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Non-Abelian topological insulators from an array of quantum wires

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We suggest a construction of a large class of topological states using an array of quantum wires. First, we show how to construct a Chern insulator using an array of alternating wires that contain electrons and holes, correlated with an alternating magnetic field. This is supported by semiclassical arguments and a full quantum-mechanical treatment of an analogous tight-binding model. We then show how electron-electron interactions can stabilize fractional Chern insulators (Abelian and non-Abelian). In particular, we construct a non-Abelian $\mathbb{Z}_3$ parafermion state. Our construction is generalized to wires with alternating spin-orbit couplings, which give rise to integer and fractional (Abelian and non-Abelian) topological insulators. The states we construct are effectively two dimensional, and are therefore less sensitive to disorder than one-dimensional systems. The possibility of experimental realization of our construction is addressed.

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Introduction. The integer quantum Hall effect (IQHE) [1] was discovered in two-dimensional (2D) systems subjected to a strong perpendicular magnetic field. The quantized conductance is a consequence of the emergence of a topological number [2], known as the Chern number. Haldane [3] showed that a graphenelike material which breaks time-reversal symmetry due to an alternating (zero average) magnetic field may have a nonzero Chern number as well. These types of materials, which have a nonzero Hall conductance with a zero total magnetic flux, are referred to as Chern insulators (CIs).

The existence of edge modes [4] in the QHE can be understood in various ways. In particular, one can understand the presence of edge modes by studying the classical curved trajectories of electrons in a magnetic field. In fact, it is possible to construct a semiclassical theory for a specific set of Chern insulators as well. Consider a system consisting of electrons and holes (whose masses differ in sign). In the presence of a magnetic field, their classical trajectories are curved in opposite directions. If, however, the electrons and the holes experience opposite magnetic fields, the trajectories will be curved in the same direction. One can imagine constructing a Chern insulator by separating the plane into regions which contain only holes and only electrons. If the magnetic field is opposite in the two regions, the classical trajectories will be similar to those of electrons in a uniform magnetic field. This suggests that, upon quantization, this system should have a nonzero Chern number [5], despite the fact that the total magnetic flux vanishes.

Motivated by this semiclassical picture, we will study in this Rapid Communication an effectively 2D system which consists of alternating wires that contain electrons and holes. Approaching the 2D problem from the quasi-one-dimensional (Q1D) limit enables a full quantum-mechanical analysis, and an analytic treatment of interaction effects using the bosonization technique.

References [6,7] argue that it is possible to understand the IQHE by considering a set of weakly coupled parallel wires. First, we will show that one can use an array of wires to construct a CI as well [see Fig. 1(a)]. We then introduce a tight-binding version of this model and obtain a phase diagram, showing the Chern number as a function of the model parameters.

Kane *et al.* [8] generalized the wires approach to the Abelian fractional quantum Hall effect (FQHE) using the bosonization technique. We will generalize our construction to a fractional CI (FCI) as well.

To do so, we introduce composite particles. This transformation maps the electrons and holes at $1/3$ filling to composite particles at filling 1. The possibility of a FCI has recently been discussed quite extensively in the literature [9]. Numerical investigations [10–15] of lattice models with nearly flat bands presented strong evidence for FCI states. More general approaches, connecting the properties of the known FQHE states and analogous FCI states, were found [16,17]. Here we present an alternative analytic approach to the subject, which may be applicable in experiments.

Teo and Kane [18] expand the approach of Ref. [8] to non-Abelian states. We will see that our results can be generalized to the non-Abelian case as well, and we will provide a detailed construction of a state similar to the $\mathbb{Z}_3$ Read-Rezay state. This state supports Fibonacci anyons, which may be used for universal quantum computation [19,20].

Using an analogy between a magnetic field and a spin-orbit coupling (in the $\hat{z}$ direction only), we will construct a topological insulator from an array of wires using an alternating spin-orbit coupling. It will then be straightforward to generalize the above model to a fractional topological insulator (FTI) [21]. Other realizations of FTI states were discussed in Refs. [22–26].

We note that the Q1D approach was recently used by various papers [27–29] to discuss a variety of topological states.

Wire construction of a CI. Motivated by the above semiclassical picture, we have designed the wire construction, shown in Fig. 1(a). In each unit cell there are four different wires. We tune the wires' chemical potentials such that wires 1 and 2 of each unit cell are near the bottom of the band, and wires 3 and 4 are near the top. Effectively, we have alternating pairs of wires that contain electrons and holes. A positive (negative) magnetic field is introduced between the pairs of electron (hole) wires. This is a Q1D version of the semiclassical picture we described above.

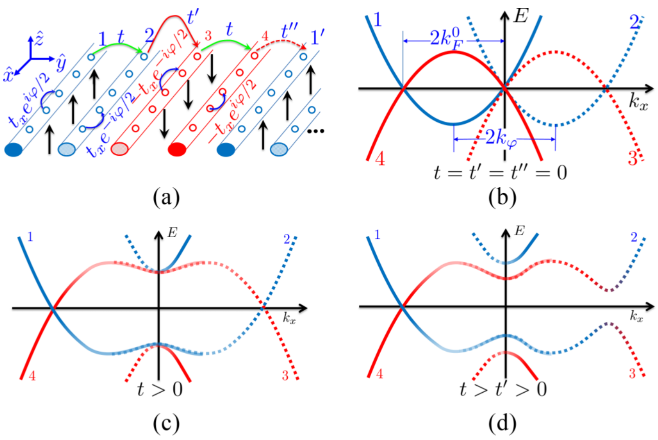

FIG. 1. (Color online) (a) Physical scheme of the Q1D model we study. Blue wires contain electrons, and red wires contain holes. The black arrows represent the magnetic field through the system. The circles represent the sites of the corresponding tight-binding model [30], and the tunneling amplitudes of the tight-binding model are represented by colored arrows. (b) The energy spectrum of the wires (as a function of k_x) near zero energy without tunneling between the wires ($t,t',t'' = 0$). The wires are tuned such that the four parabolas cross each other at zero energy, and the chemical potential is set to be zero. The spectra in blue, dashed blue, dashed red, and red correspond to wires 1, 2, 3, and 4 in a unit cell, respectively. (c) The energy spectrum when t is switched on. A gap opens near $k_x = 0$. (d) The spectrum when t' is switched on as well. This gives an additional gap at $k_x > 0$. Free chiral modes are left on wires 1 and 4. Finally, if one switches on t'', there are free chiral modes at the edge of the system, which suggests that there is a nonzero Chern number.

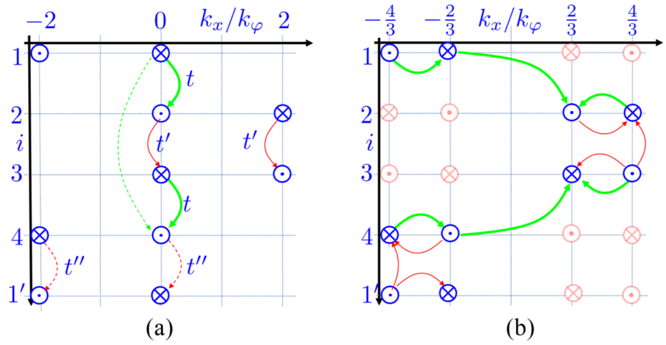

FIG. 2. (Color online) (a) A diagrammatic representation of the energy band structure in the case $\nu \equiv k_F^0/k_\varphi = 1$ [see Fig. 1 for definitions of k_φ and k_F^0]. The y axis shows the wire index inside the unit cell, and the x axis shows k_x in units of k_φ. The symbol $\odot$ ($\otimes$) represents k_i^L (k_i^R). Colored arrows represent tunneling amplitudes between the wires. (b) The same diagram for a topological insulator with $\nu = \frac{1}{3}$. Colored arrows now represent the multielectron processes responsible for the creation of Laughlin-like states. These complex processes in terms of the electrons ($\psi \sim e^{i\phi}$) for $\nu = 1/3$ are mapped to simple tunneling processes in terms of the fermions $\tilde{\psi} \sim e^{i\eta}$. Thus, $\nu = 1/3$ for ψ is equivalent to $\nu = 1$ for $\tilde{\psi}$. In the presence of spin-orbit coupling, spin up (blue) and spin down (light red) experience opposite alternating effective magnetic fields.

For illustration and simplicity it is convenient to choose a gauge in which the vector potential $\mathbf{A}$ points at the $\hat{x}$ direction. We can tune the wires' bands in such a way that all their crossing points match in energy. In this case, the energy spectra are similar to those depicted in Fig. 1(b). We define k_F^0 as the Fermi momenta in the absence of an external magnetic field. $k_\varphi = \frac{eBa}{2\hbar c}$ is the shift of the parabolas due to the magnetic fields [see Fig. 1(b)].

If, in addition, neighboring wires of the same type are weakly tunnel coupled (with an amplitude t), a gap opens between parabolas 1 and 2, and parabolas 3 and 4. The spectrum in this case is depicted in Fig. 1(c).

Introducing now a coupling between the electrons and holes *inside* a unit cell (t'), a gap will open at $k_x > 0$, and we arrive at the spectrum depicted in Fig. 1(d). If we now switch on small tunneling between different unit cells (t''), the coupling between the edges decays exponentially with the sample width, and in the thermodynamic limit we expect to find gapless edge states. The observation of gapless edge states indicates that there is a nonzero Chern number. To show this explicitly, we have constructed a 2D tight-binding model, which is the lattice version of the above continuous model. The tight-binding model enables an exact derivation of phase diagram, showing the Chern number as a function of the model parameters. For more details see the Supplemental Material [30], where the tight-binding model is defined, and its phase diagram is derived, ensuring that the results of the above discussion are valid. We note that, by construction, our model has only a single chiral edge mode, leading to the fact that the model can only have Chern numbers $C = \pm 1$. Generalization to larger Chern numbers is a possible interesting extension.

FCI. The wire construction invites us to add interactions and use bosonization techniques, similar to those used in Refs. [8,18]. This allows us to generalize the above results to FCI states. In the presence of interactions, multielectron processes may open a gap even if the Fermi point of the left movers is not equal to the Fermi points of the right movers [8,18,31,32].

To understand the required conditions for a gap opening due to multielectron scattering processes, it is useful to present the spectra of Fig. 1(b) in an alternative way. Instead of plotting the entire spectrum of the wires together, we plot only the Fermi points as a function of the wire index. Soon, we will linearize the spectra around these points. A cross ($\otimes$) denotes the Fermi point of a right mover, and a dot ($\odot$) denotes the Fermi point of a left mover. Before analyzing the fractional case, it is useful to revisit the simple $\nu = 1$ case. We will see that the main results of the tight-binding model arise naturally in the bosonization framework. Figure 2(a) shows the diagram that corresponds to this case [Fig. 1(b)].

Linearizing the spectrum around the Fermi points of each wire, and using the standard bosonization procedure, we define the two chiral bosonic fields ϕ_i^R and ϕ_i^L for each wire. In terms of these, the fermionic operators are $\psi_i^R \propto e^{i(k_i^R x + \phi_i^R)}$, $\psi_i^L \propto e^{i(k_i^L x + \phi_i^L)}$.

Without interactions, a momentum conserving single-electron tunneling between the wires [denoted in Fig. 2(a) by an arrow] is possible only when the left and right movers of adjacent wires are at the same point in k space. The single-electron tunneling operators between adjacent wires [denoted in Fig. 2(a) by green, red, and dashed red arrows] are

$$\begin{aligned} &t\psi^{R\dagger}_{1(3)}\psi^{L}_{2(4)} + \text{H.c.} \propto t\cos\left(\phi^{R}_{1(3)} - \phi^{L}_{2(4)}\right),\\ &t'\psi^{R(L)\dagger}_{2}\psi^{L(R)}_{3} + \text{H.c.} \propto t'\cos\left(\phi^{R(L)}_{2} - \phi^{L(R)}_{3}\right),\\ &t''\psi^{R(L)\dagger}_{4}\psi^{L(R)}_{1'} + \text{H.c.} \propto t''\cos\left(\phi^{R(L)}_{4} - \phi^{L(R)}_{1'}\right). \end{aligned} \tag{1}$$

We switch on the operators in the following way: First, we switch on a small $t \ll t_x$. Since this is a relevant operator, it gaps out the spectrum near $k_x = 0$. Then, we switch on smaller electron-hole couplings $t', t'' < t$. The terms $\psi_2^{R\dagger}\psi_3^{L}$ and $\psi_4^{R\dagger}\psi_{1'}^{L}$ gap out the rest of the spectrum, leaving a gapless edge mode. As we discussed before, this indicates that there is a nonzero Chern number. Note that the terms $\psi_2^{L\dagger}\psi_3^{R}$ and $\psi_4^{L\dagger}\psi_{1'}^{R}$ contain fields which are conjugate to those already pinned by t. Strong quantum fluctuations are therefore expected to suppress these terms.

We now turn to generalize this to Laughlin-like FCI states, with a filling factor $\nu = k_F^0/k_\varphi = 1/(2n+1)$, where n is a non-negative integer. For example, the k-vector pattern of the wires with $\nu = 1/3$ is shown in blue in Fig. 2(b). In this case, multielectron processes are expected to gap out the system (except for the edges). To see this, it is enlightening to define new chiral fermion operators

$$\tilde{\psi}^{R(L)}_i = \left(\psi^{R(L)}_i\right)^{(n+1)}\left(\psi^{\dagger L(R)}_i\right)^{n} \propto e^{i(q^{R(L)}_i x + \eta^{R(L)}_i)}, \tag{2}$$

with

$$\eta^{R(L)}_i = (n+1)\phi^{R(L)}_i - n\phi^{L(R)}_i, \tag{3}$$

and $q_i^{R(L)} = (n+1)k_i^{R(L)} - nk_i^{L(R)}$. A direct calculation of the commutation relations of the η fields shows that they have an additional factor of $2n\pi$ compared the ϕ fields. This gives an extra (trivial) phase factor $e^{i2\pi n}$ in the anticommutation relation of the $\tilde{\psi}$'s compared to the ψ's, ensuring that the $\tilde{\psi}$'s are fermionic operators. In addition, it can easily be checked that the resulting structure of the q's is identical to that of the k's in the case of $\nu = 1$ [Fig. 2(a)], so that $\tilde{\psi}$ can be regarded as a fermionic field with $\nu = 1$. This procedure can therefore be interpreted as an attachment of $2n$ quantum fluxes to each electron (cf. Jain's construction of composite fermions [33]).

Repeating the analysis of the $\nu = 1$ case, we can now write single $\tilde{\psi}$ tunneling operators, identical to those found in Eq. (1) (replacing $\psi \to \tilde{\psi}, \phi \to \eta$, with new tunneling amplitudes $\tilde{t}$, $\tilde{t}'$, and $\tilde{t}''$). In terms of the original electrons, these operators describe the multielectron processes shown in Fig. 2(b). Note that when the interactions are strong enough, these operators become relevant [8,18,31]. From here, the process is identical to the integer case. The gap due to the $\tilde{\psi}$ tunneling operators ensures that competing processes (for example, single-electron tunneling between wires 2 and 3, or 4 and 1′) are suppressed, as they contain fields that are conjugate to the fields pinned by $\tilde{t}$ (which is dominant by our construction).

The fact that the composite η fields (and not the original ϕ fields) are pinned leads to the various properties of these Laughlin-like states, such as the fractional charge and statistics of the excitations, in analogy to the known FQHE states [8,18].

Non-Abelian FCI. As the discussion above shows, the wire construction allowed us to create Abelian fractional Chern insulators. Reference [18] constructed non-Abelian QHE states by enlarging the unit cell, and taking a nonuniform magnetic field inside each unit cell. By our construction, any non-Abelian state constructed by Ref. [18] can be generalized to the CI case. To do so, one can take two unit cells from the construction in Ref. [18], reverse the magnetic field of the second unit cell, and use holes instead of electrons. However, the lack of a total magnetic flux in our system enables a simpler construction of non-Abelian states, which do not have a direct analog in the QHE. We now show that a slight modification of the procedure that enabled the construction of Laughlin-like states may lead to non-Abelian states. We will focus here on a state similar to the $\mathbb{Z}_3$ Read-Rezay state. Generalization to other non-Abelian states is possible.

To obtain a $\mathbb{Z}_3$ parafermion state, we take $\nu = 1/3$, and construct the $\tilde{\psi}$ operators. Let us start in the special point where $\tilde{t}$, $\tilde{t}'$, and the coupling between $\tilde{\psi}_1^R$ and $\tilde{\psi}_4^L$ are tuned to have exactly the same value, denoted by v (at the end, when the topological nature of our construction will be revealed, this strict requirement can be relaxed, as long as the bulk gap does not close). It can be shown (see the Supplemental Material [30] for more technical details) that under these assumptions our problem can be mapped to the $\beta^2 = 6\pi$ self-dual sine-Gordon model, which was studied in Refs. [34,35]. Specifically, it was shown that this model is mapped to a critical $\mathbb{Z}_3$ parafermionic field.

We have established that any unit cell has two counterpropagating $\mathbb{Z}_3$ parafermionic fields (around $k = 0$), and two counterpropagating charge modes at $k_x = -2k_\varphi$. As earlier, we can in principle gap out the spectrum by switching on specific coupling terms between different unit cells, leaving eventually a Laughlin-like charge mode and a $\mathbb{Z}_3$ parafermion mode at the edge of the sample [30,34]. However, in order to leave a chiral parafermion mode we need to consider quasiparticle tunneling terms, which are hard to realize in the present construction [30,34]. This technical problem is easily solved by adding an additional flavor quantum number to each wire, which allows one to effectively create a thin FQHE state in each unit cell (see Supplemental Material [30]). This way, in addition to supporting gapless parafermion modes in each unit cell, our construction enables their coupling. We point out that the above construction is related to the bilayer QHE system presented by Ref. [36]. Note that while the non-Abelian part is the same as the non-Abelian part of the $\mathbb{Z}_3$ Read-Rezay state, the charge mode is different.

Topological insulators from the wire approach. The entire analysis presented here can also be carried out for spinful electrons if one introduces spin-orbit interactions (in the $\hat{z}$ direction only). This can be done if an alternating electric field is introduced instead of an alternating magnetic field. For example, the electric field can be tuned in such a way that the spin-orbit coupling is positive at wires 1 and 4, and negative at wires 2 and 3. Figure 2(b) shows the appropriate Fermi momenta corresponding to $\nu = \frac{1}{3}$ (in blue for spin up and light red for spin down). If one considers only processes which conserve S_z, we get a simple construction for integer, Laughlin-like, and non-Abelian topological insulators [21], which are simply two copies of the FCI states discussed above (with opposite chiral modes for the different spin species). If we now introduce small time-reversal invariant terms which violate S_z conservation (but do not close the gap), the chiral modes remain protected.

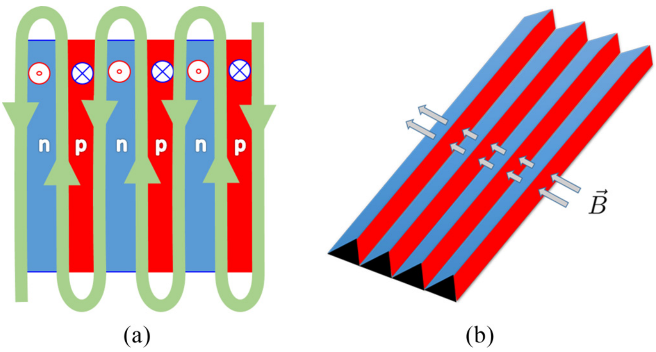

FIG. 3. (Color online) (a) A possible experimental realization of our construction. Green lines represent current carrying wires, which produce the alternating magnetic field needed for our construction. This way, the magnetic field is positive (i.e., out of the page, denoted by $\odot$) in n regions that contain electrons, and negative ($\otimes$) in p regions that contain holes. (b) Another possible realization of our construction where now the 2D plane is replaced by many V grooves connected in parallel. Again, blue regions contain electrons and red regions contain holes. The arrows represent the constant external magnetic field. In this geometry, electrons and holes experience opposite magnetic fields.

Generalization. The approach we present here can be extended to hierarchical Abelian states, as well as other non-Abelian states (such as a Moore-Read-like state). One can also study the effects of proximity to a superconductor, which is expected to yield other non-Abelian states. A detailed further study of these constructions will be performed in the future.

Experimental realizations. The above theoretical construction may also be applicable in experiments with superlattices that realize the particle-hole structure we suggest. The alternating magnetic field can be generated, for example, using a snakelike wire [37], as shown in Fig. 3(a) (for more technical details, see the Supplemental Material [30]), or using an array of V grooves [38], as shown in Fig. 3(b). We note that stripes with an alternating magnetic field can also be realized in cold atom systems [39]. By coupling many (or maybe only a few) wires together we get an effective 2D system. Our construction lacks the disadvantages of fractional 1D states [31,32], which are not topologically protected [40,41], and the need to invoke proximity to a superconductor and a strong magnetic field simultaneously in 2D [34,42,43]. As long as the width of the edge modes is smaller than the sample width, it behaves practically as a topologically protected 2D system.

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