Transport properties of a two-lead Luttinger-liquid junction out of equilibrium: Fermionic representation

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The electrical current through an arbitrary junction connecting quantum wires of spinless interacting fermions is calculated in fermionic representation. The wires are adiabatically attached to two reservoirs at chemical potentials differing by the applied voltage bias. The relevant scale-dependent contributions in perturbation theory in the interaction up to infinite order are evaluated and summed up. The result allows one to construct renormalization group equations for the conductance as a function of voltage (or temperature, wire length). There are two fixed points at which the conductance follows a power law in terms of a scaling variable Λ, which equals the bias voltage V, if V is the largest energy scale compared to temperature T and inverse wire length L^{-1}, and interpolates between these quantities in the crossover regimes.

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I. INTRODUCTION

In the past few years, exactly one-dimensional quantum wires have become available for experimental investigation in the form of carbon nanotubes, chains of metal atoms, or weakly side-coupled molecular chains in solids. The new data emerging from these experiments [1–3], in particular in nonequilibrium situations, require a more detailed and more general theoretical description than presently available. Electron transport in nanowires has been studied theoretically for more than two decades. In the first papers it was found that electron-electron interaction affects even the conductance of a clean wire [4,5]. In the case of realistic boundary conditions, namely, adiabatically attaching ideal leads to the interacting quantum wire, the two-point conductance of a clean wire is that of the leads, equal to one conductance quantum per channel, irrespective of the (forward scattering) interaction [6]. The work of Kane and Fisher [5] and Furusaki and Nagaosa [7] showed, that interaction has a dramatic effect on the conductance in the presence of a potential barrier. Namely, for repulsive interaction these authors found that the conductance tends to zero as the temperature T, or more generally, the excitation energy of the electrons approaches zero, while for attractive interaction the conductance scales to its maximum value. This behavior has been shown to carry over to the dependence on bias voltage, at sufficiently low temperatures (and for long wires). There exists a large body of theoretical work addressing different aspects of transport through Luttinger liquids without or with impurities, such as the effect of the leads on a finite length wire [8,9], the response to an ac electric field [10], the appearance of oscillatory behavior in the nonlinear conductance [11,12], and the emergence of bistability for the very strong interaction and bias voltage [13]. The transport through Luttinger-liquid junctions at not too strong interaction has also been calculated using the functional renormalization group method as reviewed in [14]. The results mentioned above have been mostly obtained within the bosonization method, which needs to be amended by a correction taking care of the physical situation of a wire of finite length attached to reservoirs (see above). Experimentally, the predictions of theory have been found to be observed, at least qualitatively [15–19].

A proper treatment of the two-point conductance in the limit of weak interaction, taking into account the gradual buildup of the Friedel oscillations around the barrier as the infrared cutoff is lowered has been given by Yue, Glazman, and Matveev [20]. These authors used the perturbative renormalization group (RG) for fermions to derive the conductance for an arbitrary (but short) potential barrier ("fermionic representation"). In this paper we extend the approach of Yue *et al.* to transport under stationary nonequilibrium conditions. Following our extensive work on transport in the linear regime through junctions of Luttinger liquids at arbitrary strength of interaction [21–28] we derive in the following RG equations for the conductance at finite bias voltage and for any interaction strength. We use the fact that the β function of the RG equation for the conductance can be obtained in very good approximation by summing a class of contributions in perturbation theory in all orders of the interaction [22]. A comparison of our previous results on the linear response conductances of two- [22] and three-lead junctions [23,25,28] with or without additional symmetries, or an applied magnetic flux [27,28], with the results of the bosonization method of conformal field theory methods, of Bethe ansatz, where available, are in full agreement provided those results were well founded. In a few cases where the conformal field theory result was based on an additional assumption we found disagreement, which we interpret as saying that the assumption was not justified.

In this paper we consider the transport of spinless fermions, which begs the question of how our results may be applied to experiment. The spinless Luttinger-liquid model has actually been used to describe transport through spin-polarized quantum wires, as considered in Ref. [2]. A generalization of our theory to spinful fermions is in progress.

II. THE MODEL

We consider a system of spinless fermions in one dimension, interacting in the region $a < |x| < L$ (the "wire"), adiabatically connected to reservoirs at $|x| > L$. There is a barrier in the narrow regime $|x| < a$, which scatters the fermions as described by the S matrix (up to overall phase factors in the individual wires)

$$S = \begin{pmatrix} r & t \\ \widetilde{t} & r \end{pmatrix} = \begin{pmatrix} \sin\theta & i\,\cos\theta e^{-i\varphi} \\ i\,\cos\theta e^{i\varphi} & \sin\theta \end{pmatrix}. \qquad (1)$$

We choose this parametrization in terms of the transmission and reflection amplitudes t,r, since it is readily generalizable to the case of multiwire junctions (n wires, $n > 2$). The above form of the S matrix is completely general.

In the continuum limit, linearizing the spectrum at the Fermi energy and adding forward scattering interaction of strength g_j in wire j, we may write the Hamiltonian $\mathcal{H}$ in the representation of incoming and outgoing waves as

$$\begin{aligned} \mathcal{H} &= \int_0^\infty dx \sum_{j=1}^{2} \left[H_j^0 + H_j^{\rm int}\Theta(a < x < L)\right], \\ H_j^0 &= v_j \psi_{j,\rm in}^\dagger i\nabla\psi_{j,\rm in} - v_j\psi_{j,\rm out}^\dagger i\nabla\psi_{j,\rm out}, \\ H_j^{\rm int} &= 2\pi v_j g_j \psi_{j,\rm in}^\dagger\psi_{j,\rm in}\psi_{j,\rm out}^\dagger\psi_{j,\rm out}. \end{aligned} \qquad (2)$$

We are using the chiral representation, labeling electrons in lead j by $(j,\eta) \equiv j_\eta$ where $\eta = +1$ for outgoing and $\eta = -1$ for incoming electrons and all position arguments x are on the positive semiaxis. The range of the interaction lies within the interval (a,L), where $a > 0$ serves as an ultraviolet cutoff (at energy scale v_j/a) and separates the domains of interaction and potential scattering on the junction; noninteracting leads are attached adiabatically at large x beyond L. In terms of the doublet of incoming fermions $\Psi_- = (\psi_{1,-},\psi_{2,-})$ the outgoing fermion operators may be expressed with the aid of the S matrix as $\Psi_+(x) = S\Psi_-(x)$. For later use we define density operators $\widehat{\rho}_{j,\eta=-1} = \psi_{j,-}^\dagger\psi_{j,-} = \Psi_-^+\rho_j\Psi_-$, and $\widehat{\rho}_{j,\eta=1} = \psi_{j,+}^\dagger\psi_{j,+} = \Psi_-^+\widetilde{\rho}_j\Psi_-$, where $\widetilde{\rho}_j = S^+\rho_j S$. The 2×2 matrices are defined by $(\rho_j)_{\alpha\beta} = \delta_{\alpha\beta}\delta_{\alpha j}$ and $(\widetilde{\rho}_j)_{\alpha\beta} = S_{\alpha j}^+ S_{j\beta}$.

III. CHARGE CURRENT OF FREE FERMIONS

The net current flowing outward through the point z in wire j is composed out of two chiral components, moving towards ($\eta = -1$) and from ($\eta = 1$) the junction,

$$J_j(z) = v_j(\langle\widehat{\rho}_{j,+}(z)\rangle - \langle\widehat{\rho}_{j,-}(z)\rangle), \qquad (3)$$

where v_j is the group velocity of the fermions. We use units where electrical charge $e = 1$, and also $\hbar = 1$ and Boltzmann's constant $k_B = 1$.

We work with the Green's functions in this chiral basis and in Keldysh formulation (we denote matrices in Keldysh space by an underbar),

$$\underline{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}. \qquad (4)$$

Here retarded, advanced, and Keldysh components of the Green's functions, in matrix form in the chiral basis, are given by [2×2 matrices in the chiral basis are denoted by a hat,

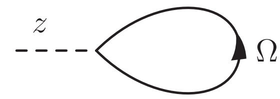

FIG. 1. The diagram showing the zeroth-order contribution to the current.

$$\begin{aligned} \widehat{G}_{\eta_l\eta_j}(l,y|j,x) &= G(l,\eta_l,y|j,\eta_j,x)] \\ \widehat{G}_\omega^R(l,y|j,x) &= -\frac{i}{\sqrt{v_l v_j}}\theta(\tau)e^{i\omega\tau}\begin{bmatrix}\delta_{lj} & 0 \\ S_{lj} & \delta_{lj}\end{bmatrix}, \\ \widehat{G}_\omega^A(l,y|j,x) &= \frac{i}{\sqrt{v_l v_j}}\theta(-\tau)e^{i\omega\tau}\begin{bmatrix}\delta_{lj} & S_{jl}^* \\ 0 & \delta_{lj}\end{bmatrix}, \\ \widehat{G}_\omega^K(l,y|j,x) &= -\frac{i}{\sqrt{v_l v_j}}e^{i\omega\tau}\begin{bmatrix}\delta_{lj}h_l & S_{jl}^*h_l \\ S_{lj}h_j & S_{jm}^*S_{lm}h_m\end{bmatrix}, \\ \tau &= \eta_l y/v_l - \eta_j x/v_j, \end{aligned} \qquad (5)$$

where $h_j(\omega) = \tanh[(\omega-\mu_j)/2T]$ is the equilibrium distribution function in the reservoir of lead j, characterized by the chemical potential μ_j. We shall assume the temperatures T in the leads to be equal.

The average density of the chiral current at point z, $\langle\rho_{j,\eta}(z)\rangle$, is represented by the diagram in Fig. 1.

In terms of the Green's function matrix and defining the external vertex by the Keldysh matrix $\underline{\gamma}_{\rm ext}$,

$$\underline{\gamma}_{\rm ext} = \frac{i}{2}\begin{pmatrix}1 & 1 \\ -1 & -1\end{pmatrix}, \qquad (6)$$

we have

$$\langle\rho_{j,\eta}(z)\rangle = \int\frac{d\Omega}{2\pi}{\rm Tr}_K[\underline{\gamma}_{\rm ext}\underline{G}_\Omega(j,\eta,z|j,\eta,z)] \qquad (7)$$

with the trace ${\rm Tr}_K$ taken over the Keldysh indices.

Using the expressions (5), we obtain

$$\begin{aligned} v_j\langle\rho_{j,-}(z)\rangle &= \frac{1}{2}\int\frac{d\Omega}{2\pi}[1 - h_j(\Omega)], \\ v_j\langle\rho_{j,+}(z)\rangle &= \frac{1}{2}\int\frac{d\Omega}{2\pi}\left[1 - \sum_m |S_{jm}|^2 h_m(\Omega)\right]. \end{aligned} \qquad (8)$$

Notice here that the incoming current in the jth wire is characterized by the distribution function referring to the same wire. The outgoing current in the jth wire is characterized by the distribution functions referring to the remaining wires. The dependence on z vanishes in the dc limit considered here.

Using the unitarity property (i.e., charge conservation), $\sum_m |S_{jm}|^2 = 1$, we may represent the net current in the form

$$J_j^{(0)}(z) = \frac{1}{2}\int\frac{d\Omega}{2\pi}\sum_m |S_{jm}|^2[h_j(\Omega) - h_m(\Omega)], \qquad (9)$$

which is a well-known expression. For the above choice of the weight function $h_j(\Omega)$, the remaining integration can be easily done with the result

$$J_j^{(0)}(z) = \frac{1}{2\pi}\sum_m |S_{jm}|^2(\mu_m - \mu_j). \qquad (10)$$

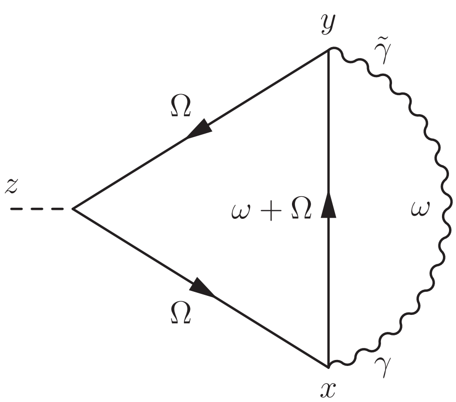

FIG. 2. The diagram providing first-order correction to the current due to interaction.

The conductance (in units of the conductance quantum $e^2/2\pi\hbar$) of a two-lead junction is in lowest order given by

$$G_0 = J/V = |S_{12}|^2 = t^2, \tag{11}$$

where $V = \mu_1 - \mu_2$ is the applied bias voltage. In the following we will find it convenient to introduce the quantity $Y = 1 - 2G_0$ characterizing the conductance.

IV. CURRENT TO FIRST ORDER IN THE INTERACTION

The first-order correction to the current in the nonequilibrium case is represented as the diagram depicted in Fig. 2. Here the wavy line stands for the electronic interaction, taking place at the point x in the wire l. The contribution to the current of chirality η_n in the nth wire can be expressed as

$$\begin{aligned} J^{(1)}_{j_\eta}(z) = v_j \int \frac{d\Omega\, d\omega}{(2\pi)^2} \int dx\, dy \sum_{\mu=1,2} \sum_{l,\eta_l} \mathrm{Tr}_K[\underline{\gamma}_{\rm ext} \\ \times \underline{G}_\Omega(j_\eta,z|l_\eta,y)\underline{\bar{\gamma}}^\mu \underline{G}_{\Omega+\omega}(l_\eta,y|l_\eta,x)\underline{\gamma}^\mu \\ \times \underline{G}_\Omega(l_\eta,x|j_\eta,z)](2\pi i g_l v_l)\delta(x-y). \end{aligned} \tag{12}$$

The trace Tr_K is over the lower (fermionic) Keldysh indices; the fermion-boson vertices $\gamma^\mu_{ij} \to \underline{\gamma}^\mu$, $\bar{\gamma}^\mu_{ij} \to \underline{\bar{\gamma}}^\mu$, tensors of rank three defined in Keldysh space, are given by

$$\gamma^1_{ij} = \bar{\gamma}^2_{ij} = \tfrac{1}{\sqrt{2}}\delta_{ij}, \quad \gamma^2_{ij} = \bar{\gamma}^1_{ij} = \tfrac{1}{\sqrt{2}}\tau^1_{ij}, \tag{13}$$

with τ^1 the first Pauli matrix.

Notice that, similarly to the case of zeroth order in the interaction, the factor v_j at the external point z is compensated by the prefactor coming from the Green's function, Eq. (5). If the point of the observation z lies outside the interacting region, $z > L$, then the dependence on z disappears in the outgoing current, $J^{(1)}_{j,+}(z > L) = J^{(1)}_j$, whereas the corrections to the incoming current are altogether absent, $J^{(1)}_{j,-}(z > L) = 0$. In what follows we discuss the corrections to the outgoing current. In view of the later generalization involving an infinite summation of higher order terms it is useful to represent the above first-order expression as

$$\begin{aligned} J^{(1)}_j = i \int \frac{d\omega}{2\pi} \int dx\, dy \sum_{l_\eta,m_\eta} \\ \times \mathrm{Tr}_K[\,\underline{T}_\omega(m_\eta,y|l_\eta,x;\, j,+,z)\underline{L}_{0,\omega}(l_\eta,x|m_\eta,y)], \end{aligned} \tag{14}$$

where we recall the definitions

$$l_\eta = (l,\eta_l),$$

etc. Here we defined a "boson propagator" representing the interaction line,

$$\underline{L}_{0,\omega}(l,\eta_l,x|m,\eta_m,y) = (2\pi g_l v_l)\delta(x-y)\delta_{lm}\tau^1_{\eta_l,\eta_m}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{15}$$

and the quantity T representing the triangle of Green's functions in Fig. 2,

$$\begin{aligned} T^{\nu\mu}_\omega&(m_\eta,y|l_\eta,x;\, j_\eta,z) \\ &= v_j \int \frac{d\Omega}{2\pi} \mathrm{Tr}_K[\underline{\gamma}_{\rm ext}\underline{\widehat{G}}_\Omega(j_\eta,z|m_\eta,y)\underline{\bar{\gamma}}^\nu \\ &\quad \times \underline{\widehat{G}}_{\Omega+\omega}(m_\eta,y|l_\eta,x)\underline{\gamma}^\mu\underline{\widehat{G}}_\Omega(l_\eta,x|j_\eta,z)]; \end{aligned} \tag{16}$$

this diagram should be combined with the one where the arrows on the fermonic lines are reverted.

The triangle diagram is characterized by two Keldysh indices and thus is subdivided into four blocks. Symbolically, we write

$$\mathrm{Tr}_K[TL] = T^{11}L^R + T^{22}L^A$$

anticipating that $T^{21} = 0, L^{21} = 0$ (to be shown later).

When integrating over Ω in (16) we find two generic integrals. One of them is

$$\int d\Omega[h_j(\Omega+\omega) - h_j(\Omega)] = 2\omega$$

and the other is

$$\int d\Omega\,[1 - h_j(\Omega+\omega)h_m(\Omega)] = 2F(\omega + \mu_m - \mu_j). \tag{17}$$

For the above form of $h_j(\Omega)$, we have $F(x) = x\,\coth(x/2T)$.

As mentioned above there are no corrections to the incoming currents. In addition to this observation we should recall Kirchhoff's law, stating the conservation of charge. Given that the total incoming current is equal to the total outgoing current, we should have $J_1 + J_2 = 0$, which is indeed confirmed by direct calculation.

Taking these facts into account, only the difference of the currents, $J^{(1)} = \frac{1}{2}(J^{(1)}_2 - J^{(1)}_1)$, is of interest. This involves the difference of the components of T belonging to different leads. Accordingly, for the case of two leads, we define the weighted difference (denoted by the same symbol, T, but dependent on fewer variables),

$$\begin{aligned} T^{\mu\nu}_\omega(m_\eta,y|l_\eta,x) = \tfrac{1}{2}\big[T^{\mu\nu}_\omega(m_\eta,y|l_\eta,x;1,+,z>L) \\ - T^{\mu\nu}_\omega(m_\eta,y|l_\eta,x;2,+,z>L)\big]. \end{aligned} \tag{18}$$

The 4×4 matrices appearing here are now direct products of 2×2 matrices in chiral space (outer block structure)

TABLE I. Convention for the indices.

$\alpha =$	1	2	3	4
Before/after	B		A	
Wire No.	1	2	1	2

and 2×2 matrices in lead space (inner block structure; see Table I):

$$T^{11} = \frac{r^2t^2}{8\pi}[F(\omega+V)-F(\omega-V)] \times \Phi_\omega(y)\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\end{bmatrix}\Phi^*_\omega(x),$$
$$T^{22} = -(T^{11})^\dagger|_{x\leftrightarrow y}, \quad T^{21}=0, \tag{19}$$
$$\Phi_\omega(x) = \mathrm{diag}\left[\frac{e^{-i\omega x/v_1}}{v_1},\frac{e^{-i\omega x/v_2}}{v_2},\frac{e^{i\omega x/v_1}}{v_1},\frac{e^{i\omega x/v_2}}{v_2}\right].$$

The vanishing of T^{21} implies that the Keldysh component of the renormalized interaction, L^K, does not enter. Inserting the components of $T^{\mu\nu}$ and L_0 into the expression (14) for the current for two equal wires, $g_j = g$, $v_j = v$, we find

$$J^{(1)} = -\frac{gt^2r^2}{\pi}\int_0^{\omega_c}\frac{d\omega}{\omega}[F(\omega+V)-F(\omega-V)]\sin^2\frac{\omega L}{v}.$$

Here we apply an upper cut-off ω_c given in the microscopic model either as $\omega_c = v/a$ as mentioned above or $\omega_c = W$, the bandwidth. The conductance as a function of voltage V, temperature T, wire length L, is found from there as

$$G^{(1)} = -2gG_0(1-G_0)\Lambda(V,T,L). \tag{20}$$

Here we introduced the scaling variable Λ,

$$\Lambda(V,T,L) = \int_0^{\omega_c}\frac{d\omega}{\omega}\frac{F(\omega+V)-F(\omega-V)}{V}\sin^2\frac{\omega L}{v}. \tag{21}$$

The factor $\sin^2(\omega L/v)]$ guarantees convergence of the integral at $\omega < 1/t_0 = \pi v/L$. At $\omega > 1/t_0$ we may average this rapidly oscillating function and approximate $\sin^2(\omega L/v)] \simeq 1/2$. Employing this and analogous procedures for the cases of small V, L^{-1} or small V,T we may approximate Λ as

$$\Lambda(V,T,L) \simeq \ln\left(\frac{\omega_c}{\max\{V,T,v/L\}}\right). \tag{22}$$

V. SCALE-DEPENDENT PART OF THE CURRENT: SUMMATION TO INFINITE ORDER IN THE INTERACTION

As shown in our previous work, the perturbative treatment may be extended into the strong-coupling regime by summing up an infinite series of relevant scale-dependent contributions to the conductance in all orders ("ladder approximation"). These represent a self-energy renormalization of the "boson propagator" L_0 introduced above. They thus technically constitute a renormalized one-loop contribution to the RG equation. This series can still be represented by the generic diagram of Fig. 2, but the wavy line of electronic interaction should be dressed by screening effects, as discussed below.

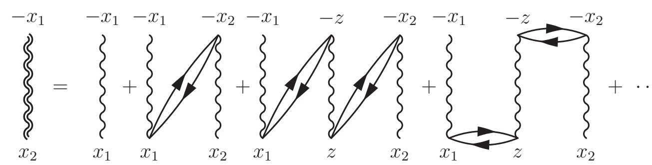

FIG. 3. A series of diagrams showing the screening. The negative sign of the coordinate corresponds to the incoming (B) electrons.

As a result of this summation the interaction line g_l acquires nonlocality and retardation effects. Moreover, if we have initially only diagonal components in Keldysh space, after the summation we generate a Keldysh component and in general a rather complicated structure. Schematically, we replace L_0 by L in Eq. (14):

$$\underline{L_0} \to \underline{L} = \begin{pmatrix} L^R_\omega(l_\eta,x|m_\eta,y) & L^K_\omega(l_\eta,x|m_\eta,y)\\ 0 & L^A_\omega(l_\eta,x|m_\eta,y)\end{pmatrix}. \tag{23}$$

We now embark on the calculation of $\underline{L}$. Introducing compact notation, we express the lowest order result $\underline{L_0}$ in the form $L_0^{\mu\nu}(l_\eta,x|m_\eta,y) = \delta_{\mu\nu}\tau^1_{\eta_l,\eta_m}\delta_{lm}\, g\delta(x-y) = \mathbf{1}_K\otimes\tau^1_C\otimes \mathbf{1}_w\, g\delta(x-y)$.

The integral equation describing the summation of the relevant infinite class of diagrams (see Fig. 3) takes the form

$$\begin{aligned}\underline{L} &= \underline{L_0} + \underline{L_0}*\underline{\Pi}*\underline{L_0} + \underline{L_0}*\underline{\Pi}*\underline{L_0}*\underline{\Pi}*\underline{L_0}+\cdots\\ &= \underline{L_0}+\underline{L_0}*\underline{\Pi}*\underline{L},\end{aligned} \tag{24}$$

where $\underline{\Pi}$ represents a fermion bubble

$$\Pi^{\mu\nu}_\omega(l_\eta,x|j_\eta,y) = i\int\frac{d\Omega}{2\pi}\mathrm{Tr}_K[\underline{\bar{\gamma}}^\mu \underline{G}_{\Omega+\omega}(l_\eta,x|j_\eta,y) \times \underline{\gamma}^\nu\underline{G}_\Omega(j_\eta,y|l_\eta,x)]. \tag{25}$$

The multiplication $*$ is defined as

$$(\Pi * L)^{\mu\nu}_\omega(j_\eta,y|n_\eta,x) = \int dz\sum_{l_\eta}\sum_{\lambda=1,2}\Pi^{\mu\lambda}_\omega(j_\eta,y|l_\eta,z) \times L^{\lambda\nu}_\omega(l_\eta,z|n_\eta,x).$$

At the level of Keldysh structure we have

$$\begin{pmatrix} L^R & L^K\\ 0 & L^A\end{pmatrix} = \begin{pmatrix} L_0 & 0\\ 0 & L_0\end{pmatrix} + \begin{pmatrix} L_0 & 0\\ 0 & L_0\end{pmatrix} * \begin{pmatrix}\Pi^R & \Pi^K\\ 0 & \Pi^A\end{pmatrix} * \begin{pmatrix} L^R & L^K\\ 0 & L^A\end{pmatrix},$$

which means that we can solve the integral equation in three steps.

First, we solve the coupled integral equations in the retarded sector

$$L^R = L_0 + L_0 * \Pi^R * L^R. \tag{26}$$

Second, considering that

$$L^A = L_0 + L_0 * \Pi^A * L^A,$$

if we are using the relation between Π^A and Π^R, we need not solve this equation separately. Third, we notice for completeness that

$$L^K = L_0 * \Pi^R * L^K + L_0 * \Pi^K * L^A$$

and hence,

$$\begin{aligned} L^K &= (1 - L_0 * \Pi^R)^{-1} * L_0 * \Pi^K * L^A \\ &= L^R * \Pi^K * L^A, \end{aligned}$$

where we used (26) to obtain the second equality. This means that once we have L^R, we can easily find the two remaining components, L^A and L^K. We recall, however, that as pointed out above the component L^K does not enter the calculation of the current.

The solution for L^R in the linear response case was presented previously in our work [26]. We follow that derivation but reformulate it somewhat for the present purposes. First we define the integral (scalar) kernel

$$\begin{aligned} P_\omega(j,x|l,z) &= (2\pi v_j v_l)^{-1}[\delta(\tau) + i\omega\theta(\tau)e^{i\omega\tau}], \\ \tau &= x/v_j - z/v_l \end{aligned} \tag{27}$$

and the matrix quantity

$$\widehat{\boldsymbol{\Pi}}^R = \begin{pmatrix} \boldsymbol{\Pi}(-x|-z), & \mathbf{Y}^T\Pi(-x|z) \\ \mathbf{Y}\Pi(x|-z), & \boldsymbol{\Pi}(x|z) \end{pmatrix}, \tag{28}$$

where $\Pi_{jl}(x|z) = \delta_{jl} P_\omega(j,x|l,z)$, $\mathbf{Y}\Pi(x|z) = Y_{jl} P_\omega(j,x|l,z)$, and $\mathbf{Y}^T\Pi(x|z) = Y_{lj} P_\omega(j,x|l,z)$ with $Y_{jl} = |S_{jl}|^2$. In the case of two leads we have $\mathbf{Y} = \mathbf{Y}^T$. Notice that $\mathbf{Y}^T\Pi(-x|z) = 0$ for $x,z > 0$, and we use the full form (28) for future reference.

Then we express the integral equation for $\mathbf{L}^R$ as a 2×2 matrix equation in the chiral space

$$\begin{aligned} \widehat{\mathbf{L}}^R(x|y) &= 2\pi\delta(x-y)\begin{pmatrix} 0 & \mathbf{g} \\ \mathbf{g} & 0 \end{pmatrix} \\ &\quad - 2\pi\int_a^L dz \begin{pmatrix} \mathbf{g}\mathbf{Y}\Pi(x|-z) & \mathbf{g}\boldsymbol{\Pi}(x|z) \\ \mathbf{g}\boldsymbol{\Pi}(-x|-z) & 0 \end{pmatrix} \widehat{\mathbf{L}}^R(z|y), \end{aligned} \tag{29}$$

Here $\widehat{\mathbf{L}}^R$ is a 4×4 (in the general case of n leads $2n \times 2n$) matrix. The elements of the 2×2 matrices in chiral space (denoted by a hat) are 2×2 matrices in the space of the two leads (denoted by bold letters). The matrix of interaction constants is then given by $\mathbf{g} = \text{diag}[g_1 v_1, g_2 v_2]$. The scattering properties of the junction are encoded in the 2×2 matrix $\mathbf{Y}$. The equation for $\mathbf{L}^A$ is similar to the above, but

$$\widehat{\boldsymbol{\Pi}}^A = (\widehat{\boldsymbol{\Pi}}^R)^\dagger\Big|_{x\leftrightarrow z} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \widehat{\boldsymbol{\Pi}}^R \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\Big|_{\omega\to-\omega,\ \mathbf{Y}\to\mathbf{Y}^T}.$$

Because L_0 does not contain ω, Y, it follows that

$$\widehat{\mathbf{L}}^A = (\widehat{\mathbf{L}}^R)^\dagger\big|_{x\leftrightarrow y}. \tag{30}$$

The Keldysh part of the kernel takes the form presented in Appendix C. We show there, that Π^K is an even function of V, and therefore L^K does not contribute to the current.

Following the method of solution of the integral equation described in [26] we first solve the equation for the case $Y_{jl} = 0$, to give a partial summation resulting in an auxiliary quantity $\mathbf{C}$,

$$\begin{aligned} \widehat{\mathbf{C}}(x|y) &= 2\pi\delta(x-y)\begin{pmatrix} 0 & \mathbf{g} \\ \mathbf{g} & 0 \end{pmatrix} \\ &\quad - 2\pi\int_a^L dz \begin{pmatrix} 0 & \mathbf{g}\boldsymbol{\Pi}(x|z) \\ \mathbf{g}\boldsymbol{\Pi}(-x|-z) & 0 \end{pmatrix} \widehat{\mathbf{C}}(z|y). \end{aligned} \tag{31}$$

In terms of $\widehat{\mathbf{C}}$ the integral equation for $\widehat{\mathbf{L}}^R$ may be expressed as

$$\begin{aligned} \widehat{\mathbf{L}}^R(x|y) &= \widehat{\mathbf{C}}(x|y) - 2\pi\int_a^L dz_1 dz_2 \widehat{\mathbf{C}}(x|z_1) \\ &\quad \times \begin{pmatrix} 0 & 0 \\ \mathbf{Y}\Pi(z_1|-z_2) & 0 \end{pmatrix} \widehat{\mathbf{L}}^R(z_2|y). \end{aligned} \tag{32}$$

The solution of the integral equation for $\widehat{\mathbf{C}}$, which is of the Wiener-Hopf type, may be found by an appropriate ansatz described in [26]. By construction, $\widehat{\mathbf{C}}(x|y)$ is diagonal in wire space, $\widehat{\mathbf{C}}_{jl}(x|y) = \delta_{jl}\widehat{C}_j(x|y)$. The explicit expressions for $\widehat{C}_j(x|y)$ are presented in Appendix D.

Returning to the integral equation (32) for $\widehat{\mathbf{L}}^R$ in terms of $\widehat{\mathbf{C}}$ we observe that its kernel is separable and thus the solution may be readily obtained. The explicit expressions and some details of the derivation of this result are given in Appendix D.

An inspection of Eqs. (14) and (19) shows that the x,y dependence of $T^{\mu\nu}$ comes only from the matrices $\Phi_\omega^*(x), \Phi_\omega(y)$. We combine these matrices with $\widehat{\mathbf{L}}^R$ and integrate over the position

$$\widehat{\mathbf{L}}^R_{\text{simple}} = \int_a^L dx\, dy\, \Phi_\omega^*(x)\widehat{\mathbf{L}}^R(x|y)\Phi_\omega(y). \tag{33}$$

Making use now of relations (19) and (30), we arrive at a much simpler algebraic expression for the current. Instead of (14) we have

$$J^{(L)} = -2\,\text{Im}\int \frac{d\omega}{2\pi}\,\text{Tr}\left[\widehat{T}^{11}_{\text{core}}\widehat{\mathbf{L}}^R_{\text{simple}}\right] \tag{34}$$

with $T^{\mu\nu}_{\text{core}}$ obtained by putting $\Phi_\omega(x) = \Phi_\omega(y) = \mathbf{1}$ in Eq. (19). We show the algebraic relation (34) diagrammatically in Fig. 4.

Introducing the quantities d_j and q_j,

$$\begin{aligned} d_j^2 &= 1 - g_j^2, \\ q_j^{-1} &= \frac{g_j}{1 + i d_j \cot\left(\frac{\omega L}{v_j d_j}\right)}, \end{aligned} \tag{35}$$

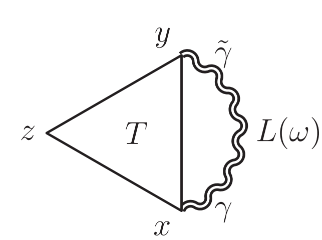

FIG. 4. The schematic diagram, with already algebraic quantity $T(\omega)$ and dressed interaction line $L(\omega)$.

we present the simpler expressions of $\widehat{C},\widehat{L}$ integrated over position.

$$\widehat{L}^R_{jk,\text{simple}} = \frac{2\pi i}{\omega}\left(\delta_{jk}\widehat{C}_{j,\text{simple}} + \Upsilon_{jk}\begin{bmatrix} V_{1,j}V_{2,k}, & V_{1,j}V_{1,k} \\ V_{2,j}V_{2,k}, & V_{2,j}V_{1,k}\end{bmatrix}\right), \tag{36}$$

where

$$\widehat{C}_{j,\text{simple}} = \left(\begin{bmatrix} -1, & 0 \\ 0, & -1 \end{bmatrix} + q_j^{-1}\begin{bmatrix} \frac{id_j e^{-iL\omega/v}}{g_j \sin(\omega L/v_j d_j)}, & 1 \\ e^{-2iL\omega/v}, & \frac{id_j e^{-iL\omega/v}}{g_j \sin(\omega L/v_j d_j)} \end{bmatrix}\right),$$

$$V_{1,j} = (\widehat{C}_{j,\text{simple}})_{12}, \quad V_{2,j} = (\widehat{C}_{j,\text{simple}})_{11},$$
$$\Upsilon_{jk} = Y_{jl}(1 - Q^{-1}Y)^{-1}_{lk}, \quad Q^{-1}_{jk} = \delta_{jk}q_j^{-1}. \tag{37}$$

Combining the above results (19), (34), (36), and (37) we find the current for two equal wires, $g_j = g$, $v_j = v$, as

$$J^{(L)} = \frac{g}{8\pi}\int_0^{\omega_c} d\omega \frac{F(\omega+V) - F(\omega-V)}{\omega} \times \text{Re}\left\{\frac{2(1-Y^2)}{1 - gY + id\cot\left(\frac{L\omega}{dv}\right)}\right\}. \tag{38}$$

Again the $\frac{1}{\omega}$ singularity of the integrand leads to a logarithmically divergent contribution, which we identify as a scaling contribution. The singularity is controlled by the largest of the three energy scales: (i) energy scale $\omega_L = v/L$ controlled by the length L, (ii) temperature $\omega_T = T$, and (iii) bias voltage $\omega_V = V$. In the limit $V \to 0, T \to 0$ we have $F(\omega + V) - F(\omega - V) = 2V\,\text{sgn}(\omega)$. The $\frac{1}{\omega}$ singularity is in this case cut off at the scale $\omega_L = dv/L$ by the $\cot(\frac{L\omega}{dv})$ term in the denominator. Above this scale we may average the rapidly oscillating function in the curly brackets in (38) over one oscillation period, $\omega_0 < \omega < \omega_0 + (\pi/t_0)$, with $t_0 = L/dv$:

$$\frac{t_0}{\pi}\int_{\omega_1}^{\omega_1+\pi/t_0} \frac{d\omega}{1 - gY \pm id\cot\omega t_0} = (1 - gY + d)^{-1}, \tag{39}$$

such that the correction to the conductance is obtained as

$$G^{(L)} = -g\frac{(1-Y^2)}{1-gY+d}\ln\frac{L}{a}$$

in agreement with [22]. In the general case we find accordingly

$$G^{(L)} = -g\frac{(1-Y^2)}{1-gY+d}\Lambda, \tag{40}$$

where $\Lambda = \ln(\omega_c/\max\{V,T,v/L\}$. In the limit of long wires, $L \to \infty$, a closed expression is found in Appendix F in the form

$$\Lambda = \ln\left(\frac{\omega_c}{2\pi T}\right) - \text{Re}\left[\psi\left(1 + \frac{iV}{2\pi T}\right)\right], \tag{41}$$

with $\psi(x)$ digamma function. This function shows a smooth interpolation between the regimes with $\ln(\frac{\omega_c}{2\pi T}) + 0.577$ at $V \ll T$ and $\ln(\omega_c/V)$ at $V \gg T$.

Further corrections not considered here are generated by the Hartree diagrams of the self-energy: In the nonequilibrium situation the local chemical potential is renormalized by a molecular field term involving the bare interaction and the local particle density. In Ref. [13] this effect is included by applying a corresponding boundary condition to the thermodynamic Bethe ansatz fields. As a consequence a bistability of the current has been found at very strong interaction. We have not included this correction term into our analysis, as it would require a separate calculation of the single particle Green's function, especially of the local chemical potential shift, which is beyond the scope of the present paper. We are therefore confining our considerations from weak up to moderately strong interaction, such that $K > 0.2$.

VI. RENORMALIZATION GROUP EQUATION FOR THE CONDUCTANCE

The above calculation of the leading scale-dependent contribution to the current allows us to derive a RG equation for the conductance $G = I/V$ as a function of the scaling variable $\Lambda = \ln(\omega_c/\max[V,T,v/L])$, $G = G(\Lambda)$. We thereby use the scaling property of G, $G(V,T,v/L,G_0;g) = G(\Lambda;g)$. In our previous works [22,24] we explicitly checked this property in the equilibrium situation. We directly calculated all the contributions to the conductance up to third order in the interaction, which involves about 10^4 diagrams. It was shown that the principal contribution near the fixed points (FPs) of this equation is obtained in one-loop order, with the interaction being dressed as described above, $g \to L$. The scaling exponents obtained this way are identical to those found earlier by the method of bosonization.

Away from the FPs one finds, in general, additional nonuniversal contributions, appearing first in the third order. These determine the prefactor in the scaling law near the FP and also fine details of the conductance in the intermediate regime. In the present study focused on the transport far out of equilibrium it would be too costly to perform a similar direct computation of all contributions up to third order. Instead we assume that even out of equilibrium we have the scaling property and the scaling exponents are fully determined by the contribution provided by the approximation of fully dressing the interaction line of the one-loop calculation.

This assumption may be justified by at least two facts. First, in the renormalized one-loop calculation presented here the bias voltage V appears as an infrared cutoff in the scale-dependent terms, replacing the cutoff energy scales temperature T or level splitting v/L present in equilibrium. It may therefore be expected that the structure of the scale-dependent terms generated by the cutoff V is analogous to that of the terms generated by the cutoff T or v/L. No additional scale-dependent terms are found in nonequilibrium and none of the scale-dependent terms present in equilibrium disappears in nonequilibrium. This suggests that the structure of the scale-dependent terms is preserved and therefore the scaling property is preserved even out of equilibrium. Secondly, as will be shown below, the results of our theory are in agreement with exact results obtained by other methods.

We now briefly review the logic by which the RG equation is derived from the perturbative result. We start from the result for the renormalized conductance G as a power series expansion in the interaction, and dependent on the scattering properties of the junction (encapsulated in the conductance G_0 in the absence of interaction) obtained above, which takes the general form

$$G = G_0 - gf(g,G_0)\Lambda + O(g^2\Lambda^2). \quad (42)$$

In the approximation of summing up the leading terms in each order, considered above, a very good approximation $f_{\rm app}$ of the function $f(g,G)$ has been obtained [see Eq. (40)]. We do not calculate the terms of order $g^2\Lambda^2$ and higher in this paper. The relation, Eq. (42), is valid in the asymptotic regime $g\Lambda \to 0$. With the aid of the scaling property we may find the analytic continuation to finite values of $g\Lambda$. To this end we first invert the relation [making use of $G = G_0 + O(g\Lambda)$] and write

$$G_0(g,G;\Lambda) = G + gf(g,G)\Lambda + O(g^2\Lambda^2).$$

Formally G_0 here is a function of G,g and Λ. We now employ the crucial property that the value of the bare conductance, G_0, should not depend on the scaling variable Λ, which means

$$0 = \frac{\partial G_0}{\partial \Lambda} + \frac{\partial G_0}{\partial G}\frac{dG}{d\Lambda} \quad (43)$$

and hence

$$\frac{dG}{d\Lambda} = -\frac{gf(g,G) + O(g^2\Lambda)}{1 + g\Lambda[\partial f(g,G)/\partial G] + O(g^2\Lambda^2)}. \quad (44)$$

The scaling property of G implies that the explicit Λ dependence in (44) cancels. This leads to the definition of the RG β function

$$\frac{dG}{d\Lambda} = \beta(g,G) = -gf(g,G). \quad (45)$$

Our earlier direct third-order calculation in [22,24] showed that the above ratio was indeed independent of Λ to the considered accuracy g^3. The function $gf(g,G)$ has been calculated beyond the ladder approximation in [22] for the present case of a two-lead junction with the result

$$\frac{d}{d\Lambda}G = -gf_{\rm app}(g,G) + c_3 g^3 G^2(1-G)^2 + O(g^4). \quad (46)$$

The second term here of order g^3originates from terms not contained in the perturbation series for L considered above. This term is subleading in the sense that it vanishes more rapidly on approach to the fixed points at $G = 0,1$ than the first term and therefore does not influence the critical properties. There are indications that this is also the case with the higher order contributions not captured by the ladder summation.

A similar conclusion regarding the relative unimportance of corrections beyond the ladder summation $g \to L$ was reached in [24] for the more general case of the three-lead Y junction. In the symmetrical setup the Y junction was characterized by two conductances, and after extensive computer analysis of perturbative corrections we arrived at a set of two coupled RG equations. We found that the three-loop corrections, not contained in the ladder series of diagrams, did not contribute to the scaling exponents.

We expect that nonuniversal contributions to the β function will also exist in the case of nonequilibrium, but those terms will again be unimportant when it comes to determining the critical behavior at the fixed points. We will therefore approximate the exact function $f(g,G)$ by the one determined in the ladder approximation and given through Eq. (42), which gives rise to the β function

$$\frac{d}{d\Lambda}G = -4g\frac{G(1-G)}{1-g(1-2G)+d}. \quad (47)$$

Introducing the Luttinger parameter $K = \sqrt{(1-g)/(1+g)}$, Eq. (47) may be reexpressed as

$$\frac{d}{d\Lambda}G = -2(1-K)\frac{G(1-G)}{K+(1-K)G}, \quad (48)$$

which is explicitly solved in the next section.

VII. SOLUTION OF RG EQUATION

Inverting Eq. (48) we write

$$2(1-K)d\Lambda = -dG\left(\frac{K}{G} + \frac{1}{1-G}\right), \quad (49)$$

which is integrated with the result

$$\frac{1-G}{G^K} = \frac{1-G_0}{G_0^K}e^{2(1-K)\Lambda}. \quad (50)$$

It is more instructive to exclude here the bare conductance G_0 and to represent our result as (cf. [3])

$$\frac{(1-G)/G^K|_{V_1,T_1}}{(1-G)/G^K|_{V_2,T_2}} = \frac{e^{2(1-K)\Lambda(V_1,T_1)}}{e^{2(1-K)\Lambda(V_2,T_2)}}. \quad (51)$$

The latter exponential can be written as

$$e^{2(1-K)\Lambda} = \left(\frac{\omega_c}{\max\{V,T,v/L\}}\right)^{2(1-K)}.$$

We see that near the two fixed points of the RG equation, $G = 0$, $G = 1$, we have the well-established scaling behavior [5]

$$\begin{aligned} G &\simeq (V/\omega_0)^{2(K^{-1}-1)}, \quad G \to 0, \\ 1-G &\simeq c_*\,(\omega_0/V)^{2(1-K)}, \quad G \to 1, \end{aligned} \quad (52)$$

with an appropriate ω_0 and where V should be replaced by $\exp[\Lambda(V,T,v_F/l)]$ in the more general situation. At the same time, (50) provides a smooth crossover between the fixed points, i.e., for those values of G which, strictly speaking, are inaccessible by the bosonization approach.

We notice further, that if the overall energy scale ω_0 is fixed near one fixed point, then the constant c_* is entirely defined by the three-loop and higher-loop terms in the RG equation. In the approximation of neglecting the three-loop terms, as in Eqs. (48) and (41), the coefficient $c_* = 1$. Keeping the three-loop terms, Eq. (50) is approximately given by [22]

$$\frac{G^K}{1-G}\left(1+G\frac{1-K}{K}\right)^{4c_3(1-K)} = \left[\frac{\max\{V,T,v/L\}}{\omega_0}\right]^{2(1-K)}, \quad (53)$$

which implies $c_* = K^{-4c_3(1-K)}$.

A. Comparison with the exact solution at $K = 1/2$

To better understand the limitations of our formula (50), we compare it with the exact result at $K = 1/2$. Explicitly, our expression in this case reads as

$$G = 1 - \frac{\sqrt{1+4x^2}-1}{2x^2}, \quad x = \frac{T}{T^*} \exp \mathrm{Re}\, \psi \left[1 + \frac{iV}{2\pi T}\right]. \tag{54}$$

The exact formula, obtained with the aid of the Bethe ansatz [29], is

$$G_{1/2} = 1 - \frac{4\pi T^*}{V} \mathrm{Im}\, \psi \left[\frac{1}{2} + \frac{T^*}{T} + i\frac{V}{4\pi T}\right], \tag{55}$$

with T^* depending on the impurity backscattering amplitude and the ultraviolet cutoff. In two important limiting cases we have for the linear conductance

$$\begin{aligned} G_{1/2}(T) &= G_{1/2}(V \to 0, T), \\ &= 1 - \frac{T^*}{T}\psi'\Big(\frac{1}{2} + \frac{T^*}{T}\Big), \\ &\simeq \frac{1}{12}\Big(\frac{T}{T^*}\Big)^2, \quad T \ll T^*, \\ &\simeq 1 - \frac{\pi^2}{2}\frac{T^*}{T}, \quad T \gg T^*. \end{aligned} \tag{56}$$

And for the nonlinear conductance

$$\begin{aligned} G_{1/2}(V) &= G_{1/2}(V, T \to 0), \\ &= 1 - \frac{4\pi T^*}{V} \arctan \frac{V}{4\pi T^*}, \\ &\simeq \frac{1}{12}\left(\frac{V}{2\pi T^*}\right)^2, \quad V \ll T^*, \\ &\simeq 1 - \pi\frac{2\pi T^*}{V}, \quad V \gg T^*. \end{aligned} \tag{57}$$

These expressions indicate the existence of nonuniversal three-loop terms in the RG β function. Indeed, fixing the overall scale at small T by $G_{1/2}(T) = (T/T_1^*)^2$ with $T_1^* = \sqrt{12}T^*$ gives the above constant $c_* = \pi^2/(4\sqrt{3}) \simeq 1.424$. At the same time, fixing the scale at small V by $G_{1/2}(V) = (V/T_2^*)^2$ with $T_2^* = 2\pi\sqrt{12}T^*$ produces $c_* = \pi/(2\sqrt{3}) \simeq 0.91$.

This means, first, that the three-loop term $\sim c_3 g^3 G^2(1-G)^2$ in the RG equation (46) has a *different* prefactor c_3, depending on whether the choice of low-energy cutoff is T or V. This fact was noted in [22] on the basis of direct computation of perturbative corrections. From the above estimate $c_* = K^{-4c_3(1-K)}$ we retrieve $c_3 \simeq 0.255$ and $c_3 \simeq -0.070$ for $G_{1/2}(T)$ and $G_{1/2}(V)$, respectively.

Secondly, in the absence of three-loop RG terms ($c_* = 1$) the ratio $G^K/(1-G)$, appearing in (51), should be a linear function of V,T at $K = 1/2$. Plotting this ratio for the functions (56) and (57), we compare it with the straight line corresponding to Eq. (54). We confirm much better agreement with the straight line in the case of the nonlinear conductance $G_{1/2}(V, T \to 0)$ (see Fig. 5).

In practical terms these observations mean the following. When fitting experimental data with one universal curve for the whole range of conductances, one should use slightly different expressions for $G(V=0,T)$ and $G(V,T=0)$. The generic formula is (53), where the value $c_3 = 1/4$ is appropriate for $G(V=0,T)$, while $c_3 \simeq -0.07$ is better suited for $G(V,T=0)$.

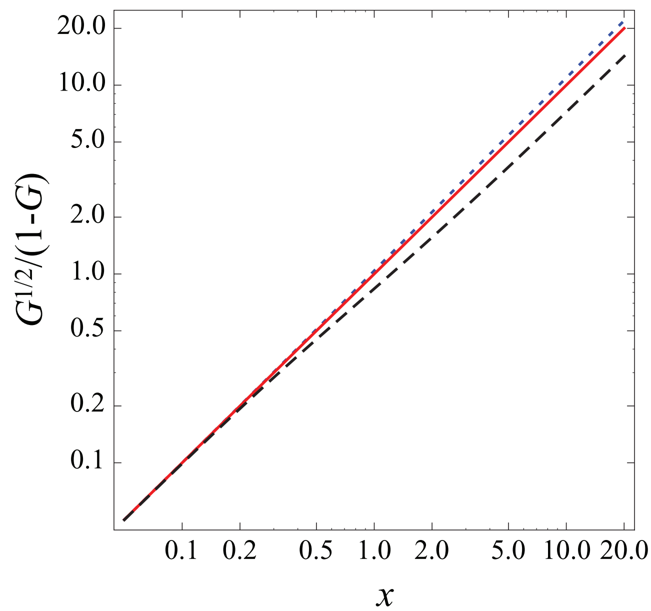

FIG. 5. (Color online) Comparison of the ratios $\sqrt{G}/(1-G)$ for (i) $G = G_{1/2}(T = xT_1^*)$, Eq. (56) (black dashed line); (ii) $G = G_{1/2}(V = xT_2^*)$, Eq. (57) (blue dotted line); and the linear dependence, $\sqrt{G}/(1-G) = x$ (red solid line) expected for the expression (54). See text for additional explanations.

B. Oscillatory nonlinear conductance

Let us also discuss the case $T = 0$ and VL finite. The expression (21) is reduced in this case to

$$\begin{aligned} \Lambda &= \ln\Big(\frac{\omega_c e}{V}\Big) + f(2VL/v), \\ f(x) &= \mathrm{Ci}(x) - \sin x/x, \\ &\simeq -\cos x/x^2, \quad x \gg 1, \\ &\simeq \gamma_E - 1 + \ln x, \quad x \ll 1, \end{aligned} \tag{58}$$

with $\gamma_E \simeq 0.577\ldots$. We see the appearance of oscillations in $2VL$, discussed in [11] for the case of weak impurity. In our treatment it corresponds to $G \simeq 1$, and from (50) may represent the conductance as follows:

$$G = 1 - c_*(\omega_0/V)^{2(1-K)} \exp[2(1-K)f(2VL/v)], \tag{59}$$

cf. (52). In the limit of large $2VL/v$ we have $|f(2VL/v)| \ll 1$ and

$$\exp[(1-K)f(2VL/v)] \simeq 1 + 2(1-K)f(2VL/v). \tag{60}$$

This is in agreement with [11] where the corresponding expression in this limit and in our notation reads as

$$1 + f_{\mathrm{BS}}^{\mathrm{osc}} \simeq 1 - 2(1-K)\frac{\cos(2VL/v)}{(2VL/v)^2}.$$

VIII. CONCLUSION

Electron transport through one-dimensional quantum wires of various types has been studied experimentally in several recent works. In a typical setup stationary charge transport is measured in a two-point geometry of a system of one or several wires connected by a junction. The quantum wires are adiabatically connected to reservoirs kept at a fixed chemical potential and temperature. These systems are described by modeling the quantum wires as Luttinger liquids (spinless, or spinful) of fermions with linear dispersion subject to pointlike interaction and treating the reservoirs as noninteracting. A useful picture of the transport process is to think of individual electrons entering the interaction region (quantum wires plus junction) from an initial reservoir and leaving as individual electrons into the final reservoir. If we model the reservoirs as noninteracting systems there is no room for collective excitations such as fractional quasiparticles or multiple quasiparticles in the final state.

Conventionally this problem has been addressed by the bosonization method, which takes advantage of the fact that the exact excitations of a clean Luttinger wire are bosons, at least in the infinite wire. The problem of including the transformation of incoming electrons into bosons has been addressed for the clean wire, and is believed to be solved. For the case of semiwires connected by a junction there is no convincing calculation of the above transformation available. In order to avoid this difficulty we are using a fermionic representation.

Our approach starts with determining the leading scale-dependent contributions to the conductances in all orders of perturbation theory. We have demonstrated in the linear response case that by summing up these terms one arrives at a description of the critical properties near the fixed points (i.e., the location of the FPs and the critical exponents describing the power laws followed by the conductances). For this it is necessary to establish the scaling property of the conductances (or else to assume its validity, which is usually done), allowing one to derive a set of renormalization group equations out of the perturbative result.

In the present paper we followed this approach for the case of stationary nonequilibrium transport. We first derived a general result for the scale-dependent terms in the conductances of an n-lead junction in first order of the interaction. Then we presented the infinite order summation for the dressed interaction. At this point we specialized our considerations to the case of two symmetric semiwires. We derived the corresponding RG equation for the conductance. In general the scaling is dependent on three energy scales: bias voltage V, temperature T, and infrared cutoff provided by the wire length v/L. Whenever one of these energy scales dominates, the scaling variable is varying logarithmically, $\Lambda = \ln(\frac{v/a}{\max\{V,T,v/L\}})$. In the case $V,T \gg v/L$ we were able to determine the form of the scaling variable describing the crossover from the regime characterized by $V \gg T$ to $V \ll T$, as well.

The intermediate results presented for the general case of an n-lead junction should be a good starting point for analyzing the behavior of nonequilibrium transport through Y junctions or even four-lead junctions. Work in this direction is in progress.

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APPENDIX A: NORMALIZATION OF WAVE FUNCTIONS

The usual summation over the quantum states in the infinite medium is done as an integration over the momentum $\int dk/(2\pi)$ or the summation $\sum_n$ over the quasimomentum $k = 2\pi n/L$ in a ring geometry with a finite length L. In our situation with a broken translational symmetry we should resort to the integration over the energy, then the correct normalization factor is given by the density of states, which is the inverse Fermi velocity in the simplest situation, $\sum_n \to \int dE/v_F$. In the case with several Fermi velocities, v_j, in different wires we shall keep the integration over the energy, and the normalization factor enters the definition of the wave functions.

Thus in the formula for the retarded Green's function,

$$\widehat{G}^R_E(l,y|j,x) = \int dE \frac{\phi_{E,l}(y)\phi^*_{E,j}(x)}{\omega - E + i0}, \tag{A1}$$

we adopt the wave functions in the jth wire in the form

$$\phi_{E,j}(y) = e^{iEy/v_j}/\sqrt{2\pi v_j} \tag{A2}$$

and come to the formulas (5). Notice also that the integration in (A1) should be restricted by the electronic bandwidth $|E| < W = E_F$, which can be modeled by introducing the density of states function, $N_j(E)$, with the property $N_j(0) = v_j^{-1}$. So strictly speaking the formulas (5) are defined at $|\omega| \ll W$, which justifies the upper cutoff in energy in the calculation of logarithmic corrections and the RG procedure.

APPENDIX B: KELDYSH STRUCTURE OF THE TRIANGLE *T*

The straightforward calculation shows that only a few terms in the complicated expression for T contribute to the final result. Let us sketch here the derivation and present arguments showing the selection of the relevant terms.

To condense our writing, we use the position-dependent notation$G^R_\omega \to R$, and position in the product denotes the position in the initial expression, (16), so that $G^R_\Omega(z|x)G^L_{\omega+\Omega}(x|y)G^A_\Omega(y|z) \leftrightarrow RLA$, etc. Up to a numerical factor we have

$$\begin{aligned} T_{11} &= RAA + (K + A)(RK - RR + KA), \\ T_{22} &= RRA + (RK + KA + AA)(K - R), \\ T_{21} &= RKA + KAA + RRK - RRR + AAA. \end{aligned} \tag{B1}$$

The combinations RRR and AAA are necessarily zero for the point z outside the interacting region. We may suggest (and it is confirmed by the direct calculation), that the contributing terms in (B1) are those which contain two Keldysh components, K. In this sense, we may keep only the terms

$$\begin{aligned} T_{11} &\simeq KRK + KKA, \\ T_{22} &\simeq RKK + KAK, \quad T_{21} \simeq 0. \end{aligned} \tag{B2}$$

Note that the notation "$\simeq$" here also means that the combination KK should be *regularized* at $\Omega \to \pm\infty$ by subtracting 1 from the product of distribution factors $h_j(\omega+\Omega)h_l(\Omega)$. This regularization is suggested by inspection of the corresponding expressions in the direct calculation. A closer inspection shows that the combinations $h_j(\Omega)h_l(\Omega)$ not containing ω do not contribute to the corrections, when multiplied by $L(\omega)$.

Thus the expressions for T_{ij} can be simplified even further:

$$T_{11} \simeq KKA, \quad T_{22} \simeq RKK, \quad T_{21} \simeq 0. \tag{B3}$$

The last expression means that the corrections to the *incoming* current are absent, because $G^A_\Omega(y|z) = 0$ and $G^R_\Omega(z|x) = 0$ in this case, due to the step functions in (5).

APPENDIX C: KELDYSH KERNEL OF INTEGRAL EQUATION

For completeness and future reference we provide here the explicit form of Keldysh kernel, appearing in Sec. V.

$$\Pi^K = \frac{i}{2\pi}\Phi^*_\omega(x)\begin{pmatrix} \mathbf{1}F(\omega) & \mathbf{Y}F(\omega) \\ \mathbf{Y}F(\omega) & \mathbf{K}(\omega)\end{pmatrix}\Phi_\omega(y) \tag{C1}$$

with

$$K_{jl}(\omega) = \int_{-\infty}^{\infty}\frac{d\Omega}{2}\sum_{m,n} S^*_{jm}S_{lm}S^*_{ln}S_{jn}\left[1 - h_m(\Omega)h_n(\Omega+\omega)\right].$$

The latter quantity may be cast in the form

$$\begin{aligned}\mathbf{K}(\omega) &= F(\omega)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + r^2t^2F_2(\omega,V)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},\\ F_2(\omega,V) &= F(\omega+V) + F(\omega-V) - 2F(\omega)\end{aligned} \tag{C2}$$

with $F(\omega)$ defined in (17). Importantly, $\mathbf{K}(\omega)$ is an even function of ω.

APPENDIX D: FULL FORM OF THE SOLUTION FOR $\widehat{L}^R_{jk}(x|y)$

The solution of (31) can be found as follows. We iterate the right-hand side of the equation once, to arrive at the diagonal kernel with components of the form

$$\frac{g^2}{v}\left[\delta\left(\frac{x-z}{v}\right) + i\frac{\omega}{2}\left(e^{i\omega(x+z)/v} + e^{i\omega|x-z|/v}\right)\right]$$

and another component obtained from here by changing $x \to L-x$, $y \to L-y$. We pick first the easier part of this iterated kernel, $\propto \delta(x-z)$, and arrive at the equation for $\widehat{C}_j(x|y)$ with more complicated inhomogeneity instead of L_0 and nonsingular kernel. This latter kernel shows a jump in its derivative at $x=z$, which we use by twice differentiating $\widehat{C}_j(x|y)$ with respect to x. We thus arrive at a second-order differential equation, similarly to what was done in [26]. The difference now is that we deal with a 2×2 matrix for $\widehat{C}_j(x|y)$ for each wire j. We determine the solution to this differential equation dependent on the x variable up to terms proportional to $e^{\pm i\omega x/(v_jd_j)}$ which are multiplied by as yet unknown matrices $\hat{A}(y)$, $\hat{B}(y)$, respectively. Considering the initial Eq. (31) for $\widehat{C}_j(x|y)$ in the simpler cases $x=0, x=L$, we form a set of two coupled (matrix) equations for $\hat{A}(y)$, $\hat{B}(y)$, which is eventually solved. As a result we obtain the quantity $\widehat{\mathbf{C}}$ diagonal in wire space, with its diagonal elements $\widehat{C}_j$ of the form

$$\begin{aligned}\widehat{C}_j(x|y) &= \frac{2\pi v_jg_j}{d_j^2}\delta(x-y)\begin{bmatrix}-g_j, & 1\\ 1, & -g_j\end{bmatrix} + \frac{i\pi\omega g_j^2}{d_j^3}\\ &\times e^{i\omega|x-y|/v_jd_j}\begin{bmatrix}d_j\mathrm{sgn}(y-x)-1, & g_j\\ g_j, & d_j\mathrm{sgn}(x-y)-1\end{bmatrix}\\ &+ \frac{i\pi\omega g_j^2}{d_j^4q_j}\left[e^{i\omega x/v_jd_j}\hat{A}_j(y) + \frac{d_je^{i\omega(2L-x)/v_jd_j}}{\sin\left(\frac{\omega L}{v_jd_j}\right)}\hat{B}_j(y)\right]\end{aligned} \tag{D1}$$

with d_j, q_j defined in (35) and

$$\begin{aligned}\hat{A}_j(y) &= \begin{bmatrix} g_j(q_j^{-1}-g_j), & g_j(1-g_jq_j^{-1})\\ (d_j-1)(q_j^{-1}-g_j), & (d_j-1)(1-g_jq_j^{-1})\end{bmatrix}\\ &\quad\times\cos\left(\frac{\omega y}{v_jd_j}\right) + id_j\sin\left(\frac{\omega y}{v_jd_j}\right)\begin{bmatrix}-q_j^{-1}, & 1\\ -q_j^{-1}, & 1\end{bmatrix},\\ \hat{B}_j(y) &= i\cos\left(\frac{\omega y}{v_jd_j}\right)\begin{bmatrix}\frac{(d_j-1)}{g_j}, & (1-d_j)\\ 1, & -g_j\end{bmatrix}\\ &\quad + d_j\sin\left(\frac{\omega y}{v_jd_j}\right)\begin{bmatrix}\frac{(d_j-1)}{g_j}, & 0\\ 1, & 0\end{bmatrix}.\end{aligned} \tag{D2}$$

We next use these expressions in Eq. (32), which can be schematically represented as

$$\begin{aligned}L &= C + C*Y*L = C + C*\Upsilon*C,\\ \Upsilon &= Y + Y*C*Y + \cdots = Y*(1-C*Y)^{-1}\end{aligned} \tag{D3}$$

and obtain finally

$$\begin{aligned}\widehat{L}^R_{jk}(x|y) &= \delta_{jk}\widehat{C}_j(x|y) - i\omega\frac{2\pi g_jg_k}{d_j^2d_k^2}\Upsilon_{jk}\\ &\quad\times\begin{pmatrix}V_{1,j}(x)V_{2,k}(y) & V_{1,j}(x)V_{1,k}(y)\\ V_{2,j}(x)V_{2,k}(y) & V_{2,j}(x)V_{1,k}(y)\end{pmatrix},\\ V_{1,j}(x) &= \left(1-g_jq_j^{-1}\right)\cos\left(\frac{\omega y}{v_jd_j}\right) + id_j\sin\left(\frac{\omega y}{v_jd_j}\right),\\ V_{2,j}(x) &= (q_j^{-1}-g_j)\cos\left(\frac{\omega y}{v_jd_j}\right) - id_jq_j^{-1}\sin\left(\frac{\omega y}{v_jd_j}\right)\end{aligned} \tag{D4}$$

with Υ given in Eq. (37).

APPENDIX E: ANALYTIC PROPERTIES OF $L(\omega)$

In contrast to our previous studies [22,26], we see now the appearance of poles in the ω plane of the quantities q_j, Eq. (35), which we further integrate over ω. Given the arbitrariness of the S matrix, reflected in $Y_{ik} = |S_{ik}|^2$, we check here the absence of singularities in $L^R(\omega)$ in the upper semiplane of complex ω.

The poles of q_j correspond to the solution of

$$\tan\bar{\omega} = -id_j,$$

where we introduced $\bar{\omega} = \omega L/v_j d_j$. The last equation means that we have an infinite sequence of roots

$$\bar{\omega} = -i \,\text{arctanh}\, d_j + \pi n, \quad n = 0, \pm 1, \pm 2, \dots,$$

hence the poles of q_j^{-1} are always in the lower semiplane of complex ω, as it should be for a retarded function.

Less trivial is the question about the position of the poles of the above expression $(1 - \mathbf{Q}^{-1} \cdot \mathbf{Y})^{-1}$. We consider it for a simpler situation with identical wires, $g_j = g$, $d_j = d = \sqrt{1-g^2}$, $v_j = v$.

The poles are defined by

$$\det(1 - \mathbf{Q}^{-1} \cdot \mathbf{Y}) = \det\left[\mathbf{1} - \frac{g\mathbf{Y}}{1 + id \cot\bar{\omega}}\right] = 0.$$

Since the denominator in the last expression cannot modify the location of poles, and $\sin\bar{\omega} = 0$ is not a solution, we can rewrite

$$\det[id \cot\bar{\omega} + 1 - g\mathbf{Y}] = 0.$$

Defining the eigenvalues of $\mathbf{Y}$ as y_j, with $j = 1,\dots,N$ (for a junction connecting N wires), the poles are defined by conditions (cf. above) $\tan\bar{\omega} = -i\frac{d}{1-gy_j}$, or

$$\bar{\omega} = -i \,\text{arctanh} \frac{d}{1-gy_j} + \pi n, \quad j = 1,\dots,N.$$

Generally we have $|y_j| \leqslant 1$ for all j. This is evident for $N = 2$, and can be easily extended to any N. The proof of this statement is as follows [24]. Consider a set of diagonal $N \times N$ matrices λ_j with values 1 in the jth row and 0 otherwise. This is a set of N generators of a Cartan subalgebra of the algebra $U(N)$, normalized according to $\text{Tr}(\lambda_j\lambda_k) = \delta_{jk}$. A rotation of these generators is defined as $\tilde{\lambda}_j = S^\dagger\lambda_j S$ where S is the unitary matrix. Obviously the new set $\tilde{\lambda}_j$ remains orthonormal $\text{Tr}(\tilde{\lambda}_j\tilde{\lambda}_k) = \delta_{jk}$. The operator P, defined by $PA = \sum_j \tilde{\lambda}_j \text{Tr}(\tilde{\lambda}_j A)$, is a projection operator, $P^2 = P$. We have $P\lambda_k = \sum_j \tilde{\lambda}_j Y_{jk}$ because $\mathbf{Y}$ can be written as $Y_{jk} = |S_{jk}|^2 = \text{Tr}(\tilde{\lambda}_j\lambda_k)$. Let $\{c_j\}$ be an eigenvector of $\mathbf{Y}$, i.e., $\sum_k Y_{jk}c_k = yc_j$. Introducing the diagonal matrix $\lambda_* = \sum_j c_j\lambda_j$, we obtain $P\lambda_* = y\tilde{\lambda}_*$, with $\tilde{\lambda}_*$ the rotated vector λ_*. Since $\|P\lambda_*\| \leqslant \|\lambda_*\|$ and $\|\tilde{\lambda}_*\| = \|\lambda_*\|$, we conclude that $|y| \leqslant 1$.

It follows that the above ratio $\frac{d}{1-gy_j}$ is always positive. As a result, the poles of $(1 - \mathbf{Q}^{-1} \cdot \mathbf{Y})^{-1}$ lie in the lower semiplane of ω.

APPENDIX F: SCALING VARIABLE Λ IN THE CROSSOVER REGIME

In the limit $L \to \infty$ we have to evaluate the integral

$$P(T,V) = \int_{-W}^{W} \frac{d\omega}{\omega}[F(\omega + V) - F(\omega - V)], \quad \text{(F1)}$$

with the upper cutoff being $W = v_F/a$ or the bandwidth.

After the rescaling $\omega = 2Tx$, $V = 2Ty$, $W = 2Tw$ we have

$$P(T,V) = 2T \int_{-w}^{w} \frac{dx}{x}\left[\frac{x+y}{\tanh(x+y)} - \frac{x-y}{\tanh(x-y)}\right].$$

We can make a shift $x \to x \pm y$ in two terms here; the contributions from the limits $\pm w$ do not cancel upon this shift, but add a constant in the limit $w \to \infty$. After simple calculation we get in this limit

$$P(T,V) = 4Ty\left(2 + \int_{-w}^{w} \frac{dx\, x \coth x}{x^2 - y^2}\right), \quad \text{(F2)}$$

where the principal value of the integral should be taken. Next we close the contour of integration in the upper semiplane of complex x, by adding a semicircle of radius w. The contribution from this semicircle vanishes as $O(w^{-2})$ and we reduce the remaining integral to the sum over the residues of $\coth x$ at $x = i\pi n$ as follows:

$$\int_{-w}^{w} \frac{dx\, x \coth x}{x^2 - y^2} \to \sum_{n=1}^{w/\pi} \frac{2n}{n^2 + (y/\pi)^2}. \quad \text{(F3)}$$

The last sum is easily evaluated with the final result of the form $P(T,V) = 4V\Lambda$ with Λ given by

$$\Lambda = \ln\left(\frac{We}{2\pi T}\right) - \frac{1}{2}\left[\psi\left(1 + \frac{iV}{2\pi T}\right) + \psi\left(1 - \frac{iV}{2\pi T}\right)\right] \quad \text{(F4)}$$

with $e = 2.718\dots$ and $\psi(x)$ digamma function.

[1] M. G. Prokudina, S. Ludwig, V. Pellegrini, L. Sorba, G. Biasiol, and V. S. Khrapai, Phys. Rev. Lett. **112**, 216402 (2014).
[2] H. T. Mebrahtu, I. V. Borzenets, H. Zheng, Y. V. Bomze, A. I. Smirnov, S. Florens, H. U. Baranger, and G. Finkelstein, Nat. Phys. **9**, 732 (2013).
[3] S. Jezouin, M. Albert, F. D. Parmentier, A. Anthore, U. Gennser, A. Cavanna, I. Safi, and F. Pierre, Nat. Commun. **4**, 1802 (2013).
[4] W. Apel and T. M. Rice, Phys. Rev. B **26**, 7063 (1982).
[5] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **46**, 15233 (1992).
[6] D. L. Maslov and M. Stone, Phys. Rev. B **52**, R5539 (1995).
[7] A. Furusaki and N. Nagaosa, Phys. Rev. B **47**, 4631 (1993).
[8] I. Safi and H. J. Schulz, Phys. Rev. B **52**, R17040 (1995).
[9] A. Furusaki and N. Nagaosa, Phys. Rev. B **54**, R5239 (1996).
[10] M. Sassetti and B. Kramer, Phys. Rev. B **54**, R5203 (1996).
[11] F. Dolcini, H. Grabert, I. Safi, and B. Trauzettel, Phys. Rev. Lett. **91**, 266402 (2003).
[12] F. Dolcini, B. Trauzettel, I. Safi, and H. Grabert, Phys. Rev. B **71**, 165309 (2005).
[13] R. Egger, H. Grabert, A. Koutouza, H. Saleur, and F. Siano, Phys. Rev. Lett. **84**, 3682 (2000).
[14] W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden, and K. Schönhammer, Rev. Mod. Phys. **84**, 299 (2012).
[15] M. Bockrath, Science **275**, 1922 (1997).
[16] F. Milliken, C. Umbach, and R. Webb, Solid State Commun. **97**, 309 (1996).

[17] A. Yacoby, H. L. Stormer, N. S. Wingreen, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **77**, 4612 (1996).
[18] S. Tarucha, T. Honda, and T. Saku, Solid State Commun. **94**, 413 (1995).
[19] S. J. Tans, M. H. Devoret, H. Dai, A. Thess, R. E. Smalley, L. J. Geerligs, and C. Dekker, Nature (London) **386**, 474 (1997).
[20] D. Yue, L. I. Glazman, and K. A. Matveev, Phys. Rev. B **49**, 1966 (1994).
[21] D. N. Aristov and P. Wölfle, Europhys. Lett. **82**, 27001 (2008).
[22] D. N. Aristov and P. Wölfle, Phys. Rev. B **80**, 045109 (2009).
[23] D. N. Aristov, A. P. Dmitriev, I. V. Gornyi, V. Y. Kachorovskii, D. G. Polyakov, and P. Wölfle, Phys. Rev. Lett. **105**, 266404 (2010).
[24] D. N. Aristov, Phys. Rev. B **83**, 115446 (2011).
[25] D. N. Aristov and P. Wölfle, Phys. Rev. B **84**, 155426 (2011).
[26] D. N. Aristov and P. Wölfle, Lith. J. Phys. **52**, 89 (2012).
[27] D. N. Aristov and P. Wölfle, Phys. Rev. B **86**, 035137 (2012).
[28] D. N. Aristov and P. Wölfle, Phys. Rev. B **88**, 075131 (2013).
[29] U. Weiss, R. Egger, and M. Sassetti, Phys. Rev. B **52**, 16707 (1995).