\Gamma_2^2+\Omega^2\Gamma_2\left(\Gamma_1^{\rm ph}\right)^{-1}}. \tag{37}$$

Notice that the frequency shift given by (36) is very similar to the ones predicted by the STM. The only difference between (36) and the STM predictions is associated with the different probability distribution assumed for the energy splitting of the resonant TLS but the shifts are completely insensitive to the presence of weak and strong fluctuators coupled to resonant TLS. As a result, the presence of strongly interacting fluctuators cannot be detected by the measurements of the frequency shifts as a function of temperature. However, as we have already shown in a previous work [14] the presence of fluctuators is revealed by the power dependence of the losses in a high quality microresonator. The fluctuators result indeed in a weaker (logarithmic) dependence of the losses on the applied power which is in very good agreement with data, in contrast with the square-root dependence predicted by the STM theory [15,18,20].

We now demonstrate that interaction between resonant TLS and strongly coupled fluctuators affects significantly the noise in a microresonator. As it is evident from (28), the noise spectrum of the microresonator is the Fourier transform of the autocorrelation function of the susceptibility. Each fluctuator that is strongly coupled to a resonant TLS contributes to the autocorrelation function of the susceptibility as $\frac{1}{4}(w_k^{\rm res} - w_k^{\rm off})^2 e^{-2\gamma_k(t_2-t_1)}$ and consequently to the noise spectrum of the microresonator as a Lorentzian. By summing over different TLSs coupled to strongly coupled fluctuators we find that the noise spectrum is

$$\frac{S_{\delta\nu}}{\nu_0^2}(\omega) = \frac{8}{15}\langle\vec{d}_0^4\rangle\mathcal{P}(\nu_0,\vec{\mathcal{E}},T)\int\frac{\gamma P(\gamma)}{\gamma^2+\omega^2}d\gamma. \tag{38}$$

Here $P(\gamma)$ is the probability distribution of the switching rates of the strongly coupled fluctuators,

$$\mathcal{P}(\nu_0,\vec{\mathcal{E}},T) = \frac{\int_{V_h} s(\nu_0,\vec{\mathcal{E}},T)|\vec{\mathcal{E}}|^4 dV}{4(\int\epsilon|\vec{\mathcal{E}}|^2 dV)^2},$$

which depends on the volume $V_h$ taken by the amorphous material and

$$s(\nu_0,\vec{\mathcal{E}},T) = \int\frac{(\nu_0-E)^2\tanh^2\left(\frac{E}{2T}\right)dE\,P(E)}{\left[(\nu_0-E)^2+\tilde{\Gamma}_2^2+\Omega^2\tilde{\Gamma}_2\left(\Gamma_1^{\rm ph}\right)^{-1}\right]^2}, \tag{39}$$

which depends on the temperature and the power applied to the microresonator.

The frequency dependence of the noise spectrum given in (38) is $1/f$ if the switching rate $\gamma$ has $P(\gamma)\sim 1/\gamma$ distribution. Such distribution is expected for practically all realistic models of fluctuators. For instance, fluctuators that represent slow TLSs flipped by phonons have $P(\Gamma_1)\sim 1/\Gamma_1$ as explained in Sec. II C. More generally, any process whose rate depends exponentially on a physical quantity $l$ with a smooth distribution is characterized by $P(\gamma)\sim 1/\gamma$ distribution in the exponentially wide range $\gamma_{\min}\ll\gamma\ll\gamma_{\max}$. For instance, such distribution for the switching rate appears for a particle trapped in a double-well potential whose quantum tunneling rate through the potential barrier depends exponentially on both the height and the width of the barrier, as well as for a thermally activated tunneling with rate $\gamma_0 e^{-E_a/K_BT}$, where $E_a$ denotes the activation energy.

The dependence of the noise spectrum on the temperature and the power applied to the microresonator can be found by performing the integral given in (39). The result has different structure at low and high temperature. Because $\sqrt{\tilde{\Gamma}_2^2+\Omega^2\tilde{\Gamma}_2(\Gamma_1^{\rm ph})^{-1}}\ll\nu_0$, at low temperature $T\ll\nu_0$ the integral is dominated by small vicinity of $\nu_0$:

$$s(\nu_0,\vec{\mathcal{E}},T)\simeq\left(\frac{\nu_0}{E_{\max}}\right)^{\mu}\frac{\bar{P}_0}{\Gamma_2\sqrt{1+|\vec{\mathcal{E}}/\mathcal{E}_c|^2}}, \tag{40}$$

where $\mathcal{E}_c$ has a physical meaning of the critical field for the TLS saturation. It is defined by

$$\mathcal{E}_c = \frac{\sqrt{\Gamma_1^{\rm ph}\Gamma_2}}{2\langle\vec{d}_0\,|\sin\theta|\rangle}. \tag{41}$$

The important property of the generalized tunneling model is that the critical electric field $\mathcal{E}_c$ is temperature dependent and it scales as as $\mathcal{E}_c\propto T^{(1+\mu)/2}$.

By substituting (40) into (38), we find that in the low-temperature limit the noise spectrum is

$$\frac{S_{\delta\nu}}{\nu_0^2}(\omega)\sim\frac{\chi}{\omega}\left(\frac{\nu_0}{E_{\max}}\right)^{\mu}\frac{U_0}{\Gamma_2}\begin{cases}\frac{\int_{V_h}\mathcal{E}_c|\vec{\mathcal{E}}|^3 dV}{4(\int_V\epsilon|\vec{\mathcal{E}}|^2 dV)^2} & \text{if } |\vec{\mathcal{E}}|\gg\mathcal{E}_c;\\ \frac{\int_{V_h}|\vec{\mathcal{E}}|^4 dV}{4(\int_V\epsilon|\vec{\mathcal{E}}|^2 dV)^2} & \text{if } |\vec{\mathcal{E}}|\ll\mathcal{E}_c.\end{cases} \tag{42}$$

At all radiation powers the spectrum of the noise has $1/f$ dependence. In a strong electric field the spectrum scales with the applied power as $\sim P^{-1/2}$ and with temperature as $\sim T^{(1-\mu)/2}$ while in the weak electric-field regime it is power independent and scales with temperature as $\sim T^{-(1+\mu)}$.

At high temperatures, $T\gg\nu_0$, the $1/f$ frequency dependence of the noise power remains intact but its temperature dependence changes. Evaluating the integral (39) in this limit we find

$$s(\nu_0,\vec{\mathcal{E}},T)\simeq c\bar{P}_0\frac{T^{\mu-1}}{E_{\max}^{\mu}}, \tag{43}$$

where $c=\int_0^\infty dx\,x^{\mu-2}\tanh^2(x/2)\approx 1.2$. By substituting (43) into (38) we find the noise spectrum in this regime,

$$\frac{S_{\delta\nu}}{\nu_0^2}(\omega)\sim\frac{\chi}{\omega}\frac{U_0}{T}\left(\frac{T}{E_{\max}}\right)^{\mu}\frac{\int_{V_h}|\vec{\mathcal{E}}|^4 dV}{4(\int_V\epsilon|\vec{\mathcal{E}}|^2 dV)^2}. \tag{44}$$

In this regime the noise spectrum has weaker temperature dependence, $\sim T^{\mu-1}$, and has no power dependence.

In the intermediate temperature $T\sim\nu_0$, one expects a smooth crossover between the limits (42) and (44), leading to predictions for the noise spectrum that is in agreement with the data.

## IV. SUMMARY OF THE ASSUMPTIONS AND APPROXIMATIONS OF THE GTM

Before moving to the discussion and conclusions of this paper it might be useful to summarize the assumptions and the approximations we made on the derivation of our results.

(i) We use a tunneling model for TLSs in an insulator that takes into account the interaction between them [26,27,31] *and* a slow power-law dependence of their density of states $\rho(E) = \rho_0(E/E_{\max})^\mu$, with parameter $\mu \approx 0.3$ derived from the experiments.

(ii) We distinguish between different TLSs: (i) coherent or quantum TLSs characterized by dephasing rate $\Gamma_2^{\rm ph} < E$ and (ii) fluctuators or classical TLSs characterized by $\Gamma_2^{\rm ph} \geqslant E$. Among the coherent TLSs, we distinguish between high-energy TLSs with $E \gg T$, and low-energy (thermally activated) TLSs with $E \leqslant T$. Resonant coherent TLSs have energy splitting $E \approx \nu_0$. Here $\mu_0$ is the frequency of the superconducting microresonator. It is assumed that $\nu_0 \gg T$.

(iii) An important quantity entering the theory is the linewidth $\Gamma_2$ of resonant TLSs due to their interaction with surrounding thermally excited TLSs that change their state emitting and absorbing photons. We calculate the width $\Gamma_2$ by assuming that the typically thermally excited TLSs are characterized by $\Delta_0 \ll \Delta \simeq E$ and by averaging over their distributions of the positions, energies, and the relaxation rates. We find that

$$\Gamma_2 = c\chi \ln\left(\frac{\Gamma_1^{\max}}{\Gamma_1^{\min}}\right)\frac{T^{1+\mu}}{E_{\max}^{\mu}}, \quad c \sim 1.$$

We use Fermi's golden rule to estimate the relaxation rate of the resonant TLSs due to the interaction with surrounding thermally excited TLSs. We find that

$$\Gamma_1^{\rm TLS} \simeq \chi^2\left(\frac{E}{E_{\max}}\right)^{2\mu}\Gamma_2.$$

In a typical low-temperature experiment, the dephasing rate $\Gamma_2$ due to the interaction with thermally activated TLSs dominates over the decoherence rate $\Gamma_2^{\rm ph} \sim \Gamma_1^{\rm ph}$ due to phonons. In contrast, the relaxation rate $\Gamma_1^{\rm TLS}$ is negligible compared to the relaxation rate $\Gamma_1^{\rm ph}$ due to phonons. We conclude that resonant TLSs are relaxed by phonons and have dephasing width $\Gamma_2 \propto T^{1+\mu}$.

(iv) The frequency noise in the superconducting microresonators is due to the switching of classical fluctuators that are strongly coupled to resonant TLSs.

A fluctuator is strongly coupled to a resonant TLS when it is located within a sphere of radius $R_0 = (\frac{U_0}{\Gamma_2})^{1/3}$ centered around the resonant TLS. Because the width $\Gamma_2$ decreases at low temperature, the volume of the sphere grows at low temperature.

Strongly coupled fluctuators induce energy drifts for the resonant TLSs larger than the broadening width $\Gamma_2$ by bringing the resonant TLSs in and out of resonance with the resonator. Each fluctuator is described as a random telegraph signal with switching rates $\gamma$. A superposition of random telegraph signals having switching rates distributed with $1/\gamma$ distribution translates into a large noise $1/f$ for the resonator.

(v) Calculations of the resonator noise spectra are done by resorting to the Bloch equations. These equations include the phenomenological description of the decay and decoherence processes with rates $\Gamma_1^{\rm ph}$ and $\Gamma_2$. The effect of the classical fluctuators is described by an additional time-dependent contribution to the effective "magnetic" field acting on the pseudospin representing the TLS. We linearize the Bloch equations by keeping terms of the first order in the electric field applied to the resonator. We solve the linearized Bloch equations in the stationary approximation by assuming that the relevant strongly coupled fluctuators are slow.

## V. DISCUSSION

The theoretical expectations derived in the previous section are in a very good agreement with main features of the data [1,3,7–10,47]. Most importantly Eqs. (42) and (44) give correct power and temperature dependence of the noise spectra. In particular, these spectra display the very unusual behavior, observed experimentally, of the noise increasing at low temperatures. There is no contradiction between this growth and the Nernst theorem, because it is due to the fact that the sensitivity of individual TLSs to the slow fluctuators increases dramatically at low temperatures.

The growth of the noise at low temperatures is a clear evidence of the importance of the interactions between TLSs. Indeed, the STM gives completely different predictions for the temperature dependence of the noise, as we show now. We focus on the weakly driven TLS in which computations are straightforward. The Bloch equations (23) become

$$\begin{aligned}\frac{dS_z^0(t)}{dt} &= -\Gamma_1\big[S_z^0(t) - m\big],\\ i\frac{dS^+(t)}{dt} &= [E - i\Gamma_2]\,S^+(t) + \Omega S_z^0(t)\cos\nu_0 t,\end{aligned} \tag{45}$$

whose solutions are

$$S_z^0(t) = m + \big[S_z^0(0) - m\big]e^{-\Gamma_1 t}, \tag{46}$$

$$\begin{aligned}S^+(t) = {}& \frac{\Omega\,[(E - i\Gamma_2)\cos\nu_0 t - i\omega\sin\nu_0 t]\,m}{\nu_0^2 - (E - i\Gamma_2)^2}\\ &+ \frac{\Omega\,[(E - i\Gamma_+)\cos\nu_0 t - i\nu_0\sin\nu_0 t]\,\delta S_t^z}{\nu_0^2 - (E - i\Gamma_+)^2},\end{aligned} \tag{47}$$

where $\delta S_t^z = [S_z^0(0) - m]e^{-\Gamma_1 t}$ and $\Gamma_+ = \Gamma_1 + \Gamma_2$. The first term in (47) describes the average response, the second the relaxation after the spin-flip process which is responsible for the noise. Because the frequency shift of the resonator is due to $\langle {\rm Re}S^+(t)\rangle$, the noise in this quantity is given by $\langle {\rm Re}S^+(t){\rm Re}S^+(0)\rangle$ which is proportional to $\langle (S_z(0) - m)^2\rangle = 1 - m^2 = \cosh^{-2}(E/2T)$. In the low-temperature regime $T \ll \nu_0$, at relevant energies $E \ll \nu_0$ and $\Gamma \ll E$, we find that the noise spectrum is

$$\frac{S_{\delta\nu}}{\nu_0^2}(\omega) \sim \frac{\bar{P}_0}{\omega} \int \frac{E^2 dE}{\nu_0^4 \cosh^2 E/2T} \frac{\int_{V_h} |\vec{\mathcal{E}}|^4 dV}{4(\int_V \epsilon |\vec{\mathcal{E}}|^2 dV)^2}$$
$$\propto [\bar{P}_0 V_h T]\frac{T^2}{\omega} = \mathcal{N}_{\mathrm{TLS}} \frac{T^2}{\omega}, \quad (48)$$

where $\mathcal{N}_{\mathrm{TLS}}$ is the number of thermally activated TLS located in the dielectric volume $V_h$. Although the noise spectrum has the correct, $1/f$, frequency dependence, its power decreases quickly at low temperatures in sharp contrast to the data.

## VI. CONCLUSIONS

The predictions of the generalized tunneling model for the noise spectra of the resonator frequency derived in the previous sections agree very well with recent detailed measurements performed in high-$Q$ superconducting microresonators [8]. Reversing the logic one can extract the phenomenological parameter $\mu$ from these data. The resulting value $\mu \approx 0.2$–$0.4$ is in a very good agreement with the value that was found in many bulk glasses [34]. This value agrees perfectly well with the direct measurements of the dephasing rate of TLSs in the insulating barrier of phase qubits that give $\Gamma_2 \propto T^{1+\mu}$ with $\mu \approx 0.24$ [22]. The absolute values of the dephasing rate observed in these experiments agree well with the theoretical expectations assuming $\chi(T/E_{\max})^\mu \sim 10^{-3}$.

As was emphasized repeatedly by Leggett the apparent universality of the dimensionless parameter $\chi \sim 10^{-3}$ in the STM is very strange and asks for theoretical explanation. In the GTM considered in this paper this puzzle becomes less striking because the parameter that controls the interaction between the TLSs has a weak energy dependence: $\chi_{\mathrm{eff}} = \chi(T/E_{\max})^\mu$. At low temperatures $T \sim 100$ mK this parameter becomes much smaller than its high energy (bare) value. Assuming that the power law $(E/E_{\max})^\mu$ extends to the atomic energy scales, $E_{\max} \sim 10^3$ K, one deduces the *bare* value of the parameter $\chi_0 \sim 10^{-1}$–$10^{-2}$. The fact that the value of $\chi_0$ at high temperature is somewhat small is not surprising, because larger values would imply melting. Indeed, the average thermal displacements, $\delta u$, of all TLSs per atomic volume is $\langle \delta u^2 \rangle_{\mathrm{Th}} \sim d^2 T P_0 a^3$ where $a$ is interatomic spacing and $d$ is a typical displacement caused by TLSs. The Lindemann melting criterion demands that $\langle \delta u^2 \rangle_{\mathrm{Th}} < (c_L a)^2$ where $c_L \approx 0.1$–$0.2$ is the Lindemann parameter. Estimating the interaction parameter $U_0 \sim \omega_D d^2 a$ we can rewrite the Lindemann melting condition as $(T/\omega_D)\chi_0 < c_L^2$ which implies that the maximal values of $\chi_0$ consistent with the glass stability are $\chi_0 \sim 10^{-1}$–$10^{-2}$.

In conclusion, the data and their theoretical analysis remove the mystery of the universality of the dimensionless parameter $\chi \sim 10^{-3}$–$10^{-4}$ at low temperatures replacing it by the phenomenological law $\rho(E) = \rho_0(E/E_{\max})^\mu$ with a small $\mu \approx 0.3$. It is very likely that this law is a consequence of a more complicated, than assumed usually, nature of the TLSs in physical glasses. The data also indicate that interaction between TLSs is responsible for their dephasing and the noise generated by them.

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