

# Symmetry-protected topological states of interacting fermions and bosons

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We study the classification for a large class of interacting fermionic and bosonic symmetry-protected topological (SPT) states, focusing on the cases where interaction reduces the classification of free-fermion SPT states. We define a SPT state as whether it is separated from the trivial state through a bulk phase transition, which is a general definition applicable to SPT states with or without spatial symmetries. We show that in all dimensions short-range interactions can reduce the classification of free-fermion SPT states, and we demonstrate these results by making a connection between the fermionic and bosonic SPT states. We first demonstrate that our formalism gives the correct classification for several known SPT states, with or without interaction, and then we generalize our method to SPT states that involve the spatial inversion symmetry.

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## I. INTRODUCTION

A symmetry-protected topological (SPT) state<sup>1</sup> is usually defined as a state with completely trivial bulk spectrum but nontrivial (e.g., gapless or degenerate) boundary spectrum when and only when the system including the boundary preserves certain symmetry [1–4]. The most well-known SPT states include the Haldane phase of spin-1 chain [5,6], quantum spin Hall insulator [7,8], topological insulator [9–11], and topological superconductor such as helium-3 B-phase [12–15]. So far all of the free-fermion SPT states have been well understood and classified in Refs. [16–20], and recent studies suggest that interaction can reduce the classification of fermionic SPT states [21–28]. Unlike fermionic systems, bosonic SPT states do need strong interaction to overcome its tendency to form a Bose-Einstein condensate. Most bosonic SPT states can be classified by symmetry group cohomology [1–4], Chern-Simons theory [29,30] and beyond [30], semiclassical nonlinear  $\sigma$  model [31], and cobordism theory [32,33].

The definition for SPT states we gave above is based on the most obvious phenomenology of the SPT states, and it gives SPT states a convenient experimental signature, which is their boundary state. Indeed, the quantum spin Hall insulator and three-dimensional (3D) topological insulator were verified experimentally by directly probing their boundary properties [34–37]. However, if a SPT state needs certain spatial symmetry [38–46], its boundary may be trivial because this spatial symmetry can be explicitly broken by its boundary. In this work we will study SPT states both with and without spatial symmetries, thus in our current work, a SPT state is simply defined as a gapped and nondegenerate state that must be separated from the trivial direct product state defined on the same Hilbert space through one or more bulk phase transitions, as long as the Hamiltonian always preserves certain symmetry.

In this work we study both strongly interacting fermionic and bosonic SPT states. In particular, we focus on the instability of free-fermionic SPT (fFSPT) states against interaction and investigate their interaction-reduced classifications. More

exotic fermionic SPT states that cannot be reduced from free-fermion descriptions [28,47,48] are not discussed in this work. The interaction-reduced classification of interacting fermionic SPT (iFSPT) states can be derived by making connection to bosonic SPT (BSPT) states with the *same* symmetry,<sup>2</sup> and we will argue that the classification of BSPT states implies the classification of their fermionic counterparts. More specifically, since BSPT states always need strong interaction, their classification tells us how interaction affects the classification of FSPT states. When describing BSPT states, we will adopt the formalism developed in Ref. [31], namely we describe a  $d$ -dimensional BSPT state using an  $O(d+2)$  nonlinear  $\sigma$  model (NLSM) field theory with a topological  $\Theta$  term, and we only focus on the stable “fixed point” states with  $\Theta = 2\pi k$ . Depending on integer  $k$ , these fixed points can correspond to either trivial or BSPT state. This formalism fits well with our definition of SPT states: It very naturally tells us whether two “fixed point” states can be connected with or without a phase transition. This is an advantage that we will fully exploit in our work.

We will first demonstrate our method in Sec. II with well-known examples such as 1D Kitaev’s Majorana chain with  $\mathbb{Z}_2^T$  symmetry [21,22], 2D  $p \pm ip$  topological superconductor (TSC) with  $\mathbb{Z}_2$  symmetry [23–26], and 3D topological superconductor <sup>3</sup>He-B with  $\mathbb{Z}_2^T$  symmetry [27,28]. Previous studies show that although all these states have  $\mathbb{Z}$  classification without interaction, their classifications will reduce to  $\mathbb{Z}_8$ ,  $\mathbb{Z}_8$ , and  $\mathbb{Z}_{16}$  under interaction. We will demonstrate that these interaction-reduced classifications naturally come from the  $\mathbb{Z}_2$  classification of the 1D Haldane spin chain, 2D Levin-Gu paramagnet [1,49], and 3D bosonic SPT state with  $\mathbb{Z}_2^T$  symmetry [50,51]. More precisely, the  $\mathbb{Z}_2$  classification of the BSPT states give us a *necessary* condition for interaction-reduced classification of their fermionic counterparts. Moreover, the analysis of BSPT states also leads to a systematic construction

<sup>1</sup>Sometimes this kind of states are also called “symmetry protected trivial” states in literature, depending on the taste and level of terminological rigor of authors.

<sup>2</sup>More precisely speaking, we consider the BSPT state with a symmetry action that is compatible with (but not identical to) the symmetry action on the FSPT state. For instance,  $\mathbb{Z}_2^T$  symmetry with  $T^2 = -1$  has no bosonic realization, so the corresponding  $\mathbb{Z}_2^T$  action on BSPT state must be derived from the  $\mathbb{Z}_2^T$  action on the FSPT state case by case.

of the specific four-fermion interaction that likely gaps out the bulk critical point between the free-fermion SPT (fFSPT) state and the trivial state in the noninteracting limit. In Secs. III–V we will generalize our method to SPT states that involve the spatial inversion symmetry. By making connection to BSPT states, we will show that interaction-reduced classification occurs very generally for inversion SPT states in all dimensions with a systematic pattern.

## II. SPT STATES WITHOUT SPATIAL SYMMETRY

### A. From the Kitaev chain to the Haldane chain

#### 1. Lattice model and bulk theory

The Kitaev Majorana chain is a 1D free-fermion SPT (fFSPT) state protected by the time-reversal symmetry  $\mathbb{Z}_2^T$  with  $\mathcal{T}^2 = +1$  (symmetry class BDI) [52]. For generality, we consider  $\nu$  copies of the Majorana chain. The model is defined on a 1D lattice with  $\nu$  flavors of Majorana fermions  $\chi_{i\alpha}$  ( $\alpha = 1, \dots, \nu$ ) on each site  $i$ ,

$$H_{\times\nu} = \sum_{\alpha=1}^{\nu} \sum_{\langle ij \rangle} i u_{ij} \chi_{i\alpha} \chi_{j\alpha}, \quad (1)$$

with the bond strength  $u_{ij} = u[1 + \delta(-)^i]$  alternating along the chain. Each unit cell contains two sites, labeled by  $A$  and  $B$ , as shown in Fig. 1. The Hamiltonian is invariant under the time-reversal symmetry  $\mathbb{Z}_2^T$ ,  $\mathcal{T} : \chi_{A\alpha} \rightarrow \chi_{A\alpha}, \chi_{B\alpha} \rightarrow -\chi_{B\alpha}, i \rightarrow -i$ , which flips the sign of the Majorana fermions on the  $B$  sublattice followed by the complexed conjugation.

Fourier transform to the momentum space and introduce the basis  $\chi_{k\alpha} = (\chi_{kA\alpha}, \chi_{kB\alpha})^T$ , the Hamiltonian Eq. (1) becomes

$$H_{\times\nu} = \frac{1}{2} \sum_{\alpha=1}^{\nu} \sum_k \chi_{-k\alpha}^\dagger \begin{bmatrix} 0 & -i u_k^* \\ i u_k & 0 \end{bmatrix} \chi_{k\alpha}, \quad (2)$$

with  $i u_k = i u(1 + \delta) - i u(1 - \delta)e^{-ik}$ , and the time-reversal symmetry acts as  $\mathcal{T} : \chi_{k\alpha} \rightarrow \sigma^3 \chi_{-k\alpha}$ . In this paper, we use  $\sigma^1, \sigma^2$ , and  $\sigma^3$  to denote the Pauli matrices. When  $\delta \ll 1$ , in the long-wavelength limit ( $k \rightarrow 0$ ),  $i u_k \rightarrow u(-k + 2i\delta)$ , so the low-energy effective Hamiltonian reads

$$H_{\times\nu} = \frac{1}{2} \sum_{\alpha=1}^{\nu} \int dx \chi_{\alpha}^\dagger (i \partial_x \sigma^1 + m \sigma^2) \chi_{\alpha}. \quad (3)$$

Here we have set  $u = 1$  and introduced  $m = 2\delta$  as the topological mass. The time-reversal symmetry operator may be written as  $\mathcal{T} = \mathcal{K} \sigma^3$ , where  $\mathcal{K}^{-1} i \mathcal{K} = -i$  implements the complex conjugation by flipping the imaginary unit.

On the free-fermion level, the 1D BDI class FSPT states are  $\mathbb{Z}$  classified [16,17] as indexed by a bulk topological integer,

$$N = \int \frac{dk}{4\pi i} \text{Tr} \sigma^3 G(k) \partial_k G(k)^{-1}, \quad (4)$$

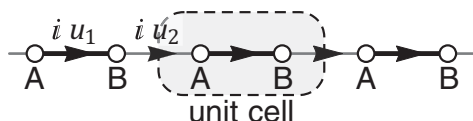


FIG. 1. Lattice model of the Kitaev Majorana chain.

where  $G(k) = -\langle \chi_k \chi_{-k}^\dagger \rangle$  is the fermion Green's function at zero frequency. Given the model in Eq. (2), the topological number  $N = \nu(1 - \text{sgn} \delta)/2$  is identical to the fermion flavor number  $\nu$  when  $\delta < 0$ . Every topological number  $N \in \mathbb{Z}$  indexes a distinct fFSPT phase, which, with respect to the symmetry, cannot be connected to each other without going through a bulk transition.

### 2. Corresponding BSPT state

However with interaction, the classification can be reduced from  $\mathbb{Z}$  to  $\mathbb{Z}_8$ , meaning that eight copies of the Majorana chain ( $N = 8$ ) can smoothly connect to the trivial state ( $N = 0$ ) without closing the bulk gap in the presence of interaction. This interaction-reduced classification was discovered in Refs. [21,22]. But we will show it again by making a connection to the boson SPT (BSPT) states, as our approach can be generalized to higher spacial dimensions.

Let us start by showing that four copies of the Majorana chain ( $\nu = 4$ ) can be connected to the Haldane spin chain [53] a  $\mathbb{Z}_2^T$  BSPT state in 1D. In this case, we have four flavors of Majorana fermion per site, denoted  $\chi_{i\alpha}$  ( $\alpha = 1, \dots, 4$ ), from which we can define a spin-1/2 object on each site, as  $\mathbf{S}_i = (S_{i1}, S_{i2}, S_{i3}) = \chi_i^\dagger (\sigma^{12}, \sigma^{20}, \sigma^{32}) \chi_i$  in the basis  $\chi_i = (\chi_{i1}, \chi_{i2}, \chi_{i3}, \chi_{i4})^T$ .<sup>3</sup> One can couple the staggered component of the spin to an O(3) order parameter  $\mathbf{n}_i$  on each site, as  $H_{BF} = -\sum_i (-)^i \mathbf{n}_i \cdot \mathbf{S}_i$ . Then the low-energy effective Hamiltonian for four copies of the Majorana chain coupled to the  $\mathbf{n}$  field reads (which we call the fermionic  $\sigma$  model or FSM)

$$H_{\times 4} = \frac{1}{2} \int dx \chi^\dagger h_{\times 4} \chi, \quad (5)$$

$$h_{\times 4} = i \partial_1 \sigma^{100} + m \sigma^{200} + n_1 \sigma^{312} + n_2 \sigma^{320} + n_3 \sigma^{332},$$

where  $\chi = (\chi_A, \chi_B)^T$ . The time-reversal symmetry operator  $\mathcal{T} = \mathcal{K} \sigma^{300}$  necessarily requires us to flip the order parameters  $\mathbf{n} \rightarrow -\mathbf{n}$  under the time reversal. Following the calculation in Ref. [54], after integrating out the fermion field  $\chi$ , we arrive at the effective theory for the boson field  $\mathbf{n}$ , which is a nonlinear  $\sigma$  model (NLSM) with a topological  $\Theta$  term at  $\Theta = 2\pi$ , given by the following action:

$$S[\mathbf{n}] = \int d\tau dx \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \frac{i\Theta}{4\pi} \epsilon^{abc} n_a \partial_\tau n_b \partial_x n_c. \quad (6)$$

$g^{-1}(\partial_\nu \mathbf{n})^2$  describes the remaining dynamics in the bosonic sector. Presumably, we work in the large  $g \rightarrow \infty$  limit, such that the  $\mathbf{n}$  field is deep in its disordered phase. With  $\Theta = 0$ , the Hamiltonian of Eq. (6) reads  $H = \int dx g \mathbf{L}^2 + \frac{1}{g} (\nabla_x \mathbf{n})^2$  [where  $\mathbf{L}(x)$  is the canonical conjugate variable of  $\mathbf{n}(x)$  at each spatial position  $x$ ], and since  $g$  flows to  $+\infty$  under coarse-graining, in the long-wavelength limit the ground-state wave function of this theory is a trivial direct product state  $|\Omega\rangle = \prod_x |l=0\rangle$  [51] with a fully gapped and nondegenerate

<sup>3</sup>Through out this paper, we use the notation  $\sigma^{ijk\dots} \equiv \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes \dots$  for the Kronecker product (direct product) of the Pauli matrices, where  $\sigma^1, \sigma^2, \sigma^3$  stands for the three Pauli matrices respectively while  $\sigma^0$  denotes the  $2 \times 2$  identity matrix.

spectrum in the bulk and at the boundary [on each coarse-grained spatial point,  $L^2 = l(l+1)$ ]. However, with  $\Theta = 2\pi$ , Eq. (6) describes a nontrivial BSPT state for the  $\mathbf{n}$  field and is equivalent [5,6,55] to the Haldane phase of spin-1 chain protected by the spin-flipping time-reversal symmetry  $T: \mathbf{n} \rightarrow -\mathbf{n}$ . In fact, the spatial boundary of Eq. (6) with  $\Theta = 2\pi$  is a  $(0+1)d$   $O(3)$  NLSM with a Wess-Zumino-Witten (WZW) term at level  $k=1$ , and by solving this theory exactly, we can demonstrate explicitly that the ground state of the spatial boundary of Eq. (6) with  $\Theta = 2\pi$  is doubly degenerate [31,55], which is equivalent to the boundary ground state of four copies of Kitaev's chain under interaction. In the low-energy limit, the boundary of four copies of the FSPT states is faithfully captured by the bosonic field  $\mathbf{n}$ . Thus we have established a connection between four copies of Majorana chain and a single copy of Haldane chain, bridging the FSPT and BSPT states in 1D.

Using the knowledge of the better-understood BSPT states, we can gain insight into the interacting fermion SPT (iFSPT) states. If eight copies of the iFSPT states is a trivial phase, then *necessarily* the bosonic theory of eight copies of the FSPT states derived using the same method must also be a trivial state. Indeed, because the Haldane phase has a well-known  $\mathbb{Z}_2$  classification [2], it is expected that two copies of the Haldane chain can be smoothly connected to the trivial state without breaking the symmetry. This can be shown by coupling two layers of the Haldane chain with a large interlayer antiferromagnetic interaction (which preserves the  $\mathbb{Z}_2^T$  symmetry), as described by the action

$$S = S[\mathbf{n}^{(1)}] + S[\mathbf{n}^{(2)}] + S_{\text{cp}}, \quad (7)$$

$$S_{\text{cp}} = \int d\tau dx A \mathbf{n}^{(1)} \cdot \mathbf{n}^{(2)},$$

when  $A \rightarrow +\infty$ ,  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  are locked into opposite directions, i.e.,  $\mathbf{n}^{(1)} = -\mathbf{n}^{(2)} = \mathbf{n}$ . Then the effective NLSM for  $\mathbf{n}$  has  $\Theta = 0$  due to the cancellation of the  $\Theta$  angles between the two layers. So two copies of the Haldane chain can be trivialized by the  $A$  coupling.

Also, when the two Haldane phases in Eq. (7) are decoupled from each other ( $A = 0$ ), both Haldane phases in Eq. (7) are separated from the trivial phase ( $\Theta = 0$ ) with a critical point at  $\Theta = \pi$ . However, with  $A \neq 0$ , this critical point is also gapped out by the  $A$  coupling;<sup>4</sup> thus with  $A > 0$ , the entire phase diagram of Eq. (7) has only one trivial phase. This observation already suggests that eight copies of Kitaev's Majorana chain is trivial under interaction.

### 3. Bulk transition

Now let us carefully investigate the interactions in the fermion model. Two copies of the Haldane chain would correspond to eight copies of the Majorana chain. Recall the relation  $\mathbf{n}_i \sim (-)^i \langle S_i \rangle$  on the mean-field level; the interlayer coupling  $A$  can be immediately ported to the fermion model

<sup>4</sup>When  $\Theta = \pi$ , both  $S[\mathbf{n}^{(1)}]$  and  $S[\mathbf{n}^{(2)}]$  can be viewed as the low energy field theory of spin-1/2 chains. Then the antiferromagnetic inter-chain coupling  $A$  will drive the system into a fully gapped state which is a direct product of inter-chain spin singlet on every site.

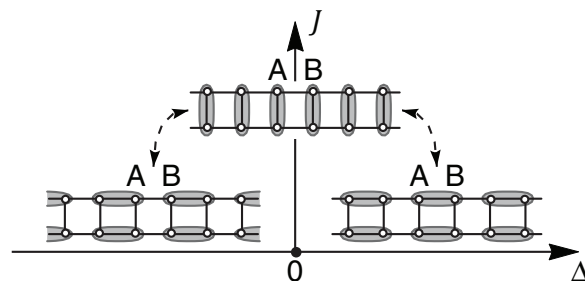


FIG. 2. Phase diagram of the interacting Majorana chain at  $\nu = 8$ . The bulk criticality at the origin can be avoided if interaction is allowed. Each horizontal chain is four copies of the Majorana chain, equivalent to a Haldane chain. The vertical bound is the on-site spin-spin interaction.  $A$  and  $B$  labels the sites in a unit cell. Gray ovals mark out the spin-singlet dimers.

as an on-site interaction among eight flavors of Majorana fermions,

$$H_{\text{int}} = \frac{J}{4} \sum_i \mathbf{S}_i^{(1)} \cdot \mathbf{S}_i^{(2)}$$

$$= J \sum_i (-\chi_{i1}\chi_{i2}\chi_{i5}\chi_{i6} + \chi_{i1}\chi_{i2}\chi_{i7}\chi_{i8} - \chi_{i1}\chi_{i3}\chi_{i5}\chi_{i7}$$

$$- \chi_{i1}\chi_{i3}\chi_{i6}\chi_{i8} - \chi_{i1}\chi_{i4}\chi_{i5}\chi_{i8} + \chi_{i1}\chi_{i4}\chi_{i6}\chi_{i7}$$

$$+ \chi_{i2}\chi_{i3}\chi_{i5}\chi_{i8} - \chi_{i2}\chi_{i3}\chi_{i6}\chi_{i7} - \chi_{i2}\chi_{i4}\chi_{i5}\chi_{i7}$$

$$- \chi_{i2}\chi_{i4}\chi_{i6}\chi_{i8} + \chi_{i3}\chi_{i4}\chi_{i5}\chi_{i6} - \chi_{i3}\chi_{i4}\chi_{i7}\chi_{i8}), \quad (8)$$

with  $J > 0$ . We should expect that eight copies of the Majorana chain can be connected to the trivial state under this interaction, as the same interaction can trivialize the BSPT in the NLSM.

Such an expectation is obvious in the spin sector. In the free-fermion limit, depending on the sign of  $\delta$ , the  $\nu = 8$  fFSPT states may correspond to two different spin-singlet dimerization patterns: the intra-unit-cell dimerization ( $\delta > 0$  trivial state) or the inter-unit-cell dimerization ( $\delta < 0$  SPT state), as shown in Fig. 2. While the strong on-site interaction in Eq. (8) will lead to a third pattern, i.e., the on-site (interlayer) dimerization, see Fig. 2. The three patterns are connected by the ring exchange of the dimers. However, it is known that the ring exchange is a smooth deformation and will not close the spin gap, so, at least in the spin sector, the  $\delta < 0$  and the  $\delta > 0$  SPT states can be smoothly connected.

To show that the charge gap also remains open, we can perform an explicit calculation based on the lattice model Eq. (1) in the strong dimerization limit  $\delta = \pm 1$ , such that the 1D chain is decoupled into independent two-site segments. In each segment, the interacting fermion system can be exact diagonalized. Then it can be shown that the charge gap indeed persists as  $u$  is tuned to zero in the presence of the interaction  $J$ , as shown in Fig. 3. So one can smoothly connect the  $N = 8$  fFSPT state to the  $N = 0$  fFSPT state in three steps: (i) turn on  $J$  and turn off  $u$ , (ii) change the sign of  $\delta$ , and (iii) turn on  $u$  and turn off  $J$ . The bulk gap will never close during this process. Thus the whole phase diagram in Fig. 2 is actually one phase.

In conclusion, we have demonstrated that the classification of the 1D FSPT states with the (BDI class) time-reversal

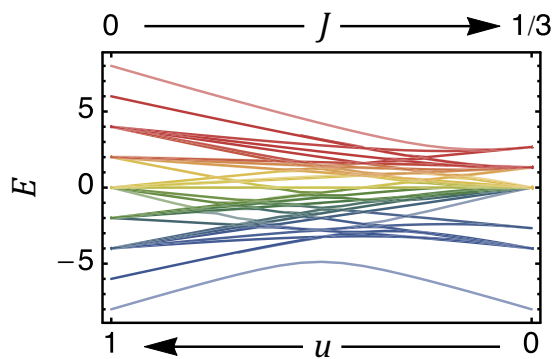


FIG. 3. (Color online) Many-body energy levels of the two-site segment by exact diagonalization. From the weak interaction limit (left) to the strong interaction (right) limit, the many-body gap never closes.

symmetry is reduced from  $\mathbb{Z}$  to  $\mathbb{Z}_8$  under interaction. We obtain the iFSPT classification by making connection to the BSPT classification. This approach can be readily generalized to higher spacial dimensions in the following. Moreover, the way that the BSPT state can be trivialized in the NLSM naturally provides us the correct fermion interaction that is needed to trivialize the FSPT states, which can be much more general than the currently known Fidkowski-Kitaev type of interaction. And this interaction can gap out the critical point  $m = 0$  in Eq. (3), which is eight copies of nonchiral 1D Majorana fermions. This bulk analysis is particularly suitable to study the crystalline SPT states, which may not have symmetry-protected physical boundary modes.

## B. From 2D TSC to Levin-Gu paramagnet

### 1. Lattice model and bulk theory

Now we turn to the 2D example of the  $p \pm ip$  TSC protected by a  $\mathbb{Z}_2$  symmetry (symmetric class D) [52]. The  $p_x \pm ip_y$  TSC can be viewed as both layers of the  $p_x + ip_y$  and the  $p_x - ip_y$  TSC's [56–58] stacked together with the  $\mathbb{Z}_2$  symmetry acts only in the  $p_x - ip_y$  layer by flipping the sign of the fermion operator (i.e., the fermion parity transform). On a 2D lattice, the model Hamiltonian can be written as

$$H = \sum_{k,l} \xi_k c_{kl}^\dagger c_{kl} + \frac{1}{2} (\Delta_k c_{-kl} c_{kl} + \text{H.c.}),$$

$$\xi_k = -2t(\cos k_1 + \cos k_2) - \mu, \quad (9)$$

$$\Delta_{kl} = -\Delta(\sin k_1 + (-)^l \sin k_2),$$

where  $l = 0, 1$  labels the two opposite layers of the chiral TSC's. The  $\mathbb{Z}_2$  symmetry acts as  $c_{kl} \rightarrow (-)^l c_{kl}$ , which prevents the mixing of fermions from different layers, such that the fermion parity is conserved in each layer independently. Depending on the chemical potential  $\mu$ , the model has two phases: the  $|\mu| > 4t$  strong pairing trivial superconductor phase and the  $|\mu| < 4t$  weak pairing topological superconductor phase. Switching to the Majorana basis  $\chi_k = (c_{k0} + c_{-k0}^\dagger, c_{k1} + c_{-k1}^\dagger, ic_{k0} - ic_{-k0}^\dagger, ic_{k1} - ic_{-k1}^\dagger)^\top / \sqrt{2}$  and in the

long-wave-length limit, the effective Hamiltonian reads

$$H_{\times 1} = \frac{1}{2} \sum_k \chi_{-k}^\top (-k_1 \sigma^{30} - k_2 \sigma^{13} + m \sigma^{20}) \chi_k, \quad (10)$$

where we have set  $\Delta = 1$  as our energy unit and defined the topological mass  $m = -4t - \mu$  (assuming  $\mu < 0$ ). The  $\mathbb{Z}_2$  symmetry acting on the Majorana fermions as  $\chi_k \rightarrow \sigma^{03} \chi_k$ . The trivial ( $m > 0$ ) and the topological ( $m < 0$ ) phases are separated by the phase transition at  $m = 0$  where the bulk gap closes. This bulk criticality is protected by the  $\mathbb{Z}_2$  symmetry.

The above  $p \pm ip$  TSC is an example of the 2D D class fFSPT states, which are known to be  $\mathbb{Z}$  classified [16,17] and are indexed by the topological number [59]

$$N = \int \frac{d^3 k}{8\pi^2} \text{Tr} \sigma^{03} G \partial_{i\omega} G^{-1} G \partial_{k_1} G^{-1} G \partial_{k_2} G^{-1}, \quad (11)$$

where  $G(k) = -\langle \chi_k \chi_{-k}^\top \rangle$  with  $k = (i\omega, \mathbf{k})$  is the fermion Green's function in the frequency-momentum space. The  $p \pm ip$  TSC in Eq. (9) corresponds to  $N = 1$ . While the other topological states in this  $\mathbb{Z}$  classification may be realized by considering multiple copies of such  $p \pm ip$  TSC's, which can be described by the following effective-field-theory Hamiltonian:

$$H_{\times v} = \frac{1}{2} \sum_{\alpha=1}^v \int d^2 \mathbf{x} \chi_{\alpha}^\top (i \partial_1 \sigma^{30} + i \partial_2 \sigma^{13} + m \sigma^{20}) \chi_{\alpha}, \quad (12)$$

with the  $\mathbb{Z}_2$ :  $\chi_{\alpha} \rightarrow \sigma^{03} \chi_{\alpha}$  symmetry protection.  $v$ -copy  $p \pm ip$  TSC would correspond to the topological number  $N = v$ .

### 2. Corresponding BSPT state

However with interaction, the classification of the  $\mathbb{Z}_2 p \pm ip$  TSC is reduced from  $\mathbb{Z}$  to  $\mathbb{Z}_8$  [23–26], meaning that eight copies of the  $p \pm ip$  TSC ( $N = 8$ ) can be smoothly connected to the trivial state ( $N = 0$ ) in the presence of interaction. This interaction-reduced classification was discussed in Refs. [23–26], but here we will provide another argument for it by making a connection to 2D BSPT states.

Let us start by showing that four copies of the  $p \pm ip$  TSC ( $v = 4$ ) can be connected [29,60] to the Levin-Gu topological paramagnet [1,49], a  $\mathbb{Z}_2$  BSPT state in 2D. We first introduce a set of interlayer  $s$ -wave pairing terms (with  $\bar{l} \equiv 1 - l$  and  $\alpha, \alpha' = 1, 2, 3, 4$  labeling the four copies)

$$\begin{aligned} \Delta_1 &= \sum_{l,\alpha,\alpha'} c_{\bar{l}\alpha} i \sigma_{\alpha\alpha'}^{12} c_{l\alpha'} + \text{H.c.} = \chi^\top \sigma^{1112} \chi, \\ \Delta_2 &= \sum_{l,\alpha,\alpha'} c_{\bar{l}\alpha} i \sigma_{\alpha\alpha'}^{20} c_{l\alpha'} + \text{H.c.} = \chi^\top \sigma^{1120} \chi, \\ \Delta_3 &= \sum_{l,\alpha,\alpha'} c_{\bar{l}\alpha} i \sigma_{\alpha\alpha'}^{32} c_{l\alpha'} + \text{H.c.} = \chi^\top \sigma^{1132} \chi, \\ \Delta_4 &= \sum_{l,\alpha} c_{\bar{l}\alpha} (-)^l c_{l\alpha} + \text{H.c.} = \chi^\top \sigma^{1200} \chi, \end{aligned} \quad (13)$$

where  $\chi = (\chi_1, \chi_2, \chi_3, \chi_4)^\top$ , and couple them to an  $O(4)$  order parameter field  $\mathbf{n} = (n_1, n_2, n_3, n_4)$ . The low-energy effective

Hamiltonian of this FSM reads

$$H_{\times 4} = \frac{1}{2} \int d^2 \mathbf{x} \chi^\dagger h_{\times 4} \chi, \\ h_{\times 4} = i \partial_1 \sigma^{3000} + i \partial_2 \sigma^{1300} + m \sigma^{2000} \\ + n_1 \sigma^{1112} + n_2 \sigma^{1120} + n_3 \sigma^{1132} + n_4 \sigma^{1200}. \quad (14)$$

Because the interlayer pairing mixes the fermions between the  $p_x + ip_y$  and the  $p_x - ip_y$  TSC's, they will gain a minus sign under the  $\mathbb{Z}_2$  symmetry transform. To preserve the  $\mathbb{Z}_2$  symmetry, we must require the order parameters to change sign as well, i.e.,  $\mathbf{n} \rightarrow -\mathbf{n}$ , under the  $\mathbb{Z}_2$  symmetry action. After integrating out the fermion field  $\chi$ , we arrive at the effective theory for the boson field  $\mathbf{n}$ , which is a NLSM with a topological  $\Theta$  term at  $\Theta = 2\pi$ , given by the following action ( $d^3 x = d\tau d^2 \mathbf{x}$ ):

$$S[\mathbf{n}] = \int d^3 x \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \frac{i\Theta}{2\pi^2} \epsilon^{abcd} n_a \partial_0 n_b \partial_1 n_c \partial_2 n_d, \quad (15)$$

which describes a nontrivial BSPT state for the  $\mathbf{n}$  field [31,61] and is equivalent to the Levin-Gu state [31] protected by the  $\mathbb{Z}_2$  symmetry  $\mathbf{n} \rightarrow -\mathbf{n}$ . This can be understood from the wave-function perspective. We first reparameterize  $\mathbf{n} = (m \cos \alpha, \phi \sin \alpha)$  where  $\mathbf{m} = (m_1, m_2, m_3)$  is an  $O(3)$  unit vector and  $\phi = \pm 1$ . Suppose the system energetically favors  $\mathbf{m}$  (i.e.,  $\alpha = 0$ ); then the wave function for the  $\mathbf{m}$  field in its paramagnetic phase ( $g \rightarrow \infty$ ) can be derived from the action Eq. (15) as [61]

$$|\Psi\rangle \sim \int \mathcal{D}[\mathbf{m}] e^{\int d^2 x \frac{i\pi}{4g} \epsilon^{abc} m_a \partial_1 m_b \partial_2 m_c} |[\mathbf{m}]\rangle \\ = \int \mathcal{D}[\mathbf{m}] (-)^{N_s[\mathbf{m}]} |[\mathbf{m}]\rangle \\ \sim \int \mathcal{D}[m_3] (-)^{N_d[m_3]} |[m_3]\rangle, \quad (16)$$

which is a superposition of all  $\mathbf{m}$  configurations with a sign factor  $(-)^{N_s[\mathbf{m}]}$  counting the parity of the Skyrmion number  $N_s$  of the  $\mathbf{m}$  field. In the Ising limit where  $m_3$  is energetically favored, the Skyrmion number  $N_s$  becomes the domain-wall number  $N_d$  of the Ising spin  $m_3$ , so the wave function becomes the superposition of Ising configurations with the domain-wall sign [61], which is exactly the wave function of the Levin-Gu state [49]. Thus we have established a connection from four copies of the  $p \pm ip$  TSC to a single copy of the Levin-Gu paramagnet, bridging the FSPT and BSPT states in 2D.

Now we can discuss the iFSPT states using the knowledge about the BSPT states: If eight copies of the TSC is trivial, then the bosonic theory derived using the same method above must necessarily be trivial. Indeed, on the BSPT side, we know that two copies of the Levin-Gu paramagnets can be smoothly connected to the trivial state without breaking the symmetry, which can be realized by coupling two layers of the Levin-Gu paramagnet with a large interlayer antiferromagnetic interaction, such that the domain-wall configuration in both layers will become identical, and the domain-wall sign from both layers will cancel out, so the resulting wave function is just a trivial Ising paramagnetic state. At the field-theory level, it can be described by the following action with interlayer

coupling:

$$S = S[\mathbf{n}^{(1)}] + S[\mathbf{n}^{(2)}] + S_{\text{cp}}, \\ S_{\text{cp}} = \int d^3 x A (n_1^{(1)} n_1^{(2)} + n_2^{(1)} n_2^{(2)} + n_3^{(1)} n_3^{(2)}) - B n_4^{(1)} n_4^{(2)}. \quad (17)$$

It is easy to check that the coupling  $S_{\text{cp}}$  respects the  $\mathbb{Z}_2$  symmetry. When  $A, B \rightarrow +\infty$ ,  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  are locked antiferromagnetically for their first three components and ferromagnetically for their last components, i.e.,  $n_a^{(1)} = -n_a^{(2)} = n_a$  ( $a = 1, 2, 3$ ) and  $n_4^{(1)} = n_4^{(2)} = n_4$ . Then the effective NLSM for the combined field  $\mathbf{n}$  has  $\Theta = 0$  due to the cancellation of the  $\Theta$  angles between the two layers. So two copies of the Levin-Gu paramagnet can be trivialized by the  $A, B \rightarrow +\infty$  coupling.<sup>5</sup> This suggests that eight copies of the original  $p \pm ip$  TSC is trivial.

### 3. Boundary modes and bulk transition

Recall the relation  $n_a \sim \langle \Delta_a \rangle$  ( $a = 1, 2, 3, 4$ ) on the mean-field level; the interlayer coupling  $S_{\text{cp}}$  can be immediately ported to the fermion model as the following four-fermion interaction (with  $A, B > 0$ ):

$$H_{\text{int}} = \int d^2 \mathbf{x} A \sum_{a=1,2,3} \Delta_a^{(1)} \Delta_a^{(2)} - B \Delta_4^{(1)} \Delta_4^{(2)}, \quad (18)$$

where  $\Delta_a$  is defined in Eq. (13). Without any interaction, eight copies of  $p \pm ip$  TSC with the  $\mathbb{Z}_2$  symmetry is separated from the trivial state through a critical point that has 16 copies of 2D massless Majorana fermions in the bulk [ $m = 0$  in Eq. (12)]. We should expect that the bulk criticality can be gapped out by the interaction Eq. (18), and eight copies of the  $p \pm ip$  TSC can be smoothly connected to the trivial state, as the same interaction can trivialize the BSPT in the NLSM.

Admittedly, in 2D (and higher dimensions), it is hard to explicitly demonstrate how the interaction gaps out the gapless bulk fermion at the critical point. Nevertheless we can show that, on an open manifold, the interaction Eq. (18) can gap out the 1D boundary states of eight copies of the  $p \pm ip$  TSC ( $N = 8$ ) without breaking the symmetry, and hence there should be no obstacle to tune the bulk system smoothly from the  $N = 8$  state to the  $N = 0$  state under interaction. The ‘‘transition’’ between the  $N = 8$  and  $N = 0$  states can be viewed as growing  $N = 0$  domains inside the  $N = 8$  state, which is equivalent to sweeping the interface between the two states through the entire bulk (this is essentially the picture of the Chalker-Coddington model [62] for the quantum Hall plateau transition), and then, as long as the interface is gapped out by interaction, the bulk gap never has to close during this ‘‘transition,’’ namely the bulk phase transition can be gapped out by the interaction. Thus all we need to show here is that

<sup>5</sup>As one can see, the design of the coupling is not unique, any inter-layer coupling that locks odd number of  $\mathbf{n}$  components antiferromagnetically will do the job to trivialize the BSPT state (for example  $A, B \rightarrow -\infty$  is also a choice), but here let us stick to our current design and focus on the  $A, B \rightarrow +\infty$  coupling.

the interaction Eq. (18) induces an effective interaction at the 1D boundary, which will gap out the boundary states.

Let us consider a boundary of the 2D system along the  $x_2$  axis, i.e., the topological mass  $m \sim x_1$  changes sign across  $x_1 = 0$ . For four copies of the  $p \pm ip$  TSC as described in Eq. (14), the boundary states are given by the projection operator  $P = (1 - i\sigma^{300}\sigma^{2000})/2$ , such that the effective FSM Hamiltonian along the boundary is given by

$$H'_{\times 4} = \frac{1}{2} \int dx_2 \eta^\top h'_{\times 4} \eta, \quad (19)$$

$$h'_{\times 4} = i\partial_2 \sigma^{300} + n_1 \sigma^{112} + n_2 \sigma^{120} + n_3 \sigma^{132} + n_4 \sigma^{200},$$

where  $\eta$  denotes the Majorana edge modes, and the  $\mathbb{Z}_2$  symmetry acts as  $\eta \rightarrow \sigma^{300}\eta$ . Under a basis transformation  $\eta \rightarrow \exp(-\frac{i\pi}{4}\sigma^{200})\eta$ , the boundary FSM Hamiltonian can be reformulated as

$$h'_{\times 4} = -i\partial_2 \sigma^{100} + n_1 \sigma^{312} + n_2 \sigma^{320} + n_3 \sigma^{332} + n_4 \sigma^{200}, \quad (20)$$

which, at the field-theory level, is equivalent to four copies of 1D (critical) Majorana chain described by Eq. (5), with the transformed  $\mathbb{Z}_2$  symmetry  $\eta \rightarrow -\sigma^{100}\eta$ . ( $n_1, n_2, n_3$ ) is the analog of the  $O(3)$  order parameter of the Majorana chain introduced in the previous section. All these order parameters are forbidden to condense by the  $\mathbb{Z}_2$  symmetry, i.e.,  $\langle n \rangle = 0$ , so the edge is gapless at the free-fermion level.

Now we consider the boundary of eight copies of the  $p \pm ip$  TSC, which is simply a doubling of Eq. (20). The field theory of this 1D boundary is equivalent to eight copies of the critical Kitaev's Majorana chain. The bulk interaction Eq. (18) will induce the interaction between Majorana surface modes, which corresponds to the coupling of  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  at the boundary:

$$S'_{\text{cp}} = \int d\tau dx_2 A' \sum_{a=1,2,3} n_a^{(1)} n_a^{(2)} - B' n_4^{(1)} n_4^{(2)}. \quad (21)$$

The  $A'$  term corresponds to exactly the same fermion interaction that trivialized eight copies of Majorana chain in the previous section, and this coupling can gap out the critical point in the previous 1D case. This means that the  $A'$  term can also gap out the boundary of the eight copies of  $2d$   $p \pm ip$  TSC without degeneracy. Once the boundary is gapped and nondegenerate, a weak  $B'$  term in Eq. (21) will not close the gap of the boundary. Since the boundary coupling Eq. (21) is induced by the bulk interaction Eq. (18), this implies that the interaction in Eq. (18) (with strong enough strength) can gap out the bulk criticality (with 16 copies of 2D massless Majorana fermions) in 2D.

### C. From $^3\text{He-B}$ to 3D bosonic SPT

#### 1. Lattice model and bulk theory

Let us go one dimension higher and consider the  $^3\text{He}$  superfluid B phase [12–15] (will be denoted as  $^3\text{He-B}$ ) which is a 3D TSC protected by the  $\mathbb{Z}_2^T$  symmetry with  $\mathcal{T}^2 = -1$  (symmetry class DIII) [52]. The  $^3\text{He-B}$  TSC is described by

the following Hamiltonian:

$$H = \sum_{\mathbf{k}} \left( \frac{k^2}{2m_{\text{He}}} - \mu \right) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} - \frac{\Delta}{2} (c_{-\mathbf{k}} i \sigma^2 \mathbf{k} \cdot \boldsymbol{\sigma} c_{\mathbf{k}} + \text{H.c.}), \quad (22)$$

where  $c_{\mathbf{k}} = (c_{\mathbf{k}\uparrow}, c_{\mathbf{k}\downarrow})^\top$  is the fermion operator for the  $^3\text{He}$  atom and  $\Delta \in \mathbb{R}$  is the  $p$ -wave pairing strength. The Hamiltonian is invariant under the time-reversal  $\mathbb{Z}_2^T$  symmetry, which acts as  $\mathcal{T} : c_{\mathbf{k}} \rightarrow i\sigma^2 c_{-\mathbf{k}}$  followed by the complex conjugation.  $^3\text{He-B}$  TSC corresponds to the  $\mu > 0$  topological phase of the model, while for  $\mu < 0$  the model describes a trivial superconductor. Switching to the Majorana basis  $\chi_{\mathbf{k}} = (c_{\mathbf{k}\uparrow} + c_{-\mathbf{k}\uparrow}^\dagger, -c_{\mathbf{k}\downarrow} - c_{-\mathbf{k}\downarrow}^\dagger, ic_{\mathbf{k}\uparrow} - ic_{-\mathbf{k}\uparrow}^\dagger, -ic_{\mathbf{k}\downarrow} + ic_{-\mathbf{k}\downarrow}^\dagger)^\top / \sqrt{2}$  and in the long-wave-length limit (to the first order in  $\mathbf{k}$ ), the effective Hamiltonian reads

$$H_{\times 1} = \frac{1}{2} \sum_{\mathbf{k}} \chi_{\mathbf{k}}^\top (-k_1 \sigma^{33} - k_2 \sigma^{10} - k_3 \sigma^{31} + m \sigma^{20}) \chi_{\mathbf{k}}, \quad (23)$$

where we have set  $\Delta = 1$  as our energy unit and defined the topological mass  $m = -\mu$  (which should not be confused with the mass of the  $^3\text{He}$  atom  $m_{\text{He}}$ ). The time-reversal operator acting on the Majorana basis is given by  $\mathcal{T} = \mathcal{K} i \sigma^{32}$ . The trivial ( $m > 0$ ) and the topological ( $m < 0$ ) phases are separated by the phase transition at  $m = 0$  where the bulk gap closes. This bulk criticality is protected by the  $\mathbb{Z}_2^T$  symmetry.

The  $^3\text{He-B}$  TSC belongs to the 3D DIII class fFSPT states, which is known to be  $\mathbb{Z}$  classified [16,17] and is indexed by the topological number [63,64]

$$N = \int \frac{d^3 \mathbf{k}}{8\pi^2} \text{Tr} \sigma^{32} G \partial_{k_1} G^{-1} G \partial_{k_2} G^{-1} G \partial_{k_3} G^{-1}, \quad (24)$$

where  $G(k) = -\langle \chi_{\mathbf{k}} \chi_{-\mathbf{k}}^\top \rangle$  is the fermion Green's function at zero frequency  $i\omega = 0$ . The  $^3\text{He-B}$  TSC in Eq. (22) corresponds to  $N = 1$ . While the other topological states in this  $\mathbb{Z}$  classification may be realized by considering multiple copies of the  $^3\text{He-B}$  TSC's, which can be described by the following effective-field-theory Hamiltonian:

$$H_{\times v} = \frac{1}{2} \sum_{\alpha=1}^v \int d^3 x \chi_{\alpha}^\top (i\partial_1 \sigma^{33} + i\partial_2 \sigma^{10} + i\partial_3 \sigma^{31} + m \sigma^{20}) \chi_{\alpha}, \quad (25)$$

with the  $\mathbb{Z}_2^T$  symmetry protection ( $\mathcal{T} = \mathcal{K} i \sigma^{32}$ ).  $v$ -copy  $^3\text{He-B}$  TSC would correspond to the topological number  $N = v$ .

#### 2. Corresponding BSPT state

However, with interaction, the classification of the 3D DIII class FSPT states is reduced from  $\mathbb{Z}$  to  $\mathbb{Z}_{16}$ , meaning that 16 copies of the  $^3\text{He-B}$  TSC ( $N = 16$ ) can be smoothly connected to the trivial state ( $N = 0$ ) in the presence of interaction. This interaction-reduced classification was discussed in Refs. [27,28], but here we will provide another argument for it by making connection to the 3D BSPT states.

Let us start by showing that eight copies of the  $^3\text{He}$  TSC ( $v = 8$ ) can be connected to the 3D BSPT state with  $\mathbb{Z}_2^T$  symmetry. Similarly to our previous approach in 1D and 2D,

here we should introduce five fermion pairing terms and couple them to an  $O(5)$  order parameter field  $\mathbf{n} = (n_1, n_2, n_3, n_4, n_5)$ , whereby the low-energy effective FSM Hamiltonian reads

$$H_{\times 8} = \frac{1}{2} \int d^3\mathbf{x} \chi^\top h_{\times 8} \chi, \quad (26)$$

$$h_{\times 8} = i\partial_1\sigma^{33000} + i\partial_2\sigma^{10000} + i\partial_3\sigma^{31000} + m\sigma^{20000} + n_1\sigma^{32212} + n_2\sigma^{32220} + n_3\sigma^{32232} + n_4\sigma^{32300} + n_5\sigma^{32100},$$

where  $\chi = (\chi_1, \chi_2, \dots, \chi_8)^\top$ . It turns out that these order parameters are spin-singlet  $s$ -wave (time-reversal broken) imaginary pairing among the eight copies of fermions. The particular form of the pairing terms given here is not a unique choice. We only require that the pairing terms anticommute with each other and also anticommute with the momentum and the topological mass terms. However, any other set of such pairing terms are related to the above choice by basis transformation among the eight copies of fermions, so we may stick to our current choice without losing any generality.

On this  $v = 8$  Majorana basis, the time-reversal operator is extended to  $\mathcal{T} = \mathcal{K}i\sigma^{32000}$ , from which it is easy to see that all five  $s$ -wave pairing terms change sign under  $\mathcal{T}$ . To preserve the  $\mathbb{Z}_2^T$  symmetry, we must require the order parameters to change sign as well, i.e.,  $\mathbf{n} \rightarrow -\mathbf{n}$ , under the  $\mathbb{Z}_2^T$  transform. After integrating out the fermion field  $\chi$ , we arrive at the effective theory for the boson field  $\mathbf{n}$ , which is a NLSM with a topological  $\Theta$  term at  $\Theta = 2\pi$ , given by the following action ( $d^4x = d\tau d^3\mathbf{x}$ ):

$$S[\mathbf{n}] = \int d^4x \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \frac{i\Theta}{\Omega_4} \epsilon^{abcde} n_a \partial_0 n_b \partial_1 n_c \partial_2 n_d \partial_3 n_e, \quad (27)$$

where  $\Omega_4 = 8\pi^2/3$  is the volume of  $S^4$ . Equation (27) describes [31] a nontrivial 3D BSPT state [50,51,53] protected by the  $\mathbb{Z}_2^T$  symmetry  $\mathbf{n} \rightarrow -\mathbf{n}$ . Thus we have established a connection from eight copies of the  ${}^3\text{He-B}$  TSC to a single copy of the 3D  $\mathbb{Z}_2^T$  BSPT state, bridging the FSPT and BSPT states in 3D.

Now we can discuss the iFSPT states using the knowledge about the BSPT states. On the BSPT side, based on the well-known  $\mathbb{Z}_2$  classification of this state [2,50], it is expected that two copies of the  $\mathbb{Z}_2^T$  BSPT state can be smoothly connected to the trivial state without breaking the symmetry. In our NLSM formalism, this conclusion can be drawn by the following interlayer coupling:

$$S = S[\mathbf{n}^{(1)}] + S[\mathbf{n}^{(2)}] + S_{\text{cp}}, \quad (28)$$

$$S_{\text{cp}} = \int d^4x A \sum_{a=1,2,3} n_a^{(1)} n_a^{(2)} - B n_4^{(1)} n_4^{(2)} - C n_5^{(1)} n_5^{(2)}.$$

It is easy to check that the coupling  $S_{\text{cp}}$  respects the  $\mathbb{Z}_2^T$  symmetry. When  $A, B, C \rightarrow +\infty$ ,  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  are locked antiferromagnetically for their first three components and ferromagnetically for their last two components, i.e.,  $n_a^{(1)} = -n_a^{(2)} = n_a$  ( $a = 1, 2, 3$ ) and  $n_b^{(1)} = n_b^{(2)} = n_b$  ( $b = 4, 5$ ). Then the effective NLSM for the combined field  $\mathbf{n}$  has  $\Theta = 0$  due to the cancellation of the  $\Theta$  angles between the two layers. So two copies of the  $\mathbb{Z}_2^T$  BSPT state can be trivialized by

the  $A, B, C \rightarrow +\infty$  coupling. Again, this is the necessary condition for interaction to reduce the classification for  ${}^3\text{He-B}$  phase to  $\mathbb{Z}_{16}$ .

### 3. Bulk phase transition under interaction

Now we argue that the quantum critical point in the noninteracting limit can be gapped out by interaction for 16 copies of  ${}^3\text{He-B}$  states. We start from the critical point  $m = 0$  in the FSM Eq. (26), where the bulk gap is closed on the free-fermion level. The field-theory Eq. (26) at  $m = 0$  has an extra inversion symmetry  $\mathcal{P} = -\mathcal{I}i\sigma^{32000}$  (where the space inversion operator  $\mathcal{I}$  sends  $\mathbf{x} \rightarrow -\mathbf{x}$ ), besides the original time-reversal symmetry  $\mathcal{T} = \mathcal{K}i\sigma^{32000}$ . Fermion interactions will be generated after integrating out dynamical field  $\mathbf{n}$ . We will argue that in this particular field-theory Eq. (26), interaction can gap out the critical point, without driving the system into either  $m < 0$  or  $m > 0$  state.

We can first gap out the fermions in the bulk by setting up a fixed configuration of the order parameter field  $\mathbf{n}$  at the cost of breaking the time-reversal symmetry. Then we restore the symmetry by proliferating the topological defects of the  $\mathbf{n}$  field, which is an approach adopted by Refs. [28,65]. Here we consider the point defect, namely the monopole configuration of  $\mathbf{n}$ , which is described by  $n_a \sim x_a$  (for  $a = 1, 2, 3$ ) and  $n_4 = n_5 = 0$  near the monopole core. This monopole breaks both  $\mathcal{T}$  and  $\mathcal{P}$ , but it preserves the combined symmetry  $\mathcal{T}' = \mathcal{P}\mathcal{T}$ . After proliferating this monopole, all the symmetries will be restored.

However, the potential obstacle is that the monopole may trap Majorana zero modes and is therefore degenerated. Proliferating such a defect will not result in a gapped and nondegenerated ground state and hence it fails to gap out the bulk criticality. So we must analyze the fermion modes at the monopole core carefully. By solving the BdG equation for a single copy of the FSM Eq. (26), it can be shown that the monopole will trap four Majorana zero modes, which transforms under  $\mathcal{T}'$  as  $\mathcal{T}' : \gamma_a \rightarrow \gamma_a$ , with  $a = 1, \dots, 4$ , followed by complex conjugation and space inversion. Thus for two copies of FSM Eq. (26), the monopole will trap eight Majorana zero modes, and the  $\mathcal{T}'$  symmetry will guarantee the spectrum of the monopole is degenerate at the noninteracting level. Nevertheless, the degeneracy can be completely lifted by interaction [21,22] without breaking  $\mathcal{T}'$ . So after the monopole proliferation, all the symmetries of Eq. (26) are restored, and the system will enter a fully gapped state which still resides on the line  $m = 0$ . Therefore with two copies of the FSM, the iFSPT state can be smoothly connected to the trivial state via strong interaction, resulting in the  $\mathbb{Z}_{16}$  classification, which is consistent with the NLSM analysis.

Later we will show that this analysis of the bulk phase transition using topological defects can be naturally generalized to all higher dimensions.

### 4. Boundary modes and bulk transition

Similarly to what has been discussed in the 1D and 2D cases, the interlayer coupling  $S_{\text{cp}}$  in Eq. (28) can be immediately ported to the fermion model as a four-fermion local interaction

(with  $A, B, C > 0$ ),

$$H_{\text{int}} = \int d^3x A \sum_{a=1,2,3} \Delta_a^{(1)} \Delta_a^{(2)} - B \Delta_4^{(1)} \Delta_4^{(2)} - C \Delta_5^{(1)} \Delta_5^{(2)}, \quad (29)$$

where  $\mathbf{\Delta} = \chi^\top (\sigma^{32212}, \sigma^{32220}, \sigma^{32232}, \sigma^{32300}, \sigma^{32100}) \chi$  are defined for both layers of the  $\nu = 8$  fermions. We should expect that 16 copies of the  ${}^3\text{He-B}$  TSC can be connected to the trivial state under this interaction, as the same interaction can trivialize the BSPT in the NLSM.

Following the same idea of the 2D case, we argue that the interaction can remove the 3D bulk criticality by showing that its 2D boundary states can be symmetrically gapped out under interaction.

Let us consider a boundary of the 3D system along the  $x_1$ - $x_3$  plane, i.e., the topological mass  $m \sim x_2$  changes sign across  $x_2 = 0$ . For eight copies of the  ${}^3\text{He-B}$  TSC as described in Eq. (26), the boundary states are given by the projection operator  $P = (1 - i\sigma^{10000}\sigma^{20000})/2$ , such that the effective FSM Hamiltonian along the boundary is given by

$$H'_{\times 8} = \frac{1}{2} \int dx_1 dx_3 \eta^\top h'_{\times 8} \eta, \\ h'_{\times 8} = i\partial_1 \sigma^{3000} + i\partial_3 \sigma^{1000} + n_1 \sigma^{2212} + n_2 \sigma^{2220} \\ + n_3 \sigma^{2232} + n_4 \sigma^{2300} + n_5 \sigma^{2100}, \quad (30)$$

where  $\eta$  denotes the Majorana surface modes and the  $\mathbb{Z}_2^T$  symmetry act as  $\mathcal{T} = \mathcal{K}i\sigma^{2000}$  on  $\eta$ . Under a series of basis transformations as follows:

$$\eta \rightarrow e^{\frac{i\pi}{4}\sigma^{1302}} e^{-\frac{i\pi}{4}\sigma^{1002}} e^{\frac{i\pi}{4}\sigma^{2300}} e^{-\frac{i\pi}{4}\sigma^{2000}} \\ \times e^{\frac{i\pi}{4}\sigma^{3102}} e^{-\frac{i\pi}{4}\sigma^{0302}} \eta, \quad (31)$$

the boundary Hamiltonian can be reformulated as

$$h'_{\times 8} = i\partial_1 \sigma^{3000} + i\partial_3 \sigma^{1300} + n_1 \sigma^{1112} + n_2 \sigma^{1120} \\ + n_3 \sigma^{1132} + n_4 \sigma^{1200} + n_5 \sigma^{2000}, \quad (32)$$

which, at the field-theory level, is equivalent to four copies of 2D (critical)  $p \pm ip$  TSC described by Eq. (14), with the transformed  $\mathbb{Z}_2^T$  symmetry  $\mathcal{T} = \mathcal{K}i\sigma^{2300}$ .  $(n_1, n_2, n_3, n_4)$  is the analog of the  $O(4)$  order parameter of the  $p \pm ip$  TSC. So, once again, the problem is reduced by one dimension to the critical 2D iFSPT states.

If we consider the boundary of 16 copies of the  ${}^3\text{He-B}$  TSC, it will simply be a doubling of Eq. (32), which is analogous to eight copies of (critical)  $p \pm ip$  TSC at the field-theory level. The bulk interaction Eq. (29) will induce the interaction between Majorana surface modes, which corresponds to the coupling of  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  at the boundary:

$$S'_{\text{cp}} = \int d\tau dx_1 dx_3 A' \sum_{a=1,2,3} n_a^{(1)} n_a^{(2)} \\ - B' n_4^{(1)} n_4^{(2)} - C' n_5^{(1)} n_5^{(2)}. \quad (33)$$

As we already argued, the  $A'$  and  $B'$  terms together can gap out the quantum critical point for eight copies of 2D  $p \pm ip$  TSC; this means that the same interaction in the field theory can also gap out the 2D boundary of 16 copies of the 3D  ${}^3\text{He-B}$

phase. And after the boundary is gapped out, adding a  $C'$  term will not close the gap at the boundary. So there should be no obstacle to smoothly connect 16 copies of  ${}^3\text{He-B}$  TSC to the trivial state, under the interaction that is ported from interlayer coupling for the corresponding 3D  $\mathbb{Z}_2^T$  BSPT states, i.e., the 3D bulk interaction in Eq. (29) can gap out the SPT to trivial state quantum critical point of 16 copies of  ${}^3\text{He-B}$ .

As one can see clearly now, the same pattern of logic will appear again and again in every spatial dimension. Using the dimension reduction argument, the boundary of the  $d$ -dimensional iFSPT state can be viewed, at the field-theory level, as the  $(d-1)$ -dimensional (critical) iFSPT. If the  $(d-1)$ -dimensional criticality can be gapped out by a  $(d-1)$ -dimensional interaction, then the same kind of interaction will be able to trivialize the boundary of the  $d$ -dimensional iFSPT state, and the  $d$ -dimensional bulk interaction that induces this  $(d-1)$ -dimensional boundary interaction likely gaps out the  $d$ -dimensional bulk criticality. Of course, one should be reminded that we are not saying that the  $d$ -dimensional iFSPT boundary has a  $(d-1)$ -dimensional lattice realization; our induction is only based on the effective-field-theory description of the long-wavelength physics. Following this induction approach, a class of the iFSPT states and their interaction-reduced classification can be studied systematically in all dimensions.

### III. SPT STATES WITH $\mathbb{Z}_2^P$ ONLY

#### A. General constructions

##### 1. Boson SPT with $\mathbb{Z}_2^P$

The boson SPT (BSPT) state with inversion symmetry exists in all dimensions. The construction is based on the  $O(d+2)$  nonlinear  $\sigma$  model (NLSM) in  $(d+1)$ -dimensional space-time with a topological  $\Theta$ -term at  $\Theta = 2\pi$ ,

$$S[\mathbf{n}] = \int d^{d+1}x \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \frac{i\Theta}{\Omega_{d+1}} \epsilon^{a_1 a_2 a_3 \dots a_{d+2}} \\ n_{a_1} \partial_0 n_{a_2} \partial_1 n_{a_3} \dots \partial_d n_{a_{d+2}}, \quad (34)$$

where  $x_0 \equiv \tau$  is the time coordinate and the rest of  $x_i$ 's ( $i = 1, \dots, d$ ) are space coordinates and  $\Omega_{d+1} = 2\pi^{\frac{d+2}{2}} / \Gamma(\frac{d+2}{2})$  is the volume of a  $(d+1)$ -hypersphere with unit radius. The action of the inversion symmetry  $\mathbb{Z}_2^P$  inverts the space and flips all components of  $\mathbf{n}$ ,

$$\mathcal{P} : \begin{cases} x_i \rightarrow -x_i & \text{for } i = 1, \dots, d \\ n_a \rightarrow -n_a & \text{for } a = 1, \dots, d+2 \end{cases}. \quad (35)$$

It is straightforward to check that the action Eq. (34) is invariant under this inversion. In the  $g \rightarrow \infty$  regime, the model has a unique gapped disordered ground state, which is a nontrivial SPT state when  $\Theta = 2\pi$  [31,51,66].

This BSPT state is  $\mathbb{Z}_2$  classified, meaning that two copies of such state can be smoothly connected to the trivial state without breaking the symmetry. To show this, we first make two copies of the model in Eq. (34), with  $\mathbf{n}$  vectors denoted by  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  in each copy, respectively, such that the total action reads  $S = S[\mathbf{n}^{(1)}] + S[\mathbf{n}^{(2)}]$ . Then we are allowed to turn on the following inversion symmetric coupling between



the two copies:

$$S_{\text{cp}} = \int d^{d+1}x A n_1^{(1)} n_1^{(2)} - B \sum_{a=2}^{d+2} n_a^{(1)} n_a^{(2)}, \quad (36)$$

In the limit of  $A, B \rightarrow +\infty$ ,  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  are locked together, and the final theory has effectively  $\Theta = 0$ , and it is a trivial state. Thus the BSPT with inversion symmetry is classified by  $\mathbb{Z}_2$  within the framework of NLSM.

However, in two-dimensional space (perhaps in some higher dimensions as well), there are additional  $\mathbb{Z}$ -classified BSPT states beyond NLSM, such as the  $E_8$  state in 2D. So the classifications in  $d = 2 \pmod{4}$  dimensions should be extended to  $\mathbb{Z}_2 \times \mathbb{Z}$ .

## 2. Free-fermion SPT with $\mathbb{Z}_2^P$

The free-fermion SPT (fFSPT) state with inversion symmetry also exists in all dimensions, described by the quadratic Majorana Hamiltonian

$$H = \frac{1}{2} \int d^d \mathbf{x} \chi^\top h_{\times 1} \chi, \quad (37)$$

$$h_{\times 1} = \sum_{i=1}^d i \partial_i \alpha^i + m \beta^0,$$

where  $\chi$  denotes the Majorana fermion operator.  $\alpha^i$  are symmetric matrices while  $\beta^0$  is antisymmetric, and they all anticommute with each other. The action of the inversion symmetry is given by the operator  $\mathcal{P} = \mathcal{I} i \beta^0$ , where  $\mathcal{I}$  is the space inversion operator such that  $\mathcal{I}^{-1} x_i \mathcal{I} = -x_i$  for all  $i = 1, \dots, d$ , and it is followed by an orthogonal transform  $i \beta^0$  in the Majorana basis, where  $\beta^0$  is just the mass matrix. Note that this inversion symmetry acts as  $\mathcal{P}^2 = -1$  on the Majorana fermions.

The  $\mathbb{Z}_2^P$  fFSPT state belongs to the symmetry class D and is  $\mathbb{Z}$  classified in general (see Table II in Ref. [67]). Because  $\mathcal{P} = \mathcal{I} i \beta^0$  rules out all the other additional mass terms that anticommute with the topological mass  $m \beta^0$ , one has to go through a bulk phase transition (by closing the single-particle gap) to drive the SPT state trivial (i.e., to change the sign of  $m$ ). The exception rests in  $d = 2 \pmod{4}$  dimensions, where the classification is extended to  $\mathbb{Z} \times \mathbb{Z}$ , which was pointed out in Ref. [67] and will be discussed in more detail later.

Although the field-theory Hamiltonian in Eq. (37) only describes the low-energy physics, it can be immediately cast into lattice models by the substitution  $i \partial_i \rightarrow \sin k_i$  and  $m \rightarrow \sum_{i=1}^d \cos k_i - d + m$ , with  $k_i$  being the quasimomentum of the fermion on the lattice. Some lattice models have been explicitly constructed in Ref. [67].

## 3. Interacting fermion SPT with $\mathbb{Z}_2^P$

The interacting fermion SPT (iFSPT) states can be obtained by introducing inversion-symmetric interaction terms to the free-fermion Hamiltonian in Eq. (37). As we discussed previously, interaction can reduce the classification of FSPT states, and the same phenomenon is expected here. To study the interaction-reduced classification, we still make use of the BSPT states discussed in the last section and connect the iFSPT

to BSPT by introducing bosonic  $\mathbf{n}$  degrees of freedom:

$$S = \int d^{d+1}x \frac{1}{2} \chi^\top (i \partial_0 + h_{\times \nu}) \chi + \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \dots, \quad (38)$$

$$h_{\times \nu} = \sum_{i=1}^d i \partial_i \alpha^i + m \beta^0 + \sum_{a=1}^{d+2} n_a \beta^a,$$

where  $h_{\times \nu}$  describes  $\nu$  copies of the fFSPT in Eq. (37) coupling to the bosonic fields  $n_a$ . Here  $\beta^a$  are anticommuting antisymmetric matrices, and they also anticommute with the all matrices  $\alpha^i$  and  $\beta^0$  [which has been enlarged from those in Eq. (37) by tensor product with  $\nu \times \nu$  identity matrix]. Of course, we need enough flavors of fermions (by making enough  $\nu$  copies of the fFSPT states) in order to support the  $d + 2$  additional  $\beta^a$  matrices. Integrating out the boson field  $\mathbf{n}$ , Eq. (38) gives a pure fermionic model with interaction. While integrating out the fermion field  $\chi$ , Eq. (38) becomes the NLSM as in Eq. (34). So Eq. (38) establishes a connection between the iFSPT and the BSPT phases [53].

The inversion symmetry act as

$$\mathcal{P} : \begin{cases} \mathbf{x} \rightarrow -\mathbf{x} \\ \chi \rightarrow i \beta^0 \chi \\ \mathbf{n} \rightarrow -\mathbf{n} \end{cases}. \quad (39)$$

It is straightforward to verify that the action in Eq. (38) respects this inversion symmetry. The inversion symmetry satisfies  $\mathcal{P}^2 = +1$  on the bosonic  $\mathbf{n}$  vector but acts projectively as  $\mathcal{P}^2 = -1$  on the Majorana fermion. With this set up, we can study the classification of iFSPT by resorting to the classification of BSPT states, which are much better understood.

## B. Examples in each dimension

### 1. $d = 1$

The  $\mathbb{Z}_2^P$  fFSPT phase in  $d = 1$  is classified by  $\mathbb{Z}$ , and its root state (Kitaev's Majorana chain) is described by the lattice model Eq. (1) (at  $\nu = 1$ ). The Hamiltonian is invariant under a bond-centered inversion symmetry  $\mathbb{Z}_2^P$  (see Fig. 1), which acts on the Majorana fermions as  $\mathcal{P} : \chi_A \rightarrow \chi_B, \chi_B \rightarrow -\chi_A$ . Because the bond is directed due to the imaginary hopping  $i u_{ij}$ , the inversion not only takes the fermion from  $A$  sites to  $B$  sites and vice versa but also must be followed by a gauge transformation, and hence we can have  $\mathcal{P}^2 = -1$  here. The inversion operator can be also written as  $\mathcal{P} = \mathcal{I} i \sigma^2$  in the basis  $\chi = (\chi_A, \chi_B)^\top$  where  $\mathcal{I}^{-1} x \mathcal{I} = -x$  implements the inversion of the spatial coordinate.

The low-energy effective Majorana Hamiltonian for the root state is given by Eq. (3) (at  $\nu = 1$ ), and we repeat here

$$h_{\times 1} = i \partial_1 \sigma^1 + m \sigma^2, \quad (40)$$

with  $\mathcal{P} = \mathcal{I} i \sigma^2$ . As long as this inversion symmetry is preserved, without interaction, the two sides of the phase diagram  $m > 0$  and  $m < 0$  are always separated by a gapless critical point at  $m = 0$ , no matter how many copies of the system we make.

To incorporate an  $O(3)$  order parameter, the model must be copied four times,

$$h_{\times 4} = i \partial_1 \sigma^{100} + m \sigma^{200} + n_1 \sigma^{312} + n_2 \sigma^{320} + n_3 \sigma^{332}, \quad (41)$$

with  $\mathcal{P} = \mathcal{I}i\sigma^{200}$  acting on the Majorana basis. Under  $\mathcal{P}$ , we must also require  $\mathbf{n} \rightarrow -\mathbf{n}$ . Then, if we integrate out the fermions, the effective theory becomes the O(3) NLSM at  $\Theta = 2\pi$  (Haldane spin chain). If we double the model  $h_{\times 4}$  again to  $h_{\times 8}$ , eight copies of this FSPT root state can be trivialized by interaction (see Sec. II for detailed analysis), as the corresponding BSPT state is trivial due to its  $\mathbb{Z}_2$  classification. Thus the  $\mathbb{Z}_2$  classification of the BSPT suggests that the  $\mathbb{Z}_2^P$  iFSPT in  $d = 1$  is  $\mathbb{Z}_8$  classified.

## 2. $d = 2$

The  $\mathbb{Z}_2^P$  fFSPT phase in  $d = 2$  is classified by  $\mathbb{Z} \times \mathbb{Z}$ , which has two root states. They are  $p + ip$  and  $p - ip$  TSC's, respectively. The free-fermion  $p + ip$  TSC in 2D is already  $\mathbb{Z}$  classified without any symmetry protection. However, with the inversion symmetry, the  $p + ip$  and  $p - ip$  TSC's are not allowed to trivialize with each other, thus we will have two independent  $\mathbb{Z}$  topological indices  $\nu_1$  and  $\nu_2$  labeling the copies of the  $p + ip$  and  $p - ip$  TSC's, respectively.

To study the interaction-reduced classification of this FSPT, we start from the special case when  $\nu_1 = \nu_2$  (i.e., the nonchiral  $p \pm ip$  TSC). The low-energy effective Majorana Hamiltonian for the nonchiral root state is given by Eq. (12) (at  $\nu = 1$ ), and we repeat here

$$h_{\times 1} = i\partial_1\sigma^{30} + i\partial_2\sigma^{13} + m\sigma^{20}, \quad (42)$$

with  $\mathcal{P} = \mathcal{I}i\sigma^{20}$ . In this case, the iFSPT can be studied by making connection to the BSPT within the scope of FSM. To incorporate an O(4) order parameter, the model must be copied four times,

$$h_{\times 4} = i\partial_1\sigma^{3000} + i\partial_2\sigma^{1300} + m\sigma^{2000} + n_1\sigma^{1112} + n_2\sigma^{1120} + n_3\sigma^{1132} + n_4\sigma^{1200}, \quad (43)$$

with  $\mathcal{P} = \mathcal{I}i\sigma^{2000}$  acting on the Majorana basis. Under  $\mathcal{P}$ , we must also require  $\mathbf{n} \rightarrow -\mathbf{n}$ . Then, if we integrate out the fermions, the effective theory becomes the O(4) NLSM at  $\Theta = 2\pi$ . If we double the model  $h_{\times 4}$  again to  $h_{\times 8}$ , eight copies of this FSPT root state can be trivialized by interaction (as was discussed in Sec. II), as the corresponding BSPT state is trivial due to its  $\mathbb{Z}_2$  classification.

When  $\nu_1 \neq \nu_2$ , the iFSPT state cannot be connected to a BSPT as discussed above, as no order parameter can be embedded no matter how many copies of the fFSPT state we make. However, we can consider such FSPT state as attaching additional layers of chiral  $p + ip$  TSC (or  $p - ip$  TSC) to the nonchiral  $p \pm ip$  TSC's. It is known that interaction cannot reduce the classification of the  $p + ip$  TSC, and the only possible effect of the interaction is to drive 16 copies of the  $p + ip$  TSC to a BSPT state known as the  $E_8$  state [53]. So we can simply extend the  $\mathbb{Z}_8$  classification of nonchiral iFSPT by attaching the  $\mathbb{Z}$ -classified chiral iFSPT, and, as a result, the  $\mathbb{Z}_2^P$  iFSPT in  $d = 2$  is  $\mathbb{Z}_8 \times \mathbb{Z}$  classified. Correspondingly, the BSPT in  $d = 2$  is  $\mathbb{Z}_2 \times \mathbb{Z}$ , in which the  $\mathbb{Z}$  index labels the number of  $E_8$  states.

## 3. $d = 3$

The  $\mathbb{Z}_2^P$  fFSPT phase in  $d = 3$  is classified by  $\mathbb{Z}$ . The low-energy effective Majorana Hamiltonian for the root state is

given by Eq. (25) (at  $\nu = 1$ ), and we repeat here

$$h_{\times 1} = i\partial_1\sigma^{33} + i\partial_2\sigma^{10} + i\partial_3\sigma^{31} + m\sigma^{20}, \quad (44)$$

with  $\mathcal{P} = \mathcal{I}i\sigma^{20}$ . To incorporate an O(5) order parameter, the model must be copied eight times,

$$h_{\times 8} = i\partial_1\sigma^{33000} + i\partial_2\sigma^{10000} + i\partial_3\sigma^{31000} + m\sigma^{20000} + n_1\sigma^{32212} + n_2\sigma^{32220} + n_3\sigma^{32232} + n_4\sigma^{32300} + n_5\sigma^{32100}, \quad (45)$$

with  $\mathcal{P} = \mathcal{I}i\sigma^{20000}$  acting on the Majorana basis. Under  $\mathcal{P}$ , we must also require  $\mathbf{n} \rightarrow -\mathbf{n}$ . Then, if we integrate out the fermions, the effective theory becomes the O(5) NLSM at  $\Theta = 2\pi$ , which is equivalent to the BSPT with  $\mathbb{Z}_2^P$  symmetry. If we double the model  $h_{\times 8}$  again to  $h_{\times 16}$ , 16 copies of this FSPT root state can be trivialized by interaction, as the corresponding BSPT state is trivial due to its  $\mathbb{Z}_2$  classification. Therefore the  $\mathbb{Z}_2^P$  iFSPT in  $d = 3$  is  $\mathbb{Z}_{16}$  classified.

## 4. Higher dimensions

The above examples can be systematically generalized to higher dimensions using the representation of Clifford algebras. In  $d$ -dimensional space, the nonchiral root state of the  $\mathbb{Z}_2^P$  fFSPT phase is described by the following Majorana Hamiltonian at low energy:

$$h_{\times 1} = \sum_{i=1}^d i\partial_i\alpha^i + m\beta^0, \quad (46)$$

with  $d$  symmetric matrices  $\alpha^i$  and one antisymmetric matrix  $\beta^0$ , which are taken from the generators of the real Clifford algebra  $\mathcal{C}\ell_{d,1}$ , i.e.,  $\{\alpha^1, \dots, \alpha^d; i\beta^0\}$  (see Appendix A for definitions). The inversion symmetry acts as  $\mathcal{P} = \mathcal{I}i\beta^0$ .

To construct the FSM in  $d$  dimensions, we must make enough copies of the fFSPT root state to incorporate the O( $d + 2$ ) order parameter. Supposing  $\nu$  is the minimal number of copies that should be made, we can write down the following Majorana Hamiltonian:

$$h_{\times \nu} = \sum_{i=1}^d i\partial_i\alpha^i + m\beta^0 + \sum_{a=1}^{d+2} n_a\beta^a, \quad (47)$$

in which the symmetric matrices  $\alpha^i$  ( $i = 1, \dots, d$ ) and the antisymmetric matrices  $\beta^a$  ( $a = 1, \dots, d + 2$ ) can be taken from the generators of the real Clifford algebra  $\mathcal{C}\ell_{d,d+2}$ , i.e.,  $\{\alpha^1, \dots, \alpha^d; i\beta^1, \dots, i\beta^{d+2}\}$  (see Appendix A for definitions), and the mass matrix  $\beta^0$  is chosen to be the pseudoscalar of  $\mathcal{C}\ell_{d,d+2}$ , i.e.,  $i\beta^0 = \prod_{i=1}^d \alpha^i \prod_{a=1}^{d+2} (i\beta^a)$ . It is straightforward to verify that the matrix  $\beta^0$  is antisymmetric by definition of  $\mathcal{C}\ell_{d,d+2}$  and hence qualified as a mass term. The inversion symmetry still acts as  $\mathcal{P} = \mathcal{I}i\beta^0$ , and under  $\mathcal{P}$ , we require  $\mathbf{n} \rightarrow -\mathbf{n}$  as well, such that  $h_{\times \nu}$  is inversion symmetric.

If we integrate out the fermions in the FSM  $H = \int d^d x \chi^\top h_{\times \nu} \chi$ , the effective theory becomes a NLSM of  $\mathbf{n}$  at  $\Theta = 2\pi$ . If we double  $h_{\times \nu}$  again to  $h_{\times 2\nu}$ ,  $2\nu$  copies of this  $d$ -dimensional FSPT root state can be trivialized by interaction, as the corresponding  $d$ -dimensional BSPT is also trivial due to its  $\mathbb{Z}_2$  classification. So the nonchiral  $\mathbb{Z}_2^P$  iFSPT is classified by  $\mathbb{Z}_{2\nu}$ . The minimal copy number  $\nu$  will be determined in the following. However, we recall that for  $d = 2$

mod 4, we also have the chiral FSPT states, which fall outside the FSM-based classification [32]. So the  $\mathbb{Z}_2^P$  iFSPT states in  $d = 2 \pmod 4$  dimension will be  $\mathbb{Z}_{2\nu} \times \mathbb{Z}$  classified, where  $\mathbb{Z}$  index labels the chiral FSPT states.

### C. The classification table

#### 1. Counting minimal copy number

The minimal copies  $\nu$  that one should make to go from the iFSPT root state to the FSM model can be simply determined from the Majorana fermion flavor numbers of both models. Given that  $h_{\times 1}$  and  $h_{\times \nu}$  are constructed using the irreducible representations of  $\mathcal{C}\ell_{d,1}$  and  $\mathcal{C}\ell_{d,d+2}$ , respectively, the minimal copy number  $\nu$  follows from

$$\nu = \frac{\dim \mathcal{C}\ell_{d,d+2}}{\dim \mathcal{C}\ell_{d,1}}, \quad (48)$$

where  $\dim \mathcal{C}\ell_{p,q}$  denotes the dimension of the irreducible real representation of the real Clifford algebra  $\mathcal{C}\ell_{p,q}$ . As concluded in Appendix A,  $\mathcal{C}\ell_{d,d+2} \cong \mathbb{H}(2^d)$ , so  $\dim \mathcal{C}\ell_{d,d+2} = 2^{d+2}$ . Therefore we have  $\nu = 2^{d+2} / \dim \mathcal{C}\ell_{d,1}$ , from which we can conclude the  $\mathbb{Z}_{2\nu}$  classification of  $\mathbb{Z}_2^P$  iFSPT states as in Table I. The classification of iFSPT also shows the eightfold Bott periodicity.

In  $d \pmod 4 = 2$  dimensions, the chiral FSPT states are not included in this classifying scheme [16–18]. The chiral FSPT states cannot be trivialized by the fermion interaction, and therefore they provide an additional  $\mathbb{Z}$  classification. So we conclude that the  $\mathbb{Z}_2^P$  iFSPT states in  $d \pmod 4 = 2$  dimensions are  $\mathbb{Z}_{2\nu} \times \mathbb{Z}$  classified.

#### 2. Bulk phase transition under interaction

Finally, we would like to mention that the same classification for the nonchiral states can be obtained by the same argument as in Sec. II C 3. The idea is that if  $2\nu$  copies of the FSPT root state can be trivialized by fermion interaction, then there must be a way to gap out its bulk phase transition with the trivial FSPT state without breaking the symmetry.

Again we start from the critical point, which corresponds to  $m = 0$  in the FSM Eq. (47). We first gap out the fermions in the bulk by setting up a fixed configuration of the order parameter field  $\mathbf{n}$  at the cost of breaking the inversion

TABLE I. The classification of  $\mathbb{Z}_2^P$  iFSPT states in each dimension  $d$ . The data of  $\mathcal{C}\ell_{d,1}$  (also see Appendix A) and the minimal copy number  $\nu$  are also listed.

$d \pmod 8$	$\dim \mathcal{C}\ell_{d,1}$	$\nu$	Classification
0	$\dim \mathbb{C}(2^{\frac{d}{2}}) = 2^{\frac{d+2}{2}}$	$2^{\frac{d+2}{2}}$	$\mathbb{Z}_2^{\frac{d+4}{2}}$
1	$\dim \mathbb{R}(2^{\frac{d+1}{2}}) = 2^{\frac{d+1}{2}}$	$2^{\frac{d+3}{2}}$	$\mathbb{Z}_2^{\frac{d+5}{2}}$
2	$\dim 2\mathbb{R}(2^{\frac{d}{2}}) = 2^{\frac{d+2}{2}}$	$2^{\frac{d+2}{2}}$	$\mathbb{Z}_2^{\frac{d+4}{2}} \times \mathbb{Z}$
3	$\dim \mathbb{R}(2^{\frac{d+1}{2}}) = 2^{\frac{d+1}{2}}$	$2^{\frac{d+3}{2}}$	$\mathbb{Z}_2^{\frac{d+5}{2}}$
4	$\dim \mathbb{C}(2^{\frac{d}{2}}) = 2^{\frac{d+2}{2}}$	$2^{\frac{d+2}{2}}$	$\mathbb{Z}_2^{\frac{d+4}{2}}$
5	$\dim \mathbb{H}(2^{\frac{d-1}{2}}) = 2^{\frac{d+3}{2}}$	$2^{\frac{d+1}{2}}$	$\mathbb{Z}_2^{\frac{d+3}{2}}$
6	$\dim 2\mathbb{H}(2^{\frac{d-2}{2}}) = 2^{\frac{d+4}{2}}$	$2^{\frac{d}{2}}$	$\mathbb{Z}_2^{\frac{d+2}{2}} \times \mathbb{Z}$
7	$\dim \mathbb{H}(2^{\frac{d-1}{2}}) = 2^{\frac{d+3}{2}}$	$2^{\frac{d+1}{2}}$	$\mathbb{Z}_2^{\frac{d+3}{2}}$

symmetry. Then we restore the symmetry by proliferating the inversion-symmetric topological defects of the  $\mathbf{n}$  field. Here we choose to focus on the point defect, namely the monopole configuration of  $\mathbf{n}$ , because such a defect generally exists in all dimensions. The monopole configuration is described by  $n_a \sim x_a$  (for  $a = 1, \dots, d$ ) and  $n_{d+1} = n_{d+2} = 0$  near the monopole core. Under inversion, both  $\mathbf{n}$  and  $\mathbf{x}$  changes sign, so the above monopole configuration is indeed inversion symmetric. Thus, if we can proliferate such monopoles, the inversion symmetry will be restored.

Again by solving the BdG equation for a single copy of the FSM, it can be shown that the monopole will always trap four Majorana zero modes no matter in which dimension  $d$ . This general property can be simply verified by counting the fermion flavors. The  $d$ -dimensional FSM has  $\dim \mathcal{C}\ell_{d,d+2} = 2^{d+2}$  flavors of Majorana fermions. Confining them to the core of a  $d$ -dimensional monopole will reduce the fermion flavor number by  $2^d$ , so the remaining flavor number is  $2^{d+2}/2^d = 4$ . Thus for two copies of FSM, the monopole will trap eight Majorana zero modes, whose degeneracy is protected on the free-fermion level by the inversion symmetry left in the monopole core, *together* with the assumption of  $m = 0$  at the critical point. Nevertheless, the degeneracy can be completely lifted by interaction [21,22] such that the monopole can be trivialized. So after the monopole proliferation, the inversion symmetry is restored, and we are left with a gapped symmetric state at  $m = 0$ . Therefore, with two copies of the FSM, the iFSPT state can be smoothly connected to the trivial state via strong interaction, resulting in the  $\mathbb{Z}_{2\nu}$  classification, which is consistent with the NLSM analysis.

### IV. SPT STATES WITH $\mathbb{Z}_2^P$ COMBINED WITH OTHER SYMMETRIES

#### A. $U(1) \times \mathbb{Z}_2^P$ SPT states

##### 1. BSPT with $U(1) \times \mathbb{Z}_2^P$

The  $U(1) \times \mathbb{Z}_2^P$  BSPT states can be studied similarly as the  $\mathbb{Z}_2^P$  BSPT states under the framework of the  $O(d+2)$  NLSM. The inversion symmetry still flips all components of  $\mathbf{n}$  as in Eq. (35). One remains to specify the  $U(1)$  symmetry action as well. Based on our experiences from lower-dimensional cases [31,50], the different ways of imposing the  $U(1)$  symmetry in the NLSM correspond to different BSPT root states.

In odd dimension  $d$ , there are  $(d+3)/2$  ways to impose the  $U(1)$  symmetry transformation, labeled by  $k = 0, \dots, (d+1)/2$  as

$$U_k : (n_{2a-1} + i n_{2a}) \rightarrow e^{i\theta} (n_{2a-1} + i n_{2a}) \quad \text{for } a = 1, 2, \dots, k, \quad (49)$$

whereas  $k = 0$  labels the case where the  $U(1)$  symmetry has no action on  $\mathbf{n}$ . Each assignment  $U_k$  of the symmetry action leads to a  $\mathbb{Z}_2$  classification of the BSPT states, as two layers of the BSPT root states can be trivialized via the interlayer coupling Eq. (36) as argued previously. So the  $U(1) \times \mathbb{Z}_2^P$  BSPT states in odd dimension is  $\mathbb{Z}_2^{(d+3)/2}$  classified.

In even dimension  $d$ , there are  $(d+4)/2$  ways to impose the  $U(1)$  symmetry transformation, labeled by  $k = 0, \dots, (d+2)/2$  following the same definition in Eq. (49). For the

TABLE II. The classification of  $U(1) \times \mathbb{Z}_2^P$  BSPT states.

$d \bmod 2$	Classification
0	$\mathbb{Z}_2^{(d+2)/2} \times \mathbb{Z}$
1	$\mathbb{Z}_2^{(d+3)/2}$

first  $(d+2)/2$  implementations  $U_k$  ( $k=1, \dots, d/2$ ), each leads to a  $\mathbb{Z}_2$  classification, respectively. However, the last implementation  $U_{(d+2)/2}$  leads to a  $\mathbb{Z}$  classification, as the coupling term Eq. (36) would necessary break the  $U(1)$  symmetry and is therefore forbidden, so there is no way to reduce the  $\mathbb{Z}$  classification. Thus the  $U(1) \times \mathbb{Z}_2^P$  BSPT states in even dimensions is  $\mathbb{Z}_2^{(d+2)/2} \times \mathbb{Z}$  classified.

We therefore conclude the classification of  $U(1) \times \mathbb{Z}_2^P$  BSPT states in Table II. However, this classification is not complete. The chiral states, such as the  $E_8$  states in 2D, are not covered by the NLSM classification.

### 2. Free-fermion SPT with $U(1) \times \mathbb{Z}_2^P$

With the  $U(1)$  symmetry, the Majorana fermions  $\chi$  can be paired up to Dirac fermions  $\psi = \chi' + i\chi''$  such that  $\psi \rightarrow e^{i\theta}\psi$  under the action of  $U(1)$ . The nonchiral  $U(1) \times \mathbb{Z}_2^P$  fFSPT root state can be described by the following Dirac Hamiltonian at low energy:

$$H = \int d^d \mathbf{x} \psi^\dagger \tilde{h}_{\times 1} \psi, \quad (50)$$

$$\tilde{h}_{\times 1} = \sum_{i=1}^d i \partial_i \gamma^i + m \gamma^{d+1},$$

in which  $\gamma^i$  ( $i=1, \dots, d+1$ ) are Hermitian complex matrices which anticommute with each other. They can be taken from the generators of the complex Clifford algebra  $\mathcal{C}\ell_{d+1}$  (in the complex representation). The action of inversion symmetry is given by the operator  $\mathcal{P} = \mathcal{I}i\gamma^{d+1}$  (if  $\mathcal{P}^2 = -1$ ) or by  $\mathcal{P} = \mathcal{I}\gamma^{d+1}$  (if  $\mathcal{P}^2 = +1$ ). In the presence of the  $U(1)$  symmetry, there is no essential difference between  $\mathcal{P}^2 = -1$  and  $\mathcal{P}^2 = +1$  [as they only differed by a  $U(1)$  rotation which is part of the symmetry], so we will focus on the former case.

The  $U(1) \times \mathbb{Z}_2^P$  fFSPT state belongs to the symmetry class A and is  $\mathbb{Z}$  classified in odd dimensions and is  $\mathbb{Z} \times \mathbb{Z}$  classified in even dimensions (see Tables I and II in Ref. [67]). Because the  $U(1)$  symmetry rules out any fermion pairing terms, and within the fermion hopping terms the inversion symmetry  $\mathcal{P}$  forbids all the other possible mass terms that anticommute with the topological mass  $m\gamma^{d+1}$ , one therefore has to go through a bulk transition (by closing the single-particle gap) to drive the SPT state trivial (i.e., to change the sign of  $m$ ). This explains the  $\mathbb{Z}$  classification in odd dimensions and one of the  $\mathbb{Z}$  classifications in even dimensions. The other  $\mathbb{Z}$  classification in even dimensions comes from the chiral state, whose root state is described by the Dirac Hamiltonian

$$\tilde{h}_{\times 1} = \sum_{i=1}^d i \partial_i \gamma^i + m' \gamma^{\text{ch}}, \quad (51)$$

where  $\gamma^{\text{ch}} \equiv \prod_{i=1}^d \gamma^i$  is the chiral matrix (which exists only for even  $d$ ). The chiral mass  $m'\gamma^{\text{ch}}$  also preserves the  $U(1) \times$

$\mathbb{Z}_2^P$  symmetry. It is impossible to find any additional  $U(1)$  preserving mass term that anticommute with the chiral mass, so the chiral states lead to the other  $\mathbb{Z}$  classification in even dimensions.

### 3. Interacting fermion SPT with $U(1) \times \mathbb{Z}_2^P$

The nonchiral  $U(1) \times \mathbb{Z}_2^P$  iFSPT states can be also studied by extending the fFSPT model to the FSM. In each dimension, the FSM is still given by Eq. (38) in the Majorana fermion basis, with the matrices  $\alpha^i$  ( $i=1, \dots, d$ ) and  $\beta^a$  ( $a=1, \dots, d+2$ ) taken from the generators of the real Clifford algebra  $\mathcal{C}\ell_{d,d+2}$  and the mass matrix  $\beta^0$  chosen to be the pseudoscalar of  $\mathcal{C}\ell_{d,d+2}$ . We must make enough copies of the  $U(1) \times \mathbb{Z}_2^P$  fFSPT root states described in Eq. (50) to obtain the FSM. To count the number  $\nu$  of copies correctly, we first rewrite the Hamiltonian in Eq. (50) in the Majorana basis, which takes the form of Eq. (37), with the matrices  $\alpha^i$  ( $i=1, \dots, d$ ) and  $\beta^0$  still taken from the generators of the complex Clifford algebra  $\mathcal{C}\ell_{d+1}$  but using its real representation. Then the minimal copy number  $\nu$  is given by

$$\nu = \frac{\dim \mathcal{C}\ell_{d,d+2}}{\dim \mathcal{C}\ell_{d+1}}, \quad (52)$$

where  $\dim \mathcal{C}\ell_n$  denotes the dimension of the irreducible real representation of the complex Clifford algebra  $\mathcal{C}\ell_n$ . Given that  $\dim \mathcal{C}\ell_{d,d+2} = 2^{d+2}$  (see Appendix A),  $\nu$  in each dimension  $d$  can be calculated in Table III. We conclude that the nonchiral  $U(1) \times \mathbb{Z}_2^P$  fFSPT root state can be made into a FSM incorporating the  $O(d+2)$  order parameters by copying  $\nu$  times, such that their corresponding iFSPT states can be classified by making a connection to the BSPT classifications. However, the chiral fFSPT root state (which appears in even dimensions) cannot be connected to the FSM without breaking the  $U(1)$  symmetry and should be classified separately.

Integrating out the fermions in the FSM, we arrive at the  $O(d+2)$  NLSM in Eq. (34) at  $\Theta = 2\pi$ , with the symmetry action inherited from the FSM such that the inversion symmetry flips all components of  $\mathbf{n}$ , while the  $U(1)$  symmetry rotates two and only two components of  $\mathbf{n}$ , say,  $(n_1 + in_2) \rightarrow e^{i\theta}(n_1 + in_2)$  (see Appendix B for examples). The action of  $U(1)$  here corresponds to the  $U_1$  implementation as defined in Eq. (49), which gives a  $\mathbb{Z}_2$  classification of the BSPT states, meaning that two copies of the FSM can be trivialized by the interaction which also trivialize the corresponding BSPT states. As have been counted in Eq. (52), each copy of the FSM corresponds to  $\nu$  copies of the fFSPT root states, so the nonchiral  $U(1) \times \mathbb{Z}_2^P$  iFSPT states are  $\mathbb{Z}_{2\nu}$  classified (see Table III). For the chiral fFSPT states, it is not possible to extend them to the FSM without breaking the  $U(1)$  symmetry,

 TABLE III. The classification of  $U(1) \times \mathbb{Z}_2^P$  iFSPT states in each dimension  $d$ . The data of  $\mathcal{C}\ell_{d+1}$  (also see Appendix A) and the minimal copy number  $\nu$  are also listed.

$d \bmod 2$	$\dim \mathcal{C}\ell_{d+1}$	$\nu$	Classification
0	$\dim 2\mathbb{C}(2^{\frac{d}{2}}) = 2^{\frac{d+4}{2}}$	$2^{\frac{d}{2}}$	$\mathbb{Z}_2^{\frac{d+2}{2}} \times \mathbb{Z}$
1	$\dim \mathbb{C}(2^{\frac{d+1}{2}}) = 2^{\frac{d+3}{2}}$	$2^{\frac{d+1}{2}}$	$\mathbb{Z}_2^{\frac{d+3}{2}}$

thus their classification cannot be reduced by the interaction which is still  $\mathbb{Z}$ . In conclusion, the  $U(1) \times \mathbb{Z}_2^P$  iFSPT states are  $\mathbb{Z}_2^{d+3}$  classified in odd  $d$  dimensions and are  $\mathbb{Z}_2^{d+2} \times \mathbb{Z}$  classified in even  $d$  dimensions.

## B. $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$ SPT states

### 1. Boson SPT with $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$

We study the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  BSPT states under the framework of the  $O(d+2)$  NLSM. The inversion symmetry  $\mathbb{Z}_2^P$  always flips all components of  $\mathbf{n}$  as in Eq. (35), while the time-reversal symmetry  $\mathbb{Z}_2^T$  must flip odd numbers of  $\mathbf{n}$  components to keep the  $\Theta$ -term invariant. Based on our experiences gained from lower-dimensional cases [31], the different ways of imposing the  $\mathbb{Z}_2^T$  symmetry in the NLSM correspond to different of BSPT root states.

In odd dimension  $d$ , there are  $(d+3)/2$  ways to impose the  $\mathbb{Z}_2^T$  symmetry transformation, labeled by  $k = 0, \dots, (d+1)/2$  as

$$\mathcal{T}_k : n_i \rightarrow -n_i \quad \text{for } i = 1, \dots, 2k+1, \quad (53)$$

which flips the first  $(2k+1)$  components of  $\mathbf{n}$ , leaving the rest of the components unchanged. Each assignment  $\mathcal{T}_k$  of the symmetry action leads to a  $\mathbb{Z}_2$  classification of the BSPT states, as two layers of the BSPT root states can be trivialized via the interlayer coupling Eq. (36), as argued previously. So the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  BSPT states in odd dimension is  $\mathbb{Z}_2^{(d+3)/2}$  classified.

In even dimension  $d$ , there are  $(d+2)/2$  ways to impose the  $U(1)$  symmetry, labeled by  $k = 0, \dots, d/2$  following the same definition in Eq. (53). Each assignment  $\mathcal{T}_k$  still leads to a  $\mathbb{Z}_2$  classification. Thus the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  BSPT states in even dimension is  $\mathbb{Z}_2^{(d+2)/2}$  classified.

We therefore conclude the classification of  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  BSPT states in Table IV. Again this is not a complete classification, as the analogs of the  $E_8$  states are not considered here.

### 2. Free-fermion SPT with $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$

According to Ref. [67], there are no  $\mathbb{Z}$ -classified free-fermion  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  SPT states if the time-reversal  $\mathcal{T}$  and the inversion  $\mathcal{P}$  commute with each other. For our purposes, to study the interaction-reduced classification of FSPT states, we wish to start with  $\mathbb{Z}$ -classified fFSPT states. Therefore we consider a peculiar setting where  $\mathcal{T}$  and  $\mathcal{P}$  do not commute, and the symmetry group (acting on the fermions) is defined by

$$\mathcal{T}^2 = \mathcal{P}^2 = -1, \quad \mathcal{T}\mathcal{P}\mathcal{T}\mathcal{P} = -1. \quad (54)$$

This is a projective representation of the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  symmetry, which can be realized as a projective symmetry group [68] if the fermions are coupled to a  $\mathbb{Z}_2$  gauge field (such as spinons in the  $\mathbb{Z}_2$  spin-liquid).

TABLE IV. The classification of  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  BSPT states.

$d \bmod 2$	Classification
0	$\mathbb{Z}_2^{(d+2)/2}$
1	$\mathbb{Z}_2^{(d+3)/2}$

In the presence of the time-reversal symmetry, the chiral SPT states are ruled out. The  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  fFSPT root state is nonchiral and can be described by the following Majorana Hamiltonian at low energy:

$$H = \frac{1}{2} \int d^d \mathbf{x} \chi^\top h_{\times 1} \chi, \quad (55)$$

$$h_{\times 1} = \sum_{i=1}^d i \partial_i \alpha^i + m \beta^1,$$

with the inversion symmetry  $\mathcal{P} = \mathcal{I}i\beta^1$  and time-reversal symmetry  $\mathcal{T} = \mathcal{K}i\beta^2$ . Here the symmetric matrices  $\alpha^i$  ( $i = 1, \dots, d$ ) and the antisymmetric matrices  $\beta^1, \beta^2$  are taken from the generators of the real Clifford algebra  $\mathcal{C}\ell_{d,2}$ , i.e.,  $\{\alpha^1, \dots, \alpha^d; i\beta^1, i\beta^2\}$  (see Appendix A for definitions). It worth mentioning that at dimensions  $d = 3, 7 \pmod{8}$ , the representations  $\mathcal{C}\ell_{3,2} \cong \mathbb{R}(4) \oplus \mathbb{R}(4)$  and  $\mathcal{C}\ell_{7,2} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$  can be split into two subalgebras. Each subalgebra is sufficient to faithfully represent the anticommutation relations among the generators. So the minimal fermion flavor is only half of  $\dim \mathcal{C}\ell_{d,2}$  when  $d = 3, 7 \pmod{8}$ . For later convenience, we define the reduced dimension  $\text{rdim}$  as the dimension of the minimal faithful representation of the anticommuting generators (but not the whole algebra), which follows

$$\text{rdim} \mathcal{C}\ell_{p,q} \equiv \begin{cases} \frac{1}{2} \dim \mathcal{C}\ell_{p,q} & p - q = 1, 5 \pmod{8}, \\ \dim \mathcal{C}\ell_{p,q} & \text{otherwise.} \end{cases} \quad (56)$$

Thus in terms of the reduced dimension, the Majorana fermion flavor number of the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  root state in Eq. (55) is simply given by  $\text{rdim} \mathcal{C}\ell_{d,2}$  in dimension  $d$ .

Because the inversion symmetry  $\mathcal{P}$  has ruled out all the other possible mass terms that anticommute with the topological mass  $m\beta^1$ , which already leads to the  $\mathbb{Z}$  classification, the additional time-reversal symmetry will not change the classification. So the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  symmetry defined in the Eq. (54) fFSPT states is  $\mathbb{Z}$  classified.

### 3. Interacting fermion SPT with $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$

The (projective)  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  iFSPT states can be also studied by extending the fFSPT model to the FSM. In each dimension, the FSM is still given by Eq. (38) in the Majorana fermion basis, with the matrices  $\alpha^i$  ( $i = 1, \dots, d$ ) and  $\beta^a$  ( $a = 1, \dots, d+2$ ) taken from the generators of the real Clifford algebra  $\mathcal{C}\ell_{d,d+2}$ , and the mass matrix  $\beta^0$  is chosen to be the pseudoscalar of  $\mathcal{C}\ell_{d,d+2}$ . We must make enough copies of the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  fFSPT root states described in Eq. (55) to obtain the FSM. Then the minimal copy number  $\nu$  is given by

$$\nu = \frac{\dim \mathcal{C}\ell_{d,d+2}}{\text{rdim} \mathcal{C}\ell_{d,2}}, \quad (57)$$

where  $\text{rdim} \mathcal{C}\ell_{d,2}$  is the reduced dimension defined in Eq. (56). Given that  $\dim \mathcal{C}\ell_{d,d+2} = 2^{d+2}$  (see Appendix A),  $\nu$  in each dimension  $d$  can be calculated in Table V. We conclude that the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  fFSPT root state can be made into a FSM incorporating the  $O(d+2)$  order parameters by copying  $\nu$  times, such that their corresponding iFSPT states can be classified by making a connection to the BSPT classifications.

TABLE V. The classification of  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  iFSPT states in each dimension  $d$ . The data of  $\mathcal{C}\ell_{d,2}$  (also see Appendix A) and the minimal copy number  $\nu$  are also listed.

$d \bmod 8$	$\text{rdim } \mathcal{C}\ell_{d,2}$	$\nu$	Classification
0	$\text{rdim } \mathbb{H}(2^{\frac{d}{2}}) = 2^{\frac{d+4}{2}}$	$2^{\frac{d}{2}}$	$\mathbb{Z}_2^{\frac{d+2}{2}}$
1	$\text{rdim } \mathbb{C}(2^{\frac{d+1}{2}}) = 2^{\frac{d+3}{2}}$	$2^{\frac{d+1}{2}}$	$\mathbb{Z}_2^{\frac{d+3}{2}}$
2	$\text{rdim } \mathbb{R}(2^{\frac{d+2}{2}}) = 2^{\frac{d+2}{2}}$	$2^{\frac{d+2}{2}}$	$\mathbb{Z}_2^{\frac{d+4}{2}}$
3	$\text{rdim } 2\mathbb{R}(2^{\frac{d+1}{2}}) = 2^{\frac{d+1}{2}}$	$2^{\frac{d+3}{2}}$	$\mathbb{Z}_2^{\frac{d+5}{2}}$
4	$\text{rdim } \mathbb{R}(2^{\frac{d+2}{2}}) = 2^{\frac{d+2}{2}}$	$2^{\frac{d+2}{2}}$	$\mathbb{Z}_2^{\frac{d+4}{2}}$
5	$\text{rdim } \mathbb{C}(2^{\frac{d+1}{2}}) = 2^{\frac{d+3}{2}}$	$2^{\frac{d+1}{2}}$	$\mathbb{Z}_2^{\frac{d+3}{2}}$
6	$\text{rdim } \mathbb{H}(2^{\frac{d}{2}}) = 2^{\frac{d+4}{2}}$	$2^{\frac{d}{2}}$	$\mathbb{Z}_2^{\frac{d+2}{2}}$
7	$\text{rdim } 2\mathbb{H}(2^{\frac{d-1}{2}}) = 2^{\frac{d+3}{2}}$	$2^{\frac{d+1}{2}}$	$\mathbb{Z}_2^{\frac{d+3}{2}}$

Integrating out the fermions in the FSM, we arrive at the  $O(d+2)$  NLSM in Eq. (34) at  $\Theta = 2\pi$ , with the symmetry action inherited from the FSM such that the inversion symmetry flips all components of  $\mathbf{n}$ , while the  $\mathbb{Z}_2^T$  symmetry always flips an odd number of  $\mathbf{n}$  components (see Appendix B for examples). Such an implementation of the symmetry action always results in  $\mathbb{Z}_2$ -classified BSPT states, meaning that two copies of the FSM can be trivialized by the interaction which also trivializes the corresponding BSPT states. As have been counted in Eq. (57), each copy of the FSM corresponds to  $\nu$  copies of the fFSPT root states, so the  $\mathbb{Z}_2^T \times \mathbb{Z}_2^P$  iFSPT states are  $\mathbb{Z}_{2\nu}$  classified (see Table V).

## V. SUMMARY

In this paper we systematically studied the classification of a large class of strongly interacting fermionic and bosonic SPT states in all dimensions. And for all the examples we considered in this paper, we argue that the classification of the BSPT states implies that short-range interactions can reduce the classification of FSPT states with the same symmetry. Furthermore, using different methods, we argue that certain interactions can gap out the critical point between the FSPT state and the trivial state in the noninteracting limit, which implies that under interaction some FSPT states are driven trivial, and it can be connected to the trivial state without the bulk phase transition.

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## APPENDIX A: IRREDUCIBLE REPRESENTATION OF CLIFFORD ALGEBRA

The generators of the real Clifford algebra  $\mathcal{C}\ell_{p,q}$  can be represented by a set of real matrices  $\{\alpha^1, \dots, \alpha^p; i\beta^1, \dots, i\beta^q\}$

anticommuting with each other,

$$\begin{aligned} \alpha^i \alpha^j &= -\alpha^j \alpha^i, & \beta^i \beta^j &= -\beta^j \beta^i & \text{for } i \neq j, \\ \alpha^i \beta^j &= -\beta^j \alpha^i & \text{for any } i, j, \end{aligned} \quad (\text{A1})$$

among which the  $\alpha^i$  matrices square to 1 (as  $\alpha^i \alpha^i = 1$ ) and the  $i\beta^i$  matrices squares to  $-1$  (as  $i\beta^i i\beta^i = -1$ ). We adopt this notation such that both  $\alpha^i$  and  $\beta^j$  matrices are Hermitian and can be expressed as the direction product of Pauli matrices.  $\alpha^i$ 's are real and transpose symmetric, and there are  $p$  of them in the generators of  $\mathcal{C}\ell_{p,q}$ ; while  $\beta^i$ 's are imaginary and transpose antisymmetric, and there are  $q$  of them in the generators of  $\mathcal{C}\ell_{p,q}$ .

The real Clifford algebras are isomorphic to the matrix algebras of real numbers  $\mathbb{R}$ , complex numbers  $\mathbb{C}$ , or quaternions  $\mathbb{H}$ . The first several examples include  $\mathcal{C}\ell_{0,0} \cong \mathbb{R}$ , whose irreducible representation is one dimensional (just a real number);  $\mathcal{C}\ell_{0,1} \cong \mathbb{C}$ , generated by  $\{i\sigma^2\}$  giving a two-dimensional irreducible representation ( $2 \times 2$  real matrix); and  $\mathcal{C}\ell_{0,2} \cong \mathbb{H}$ , generated by  $\{i\sigma^{12}, i\sigma^{32}\}$  giving a four-dimensional irreducible representation ( $4 \times 4$  real matrix). Here  $\sigma^{ijk\dots} = \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes \dots$  denotes the direct product of a series of Pauli matrices  $\sigma^0, \sigma^1, \sigma^2$ , or  $\sigma^3$ . The irreducible representations of the real Clifford algebra are concluded in Table VI, where  $\mathbb{R}(N)$ ,  $\mathbb{C}(N)$ , and  $\mathbb{H}(N)$  denote the algebras of  $N \times N$  matrix over  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$ , respectively, and  $2\mathbb{R}(N)$ ,  $2\mathbb{H}(N)$  are shorthand notations of  $\mathbb{R}(N) \oplus \mathbb{R}(N)$ ,  $\mathbb{H}(N) \oplus \mathbb{H}(N)$ . The larger Clifford algebra lying outside the table can be obtained by the eightfold Bott periodicity, namely  $\mathcal{C}\ell_{p+8,q} \cong \mathcal{C}\ell_{p,q+8} \cong \mathcal{C}\ell_{p,q} \otimes \mathbb{R}(16)$ . From Table VI, the dimension of the (real) irreducible representation of the Clifford algebra can be easily read out as  $\dim \mathbb{R}(N) = N$ ,  $\dim \mathbb{C}(N) = 2N$ ,  $\dim \mathbb{H}(N) = 4N$ ,  $\dim 2\mathbb{R}(N) = 2N$ , and  $\dim 2\mathbb{H}(N) = 8N$ .

The complex Clifford algebra  $\mathcal{C}\ell_n$  is much simpler, whose generators can be represented by a set of complex matrices  $\{\gamma^1, \dots, \gamma^n\}$ , satisfying

$$\gamma^i \gamma^j = -\gamma^j \gamma^i \text{ (for } i \neq j), \quad \gamma^i \gamma^i = 1. \quad (\text{A2})$$

The complex Clifford algebras are isomorphic to the matrix algebras of complex numbers  $\mathbb{C}$ . For even  $n$ ,  $\mathcal{C}\ell_{2m} \cong \mathbb{C}(2^m)$ , and for odd  $n$ ,  $\mathcal{C}\ell_{2m+1} \cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$  [or shorthanded as  $2\mathbb{C}(2^m)$ ]. Their real irreducible representations are of the dimensions  $\dim \mathbb{C}(2^m) = 2^{m+1}$  and  $\dim 2\mathbb{C}(2^m) = 2^{m+2}$ .

TABLE VI. Periodic table of real Clifford algebras.

$\mathcal{C}\ell_{p,q}$	$q$							
	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$2\mathbb{R}(8)$
1	$2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
2	$\mathbb{R}(2)$	$2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$2\mathbb{H}(8)$	$\mathbb{H}(16)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$2\mathbb{H}(16)$
5	$2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
6	$\mathbb{H}(4)$	$2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$2\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$2\mathbb{R}(64)$	$\mathbb{R}(128)$

**APPENDIX B: FERMION  $\sigma$ -MODEL IN EACH DIMENSION**

Here we enumerate the examples of the fermion  $\sigma$  model (FSM) in each dimension with explicit matrix representation and show how the various symmetry action is embedded in the FSM. The action of the FSM in  $d$  dimensions takes the following general form:

$$S = \int d^{d+1}x \frac{1}{2} \chi^\top (i\partial_0 + h^{(d)}) \chi + \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \dots, \quad (B1)$$

$$h^{(d)} = \sum_{i=1}^d i \partial_i \alpha^i + m \beta^0 + \sum_{a=1}^{d+2} n_a \beta^a,$$

where  $\alpha^i$  ( $i = 1, \dots, d$ ) are transpose-symmetric Hermitian matrices and  $\beta^a$  ( $a = 0, \dots, d+2$ ) are transpose-antisymmetric Hermitian matrices. We consider that the inversion symmetry always act as  $\mathcal{P} = \mathcal{I}i\beta^0$ . We will provide the explicit examples of these matrices in the fermion Hamiltonian  $h^{(d)}$ . In general, the Majorana fermion  $\chi$  is of  $2^{d+2}$  flavors, meaning that the dimensions of the matrices  $\alpha^i$  and  $\beta^a$  are  $2^{d+2}$ . We will use the notation  $\sigma^{ijk\dots} = \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes \dots$  to denote the direct product of Pauli matrices  $\sigma^1, \sigma^2, \sigma^3$  as well as the  $2 \times 2$  identity matrix  $\sigma^0$ . Each Pauli matrix  $\sigma^i$  acts on a two-dimensional single-particle Hilbert space, which is also the size of a qubit. We may thus count the dimension of the single-particle Hilbert space (which is also the Majorana fermion flavor number) by qubits. A Hamiltonian made of matrices like  $\sigma^{ijk\dots}$  with  $n$  indices acts in the Hilbert space of  $n$  qubit which is of the dimension  $2^n$ . In the following, we will use examples to demonstrate both the FSM and the FSPT root-state model with various symmetries.

In the  $d = 1$  spacial dimension, the Hamiltonian  $h^{(1)}$  is defined on a three-qubit single-particle Hilbert space,

$$h^{(1)} = i \partial_1 \sigma^{100} + m \sigma^{200} + n_1 \sigma^{312} + n_2 \sigma^{332} + n_3 \sigma^{320}. \quad (B2)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i \sigma^{200} \chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first one-qubit subspace and must be 4-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi \sigma^{020}) \chi, (n_1 + i n_2) \rightarrow e^{2i\varphi} (n_1 + i n_2)$  symmetry, the root-state Hamiltonian takes the first two-qubit subspace and must be doubled to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i \sigma^{320} \chi, n_3 \rightarrow -n_3$  symmetry, the root-state Hamiltonian takes the first two-qubit subspace and must be doubled to form the FSM.

In  $d = 2$  spacial dimension, the Hamiltonian  $h^{(2)}$  is defined on a four-qubit single-particle Hilbert space,

$$h^{(2)} = i \partial_1 \sigma^{1000} + i \partial_2 \sigma^{3100} + m \sigma^{2000} + n_1 \sigma^{3312} + n_2 \sigma^{3332} + n_3 \sigma^{3320} + n_4 \sigma^{3200}. \quad (B3)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i \sigma^{2000} \chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first two-qubit subspace and must be 4-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi \sigma^{0020}) \chi, (n_1 + i n_2) \rightarrow e^{2i\varphi} (n_1 + i n_2)$  symmetry, the root-state Hamiltonian takes the first three-qubit subspace and must be doubled to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i \sigma^{3200} \chi, n_4 \rightarrow -n_4$  symmetry, the root-state Hamiltonian takes the first two-qubit subspace and must be 4-multiplied to form the FSM.

In the  $d = 3$  spacial dimension, the Hamiltonian  $h^{(3)}$  is defined on a five-qubit single-particle Hilbert space,

$$h^{(3)} = i \partial_1 \sigma^{10000} + i \partial_2 \sigma^{31000} + i \partial_3 \sigma^{33000} + m \sigma^{20000} + n_1 \sigma^{32100} + n_2 \sigma^{32300} + n_3 \sigma^{32212} + n_4 \sigma^{32232} + n_5 \sigma^{32220}. \quad (B4)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i \sigma^{20000} \chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first two-qubit subspace and must be 8-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi \sigma^{00200}) \chi, (n_1 + i n_2) \rightarrow e^{2i\varphi} (n_1 + i n_2)$  symmetry, the root-state Hamiltonian takes the first three-qubit subspace and must be 4-multiplied to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i \sigma^{32000} \chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry, the root-state Hamiltonian takes the first two-qubit subspace and must be 8-multiplied to form the FSM.

In the  $d = 4$  spacial dimension, the Hamiltonian  $h^{(4)}$  is defined on a six-qubit single-particle Hilbert space,

$$h^{(4)} = i \partial_1 \sigma^{100000} + i \partial_2 \sigma^{310000} + i \partial_3 \sigma^{330000} + i \partial_4 \sigma^{322000} + m \sigma^{200000} + n_1 \sigma^{321100} + n_2 \sigma^{321300} + n_3 \sigma^{321212} + n_4 \sigma^{321232} + n_5 \sigma^{321220} + n_6 \sigma^{323000}. \quad (B5)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i \sigma^{200000} \chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first three-qubit subspace and must be 8-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi \sigma^{000200}) \chi, (n_1 + i n_2) \rightarrow e^{2i\varphi} (n_1 + i n_2)$  symmetry, the root-state Hamiltonian takes the first four-qubit subspace and must be 4-multiplied to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i \sigma^{323000} \chi, n_6 \rightarrow -n_6$  symmetry, the root-state Hamiltonian takes the first three-qubit subspace and must be 8-multiplied to form the FSM.

In the  $d = 5$  spacial dimension, the Hamiltonian  $h^{(5)}$  is defined on a seven-qubit single-particle Hilbert space,

$$h^{(5)} = i \partial_1 \sigma^{1000000} + i \partial_2 \sigma^{3100000} + i \partial_3 \sigma^{3300000} + i \partial_4 \sigma^{3212000} + i \partial_5 \sigma^{3232000} + m \sigma^{2000000} + n_1 \sigma^{3201000} + n_2 \sigma^{3203000} + n_3 \sigma^{3222100} + n_4 \sigma^{322300} + n_5 \sigma^{3222212} + n_6 \sigma^{322232} + n_7 \sigma^{3222220}. \quad (B6)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i \sigma^{2000000} \chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first four-qubit subspace and must be 8-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi \sigma^{0002000}) \chi, (n_1 + i n_2) \rightarrow e^{2i\varphi} (n_1 + i n_2)$  symmetry, the root-state Hamiltonian takes the first four-qubit subspace and must be 8-multiplied to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i \sigma^{3201000} \chi, n_1 \rightarrow -n_1$  symmetry, the root-state Hamiltonian takes the first four-qubit subspace and must be 8-multiplied to form the FSM.

In the  $d = 6$  spacial dimension, the Hamiltonian  $h^{(6)}$  is defined on an eight-qubit single-particle Hilbert space,

$$h^{(6)} = i \partial_1 \sigma^{10000000} + i \partial_2 \sigma^{31000000} + i \partial_3 \sigma^{33000000} + i \partial_4 \sigma^{32120000} + i \partial_5 \sigma^{32320000} + i \partial_6 \sigma^{32222000} + m \sigma^{20000000} + n_1 \sigma^{32221000} + n_2 \sigma^{32223000} + n_3 \sigma^{32010100} + n_4 \sigma^{32010300} + n_5 \sigma^{32010212} + n_6 \sigma^{32010232} + n_7 \sigma^{32010220} + n_8 \sigma^{32030000}. \quad (B7)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i\sigma^{20000000}\chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first five-qubit subspace and must be 8-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi\sigma^{00002000})\chi, (n_1 + in_2) \rightarrow e^{2i\varphi}(n_1 + in_2)$  symmetry, the root-state Hamiltonian takes the first five-qubit subspace and must be 8-multiplied to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i\sigma^{32030000}\chi, n_6 \rightarrow -n_6$  symmetry, the root-state Hamiltonian takes the first five-qubit subspace and must be 8-multiplied to form the FSM.

In the  $d = 7$  spacial dimension, the Hamiltonian  $h^{(7)}$  is defined on a nine-qubit single-particle Hilbert space,

$$\begin{aligned} h^{(7)} = & i\partial_1\sigma^{10000000} + i\partial_2\sigma^{31000000} + i\partial_3\sigma^{33000000} + i\partial_4\sigma^{32120000} + i\partial_5\sigma^{32320000} + i\partial_6\sigma^{32012000} + i\partial_7\sigma^{32032000} \\ & + m\sigma^{20000000} + n_1\sigma^{322010002} + n_2\sigma^{322030002} + n_3\sigma^{322221200} + n_4\sigma^{322223200} + n_5\sigma^{322220120} + n_6\sigma^{322220320} \\ & + n_7\sigma^{322222010} + n_8\sigma^{322222030} + n_9\sigma^{322222220}. \end{aligned} \quad (B8)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i\sigma^{20000000}\chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first five-qubit subspace and must be 16-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi\sigma^{00002000})\chi, (n_1 + in_2) \rightarrow e^{2i\varphi}(n_1 + in_2)$  symmetry, the root-state Hamiltonian takes the first five-qubit subspace and must be 16-multiplied to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i\sigma^{32220000}\chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry, the root-state Hamiltonian takes the first five-qubit subspace and must be 16-multiplied to form the FSM.

In the  $d = 8$  spacial dimension, the Hamiltonian  $h^{(8)}$  is defined on a 10-qubit single-particle Hilbert space,

$$\begin{aligned} h^{(8)} = & i\partial_1\sigma^{100000000} + i\partial_2\sigma^{310000000} + i\partial_3\sigma^{330000000} + i\partial_4\sigma^{321200000} + i\partial_5\sigma^{323200000} + i\partial_6\sigma^{320120000} + i\partial_7\sigma^{320320000} \\ & + i\partial_8\sigma^{322200000} + m\sigma^{200000000} + n_1\sigma^{3220310002} + n_2\sigma^{3220330002} + n_3\sigma^{3220101200} + n_4\sigma^{3220103200} + n_5\sigma^{3220100120} \\ & + n_6\sigma^{3220100320} + n_7\sigma^{3220102010} + n_8\sigma^{3220102030} + n_9\sigma^{3220102220} + n_{10}\sigma^{3220320000}. \end{aligned} \quad (B9)$$

With the  $\mathbb{Z}_2^P : \chi \rightarrow i\sigma^{200000000}\chi, \mathbf{n} \rightarrow -\mathbf{n}$  symmetry only, the root-state Hamiltonian takes the first five-qubit subspace and must be 32-multiplied to form the FSM. With the additional  $U(1) : \chi \rightarrow \exp(i\varphi\sigma^{000020000})\chi, (n_1 + in_2) \rightarrow e^{2i\varphi}(n_1 + in_2)$  symmetry, the root-state Hamiltonian takes the first six-qubit subspace and must be 16-multiplied to form the FSM. With the additional  $\mathbb{Z}_2^T : \chi \rightarrow i\sigma^{3220320000}\chi, n_{10} \rightarrow -n_{10}$  symmetry, the root-state Hamiltonian takes the first six-qubit subspace and must be 16-multiplied to form the FSM.

For higher spacial dimensions ( $d > 8$ ), due to the Bott periodicity of the Clifford algebra, the Hamiltonian  $h^{(d)}$  can be extended systematically from the Hamiltonian  $h^{(d-8)}$  with eight dimensions lower. Supposing that  $h^{(d)} = \sum_{i=1}^d i\partial_i\alpha^i + m\beta^0 + \sum_{a=1}^{d+2} n_a\beta^a$  and  $h^{(d-8)} = \sum_{i=1}^{d-8} i\partial_i\tilde{\alpha}^i + m\tilde{\beta}^0 + \sum_{a=1}^{d-6} n_a\tilde{\beta}^a$ , the extension is given by

$$\begin{aligned} \alpha^i &= \tilde{\alpha}^i \otimes \sigma^{00000000}, & (i = 1, \dots, d-9) \\ \beta^a &= \tilde{\beta}^a \otimes \sigma^{00000000}, & (a = 0, \dots, d-7) \\ \alpha^{d-8} &= \tilde{\alpha}^{d-8} \otimes \sigma^{22220000}, & \beta^{d-6} = \tilde{\beta}^{d-6} \otimes \sigma^{00002222}, \\ \alpha^{d-7} &= \tilde{\alpha}^{d-8} \otimes \sigma^{10000000}, & \beta^{d-5} = \tilde{\beta}^{d-6} \otimes \sigma^{00001000}, \\ \alpha^{d-6} &= \tilde{\alpha}^{d-8} \otimes \sigma^{30000000}, & \beta^{d-4} = \tilde{\beta}^{d-6} \otimes \sigma^{00003000}, \\ \alpha^{d-5} &= \tilde{\alpha}^{d-8} \otimes \sigma^{21200000}, & \beta^{d-3} = \tilde{\beta}^{d-6} \otimes \sigma^{00002120}, \\ \alpha^{d-4} &= \tilde{\alpha}^{d-8} \otimes \sigma^{23200000}, & \beta^{d-2} = \tilde{\beta}^{d-6} \otimes \sigma^{00002320}, \\ \alpha^{d-3} &= \tilde{\alpha}^{d-8} \otimes \sigma^{20120000}, & \beta^{d-1} = \tilde{\beta}^{d-6} \otimes \sigma^{00002012}, \\ \alpha^{d-2} &= \tilde{\alpha}^{d-8} \otimes \sigma^{20320000}, & \beta^d = \tilde{\beta}^{d-6} \otimes \sigma^{00002032}, \\ \alpha^{d-1} &= \tilde{\alpha}^{d-8} \otimes \sigma^{22010000}, & \beta^{d+1} = \tilde{\beta}^{d-6} \otimes \sigma^{00002201}, \\ \alpha^d &= \tilde{\alpha}^{d-8} \otimes \sigma^{22030000}, & \beta^{d+2} = \tilde{\beta}^{d-6} \otimes \sigma^{00002203}. \end{aligned} \quad (B10)$$

Every symmetry transform matrix  $O$  is extended to  $O \otimes \sigma^{00000000}$ . The structure of the Hamiltonian and the symmetry action remains the same under the extension. The single-particle Hilbert space of every root state is enlarged by four qubits, while that of the FSM is enlarged by eight qubits, so the minimal copy number to obtain the FSM from the root state is always 16 times multiplied in every eight dimensions higher.

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