Spin fluctuations in quantum dots

A. U. Sharafutdinov, D. S. Lyubshin, and I. S. Burmistrov
L.D. Landau Institute for Theoretical Physics RAS, Kosygina street 2, 119334 Moscow, Russia
and Moscow Institute of Physics and Technology, 141700 Moscow, Russia
(Received 25 August 2014; revised manuscript received 2 November 2014; published 17 November 2014)

We explore the static longitudinal and dynamic transverse spin susceptibilities in quantum dots and nanoparticles within the framework of the Hamiltonian that extends the universal Hamiltonian to the case of uniaxial anisotropic exchange. For the limiting cases of Ising and Heisenberg exchange interactions, we ascertain how fluctuations of single-particle levels affect the Stoner instability in quantum dots. We reduce the problem to the statistics of extrema of a certain Gaussian process. We prove that, despite possible strong randomness of the single-particle levels, the spin susceptibility and all its moments diverge simultaneously at the point which is determined by the standard criterion of the Stoner instability involving the mean level spacing only.

DOI: 10.1103/PhysRevB.90.195308 PACS number(s): 75.75.−c, 73.23.Hk, 73.63.Kv

I. INTRODUCTION

The physics of quantum dots continuously attracted a lot of experimental and theoretical interest [1–5]. Within the assumption that the Thouless energy (E_{Th}) is much larger than mean single-particle level spacing (δ), $E_{\text{Th}}/\delta \gg 1$, an effective zero-dimensional Hamiltonian has been derived [6]. In this so-called universal Hamiltonian, the electron-electron interaction that involves a set of matrix elements in the single-particle basis is reduced to just three parameters: the charging energy (E_c), the ferromagnetic Heisenberg exchange ($J > 0$), and the interaction in the Cooper channel. The single-particle energies are random quantities with Wigner-Dyson statistics. Thus, the universal Hamiltonian provides a convenient framework for the theoretical description of quantum dots.

The charging energy (typically $E_c \gg \delta$) restricts the probability of real electron tunneling through a quantum dot at low temperatures $T \ll E_c$ [7]. This phenomenon of the Coulomb blockade leads to suppression of the tunneling density of states in quantum dots at low temperatures [8,9]. As it is well known, strong enough exchange interaction in bulk materials leads to a Stoner instability at the value of the Fermi-liquid interaction parameter $F_0^\sigma = -1$ and a corresponding quantum phase transition between a paramagnet and a ferromagnet. For a quantum dot of size $L \gg \lambda_F$ (λ_F stands for the Fermi wavelength) the exchange interaction can be estimated by bulk value of the Fermi-liquid interaction parameter: $J/\delta = -F_0^\sigma$. Therefore, for such quantum dots one can expect that at $J = \delta$ the ground state becomes fully spin polarized. In addition, in quantum dots it is possible to realize an interesting situation in which the ground state has a finite total spin, i.e., partial spin polarization [6]. In the case of the equidistant single-particle spectrum, partial spin polarization occurs in the range $\delta/2 \lesssim J < \delta$. As J increases towards δ, the total spin in the ground state increases and at $J = \delta$ all electrons in a quantum dot become spin polarized. We emphasize that the mesoscopic Stoner instability is specific to finite-size systems and disappears in the thermodynamic limit $\delta \to 0$.

The finite value of the total spin in the ground state results in the Curie-type behavior of the static spin susceptibility at low temperatures. The dynamic spin susceptibility is trivial and its imaginary part reduces to the Dirac delta function. Due to the entanglement of the charge and spin degrees of freedom within the universal Hamiltonian, the mesoscopic Stoner instability affects the electron transport through a quantum dot. For example, it leads to an additional nonmonotonicity of the energy dependence of the tunneling density of states [10–12] and to the enhancement of the shot noise [13]. Although it was demonstrated [14] that exchange interaction $J \lesssim \delta/2$ is important for a quantitative description of the experiments on low-temperature ($T \lesssim \delta$) transport through quantum dots fabricated in a two-dimensional electron gas, we are not aware of experiments indicating the mesoscopic Stoner instability.

The case of Heisenberg exchange in the universal Hamiltonian can be contrasted with the case of Ising exchange. Such situation is realized in a quantum dot in a two-dimensional electron gas with strong spin-orbit coupling. In the presence of a spin-orbit coupling, the description of a quantum dot in the framework of the universal Hamiltonian breaks down. Even for a weak spin-orbit coupling (large spin-orbit length $\lambda_{\text{SO}} \gg L$), fluctuations of the matrix elements of the electron-electron interaction cannot be neglected in spite of the condition $\delta/E_{\text{Th}} \ll 1$ [15,16]. For a quantum dot in a two-dimensional electron gas, the orbital degrees of freedom are coupled to in-plane components of the spin. Then, in the regime $(\lambda_{\text{SO}}/L)^2 \gg (E_{\text{Th}}/\delta)(L/\lambda_{\text{SO}})^4 \gg 1$, the low-energy description is again possible in terms of the universal Hamiltonian but with the Ising exchange interaction ($J_z > 0$) [15,17]. In this case, mesoscopic Stoner instability is absent for the equidistant single-particle spectrum: the total spin in the ground state is zero for $J_z < \delta$ [6]. As a consequence, the tunneling density of states is almost independent of J_z while the longitudinal spin susceptibility χ_{zz} is independent of T as in a clean Fermi liquid [10,18]. However, the transverse dynamical spin susceptibility $\chi_\perp(\omega)$ is nontrivial. Its imaginary part is odd in frequency and has the maximum and the minimum [18].

The simplest model (albeit not microscopically justified) interpolating between the cases of the Heisenberg and Ising exchange interactions is extension of the universal Hamiltonian to the case of an anisotropic exchange interaction. Within this model with the equidistant spectrum the mesoscopic Stoner instability should disappear as the exchange interaction becomes anisotropic. We note that the experiments on tunneling spectra in nanometer-scale ferromagnetic nanoparticles revealed the presence of an exchange interaction with significant anisotropy [19]. The simplest model which allows us to explain the main features of experimentally measured excitation spectra of ferromagnetic nanoparticles resembles the universal Hamiltonian with uniaxial anisotropy in exchange interaction [20]. Such modification of exchange interaction can arise due to shape, surface, or bulk magnetocrystalline anisotropy. In addition, in the presence of spin-orbit scattering, the anisotropic part of the exchange interaction can experience large mesoscopic fluctuations [21,22]. The alternative reason for appearance of anisotropy in the exchange interaction in quantum dots is the presence of ferromagnetic leads [23].

So far, we avoid discussion of randomness of the single-particle levels in the universal Hamiltonian due to disorder in a quantum dot. As known, in low dimensions $d \leqslant 2$ interaction and disorder can induce a transition between paramagnetic and ferromagnetic phases at a finite temperature T [24,25]. In $d = 3$, the Stoner instability can be promoted by disorder and occurs at smaller values of exchange interaction [26]. Within the universal Hamiltonian [6,12], level fluctuations affect the temperature dependence of the average static spin susceptibility χ_{zz} in the case of the Heisenberg exchange. For the case of the Ising exchange, the role of disorder is even more dramatic. Due to level fluctuations, the average spin susceptibility acquires a Curie-type T dependence dominating at low enough T and for $\delta - J_z \ll \delta$ [6]. However, in this regime of strong (with respect to the small distance $\delta - J_z \ll \delta$ to the average position of the Stoner instability at $J_z = \delta$) level fluctuations it is far from being obvious why the average spin susceptibility is an adequate quantity for characterization of the spin state of a quantum dot. At $\delta - J_z \ll \delta$, a quantum dot is in the paramagnetic phase on average but, for example, it can be fully spin polarized for a particular realization of the single-particle levels. Such fully spin-polarized realizations might affect the spin susceptibility distribution function such that it becomes wide and non-Gaussian. At zero temperature, the level fluctuations might shift the position of the Stoner instability from its average position $J_z = \delta$ and lead to the existence of a finite-temperature transition between the paramagnetic and the ferromagnetic phases in quantum dots. Of course, the very same scenario might be relevant for the case of the Heisenberg exchange.

In this paper, we study the universal Hamiltonian extended to the case of exchange interaction with uniaxial anisotropy. Within this model we derive exact analytical results for the longitudinal static and transverse dynamic spin susceptibilities for arbitrary single-particle spectrum, temperature, exchange, and Coulomb interactions. This allows us to answer the following questions:

(1) Does the total spin in the ground state vanish continuously or discontinuously as the anisotropy increases?

(2) How is the delta function in the imaginary part of the dynamical spin susceptibility broaden with increase of exchange anisotropy?

(3) How is the distribution function for the spin susceptibility (in the cases of Heisenberg and Ising exchange interactions) changed with increase of exchange towards the Stoner instability?

(4) Might it be possible that at zero temperature the level fluctuations shift the position of the Stoner instability from its average position determined by the mean level spacing?

To answer the question (i), we compute the temperature and magnetic field dependence of the static longitudinal spin susceptibility χ_{zz} for equidistant single-particle spectrum. Except the case of the Ising exchange it always has a nonzero temperature-dependent contribution of Curie type $(1/T)$ or of $1/\sqrt{T}$ type. This indicates that destruction of the mesoscopic Stoner instability by uniaxial anisotropy is not abrupt.

To resolve the issue (ii), we also compute the transverse spin susceptibility for equidistant single-particle levels. Its imaginary part as a function of frequency has always the maximum and the minimum whose positions tend to zero linearly with decrease of anisotropy.

To answer the questions (iii) and (iv), we utilize that at low temperatures and for $\delta - J_z \ll \delta$, the statistical properties of the longitudinal spin susceptibility (both for the Ising and Heisenberg exchanges) are determined by the statistics of the extrema of a certain Gaussian process with a drift. This random process resembles locally a fractional Brownian motion with the Hurst exponent $H = 1 - \epsilon$ where $\epsilon \to 0$. We recall that the fractional Brownian motion with the Hurst exponent H is the Gaussian process $B_H(t)$ with zero mean $\overline{B_H(t)} = 0$ and the two-point correlation function $\overline{[B_H(t) - B_H(t')]^2} = |t - t'|^{2H}$. We rigorously prove that in the case of Ising (Heisenberg) exchange, all moments of static longitudinal spin susceptibility χ_{zz} are finite for $J_z < \delta$ $(J < \delta)$. For the Ising exchange, we argue also that all moments of dynamic transverse spin susceptibility $\chi_\perp(\omega)$ do not diverge for $J_z < \delta$. We estimate the tail of the complementary cumulative distribution function for χ_{zz} for both Ising and Heisenberg exchange interactions. Our results mean that the level fluctuations do not shift the Stoner instability from its average position and do not induce a finite-temperature transition between the paramagnetic and the ferromagnetic phases.

In our study, we omit the Cooper channel interaction which within the universal Hamiltonian framework is responsible for superconducting correlations in quantum dots [27]. This is possible under assumption that the Cooper channel interaction is repulsive and, therefore, renormalizes to zero [6]. We also neglect corrections to the universal Hamiltonian due to the fluctuations in the matrix elements of the electron-electron interaction [28,29]. They are small in the regime $\delta/E_{\mathrm{Th}} \ll 1$ but lead to interesting physics beyond the universal Hamiltonian [5]. Finally, we mention that although we report the exact analytical results for static and dynamical spin susceptibilities for a given single-particle spectrum valid for arbitrary temperature, further analysis is performed for temperatures $T \gg \delta$.

The outline of the paper is as follows. In Sec. II, we introduce the model Hamiltonian, and derive exact analytical expressions for the corresponding grand canonical partition function and longitudinal static spin susceptibility. In Sec. III, we analyze the temperature and magnetic field dependence of longitudinal static spin susceptibility in the case of equidistant single-particle spectrum and anisotropic exchange interaction. In Sec. IV, we present a detailed analysis of the effect of level fluctuations on the longitudinal static spin susceptibility for the cases of Ising and Heisenberg exchange interactions. In Sec. V, we compute the transverse dynamical spin susceptibility in the case of equidistant single-particle spectrum and anisotropic exchange interaction and analyze the effect of level fluctuations in the case of Ising exchange interaction. We conclude the paper with summary of the main results and discussion of how our predictions can be experimentally verified (Sec. VI). Some of the results were published in a brief form in Ref. [30].

II. HAMILTONIAN AND PARTITION FUNCTION

A. Hamiltonian

We consider the following Hamiltonian with direct Coulomb and anisotropic exchange interactions:

$$H = H_0 + H_C + H_S. \tag{1}$$

The noninteracting Hamiltonian

$$H_0 = \sum_{\alpha,\sigma} \epsilon_{\alpha,\sigma} a^\dagger_{\alpha\sigma} a_{\alpha\sigma} \tag{2}$$

is given as usual in terms of the single-particle creation ($a^\dagger_{\alpha\sigma}$) and annihilation ($a_{\alpha\sigma}$) operators. It involves the spin-dependent ($\sigma = \pm$) single-particle energy levels $\epsilon_{\alpha,\sigma}$. In what follows, we assume that they depend on applied magnetic field B via the Zeeman splitting $\epsilon_{\alpha,\sigma} = \epsilon_\alpha + g_L \mu_B B\sigma/2$. Here, g_L and μ_B stand for the Landé g factor and the Bohr magneton, respectively. The charging interaction part of the Hamiltonian

$$H_C = E_c(\hat n - N_0)^2 \tag{3}$$

describes the direct Coulomb interaction in a quantum dot in the zero-dimensional approximation $E_{\mathrm{Th}}/\delta \gg 1$. Here,

$$\hat n = \sum_\alpha n_\alpha = \sum_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma} \tag{4}$$

denotes the particle-number operator, and N_0 is the background charge. The term

$$H_S = -J_\perp\left(\hat S_x^2 + \hat S_y^2\right) - J_z \hat S_z^2 \tag{5}$$

represents the anisotropic exchange interaction within the quantum dot (QD). The total spin operator

$$\hat{\boldsymbol S} = \frac{1}{2}\sum_{\sigma\sigma'} a^\dagger_{\alpha\sigma} \boldsymbol\sigma_{\sigma\sigma'} a_{\alpha\sigma'} \tag{6}$$

is defined in terms of the standard Pauli matrices $\boldsymbol\sigma$.

In the case of isotropic Heisenberg exchange $J_\perp = J_z$, the Hamiltonian (1) reduces to the universal Hamiltonian which describes a quantum dot in the limit $E_{\mathrm{Th}}/\delta \gg 1$ [6]. In this case, the single-particle levels ϵ_α are random. Their statistics (in the absence of magnetic field $B = 0$) is described by the orthogonal Wigner-Dyson ensemble. The Hamiltonian (1) with the Ising exchange $J_\perp = 0$ and $B = 0$ can be used for description of lateral quantum dots with spin-orbit coupling [15,17]. In this case, the statistics of ϵ_α is described by the unitary Wigner-Dyson ensemble.

B. Exact expression for the grand canonical partition function

The grand canonical partition function for the Hamiltonian (1) is defined as $Z = \mathrm{Tr}\, e^{-\beta H + \beta\mu\hat n}$ (μ denotes the chemical potential). It can be found by using the following trick. Let us separate H_S into the Heisenberg and Ising parts:

$$H_S = -J_\perp \hat{\boldsymbol S}^2 - (J_z - J_\perp)\hat S_z^2. \tag{7}$$

Then, the time-evolution operator in the imaginary time can be rewritten as

$$e^{-\tau H_S} = \frac{\sqrt{\tau}}{2\sqrt{\pi|J_z - J_\perp|}} \int_{-\infty}^{\infty} d\mathcal B \, \exp\left(-\frac{\tau \mathcal B^2}{4|J_z - J_\perp|}\right) \times e^{\tau J_\perp \hat{\boldsymbol S}^2 - \eta \mathcal B \hat S_z}, \tag{8}$$

where $\eta = \sqrt{\mathrm{sgn}(J_z - J_\perp)}$. The exponent in the second line of Eq. (8) indicates that the grand canonical partition function for the Hamiltonian (1) can be found in two steps. At first, one can use well-known results for the partition function for the case of isotropic exchange and effective magnetic field $B + \eta\mathcal B/(g_L\mu_B)$ [12,31]. Second, one needs to integrate over the effective magnetic field $\mathcal B$ with the kernel given in the first line of Eq. (8). Thus, we obtain the following exact result for the grand canonical partition function of Hamiltonian (1):

$$Z(b) = \sum_{n_\uparrow,n_\downarrow} Z_{n_\uparrow} Z_{n_\downarrow} e^{-\beta E_c(n-N_0)^2 + \beta J_\perp m(m+1) + \beta\mu n} \times \mathrm{sgn}(2m+1) \sum_{l=-|m+1/2|+1/2}^{|m+1/2|-1/2} e^{\beta(J_z - J_\perp)l^2 - \beta b l}. \tag{9}$$

Here, $b = g_L\mu_B B/2$. The integers $n_\uparrow$ and $n_\downarrow$ represent the number of spin-up and spin-down electrons, respectively. The total number of electrons is $n = n_\uparrow + n_\downarrow$, and $m = (n_\uparrow - n_\downarrow)/2$. We note that for a configuration with given $n_\uparrow$ and $n_\downarrow$ electrons the total spin equals $S = |m + 1/2| - 1/2$. The integers l denote z projection of the total spin $\boldsymbol S$. The factors $Z_{n_\uparrow}$ and $Z_{n_\downarrow}$ are canonical partition functions for $n_\uparrow$ and $n_\downarrow$ noninteracting spinless electrons, respectively. The canonical partition function takes into account the contributions from the single-particle energies and is given by Darwin-Fowler integral

$$Z_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} \prod_\gamma (1 + e^{i\theta - \beta\epsilon_\gamma}). \tag{10}$$

For the Heisenberg exchange interaction $J_\perp = J_z$ our result (9) coincides with the result known in the literature [11,12,31]. In the case of purely Ising exchange interaction $J_\perp = 0$, our result (9) agrees with the result obtained in Ref. [18]. We note that the result (9) can be also derived directly from Hamiltonian (1) with the help of Wei-Norman-Kolokolov transformation (see Appendix A).

In order to analyze the exact result (9) for the grand canonical partition function, it will be convenient to use the following completely equivalent integral representation:

$$Z(b) = \frac{e^{-\beta J_\perp/4}}{2\pi\sqrt{J_\perp|J_z - J_\perp|}} \int_{-\infty}^{\infty} dh\, d\mathcal B\, e^{-\frac{h^2}{\beta J_\perp}} e^{-\frac{\beta(b+\eta\mathcal B)^2}{4J_\perp}} \times \frac{\sinh(h)\sinh[(b+\eta\mathcal B)h/J_\perp]}{\sinh[\beta(b+\eta\mathcal B)/2]} \sum_{k\in\mathbb Z} e^{-\beta E_c(k-N_0)^2} \times e^{-\frac{\beta\mathcal B^2}{4|J_z - J_\perp|}} \int_{-\pi}^{\pi} \frac{d\phi_0}{2\pi} e^{i\phi_0 k} \prod_\sigma e^{-\beta\Omega_0(\mu - i\phi_0 T + \sigma h T)}. \tag{11}$$

The grand canonical partition function for noninteracting spinless electrons is defined in a standard way:

$$\Omega_0(\mu) = -T \ln \prod_\gamma (1 + e^{-\beta(\epsilon_\gamma - \mu)}). \tag{12}$$

The variables ϕ_0 and h have the meaning of the zero-frequency Matsubara components of an electric potential and a magnetic field which can be used to decouple the direct Coulomb [8] and exchange interaction [18,32] terms, respectively.

C. Longitudinal spin susceptibility

The general expressions (9) and (11) for the grand partition function Z allow us to extract the results for the longitudinal spin susceptibility:

$$\chi_{zz}(T,b) = T \frac{\partial^2}{\partial b^2} \ln Z. \tag{13}$$

It is worthwhile to mention that in zero magnetic field one can use the equivalent formula

$$\chi_{zz}(T,b=0) = \partial \ln Z / \partial J_z \tag{14}$$

to simplify calculations. As it is well known [8,33], at $T \gg \delta$ (the regime we are interested in) we can perform integration over ϕ_0 in Eq. (11) in the saddle-point approximation. Then, the grand canonical partition function is factorized into two multipliers:

$$Z = Z_C Z_S, \tag{15}$$

where

$$Z_C = \sqrt{\frac{\beta\Delta}{4\pi}} \sum_{n\in\mathbb{Z}} e^{-\beta E_c (n-N_0)^2 + \beta(\mu-\mu_n)n - 2\beta\Omega_0(\mu_n)} \tag{16}$$

describes the effect of charging energy. Here, μ_n is the solution of the saddle-point equation $n = -2\partial\Omega_0(\mu)/\partial\mu$ and

$$\Delta^{-1} = -\frac{\partial^2 \Omega_0(\mu)}{\partial \mu^2}\Big|_{\mu=\mu_n} \tag{17}$$

stands for the thermodynamic density of states at the Fermi level. We note that in the regime $T \ll E_c$ (which we are interested in) one can approximate μ_n by $\tilde\mu = \mu_{N_0}$. The term

$$Z_S = \frac{e^{-\beta J_\perp/4}}{2\pi\sqrt{J_\perp |J_z - J_\perp|}} \int_{-\infty}^{\infty} dh\, d\mathcal{B}\, e^{-\frac{1}{4\beta J_\perp}[4h^2 + \beta^2 (b+\eta\mathcal{B})^2]} \times \frac{\sinh(h)\sinh[(b+\eta\mathcal{B})h/J_\perp]}{\sinh[\beta(b+\eta\mathcal{B})/2]} e^{-\frac{\beta\mathcal{B}^2}{4|J_z - J_\perp|}]} \times \prod_\sigma e^{\beta\Omega_0(\tilde\mu) - \beta\Omega_0(\tilde\mu + h\sigma/\beta)} \tag{18}$$

describes the contribution due to exchange interaction. The function

$$\beta \sum_\sigma [\Omega_0(\tilde\mu) - \Omega_0(\tilde\mu + h\sigma/\beta)] = \int_{-\infty}^{\infty} dE\, \nu_0(E) \ln\left[1 + \frac{\sinh^2(h/2)}{\cosh^2(E/2T)}\right] \tag{19}$$

that appears in Eq. (18) depends on a particular realization of the single-particle spectrum via the single-particle density of states $\nu_0(E) = \sum_\alpha \delta(E + \tilde\mu - \epsilon_\alpha)$. Provided $h^2 \ll \exp(\beta\tilde\mu)$, we can write

$$\beta \sum_\sigma [\Omega_0(\tilde\mu) - \Omega_0(\tilde\mu + h\sigma/\beta)] = \frac{h^2}{\beta\delta} - V(h), \tag{20}$$

where

$$V(h) = -\int_{-\infty}^{\infty} dE\, \delta\nu_0(E) \ln\left[1 + \frac{\sinh^2(h/2)}{\cosh^2(E/2T)}\right]. \tag{21}$$

Here, $\delta\nu_0(E)$ stands for the deviation of the single-particle density of states $\nu_0(E)$ from its average (over realizations of the single-particle spectrum) value: $1/\delta = \overline{1/\Delta} = \overline{\nu_0(E)}$.

The charging energy contribution Z_C is independent of the magnetic field and therefore does not affect the spin susceptibility. We note that the normalization is such that $Z_S = 1$ for $b = J_\perp = J_z = 0$. In what follows, we will discuss Z_S only.

III. LONGITUDINAL SPIN SUSCEPTIBILITY: EQUIDISTANT SINGLE-PARTICLE SPECTRUM

We start our analysis from the case of the equidistant single-particle spectrum, i.e., we completely neglect the effect of level fluctuations [we set $V(h)$ in Eq. (20) to zero]. We discuss the role of level fluctuations in Sec. IV.

A. Case of an easy axis: $J_z \geqslant J_\perp$

Using the integral representations (18) and (20) we can perform integration over h and find

$$Z_S = \left(\frac{\delta}{\delta - J_z}\right)^{1/2} e^{\frac{\beta J_\perp^2}{4(\delta - J_\perp)}} e^{-\frac{\beta b^2}{4(J_z - J_\perp)}} \times \frac{1}{2}\sum_{p=\pm} F_1\left(\frac{\delta}{\delta - J_\perp} + \frac{pb}{J_z - J_\perp}, \sqrt{\beta J_*}\right). \tag{22}$$

Here, $J_* = (\delta - J_\perp)(J_z - J_\perp)/(\delta - J_z)$ is the energy scale specific for the anisotropic problem that interpolates between 0 (for $J_z = J_\perp$) and $\delta J_z/(\delta - J_z)$ (for $J_\perp = 0$). The function $F_1(x,y)$ is defined as follows:

$$F_1(x,y) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt\pi} \frac{\sinh(xyt)}{\sinh(yt)} e^{-t^2}. \tag{23}$$

With the help of Eqs. (13) and (22) one can find dependence of the longitudinal static spin susceptibilities on magnetic field and temperature. For a given values of J_z and $J_\perp$ it is shown in Fig. 1.

Using Eq. (14), the zero-field longitudinal spin susceptibility can be written as

$$\chi_{zz}(T) = \frac{1}{2(\delta - J_z)} + \frac{1}{2}\left(\frac{\delta - J_\perp}{\delta - J_z}\right)^2 \times \frac{\partial}{\partial J_*} \ln F_1\left(\frac{\delta}{\delta - J_\perp}, \sqrt{\beta J_*}\right). \tag{24}$$

At high temperatures $T \gg \max\{\delta, \frac{\delta^2(J_z - J_\perp)}{(\delta - J_\perp)(\delta - J_z)}\}$, the result (24) for the zero-field longitudinal static spin susceptibility

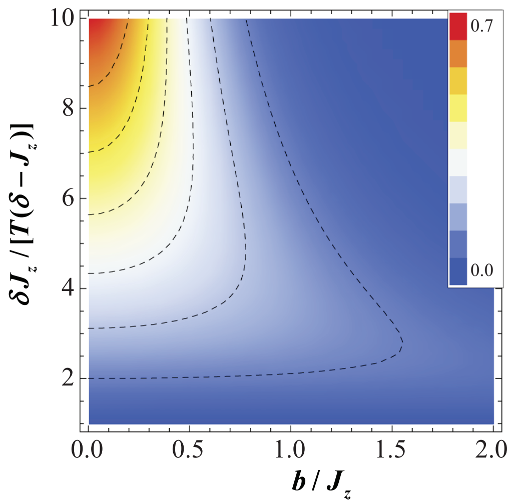

FIG. 1. (Color online) Dependence of the relative correction $[2(\delta-J_z)\chi_{zz}-1]$ to the Fermi-liquid-like result on dimensionless magnetic field and inverse temperature b/J_z and $\delta J_z/(\delta-J_z)T$. We chose $J_z=0.94\delta$ and $J_\perp=0.3\delta$.

can be simplified [cf. Eq. (B1)]. Then, we obtain

$$\chi_{zz}(T)=\frac{1}{2(\delta-J_z)}+\frac{\beta}{12}\frac{(2\delta-J_\perp)J_\perp}{(\delta-J_z)^2}. \tag{25}$$

Away from the isotropic case ($J_z=J_\perp$) a set of temperature intervals with different temperature behavior of the longitudinal spin susceptibility exists. Following, we use the asymptotic result (B2) from Appendix B. At temperatures $\max\{\delta,\frac{\delta(J_z-J_\perp)}{(\delta-J_z)}\}\ll T\ll\frac{\delta^2(J_z-J_\perp)}{(\delta-J_\perp)(\delta-J_z)}$, we find

$$\chi_{zz}(T)=\frac{1}{2(\delta-J_z)}+\frac{\beta}{4}\frac{\delta^2}{(\delta-J_z)^2}. \tag{26}$$

For the temperature range $\max\{\delta,\frac{(\delta-J_\perp)(J_z-J_\perp)}{(\delta-J_z)}\}\ll T\ll\frac{\delta(J_z-J_\perp)}{(\delta-J_z)}$, we obtain

$$\chi_{zz}(T)=\frac{1}{2(\delta-J_z)}+\frac{\beta}{4}\frac{J_\perp^2}{(\delta-J_z)^2}. \tag{27}$$

If the temperature is within the interval $\max\{\delta,\frac{J_\perp^2(J_z-J_\perp)}{(\delta-J_\perp)(\delta-J_z)}\}\ll T\ll\frac{(\delta-J_\perp)(J_z-J_\perp)}{(\delta-J_z)}$, the zero-field longitudinal static spin susceptibility becomes

$$\chi_{zz}(T)=\frac{1}{2(\delta-J_z)}+\frac{1}{2\sqrt{\pi}}\frac{J_\perp\sqrt{\beta J_*}}{(\delta-J_z)(J_z-J_\perp)}. \tag{28}$$

Finally, for the lowest-temperature range $\delta\ll T\ll\min\{\frac{J_\perp^2(J_z-J_\perp)}{(\delta-J_\perp)(\delta-J_z)},\frac{(\delta-J_\perp)(J_z-J_\perp)}{(\delta-J_z)}\}$ we find [cf. Eq. (B3)]

$$\chi_{zz}(T)=\frac{1}{2(\delta-J_z)}+\frac{\beta}{4}\frac{J_\perp^2}{(\delta-J_z)^2}. \tag{29}$$

We mention that χ_{zz} consists of two contributions [see Eqs. (25)–(27) and (29)]: the one which resembles the Fermi-liquid result for spin susceptibility, $\propto 1/(\delta-J_z)$, and the other which is of Curie type, $\propto\beta\delta^2/(\delta-J_z)^2$. Such behavior is illustrated in Fig. 2 where the dependence of longitudinal spin susceptibility (24) on temperature and J_z at a fixed ratio $J_\perp/\delta$ is shown. We emphasize that longitudinal spin susceptibility diverges at $J_z=\delta$ regardless of the value of $J_\perp$.

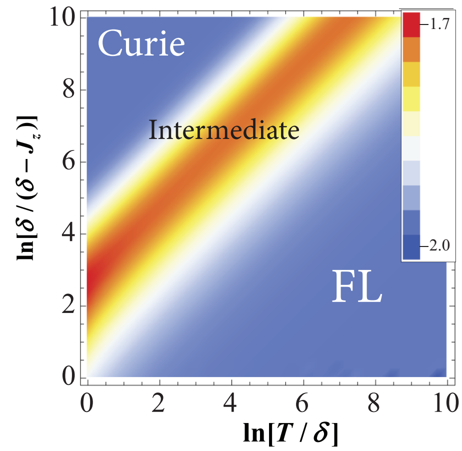

FIG. 2. (Color online) Dependence of $-\frac{d\ln[\chi_{zz}-\frac{\delta}{2(\delta-J_z)}]}{d(\delta-J_z)}$ on $\ln\frac{T}{\delta}$ and $\ln\frac{\delta}{\delta-J_z}$ for $J_\perp=0.3\delta$. In the left upper region, the Curie-type behavior dominates the Fermi-liquid-like result. In the right lower region, the Curie-type correction to the Fermi-liquid-like result $[\chi_{zz}-\frac{\delta}{2(\delta-J_z)}]\propto\frac{1}{(\delta-J_z)^2}$ is small. Red region corresponds to the intermediate regime in which there is a correction $[\chi_{zz}-\frac{\delta}{2(\delta-J_z)}]\propto\frac{1}{T^{1/2}(\delta-J_z)^{3/2}}$ to the Fermi-liquid result due to transverse degrees of freedom.

To understand the origin of such interesting behavior of the zero-field longitudinal spin susceptibility, it is useful to rewrite Eq. (18) in terms a series form again:

$$Z_S=\sqrt{\frac{\beta\delta}{\pi}}e^{\frac{\beta J_\perp^2}{4(\delta-J_\perp)}}\sum_{S_z=-\infty}^{\infty}\sum_{S=|S_z|}^{\infty}\left(e^{-\beta(\delta-J_\perp)(S-\frac{J_\perp}{2(\delta-J_\perp)})^2}-e^{-\beta(\delta-J_\perp)(S+1+\frac{J_\perp}{2(\delta-J_\perp)})^2}\right)e^{\beta(J_z-J_\perp)S_z^2}. \tag{30}$$

Here, we used the following result:

$$Z_{n_\uparrow}Z_{n_\downarrow}\approx\sqrt{\frac{\beta\delta}{4\pi}}e^{-\beta\mu_n n-2\beta\Omega_0(\mu_n)}\sqrt{\frac{\beta\delta}{\pi}}e^{-\beta\delta m^2} \tag{31}$$

which is valid provided the following conditions hold: $\delta\ll T$ and $n\gg|m|$ (see Appendix C).

In the case of large temperatures $T\gg J_\perp^2/(\delta-J_z)$, our results (25)–(29) imply the Fermi-liquid behavior of $\chi_{zz}(T)$. In this temperature range, all terms except the first one with $S=|S_z|$ in the sum over S in Eq. (30) cancel each other. Then, we find

$$Z_S=\sqrt{\frac{\beta\delta}{\pi}}\sum_{S_z=-\infty}^{\infty}e^{-\beta(\delta-J_z)S_z^2}=\left(\frac{\delta}{\delta-J_z}\right)^{1/2} \tag{32}$$

and, consequently, $\chi_{zz}(T)=1/[2(\delta-J_z)]$. This result implies that the average value of S_z^2 is of the order of $1/[2\beta(\delta-J_z)]\gg 1$ regardless of $J_\perp$. At the same time, the average value of the squared total spin $\boldsymbol{S}^2$ is of the order of $1/[2\beta(\delta-J_z)]+1/[\beta(\delta-J_\perp)]$. Therefore, at $J_\perp\lesssim J_z$, the total spin strongly fluctuates in all three directions so that $\boldsymbol{S}^2\approx 3S_z^2$ whereas for $J_\perp\ll J_z$ the total spin fluctuates along the z axis only so that $\boldsymbol{S}^2\approx S_z^2$.

We mention the unusual (inverse square-root) temperature dependence of the longitudinal spin susceptibility in Eq. (28). However, the result (28) is valid in a temperature range that exists only if $J_\perp \ll J_z \lesssim \delta$. Then, the restrictions for the temperature become $\max\{\delta, J_\perp^2/(\delta - J_z)\} \ll T \ll \delta^2/(\delta - J_z)$. Therefore, we can use the arguments from the previous paragraph. In order to explain the $\sqrt{\beta}$ dependence of χ_{zz}, one needs to perform the perturbation expansion in $J_\perp|S_z| \sim J_\perp/\sqrt{\beta(\delta - J_z)}$ for Eq. (30).

At low temperatures $T \ll J_\perp^2/(\delta - J_z)$, our results (25)–(27) and (29) imply a Curie-type longitudinal spin susceptibility. In this case, the second term in brackets in the right-hand side of Eq. (30) can be neglected. The sum over S can be estimated by the integral which is dominated by $S \sim |S_z|$. Then, we find

$$Z_S = \sqrt{\frac{\beta\delta}{\pi}}\, e^{\frac{\beta J_\perp^2}{4(\delta-J_z)}} \sum_{S_z=-\infty}^{\infty} \frac{e^{-\beta(\delta-J_z)(|S_z|-\frac{J_\perp}{2(\delta-J_z)})^2}}{2\beta(\delta-J_\perp)|S_z| - \beta J_\perp}. \quad (33)$$

This estimate yields the typical value of $|S_z| = J_\perp/[2(\delta - J_z)]$ and, thus, the Curie-type behavior of the longitudinal spin susceptibility: $\chi_{zz} = \beta|S_z|^2 = \beta J_\perp^2/[2(\delta - J_z)]^2$. Therefore, at relatively low temperatures $\delta \ll T \ll J_\perp^2/(\delta - J_z)$, the configuration with a nonzero total spin $S = |S_z| = J_\perp/[2(\delta - J_z)]$ gives the main contribution to the thermodynamic quantities.

For small magnetic fields $b \ll \delta(J_z - J_\perp)/(\delta - J_\perp)$, the longitudinal spin susceptibility $\chi_{zz}(T,b)$ can be well approximated by the zero-field result. For larger magnetic fields $b \gg \delta(J_z - J_\perp)/(\delta - J_\perp)$, there are two regions of temperature with different behavior. In the range of temperatures $b(\delta - J_\perp)/(\delta - J_z) \ll T \ll b\delta/(\delta - J_z)$, the longitudinal spin susceptibility becomes linear in temperature [cf. Eq. (B1)]:

$$\chi_{zz}(T,b) = \frac{1}{2(\delta - J_z)} + \frac{T}{b^2}. \quad (34)$$

At higher temperatures $T \gg b\delta/(\delta - J_z)$, the temperature dependence of the longitudinal spin susceptibility saturates:

$$\chi_{zz}(T,b) = \frac{1}{2(\delta - J_z)}. \quad (35)$$

In the limit of large magnetic fields, the ground-state energy for the configuration with the total spin projection S_z is equal to $(\delta - J_z)S_z^2 - bS_z$. Thus, the projection of the total spin in the ground state is $S_z = b/[2(\delta - J_z)]$. It allows us to estimate the longitudinal spin susceptibility as $\chi_{zz}=dS_z/db=1/[2(\delta - J_z)]$ in agreement with Eq. (35).

B. Case of the easy plane ($J_z < J_\perp$)

Using the integral representations (18) and (20), we integrate over h and obtain

$$Z_S = \left(\frac{\delta}{\delta - J_z}\right)^{1/2} e^{\frac{\beta(J_\perp^2+b^2)}{4(\delta-J_\perp)}} \int_{-\pi/2}^{\pi/2} \frac{dt}{\sqrt{\pi}}\, e^{-\frac{t^2}{\beta|J_*|}+\frac{ibt}{\delta-J_\perp}} \times \frac{\sinh\left(\frac{\delta(\beta b+2it)}{2(\delta-J_\perp)}\right)}{\sqrt{\beta|J_*|}\sinh\left(\frac{\beta b+2it}{2}\right)} \vartheta_3\left(e^{-\frac{\pi^2}{\beta(J_\perp-J_z)}}, \frac{i\pi t}{\beta(J_\perp - J_z)}\right). \quad (36)$$

Here, $\vartheta_3(q,z) = \sum_m q^{m^2} e^{2imz}$ stands for the Jacobi theta function. Since $T \gg \delta \geqslant J_\perp - J_z$, the Jacobi theta function ϑ_3 becomes equal to unity. Then, for $b = 0$ we find

$$Z = \left(\frac{\delta}{\delta - J_z}\right)^{1/2} e^{\frac{\beta J_\perp^2}{4(\delta-J_\perp)}} F_2\left(\frac{\delta}{\delta - J_\perp}, \sqrt{\beta|J_*|}\right), \quad (37)$$

where

$$F_2(x,y) = \int_{-\pi/2y}^{\pi/2y} \frac{dt}{\sqrt{\pi}} \frac{\sin(xyt)}{\sin(yt)} e^{-t^2}. \quad (38)$$

At temperatures $T \gg \max\{\delta, \frac{\delta^2(J_\perp-J_z)}{(\delta-J_\perp)(\delta-J_z)}\}$, with the help of Eq. (B5) we obtain that the longitudinal spin susceptibility is given by Eq. (25). In the temperature range $\delta \ll T \ll \frac{\delta^2(J_\perp-J_z)}{(\delta-J_\perp)(\delta-J_z)}$, the behavior of χ_{zz} is described by Eq. (26). In the case of an easy-plane anisotropy, the interplay between Fermi-liquid and Curie-type temperature dependencies of the longitudinal spin susceptibility can be explained in exactly the same way as it was done for the case of an easy-axis anisotropy.

The longitudinal static spin susceptibility is almost insensitive to the presence of a small magnetic field $b \ll \delta(J_\perp - J_z)/(\delta - J_z)$. In the opposite case $b \gg \delta(J_\perp - J_z)/(\delta - J_z)$, one can neglect t in the sinh's arguments in Eq. (36). Then, at $b \gg \delta(J_\perp - J_z)/(\delta - J_z)$ we find

$$Z_S = \left(\frac{\delta}{\delta - J_z}\right)^{1/2} e^{\frac{\beta J_\perp^2}{4(\delta-J_\perp)}+\frac{\beta b^2}{4(\delta-J_z)}} \frac{\sinh\frac{\delta\beta b}{2(\delta-J_\perp)}}{\sinh\frac{\beta b}{2}}. \quad (39)$$

The result (39) implies that for magnetic fields in the range $(\delta - J_\perp)T/\delta \ll b \ll T$, the longitudinal spin susceptibility is described by Eq. (34) whereas for $b \gg T$, χ_{zz} is given by Eq. (35).

IV. LONGITUDINAL SPIN SUSCEPTIBILITY: THE EFFECT OF LEVEL FLUCTUATIONS

As it was explained above, the Hamiltonian (1) describes a quantum dot in the zero-dimensional limit for the Ising and Heisenberg exchange interactions only. Therefore, it is reasonable to study the effect of level fluctuations on the results obtained above for $J_\perp = 0$ and J_z. We start with the case of Ising exchange.

A. Ising exchange

To simplify the general result (18) in the case of the Ising exchange, it is convenient to make a change of variable $\mathcal{B} \to \mathcal{B} - 2h(J_z - J_\perp)T/J_z$, to take the limit $J_\perp \to 0$, and then to integrate over $\mathcal{B}$. Thus, we find

$$Z_S = \left(\frac{\delta}{\delta - J_z}\right)^{1/2} e^{\frac{\beta b^2}{4(\delta-J_z)}}\, \Xi\left(\frac{b}{J_z}, \frac{\beta J_z\delta}{\delta - J_z}\right), \quad (40)$$

where

$$\Xi(x,y) = \int_{-\infty}^{\infty} \frac{dh}{\sqrt{\pi}} e^{-h^2 - V(h\sqrt{y}+xy/2)}. \quad (41)$$

The information on fluctuations of single-particle levels is encoded in the even random function $V(h)$ via the density of states [see Eq. (21)]. We remind that the single-particle density of states ν_0(E) has non-Gaussian statistics [34]. However, for

$\max\{|h|,T/\delta\} \gg 1$ the function $V(h)$ is a Gaussian random variable with zero mean value (see, e.g., Ref. [35]). The two-point correlation function of V can be written as follows (see Appendix D):

$$\overline{V(h_1)V(h_2)} = \sum_{\sigma=\pm} L(h_1+\sigma h_2) - 2L(h_1) - 2L(h_2),$$
$$L(h) = \frac{2}{\pi^2 \boldsymbol{\beta}} \int_0^{|h|} dt\, t \left[\mathrm{Re}\,\psi\left(1+\frac{it}{2\pi}\right) + \gamma\right]. \tag{42}$$

Here, $\psi(z)$ is the Euler digamma function and $\gamma = -\psi(1)$ is the Euler-Mascheroni constant. In the case of the Ising exchange, the parameter $\boldsymbol{\beta}$ in Eq. (42) is equal to $\boldsymbol{\beta}=2$ since the energy levels ϵ_α in the Hamiltonian (1) are described by the unitary Wigner-Dyson ensemble (class A) [17]. The asymptotics of $L(h)$ are as follows [12]:

$$L(h) = \frac{4}{\boldsymbol{\beta}}\left(\frac{h}{2\pi}\right)^2 \begin{cases} \frac{\zeta(3)}{2}\left(\frac{h}{2\pi}\right)^2 - \frac{\zeta(5)}{3}\left(\frac{h}{2\pi}\right)^4, & \frac{|h|}{2\pi} \ll 1 \\ \ln\frac{|h|}{2\pi} + \gamma - \frac{1}{2}, & \frac{|h|}{2\pi} \gg 1. \end{cases} \tag{43}$$

1. *Perturbation expansion for* $\overline{\chi}_{zz}$

According to Eq. (40), the average longitudinal spin susceptibility $\overline{\chi}_{zz}$ is determined by the quantity $\overline{\ln \Xi(x,y)}$. Although $V(h)$ is a Gaussian random variable, exact evaluation of $\overline{\ln \Xi(x,y)}$ for arbitrary values of x and y is a complicated problem. We start from the perturbation theory in the correlation function $\overline{V(h)V(h')}$. Expanding expression (40) for $\Xi(x,y)$ to the second order in V and performing the averaging of $\ln \Xi(x,y)$ with the help of Eq. (42), we find

$$\overline{\ln \Xi(x,y)} = \int_0^\infty \frac{du}{\sqrt{\pi}} e^{-u^2}[e^{-x^2y/4}\cosh(ux\sqrt{y})L(2u\sqrt{y}) - (e^{-x^2y/2}\cosh(ux\sqrt{2y})+1)L(u\sqrt{2y})]. \tag{44}$$

There exist four regions of different behavior of $\overline{\ln \Xi(x,y)}$. They are shown in Fig. 3. It is convenient to introduce the renormalized exchange $\bar{J}_z = \delta J_z/(\delta - J_z)$.

In the region I, $\bar{J}_z \max\{1,(b/J_z)\} \ll T$, the arguments of $\Xi(x,y)$ satisfy the condition $y \ll \min\{1,1/x\}$. The latter allows one to use the asymptotics of $L(h)$ for $|h| \ll 1$ [see Eq. (43)]. Then, we find

$$\overline{\ln \Xi(x,y)} = \frac{3\zeta(3)y^2}{8\pi^4\boldsymbol{\beta}}\left[1 + yx^2 - \frac{5\zeta(5)y}{2\pi^2\zeta(3)} \times \left(1+\frac{3}{2}yx^2+\frac{y^2x^4}{6}\right)\right]. \tag{45}$$

Hence, we obtain the following result for the average longitudinal spin susceptibility at temperatures $T \gg \bar{J}_z \max\{1,(b/J_z)\}$:

$$\overline{\chi}_{zz} = \frac{1}{2(\delta - J_z)} + \frac{3\zeta(3)}{4\pi^4\boldsymbol{\beta}}\frac{\delta^3 J_z}{(\delta-J_z)^3T^2} - \frac{45\zeta(5)}{16\pi^6\boldsymbol{\beta}}\frac{\delta^4 J_z^2}{(\delta-J_z)^4T^3}\left[1+\frac{2}{3}\frac{\delta b^2}{J_zT(\delta-J_z)}\right]. \tag{46}$$

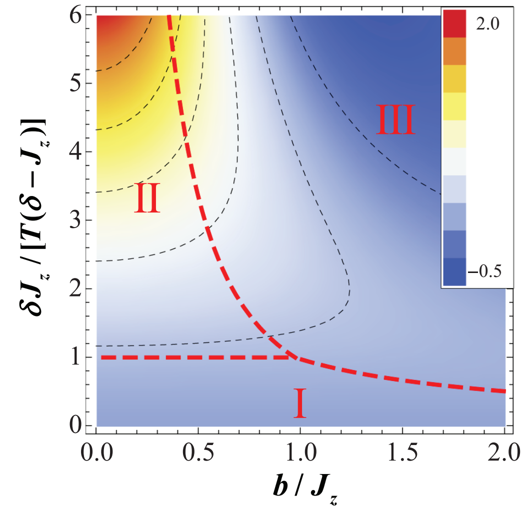

FIG. 3. (Color online) Different regions of behavior of the relative correction to $\overline{\chi}_{zz}$ due to fluctuations for the case of Ising exchange in the plane of dimensionless magnetic field and inverse temperature b/J_z and $\delta J_z/T(\delta - J_z)$. Note that in our analysis we assume $T \gg \delta$.

In the region I, the corrections to the longitudinal spin susceptibility are always small and, therefore, the perturbation theory is well justified. We present a more transparent way for derivation of Eq. (46). At first, one can substitute $1/\Delta$ for $1/\delta$ in the expression (25) (with $J_\perp = 0$) for the equidistant spectrum. Second, we expand χ_{zz} to the second order in the deviation $\Delta - \delta$. Finally, one can perform averaging with the help of the relation [12]

$$\overline{(\Delta-\delta)^2} = \frac{3\zeta(3)}{2\pi^4\boldsymbol{\beta}}\frac{\delta^4}{T^2}, \quad \delta \ll T \tag{47}$$

and obtain the result (46) (with $b=0$).

In the region II, $\bar{J}_z \gg T \gg \max\{\delta,\bar{J}_z(b/J_z)^2\}$, one can perform an expansion in x^2y in the right-hand side of Eq. (44) since the condition $1 \ll y \ll 1/x^2$ holds. However, the argument of L is typically large and we need to use its asymptotics for $|h| \gg 1$ [see Eq. (43)]. Then, we obtain

$$\overline{\ln \Xi(x,y)} = \frac{y\ln 2}{4\pi^2\boldsymbol{\beta}}(2+yx^2) - \frac{y^3x^4}{48\pi^2\boldsymbol{\beta}}. \tag{48}$$

Therefore, the average longitudinal spin susceptibility in the region II $[\bar{J}_z \gg T \gg \max\{\delta,\bar{J}_z(b/J_z)^2\}]$ is as follows:

$$\overline{\chi}_{zz} = \frac{1}{2(\delta-J_z)} + \frac{\ln 2}{2\boldsymbol{\beta}\pi^2}\frac{\delta^2}{T(\delta-J_z)^2} - \frac{1}{4\pi^2\boldsymbol{\beta}}\frac{\delta^3b^2}{J_z(\delta-J_z)^3T^2}. \tag{49}$$

At zero magnetic field, we check that the contribution of the second order in L to $\overline{\ln \Xi(0,y)}$ is of order of $[y/(\pi^2\boldsymbol{\beta})]^2$ (see Appendix E). Therefore, the perturbation theory in the two-point correlation function of V is justified for $T \gg \bar{J}_z/(\pi^2\boldsymbol{\beta})$ only. In this regime, the variance of χ_{zz} is small $[\overline{(\chi_{zz})^2} - \overline{\chi}_{zz}^2]/\overline{\chi}_{zz}^2 \sim \bar{J}_z/(\pi^2\boldsymbol{\beta}T) \ll 1$ (see Appendix E). Therefore, at $T \gg \bar{J}_z/(\pi^2\boldsymbol{\beta})$ one can expect the normal distribution of χ_{zz}.

Finally, in the region III, $\delta \ll T \ll \bar{J}_z \min\{(b/J_z),(b/J_z)^2\}$, the typical value of u contributing to the integral in the right-hand side of Eq. (44) can be not only of the order unity but also of the order of $x\sqrt{y} \gg 1$. In the latter case, since $yx \gg 1$ one needs to use the asymptotics of $L(h)$ for $|h| \gg 1$ [see Eq. (43)]. Then, we find

$$\overline{\ln \Xi(x,y)} = \frac{y}{2\pi^2\beta}\left(\ln\frac{|x|y}{2} + c_2\right) - \int_0^\infty \frac{du}{\sqrt{\pi}} e^{-u^2} L(u\sqrt{2y}). \qquad (50)$$

We thus obtain the average longitudinal spin susceptibility in the region III [$\delta \ll T \ll \bar{J}_z \min\{(b/J_z),(b/J_z)^2\}$]:

$$\overline{\chi}_{zz} = \frac{1}{2(\delta - J_z)} - \frac{1}{2\beta\pi^2}\frac{\delta J_z}{(\delta - J_z)b^2}. \qquad (51)$$

For magnetic fields $b \gg J_z$, the effect of level fluctuations is suppressed and the perturbation theory is justified. At $b \sim J_z\sqrt{T/\bar{J}_z} \ll J_z$, the result (51) agrees with the result (46) whereas at $T \sim \bar{J}_z b/J_z \gg \bar{J}_z$ the corrections due to level fluctuations in (51) and (49) become of the same order.

Results (49) and (51) imply nonmonotonous behavior of the average longitudinal spin susceptibility with magnetic field b in the temperature range $\bar{J}_z/(\pi^2\beta) \ll T \ll \bar{J}_z$ (see Fig. 4). The susceptibility $\overline{\chi}_{zz}(b)$ as a function of b has a minimum at $b \sim T J_z/\bar{J}_z$. In the region of strong fluctuations $\delta \ll T \ll \bar{J}_z/(\pi^2\beta)$, we expect similar behavior of the average longitudinal spin susceptibility.

Although the result (51) is derived for $T \gg \delta$, for $\delta - J_z \ll J_z$ it can be obtained from the following zero-temperature arguments. The difference in the ground-state energies for the state with projections $S_z + 1$ and S_z of the total spin can be estimated as

$$E_g(S_z+1) - E_g(S_z) = 2(\delta - J_z)S_z - bS_z + \Delta E_{2S_z}. \qquad (52)$$

Here, ΔE_{2S_z} is the fluctuation of the energy window in which there are $2S_z$ levels on average. It can be expressed as $\Delta E_{2S_z} = \delta\, \Delta n_{2S_z}$ where Δn_{2S_z} is the fluctuation of the number of single-particle levels in the strip with $2S_z$ levels in average. From the random matrix theory it is well known that [34]

$$\overline{(\Delta n_{2S_z})^2} = \frac{2}{\pi^2\beta}[\ln 2S_z + \text{const}]. \qquad (53)$$

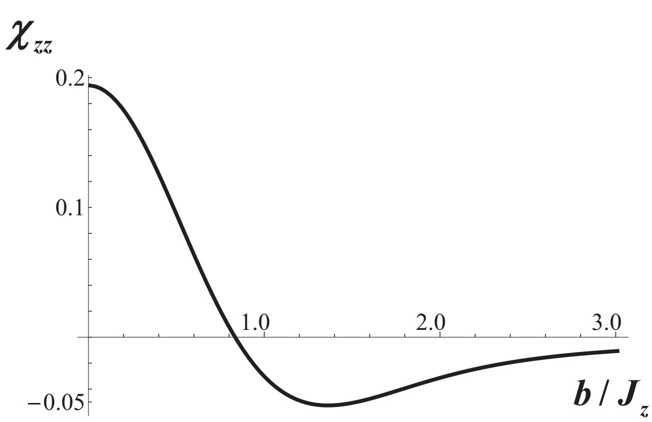

FIG. 4. Dependence of relative correction $\delta\chi_{zz} = \overline{\chi}_{zz} - 1/[2(\delta - J_z)]$ due to fluctuations on b/J_z [see Eqs. (49) and (51)]. The temperature $T = \delta J_z/[6(\delta - J_z)]$.

Comparing the energies of the ground states with total spin projections $S_z + 1$ and S_z, we find from Eq. (52) that

$$S_z = \frac{1}{2(\delta - J_z)}[b - \delta\, \Delta n_{2S_z}]. \qquad (54)$$

Hence, the average longitudinal spin susceptibility can be estimated as

$$\overline{\chi}_{zz} \sim \frac{\partial \overline{S}_z}{\partial b} = \frac{1}{2(\delta - J_z)}\left[1 + \frac{\delta^2}{2(\delta - J_z)^2}\frac{d^2\overline{(\Delta n_z)^2}}{dz^2}\right], \qquad (55)$$

where $z = 2\overline{S}_z \approx b/(\delta - J_z)$. Using Eq. (53), we reproduce the result (51).

2. Distribution function for χ_{zz}

The average longitudinal spin susceptibility is mostly affected by the level fluctuations in the region II [$\bar{J}_z \gg T \gg \bar{J}_z(b/J_z)^2$]. The perturbative result (46) loses its validity at $\bar{J}_z/(\pi^2\beta) \gg T \gg \delta$. Such a regime is realized in the close vicinity of the Stoner instability $\delta - J_z \ll \delta/(\pi^2\beta)$. In this case of strong fluctuations it is useful to know the distribution function of χ_{zz} rather than the average value.

In the range of temperatures $\delta \ll T \ll \bar{J}_z$, the integral in the right-hand side of Eq. (41) is dominated by large values of $|h|$. Then, using the asymptotic expression (43), one can check that for $|h_1|,|h_2| \gg 1$ the two-point correlation function (42) is homogeneous of degree two [6]:

$$\overline{V(uh_1)V(uh_2)} = u^2\overline{V(h_1)V(h_2)}. \qquad (56)$$

With the help of Eq. (56), at zero magnetic field $b = 0$ and for $\delta \ll T \ll \bar{J}_z/(\pi^2\beta)$, Eqs. (40) and (41) can be simplified to

$$Z_S = \left(\frac{\delta}{\delta - J_z}\right)^{1/2}\int_{-\infty}^{\infty}\frac{dh}{\sqrt{\pi}}e^{-h^2 - zv(h)}. \qquad (57)$$

We remind that the normalization is such that Z_S=1 at J_z=0. According to Eq. (9) for $J_\perp = b = 0$, the grand canonical partition function increases as J_z increases. Hence, it follows that $Z_S \geqslant 1$. According to Eq. (57), the statistics of the zero-field longitudinal spin susceptibility is determined by the single parameter $z = [\beta\bar{J}_z/(\pi^2\beta)]^{1/2}$. The Gaussian random process $v(h)$ has zero mean and is even in h, $v(h) = v(-h)$. Its two-point correlation function reads as

$$\overline{v(h_1)v(h_2)} = \frac{1}{2}\sum_{\sigma=\pm}(h_1 + \sigma h_2)^2\ln(h_1 + \sigma h_2)^2 - h_1^2\ln h_1^2 - h_2^2\ln h_2^2. \qquad (58)$$

Hence, we find that

$$\overline{[v(h+u) - v(h)]^2} = -2u^2\ln|u| + O(u^2) = O(u^{2H}) \qquad (59)$$

for any $H = 1 - \epsilon < 1$. Thus, the trajectories of $v(h)$ are continuous and its increments are strongly positively correlated (see Fig. 5). In fact, the process $v(h)$ is in many aspects close to the ballistic one $\tilde{v}(h) = \xi|h|$ with ξ being a Gaussian random variable recall that $\tilde{v}(h)$ is the unique process with $H = 1$]. The process $v(h)$ has arisen before in a seemingly unrelated context [36].

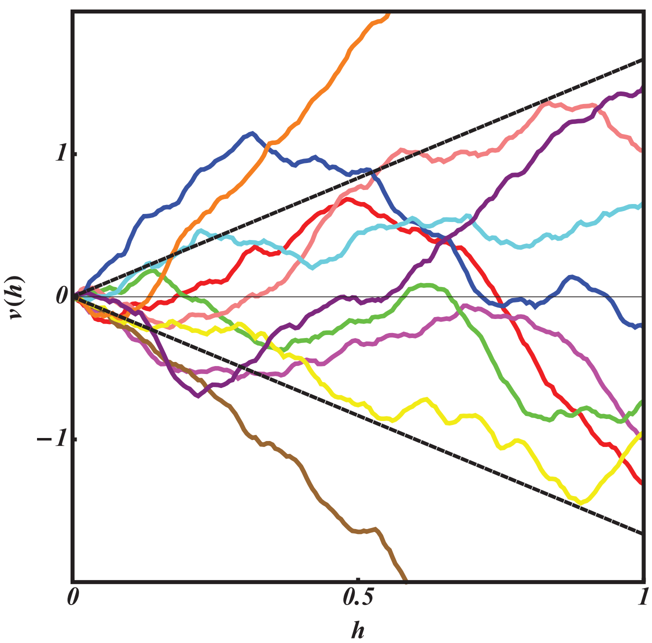

FIG. 5. (Color online) Several realizations of the process $v(h)$; dashed lines $\pm 2h\sqrt{\ln 2}$ are guides for the eye.

We are interested in the complementary cumulative distribution function $\mathcal{P}(W)$, i.e., the probability that $\ln Z_S$ exceeds W: $\mathcal{P}(W) \equiv \text{Prob}\{\ln Z_S > W\}$. It has the following properties: $\mathcal{P}(0) = 1$, $\mathcal{P}(\infty) = 0$, and $\mathcal{P}(W)$ is monotonously decreasing as W increases. The average moments of $\ln Z_S$ can be conveniently written as $\overline{[\ln Z_S]^k} = k\int_0^\infty dW\, W^{k-1}\mathcal{P}(W)$. Although a closed analytical expression for the complementary cumulative distribution function is not known, we bound $\mathcal{P}(W)$ from above to prove that all moments of $\ln Z_S$ (and consequently all moments of χ_{zz}) are finite for $J_z < \delta$. At first, we split the Gaussian weight $\exp(-h^2)$ in the integral in the right-hand side of Eq. (40) and obtain ($0 < \gamma < 1$ is an arbitrary splitting parameter)

$$Z_S \leqslant \frac{2\sqrt{\bar{J}_z}}{\sqrt{\pi\gamma J_z}}\int_0^\infty dh\, e^{-(1-\gamma)h^2/\gamma} \max_{h\geqslant 0}\{e^{-h^2 - zv(h)/\sqrt{\gamma}}\}. \quad (60)$$

The inequality (60) allows us to reduce the problem of finding an upper bound for $\mathcal{P}(W)$ to the statistics of the maxima of the Gaussian process $Y_\gamma(h) = -h^2 - (z/\sqrt{\gamma})v(h)$ which locally resembles a fractional Brownian motion with a drift. Indeed, from Eq. (60) we find

$$\mathcal{P}(W) \leqslant \text{Prob}\left\{\max_{h\geqslant 0} Y_\gamma(h) > W + \frac{1}{2}\ln\frac{(1-\gamma)J_z}{\bar{J}_z}\right\}. \quad (61)$$

To give an upper bound for the probability $\text{Prob}\{\max_{h\geqslant 0} Y_\gamma(h) > w\}$ we employ the Slepian's inequality [37]. Let us consider an auxiliary Gaussian process $X(h) = -h^2 + (2z\sqrt{\ln 2}/\sqrt{\gamma})B(h^2)$ where $B(h)$ is the standard Brownian motion [$\overline{B(h)} = 2h$; the Hurst exponent $H = 1/2$]. For any interval $\mathcal{T}$ the sample paths $\{X(h), h \in \mathcal{T}\}$ and $\{Y_\gamma(h), h \in \mathcal{T}\}$ are bounded. The following relations hold:

$$\overline{X(h)} = \overline{Y_\gamma(h)}, \qquad \overline{X^2(h)} = \overline{Y_\gamma^2(h)},$$
$$\overline{[X(h_1) - X(h_2)]^2} \geqslant \overline{[Y_\gamma(h_1) - Y_\gamma(h_2)]^2}. \quad (62)$$

The first two equalities are trivially satisfied while the last inequality follows from an easily verifiable inequality $\overline{[v(1/2 + r) - v(1/2 - r)]^2} \leqslant 8r\ln 2$ for $|r| \leqslant 1/2$. Then, the processes $Y_\gamma(h)$ and $X(h)$ satisfy the Slepian's inequality

$$\text{Prob}\left\{\max_{h\geqslant 0} Y_\gamma(h) > w\right\} \leqslant \text{Prob}\{\max_{h\geqslant 0} X(h) > w\} \quad (63)$$

for all real w. Using a well-known result for the Brownian motion with a linear drift (see, e.g., Ref. [38])

$$\text{Prob}\left\{\max_{h\geqslant 0} X(h) > w\right\} = \exp\left(-\frac{\gamma w}{2z^2\ln 2}\right), \quad w > 0 \quad (64)$$

we find the following upper bound for the complementary cumulative distribution function:

$$\mathcal{P}(W) \leqslant \exp\left\{-\frac{\gamma}{2z^2\ln 2}\left[W + \frac{1}{2}\ln\frac{(1-\gamma)J_z}{\bar{J}_z}\right]\right\}. \quad (65)$$

From Eq. (65) it follows that for $\bar{J}_z/(\pi^2\beta) \gg T \gg \delta$ all moments of $\ln Z_S$ (and hence all moments of χ_{zz}) are finite for $J_z < \delta$. Therefore, even in the presence of the strong level fluctuations the Stoner instability occurs at $J_z = \delta$ only. For $J_z < \delta$ and for temperatures $T \gg \delta$, the quantum dot is in the paramagnetic state.

For $z \gg 1$ the saddle-point approximation in Eq. (40) becomes exact and the statistics of $\ln Z_S$ reduces to the statistics of maxima of the process $Y(h) = -h^2 - zv(h)$. As it can be seen from rescaling of h, the probability that the maximum of $Y(h)$ exceeds w equals the probability that the maximum of $\widetilde{Y}(s) = v(s)/(1 + s^2)$ defined on $s \geqslant 0$ exceeds $\sqrt{w}/z$. From the results of Hüsler and Piterbarg [39] it follows that the large-w tail of $\text{Prob}\{\max_{h\geqslant 0} Y(h) > w\}$ is determined by a small vicinity of the point $s^* = 1$ where the variance of $\widetilde{Y}(s)$ attains its maximum $\ln 2$. Furthermore, should we have a finite limit

$$\lim_{s,t\to s_*}\frac{\overline{[\widetilde{Y}(s) - \widetilde{Y}(t)]^2}}{K^2(s-t)} > 0 \quad (66)$$

for some function $K(x)$ regularly varying at 0 with index $\alpha \in (0,1)$, the precise asymptotics would read as

$$\text{Prob}\left\{\max_{h\geqslant 0} Y(h) > W\right\} \sim \text{const}(\alpha) \times \frac{(z^2/W)^{-1}}{K^{-1}(\sqrt{z^2/W})} \times \exp\left[-\frac{W}{2z^2\ln 2}\right], \quad W/z^2 \gg 1. \quad (67)$$

Here, $K^{-1}(x)$ stands for the functional inverse of $K(x)$. In our case, Eq. (59) translates into $K(x) = x\sqrt{\ln(1/x)}$ which is regularly varying with index $\alpha = 1$ [recall that a function $f(x)$ is regular varying at $x = 0$ with index α if $\lim_{t\to 0} f(at)/f(t) = a^\alpha$ for any $a > 0$]. The result of Ref. [39] is therefore not directly applicable, but we believe this to be a technicality. In analogy with a similar situation for fractional Brownian motion, we expect the asymptotics (67) to hold with only the W-independent factor const(α) modified. Note that the exponential part can be tracked to be the tail of a normal distribution with variance $\ln 2$ taken at $\sqrt{W/z}$, and that it had been correctly reproduced by our initial estimate. Therefore, we find with logarithmic accuracy that the tail of the complementary cumulative distribution function is given by $\{W \gg [\ln 2/(\pi^2 \boldsymbol{\beta})]\delta J_z/[T(\delta - J_z)]\}$

$$\mathcal{P}(W) \propto \mathcal{P}_{\rm tail}\left(\frac{\pi^2 \boldsymbol{\beta} T(\delta - J_z)W}{\delta J_z}\right),$$
$$\mathcal{P}_{\rm tail}(x) = \frac{\sqrt{\ln x}}{\sqrt{x}} \exp\left(-\frac{x}{2\ln 2}\right). \quad (68)$$

This result is valid in the temperature range $\bar{J}_z/(\pi^2 \boldsymbol{\beta}) \gg T \gg \delta$ and is consistent with the upper bound (65).

To illustrate the result (68) we approximate the Gaussian process $v(h)$ by a degenerate one $\tilde{v}(h) = \xi|h|$, where ξ is the Gaussian random variable with zero mean $\overline{\xi} = 0$ and variance $\overline{\xi^2} = 4\ln 2$. Substituting the process $\tilde{v}(h)$ for $v(h)$ into the right-hand side of Eq. (57), we estimate the partition function as $Z_S \simeq \sqrt{\bar{J}_z/J_z}\exp(z^2\xi^2/4)[1 - \text{erf}(z\xi/2)]$. The large values of Z_S correspond to large negative values of ξ such that $\ln Z_S \approx z^2\xi^2/4$. Therefore, the tail of distribution of $\ln Z_S$ is simple exponential. Hence, we find that for $z \gg 1$ the tail of the complementary cumulative distribution function $\mathcal{P}(W)$ is given by Eq. (68) without the logarithm in the preexponent. As shown in Fig. 6, the overall behavior of $\mathcal{P}(W)$ for $z \gg 1$ is well enough approximated by the complementary cumulative distribution function for the degenerate process $\tilde{v}(h)$. Also, we mention that the behavior of $\mathcal{P}(W)$ for $z \gg 1$ is very different from its behavior at $z \lesssim 1$. For the latter, $\mathcal{P}(W)$ is given by the complementary cumulative distribution function of the normal distribution (see Fig. 6).

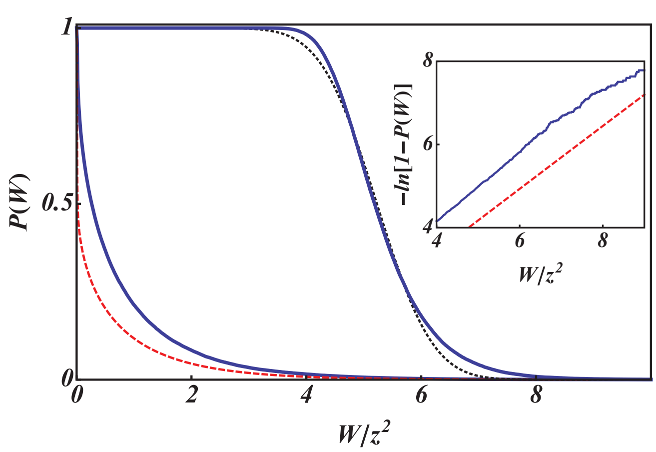

FIG. 6. (Color online) The dependence of $\mathcal{P}(W)$ on W/z^2 at $T = 3\delta$ computed numerically for $J_z/\delta = 0.94$ ($z \approx 0.5$) (upper solid curve) and $J_z/\delta = 0.999\,94$ ($z \approx 16.8$) (lower solid curve). The black dotted curve is the complementary cumulative distribution function for the normal distribution with mean and variance as one can find from the lowest-order perturbation theory in V for $T = 3\delta$ and $J_z/\delta = 0.94$ [cf. Eqs. (E7) and (E8)]. The red dashed curve is the complementary cumulative distribution function of the degenerate process $\tilde{v}(h)$ for $T = 3\delta$ and $J_z/\delta = 0.999\,94$. Inset: Comparison of the tail of $\mathcal{P}(W)$ computed numerically for $J_z/\delta = 0.999\,94$ ($z \approx 16.8$) and asymptotic result (68).

Equation (68) implies that the average moments of $\ln Z_S$ scale as $\overline{(\ln Z_S)^k} \sim z^{2k}$ for $z \gg 1$. Hence, for $\delta \ll T \ll \bar{J}_z/(\pi^2 \boldsymbol{\beta})$ the kth moment of the spin susceptibility is given by

$$\overline{\chi_{zz}^k} \propto \left[\frac{\delta^2}{\pi^2 \boldsymbol{\beta}(\delta - J_z)^2 T}\right]^k, \quad k = 1,2,\ldots . \quad (69)$$

The result (69) can be obtained from the saddle-point analysis of the integral in the right-hand side of Eq. (57), i.e., in essence, by Larkin-Imry-Ma–type arguments [40,41]. The scaling of the average spin susceptibility [Eq. (69) with $k = 1$] was proposed in Ref. [6] using arguments of Larkin-Imry-Ma type.

B. Heisenberg exchange

For the case of the isotropic exchange $J_\perp = J_z \equiv J$, the integration over $\mathcal{B}$ in Eq. (18) becomes trivial. Then, for $T \gg \delta$ we obtain [12]

$$Z_S = \left(\frac{\delta}{\delta - J}\right)^{1/2} \frac{e^{\frac{\beta b^2}{4J}}}{\sinh(\beta b/2)}\tilde{\Xi}\left(\frac{b}{J}, \frac{\beta J\delta}{\delta - J}\right), \quad (70)$$

where

$$\tilde{\Xi}(x,y) = \int_{-\infty}^{\infty}\frac{dh}{\sqrt{\pi}}\sinh(hx\sqrt{y})e^{-h^2+h\sqrt{y}-V(h\sqrt{y})}. \quad (71)$$

Since in the absence of magnetic field Z grows with increase of J [see Eq. (9)], one can check that for the Heisenberg exchange $Z_S \geqslant 1$. The detailed results of the perturbative expansion in V for the longitudinal spin susceptibility can be found in Ref. [12]. Similarly to the case of the Ising exchange, the effect of fluctuations is important at $b = 0$ and $\delta \ll T \ll J\delta/[\pi^2 \boldsymbol{\beta}(\delta - J)]$. In this range of parameters, the typical value of $|h|$ in the integral in the right-hand side of Eq. (71) is large $|h| \sim \sqrt{\beta \bar{J}} \gg 1$, where $\bar{J} = \delta J/(\delta - J)$. Then, for $b = 0$ Eq. (9) can be rewritten as

$$Z_S = \frac{2}{\sqrt{\beta \bar{J}}}\frac{\delta}{\delta - J}\int_{-\infty}^{\infty}\frac{dh}{\sqrt{\pi}}\, h\, e^{-h^2+h\sqrt{y}-zv(h)}, \quad (72)$$

where $z = [\beta \bar{J}/(\pi^2 \boldsymbol{\beta})]^{1/2}$. Here, $\boldsymbol{\beta} = 1$ which corresponds to the orthogonal Wigner-Dyson ensemble. The complementary cumulative distribution function $\mathcal{P}(W) = \text{Prob}\{\ln Z_S > W\}$ can be estimated in a similar way as in the previous section. Writing

$$Z_S \leqslant \frac{2}{\sqrt{\beta \bar{J}}}\left(\frac{\delta}{\delta - J}\right)\left[2\int_0^{\infty}\frac{dh\, h \sinh(h\sqrt{y})}{\sqrt{\pi\gamma}}e^{-(1-\gamma)h^2/\gamma}\right]$$
$$\times \max_{h \geqslant 0}\{e^{-h^2-(z/\sqrt{\gamma})v(h)}\}, \quad (73)$$

with arbitrary splitting parameter γ ($0 < \gamma < 1$), we obtain the following upper bound:

$$\mathcal{P}(W) \leqslant a_\gamma \exp\left\{-\frac{\gamma}{2z^2\ln 2}\left[W + \frac{3}{2}\ln\frac{(1-\gamma)J}{\bar{J}}\right]\right\}, \quad (74)$$

where $a_\gamma = \exp\{(\pi^2 \boldsymbol{\beta}\gamma)/[8(1-\gamma)\ln 2]\}$. This upper bound implies that all moments of $\ln Z_S$ (and of χ_{zz}) are finite for $J < \delta$. At $z \gg 1$ the integral in the right-hand side of Eq. (71)

can be evaluated in the saddle-point approximation, reducing the statistics of $\ln Z_S$ to the statistics of maxima of the process $Y(h) = -h^2 + h\sqrt{y} - zv(h)$. Then, using as in the previous section the results of Hüsler and Piterbarg [39], we find the tail of the complementary cumulative distribution function at $W \gg \delta J/[T(\delta - J)]$ is given by $\mathcal{P}_{\rm tail}[\pi^2 \boldsymbol{\beta} T(\delta - J)W/(\delta J)]$ [see Eq. (68)]. We note that for this tail the drift term $h\sqrt{y}$ in the process $Y(h)$ is not important.

The typical value of h contributing to the integral in the right-hand side of Eq. (72) is $\sqrt{y}/2$. Then, for $z \gg 1$ we find, with logarithmic accuracy, $\ln Z_s - y/4 = (z\sqrt{y}/2)v(1)$. Hence, for $\delta \ll T \ll \bar{J}/(\pi^2\boldsymbol{\beta})$ the average kth moment of the longitudinal spin susceptibility can be estimated as

$$\overline{\left(\chi_{zz} - \chi_{zz}^{(0)}\right)^k} \propto \left(\frac{\delta^2}{\sqrt{\pi^2\boldsymbol{\beta}}T(\delta - J)^2}\right)^k, \tag{75}$$

where $\chi_{zz}^{(0)} = \delta^2/[12T(\delta - J)^2]$ is the spin susceptibility in the absence of level fluctuations. We note that for $\delta \ll T \ll \bar{J}/(\pi^2\boldsymbol{\beta})$ the scaling of the average longitudinal spin susceptibility similar to Eq. (75) with $k = 1$ was derived in Ref. [6].

V. TRANSVERSE SPIN SUSCEPTIBILITY

The transverse spin susceptibility is defined as follows (see, e.g., Ref. [42]):

$$\chi_\perp(\omega) = \frac{i}{Z}\int_0^\infty dt\, e^{i(\omega + i0^+)t}\,\mathrm{Tr}([\hat{S}_+(t), \hat{S}_-(0)]e^{-\beta H}), \tag{76}$$

where $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$. Since, in contrast with $\hat{S}_z$, the operators $\hat{S}_x$, $\hat{S}_y$ of the total spin do not commute with the Hamiltonian H (for $J_z \neq 0$), the transverse spin susceptibility can acquire nontrivial frequency dependence.

In order to find the dynamic transverse spin susceptibility (76) we use the Heisenberg equations of motion for the spin operators: $d\hat{\boldsymbol{S}}/dt = i[H, \boldsymbol{S}]$. Since the operator S_z commutes with the Hamiltonian, it has no dynamics, $\hat{S}_z(t) = \hat{S}_z$. For the other components of the total spin we find

$$\begin{aligned}\hat{S}^\pm(t) &= e^{\mp 2i(J_\perp - J_z)\hat{S}_z t}\,\hat{S}^\pm(0)e^{-i(J_\perp - J_z)t \pm ibt}\\ &\equiv \hat{S}^\pm(0)e^{\mp 2i(J_\perp - J_z)\hat{S}_z t}\,e^{i(J_\perp - J_z)t \pm ibt}.\end{aligned} \tag{77}$$

Using expressions (77), we integrate over time in Eq. (76) and obtain the following operator expression for the transverse spin susceptibility:

$$\chi_\perp(\omega) = \frac{1}{Z}\sum_{\sigma=\pm}\mathrm{Tr}\,\frac{\left(\sigma\left[\hat{\boldsymbol{S}}^2 - \hat{S}_z^2\right] - \hat{S}_z\right)e^{-\beta H}}{\omega + b + (J_\perp - J_z)(2\hat{S}_z + \sigma) + i0^+}. \tag{78}$$

Since operators $\hat{S}_z$ and $\hat{\boldsymbol{S}}^2$ commute with H, one easily evaluates the trace in Eq. (78) with the help of Eq. (9). Thus, we derive the exact result for the dynamic transverse spin susceptibility:

$$\begin{aligned}\chi_\perp(\omega) = \frac{1}{Z}\sum_{n_\uparrow, n_\downarrow} & Z_{n_\uparrow}Z_{n_\downarrow}e^{-\beta E_c(n - N_0)^2 + \beta J_\perp m(m+1) + \beta\mu n}\\ &\times \mathrm{sgn}\,(2m+1)\sum_{l=-|m+1/2|+1/2}^{|m+1/2|-1/2} e^{\beta(J_z - J_\perp)l^2 - \beta bl}\\ &\times \sum_{\sigma=\pm}\frac{\sigma[m(m+1) - l^2] - l}{\omega + b + (J_\perp - J_z)(2l + \sigma) + i0^+}.\end{aligned} \tag{79}$$

In what follows, we will be interested in the imaginary part of $\chi_\perp(\omega)$. The real part can be restored from the Kramers-Kronig relations. Using Eq. (79), the imaginary part of the dynamic transverse spin susceptibility can be written as

$$\begin{aligned}\mathrm{Im}\,\chi_\perp(\omega) = -\frac{\pi}{Z}\sum_{n\in\mathbb{Z}}\sum_{\sigma=\pm} & \delta(\omega + b + (2n - \sigma)(J_z - J_\perp))\\ &\times \left(n + \sigma T\frac{\partial}{\partial J_\perp}\right)Z(n).\end{aligned} \tag{80}$$

Here, we introduce the Fourier transform of the partition function $Z(b + i\lambda T)$ in the complex magnetic field $b + i\lambda T$:

$$Z(n) = \int_{-\pi}^{\pi}\frac{d\lambda}{2\pi}\,e^{-i\lambda n}Z(b + i\lambda T). \tag{81}$$

As it follows from Eq. (80), the imaginary part of the transverse spin susceptibility obeys the sum rule

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathrm{Im}\,\chi_\perp(\omega) = M, \tag{82}$$

where the magnetization $M = -\langle\hat{S}_z\rangle = T\partial\ln Z/\partial b$. Since at $b = 0$ the function $Z(n)$ is even, the imaginary part of the zero-field transverse spin susceptibility is odd in frequency: $\mathrm{Im}\,\chi_\perp(-\omega) = -\mathrm{Im}\,\chi_\perp(\omega)$, so the sum rule (82) is trivially satisfied.

We mention that in the case of an isotropic exchange $J_z = J_\perp$, Eq. (80) becomes trivial, $\mathrm{Im}\,\chi_\perp(\omega) = 2\pi M\delta(\omega - b)$. In this case, the behavior of the transverse spin susceptibility is fully determined by the behavior of the magnetization M. Therefore, in what follows we shall not discuss the transverse spin susceptibility for the isotropic exchange.

A. Equidistant single-particle spectrum

At first, we consider the case of an equidistant single-particle spectrum and, therefore, neglect effects related to the level fluctuations. As it was discussed in Sec. II C, for $\delta \ll T$ the partition function can be factorized in accordance with Eq. (15). Since the factor Z_C does not depend on the magnetic field, it does not influence the results for $\chi_\perp(\omega)$ and we omit it below in this section. It implies that Z_S, $Z_S(n)$, and $Z_S(b + i\lambda T)$ should be substituted for Z, $Z(n)$, and $Z(b + i\lambda T)$ in Eqs. (80) and (81), respectively. Using Eq. (31) for the equidistant single-particle spectrum, we can rewrite $Z_S(n)$ in the following way:

$$Z_S(n) = \sqrt{\frac{\beta\delta}{\pi}} \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{-i\lambda n} \sum_m e^{-\beta(\delta - J_\perp)m^2 + \beta J_\perp m} \times \operatorname{sgn}(2m+1) \sum_{l=-|m+1/2|+1/2}^{|m+1/2|-1/2} e^{\beta(J_z - J_\perp)l^2 - \beta b l - i\lambda l}. \tag{83}$$

Next, performing integration over λ, we obtain the following result:

$$Z_S(n) = \sqrt{\frac{\beta\delta}{\pi}} e^{\beta(J_z - J_\perp)n^2 + \beta b n} \left[\sum_{m=|n|} e^{-\beta(\delta - J_\perp)m^2 + \beta J_\perp m} - \sum_{m=|n|+1} e^{-\beta(\delta - J_\perp)m^2 - \beta J_\perp m} \right]. \tag{84}$$

In the case $\beta(\delta - J_\perp)|n| \ll 1$, applying the Euler-Maclaurin formula to estimate the sums over m, we find

$$Z_S(n) = \frac{1}{2}\left(\frac{\delta}{\delta - J_\perp}\right)^{1/2} e^{\frac{\beta J_\perp^2}{4(\delta - J_\perp)}} e^{\beta(J_z - J_\perp)n^2 + \beta b n} \times \sum_{s=\pm} \operatorname{erf}\left[\sqrt{\beta(\delta - J_\perp)}\left(s|n| + \frac{J_\perp}{2(\delta - J_\perp)}\right)\right] + \sqrt{\frac{\beta\delta}{\pi}} e^{-\beta(\delta - J_z)n^2 + \beta b n} \cosh(\beta J_\perp |n|). \tag{85}$$

In the opposite case $\beta(\delta - J_\perp)|n| \gg 1$, the term with $m = |n|$ in the right-hand side of Eq. (84) provides the main contribution. Then, we obtain

$$Z_S(n) = \sqrt{\frac{\beta\delta}{\pi}} e^{-\beta(\delta - J_z)n^2 + \beta J_\perp |n| + \beta b n}. \tag{86}$$

We note that for $J_\perp = 0$, both expressions (85) and (86) coincide and are valid, in fact, for arbitrary n.

According to Eq. (80), $\operatorname{Im}\chi_\perp(\omega)$ is represented as the sum of delta peaks. Since their positions are independent of the realization of single-particle levels, the delta peaks survive averaging of $\operatorname{Im}\chi_\perp(\omega)$ over level fluctuations. Therefore, in order to discuss the frequency dependence of the transverse spin susceptibility in a form of a smooth curve, we assume some natural broadening $\Gamma \gg |J_z - J_\perp|$ for these delta peaks. Then, the sum over n in Eq. (80) can be replaced by an integral and we obtain

$$\operatorname{Im}\chi_\perp(\omega) = -\frac{\pi}{2|J_z - J_\perp| Z_S} \times \sum_{\sigma=\pm}\left(n + \sigma T \frac{\partial}{\partial J_\perp}\right) Z_S(n)\Big|_{n=-\varpi+\sigma/2}, \tag{87}$$

where $\varpi = (\omega + b)/[2(J_z - J_\perp)]$.

In the limit of large frequencies or large magnetic fields $\beta(\delta - J_\perp)|\varpi| \gg 1$, the imaginary part of the transverse spin susceptibility is exponentially small:

$$\operatorname{Im}\chi_\perp(\omega) = \frac{\varpi\sqrt{\pi\beta\delta}}{|J_z - J_\perp| Z_S} \exp[-\beta(\delta - J_z)|\varpi|(|\varpi| + 1) + \beta J_\perp |\varpi| - \beta b \varpi]. \tag{88}$$

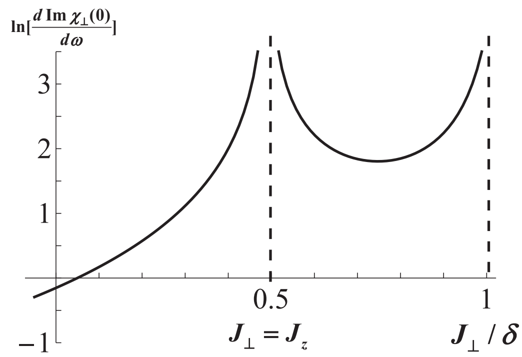

FIG. 7. Dependence of the slope $\frac{d \operatorname{Im}\chi_\perp(0)}{d\omega}$ of the traverse spin susceptibility on $J_\perp/\delta$ for $J_z = \delta/2$.

In the absence of magnetic field $b = 0$, $\operatorname{Im}\chi_\perp$ is an odd function of the frequency ω. For $\omega \to 0$ the imaginary part of the transverse spin susceptibility has linear behavior:

$$\operatorname{Im}\chi_\perp(\omega) = \frac{\omega\sqrt{\pi\beta\delta}}{2|J_z - J_\perp|(\delta - J_\perp)Z_S}\left[\frac{2\delta - J_\perp}{2(\delta - J_\perp)} + \frac{\sqrt{\pi}}{2\sqrt{\beta(\delta - J_\perp)}}\mathcal{G}\left(\frac{\beta J_\perp^2}{4(\delta - J_\perp)}\right)\right], \tag{89}$$

where

$$\mathcal{G}(x) = (1 + 2x)e^x \operatorname{erf}(\sqrt{x}). \tag{90}$$

The slope of $\operatorname{Im}\chi_\perp(\omega)$ at $\omega = 0$ has different behaviors for $J_\perp < J_z$ and for $J_\perp > J_z$. In the interval $0 \leqslant J_\perp \leqslant J_z$, the slope grows monotonously with the increase of $J_\perp$ and diverges at $J_\perp = J_z$. In the range $J_z < J_\perp < \delta$, the slope has a minimum (see Fig. 7). The imaginary part of the zero-field transverse spin susceptibility has two extrema (a minimum at a negative frequency and a maximum at a positive frequency). In the case $\delta - J_z, J_\perp \ll \delta$ and $\delta \ll T \ll \delta^2/(\delta - J_z)$, the positions of the extrema can be estimated as

$$\omega_{\text{ext}} \approx \pm\frac{2(J_z - J_\perp)}{\sqrt{2\beta(\delta - J_z)}}\left[\left(1 + \frac{\beta J_\perp^2}{8(\delta - J_z)}\right)^{1/2} + \left(\frac{\beta J_\perp^2}{8(\delta - J_z)}\right)^{1/2}\right]. \tag{91}$$

The behavior of $\chi_\perp(\omega)$ as a function of frequency is shown in Fig. 8. In the presence of a magnetic field $\operatorname{Im}\chi_\perp(\omega)$ is shifted along the frequency axis and becomes asymmetric (see Fig. 8).

It is worthwhile to discuss the case of the Ising exchange ($J_\perp = 0$) in more detail. In the regime of small frequencies and magnetic fields $|\omega|, |b| \ll T J_z/\delta$, the imaginary part of the dynamic spin susceptibility reads as

$$\operatorname{Im}\chi_\perp(\omega) = \frac{\omega\sqrt{\pi\beta(\delta - J_z)}}{2J_z\delta}\exp\left\{-\frac{\beta[(\delta - J_z)\omega + \delta b]^2}{4J_z^2(\delta - J_z)}\right\}. \tag{92}$$

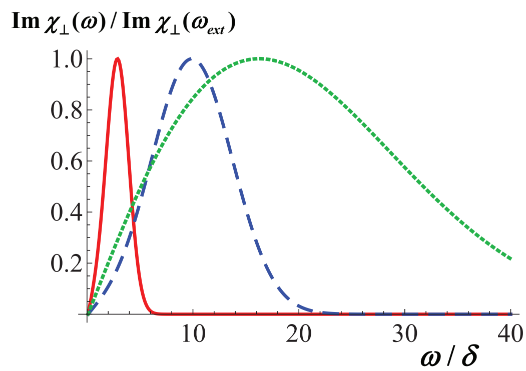

FIG. 8. (Color online) Dependence of Im $\chi_\perp(\omega)$ on ω for $J_z = 0.98\delta$ and several values of $J_\perp$: $J_\perp = 0.92\delta$ (red solid line), $J_\perp = 0.75\delta$ (blue dashed line), and $J_\perp = 0$ (green dotted line). The curves shrink to $\omega = 0$ as one moves closer to the isotropic case.

Although Im $\chi_\perp(\omega)$ is asymmetric in the presence of magnetic field, it still vanishes at zero frequency Im $\chi_\perp(\omega = 0) = 0$. In the opposite limit $|\omega|,|b| \gg TJ_z/\delta$, from Eq. (88) we find

$$\text{Im}\,\chi_\perp(\omega) = \frac{(\omega+b)\sqrt{\pi\beta(\delta-J_z)}}{2J_z^2}\exp\left\{-\frac{\beta(\delta-J_z)}{2J_z}|\omega+b|\right\} \times \exp\left\{-\frac{\beta[(\delta-J_z)\omega+\delta b]^2}{4J_z^2(\delta-J_z)}\right\}. \tag{93}$$

In the case $b = 0$, our results (92) and (93) coincide with the small- and large-frequency asymptotics of the result obtained in Ref. [18]. The presence of a magnetic field leads to a shift of the extrema of the imaginary part of the dynamic spin susceptibility according to

$$\omega_{\text{ext}} \approx \pm\sqrt{\frac{2J_z^2}{\beta(\delta-J_z)}}\left\{\left(1+\frac{\beta b^2}{8(\delta-J_z)}\right)^{1/2} \mp \frac{b\sqrt{\beta}}{8\sqrt{(\delta-J_z)}}\right\}. \tag{94}$$

B. Effect of level fluctuations (Ising case)

The above results for the dynamic spin susceptibility have been obtained without taking the level fluctuations into account. Following, we consider how the level fluctuations affect the dynamic spin susceptibility in the case of the Ising exchange. As we shall demonstrate, the effect of level fluctuations on Im $\chi_\perp(\omega)$ is small in most cases. Since the effect of level fluctuations is suppressed by the magnetic field, following we consider only the case $b = 0$.

We start from a generalization of Eq. (31) to an arbitrary spectrum (see Appendix C):

$$Z_{n_\uparrow}Z_{n_\downarrow} \approx \sqrt{\frac{\beta\delta}{4\pi}}e^{-\beta\mu_n n-2\beta\Omega_0(\mu_n)}\int_{-\infty}^{\infty}\frac{d\theta}{\pi}e^{-2mi\theta}e^{-\frac{\theta^2}{\beta\delta}-V(i\theta)}. \tag{95}$$

With the help of Eqs. (79), (87), and (95), we rewrite the imaginary part of the dynamic spin susceptibility as follows:

$$\text{Im}\,\chi_\perp(\omega) = -\frac{\sqrt{\pi\beta(\delta-J_z)}}{2J_z}\sum_{\sigma=\pm}e^{\beta J_z n^2}\left[\sum_{m=|n|+1}^{\infty}2\sigma m \times e^{-\beta\delta m^2}F_\chi\left(m,\beta\delta,\frac{\beta\delta J_z}{\delta-J_z}\right)+(n+\sigma|n|)e^{-\beta\delta n^2} \times F_\chi\left(|n|,\beta\delta,\frac{\beta\delta J_z}{\delta-J_z}\right)\right]\Bigg|_{n=(\sigma J_z-\omega)/(2J_z)}. \tag{96}$$

Here, the random function

$$F_\chi(m,x,y) = \frac{\int_{-\infty}^{\infty}d\theta\, e^{-\theta^2-V(xm+i\theta\sqrt{x})}}{\int_{-\infty}^{\infty}dh\, e^{-h^2-V(h\sqrt{y})}} \tag{97}$$

is equal to unity in the absence of level fluctuations (for $V = 0$).

Expanding the right-hand side of Eq. (97) to the second order in V we find

$$\overline{F_\chi(m)} = \int_{-\infty}^{\infty}\frac{dh_1dh_2}{\pi}e^{-h_1^2-h_2^2}\left\{1+\frac{1}{2}L(2xm+2ih_1\sqrt{x}) - 2L(xm+ih_1\sqrt{x}+h_2\sqrt{y})-\frac{1}{2}L(2h_1\sqrt{y}) + 2L(h_1\sqrt{y}+h_2\sqrt{y})\right\}. \tag{98}$$

In the high-temperature regime $T \gg \delta J_z/(\delta - J_z)$, and for $|m| \ll T/\delta$, all three integrals in the right-hand side of Eq. (98) are of the same order. Using the asymptotic expression (43) for the function $L(h)$ at $|h| \ll 1$, we obtain the following result for the imaginary part of the average dynamic spin susceptibility at low frequencies $\delta|\omega|/(2J_z) \ll T$ and high temperatures $T \gg \delta J_z/(\delta - J_z)$:

$$\frac{\text{Im}\,\overline{\chi_\perp(\omega)}}{\text{Im}\,\chi_\perp^{(0)}(\omega)} = 1+\frac{3\zeta(3)\delta^2}{16\pi^4\boldsymbol{\beta}T^2}\left[-\frac{\delta^2}{(\delta-J_z)^2} - \frac{\delta^2\omega^2}{TJ_z^2(\delta-J_z)}+\frac{\delta^2\omega^4}{4T^2J_z^4}\right]. \tag{99}$$

Here, Im $\chi_\perp^{(0)}(\omega)$ is given by Eq. (92) with $b = 0$. We mention that Eq. (99) can be obtained from Eq. (92) if one substitutes $1/\Delta$ for $1/\delta$ and performs averaging with the help of Eq. (47). In the regime of low frequencies and high temperatures, the effect of level fluctuations is small.

In the case of high frequencies and high temperatures $\delta|\omega|/(2J_z) \gg T \gg \delta J_z/(\delta - J_z)$, the first and second lines in the right-hand side of Eq. (98) provide the main contribution. Then, with the help of the asymptotic expression (43) for $L(h)$ at $|h| \gg 1$ we find that for $|\omega|/(2J_z) \gg T/\delta \gg J_z/(\delta - J_z)$ the imaginary part of the average dynamic spin susceptibility can be written as

$$\frac{\text{Im}\,\overline{\chi_\perp(\omega)}}{\text{Im}\,\chi_\perp^{(0)}(\omega)} = 1+\frac{\ln 2}{2\pi^2\boldsymbol{\beta}}\frac{\omega^2\delta^2}{T^2J_z^2}. \tag{100}$$

Here, Im $\chi_\perp^{(0)}(\omega)$ is given by Eq. (93) with $b = 0$. We note that the result (100) is valid provided $[\omega\delta/(TJ_z)]^2 \ll \pi^2\boldsymbol{\beta}$ so that the perturbation theory in V is justified. We emphasize that although the result Eq. (100) is valid at high temperatures $T \gg \delta J_z/(\delta - J_z)$, it cannot be obtained from Eq. (93) by a substitution of $1/\Delta$ for $1/\delta$ and averaging with the help of Eq. (47).

In the case of low temperatures $T \ll \delta J_z/(\delta - J_z)$, the m-independent contributions in the right-hand side of Eq. (98) vanish in the leading order. Using the asymptotic result for $L(h)$ at $|h| \gg 1$ [see Eq. (43)], we obtain

$$\overline{F_\chi(m)} = 1 - \frac{x}{\pi^2 \boldsymbol{\beta}} \begin{cases} (xm^2 - \frac{1}{2}) \ln y, & x|m| \ll 1 \\ (xm^2 - \frac{1}{2}) \ln \frac{y}{x^2 m^2}, & 1 \ll x|m| \ll \sqrt{y} \\ \frac{y}{2x} \ln \frac{x^2 m^2}{y}, & \sqrt{y} \ll x|m|. \end{cases} \tag{101}$$

Hence, we find the following result for the imaginary part of the average dynamical spin susceptibility at low frequencies $|\omega|/J_z^2 \ll T/\delta \ll J_z/(\delta - J_z)$:

$$\frac{\operatorname{Im} \overline{\chi_\perp(\omega)}}{\operatorname{Im} \chi_\perp^{(0)}(\omega)} = 1 - \frac{\delta}{\boldsymbol{\beta} \pi^2 T} \left(\frac{\delta \omega^2}{4 T J_z^2} + \frac{1}{2} \right) \ln \frac{\delta J_z}{(\delta - J_z) T}. \tag{102}$$

Here, $\operatorname{Im} \chi_\perp^{(0)}(\omega)$ is given by Eq. (92) with $b = 0$. In the temperature range $|\omega|/J_z^2 \ll T/\delta \ll J_z/(\delta - J_z)$ the effect of level fluctuations is suppressed by an additional small factor $\delta/T \ll 1$. Thus, we expect that the perturbation theory is valid even for $T \ll \delta J_z/[\pi^2 \boldsymbol{\beta}(\delta - J_z)]$.

In the high-frequency regime $1 \ll [\omega\delta/(J_z T)]^2$, and at low temperatures $T \ll \delta J_z/(\delta - J_z)$ we obtain from Eq. (101) the following result for the average dynamical spin susceptibility:

$$\frac{\operatorname{Im} \overline{\chi_\perp(\omega)}}{\operatorname{Im} \chi_\perp^{(0)}(\omega)} = 1 + \frac{1}{2\pi^2 \boldsymbol{\beta}} \min \left\{ \frac{\omega^2 \delta^2}{2 J_z^2 T^2}, \frac{\delta J_z}{T(\delta - J_z)} \right\} \times \ln \max \left\{ \frac{\omega^2 \delta(\delta - J_z)}{4 J_z^3 T}, \frac{4 J_z^3 T}{\omega^2 \delta(\delta - J_z)} \right\}. \tag{103}$$

Here, $\operatorname{Im} \chi_\perp^{(0)}(\omega)$ is given by Eq. (93) for $b = 0$. The perturbation theory is justified for $\max\{[\omega\delta/(J_z T)]^2, \delta J_z/[T(\delta - J_z)]\} \ll \pi^2 \boldsymbol{\beta}$. We remind that the maximum of $\operatorname{Im} \chi_\perp^{(0)}(\omega)$ is close to the frequency $\omega_{\text{ext}} \approx \sqrt{2 J_z^2 T/(\delta - J_z)}$. Then, as it follows from Eq. (103), the fluctuations yield an enhancement of the maximal value of the average dynamical spin susceptibility of the relative order $\{\delta J_z/[\pi^2 \boldsymbol{\beta} T(\delta - J_z)]\}$. Due to fluctuations, there is a small shift of the maximum towards zero frequency $\delta\omega_{\text{ext}}/\omega_{\text{ext}} \sim -\delta^2/(\pi^2 \boldsymbol{\beta} T^2)$.

Since $Z_S \leqslant 1$, we can bound the function $F_\chi(m)$ from above as

$$F_\chi(m) \leqslant \left(\frac{\delta}{\delta - J_z} \right)^{1/2} \int_{-\infty}^{\infty} d\theta \, e^{-\theta^2 - V(xm + i\theta\sqrt{x})}. \tag{104}$$

Therefore, $F_\chi(m)$ remains finite for $J_z < \delta$. Thus, in spite of the level fluctuations, the Stoner instability in $\operatorname{Im} \chi_\perp(\omega)$ emerges only at $J_z = \delta$.

According to Eq. (104), averaging over level fluctuations keeps $\operatorname{Im} \chi_\perp(\omega)$ finite. However, the form of the curve can be changed drastically in the regime of strong fluctuations. To estimate $\operatorname{Im} \chi_\perp(\omega)$ at $\delta \ll T \ll \delta J_z/[\pi^2 \boldsymbol{\beta}(\delta - J_z)]$, we substitute the degenerate process $\tilde{v}(h)$ for $V(h)$ into Eq. (97). Then, a straightforward calculation yields

$$\overline{F_\chi(m)} = \frac{e^{\beta(\delta - J_z)m^2}}{\sqrt{8z^2 \ln 2}} \exp\left[-\frac{\beta(\delta - J_z)m^2}{2z^2 \ln 2} \right] \tag{105}$$

for $\beta(\delta - J_z)m^2 \gg 1$. We recall that $z^2 = \delta^2/[\pi^2 \boldsymbol{\beta} T(\delta - J_z)]$. This result implies that $\operatorname{Im} \overline{\chi_\perp(\omega)}$ has a minimum and a maximum at frequencies

$$\omega_{\text{ext}} = \pm \frac{2\sqrt{\ln 2}}{\sqrt{\pi^2 \boldsymbol{\beta}}} \frac{\delta^2}{\delta - J_z}. \tag{106}$$

Due to strong fluctuations of the single-particle levels, the frequency of the extremum shifts towards higher frequencies (in comparison with the corresponding result without fluctuations) and becomes temperature independent. The fluctuations do not affect considerably the values of $\operatorname{Im} \overline{\chi_\perp(\omega)}$ at the extrema. Therefore, the slope at $\omega = 0$ becomes smaller $\operatorname{Im} \overline{\chi_\perp(\omega)}/\operatorname{Im} \chi_\perp^{(0)}(\omega) \propto 1/z \ll 1$.

VI. CONCLUSIONS

In this paper, we have addressed the spin fluctuations and dynamics in quantum dots and ferromagnetic nanoparticles. Within the framework of the model Hamiltonian which is an extension of the universal Hamiltonian to the case of uniaxial anisotropic exchange interaction, we have derived exact analytic expressions for the static longitudinal and dynamic transverse spin susceptibilities for arbitrary single-particle spectrum.

For the equidistant single-particle levels, we analyzed the temperature and magnetic field dependence of χ_{zz}. For $J_\perp \neq 0$, the zero-field longitudinal spin susceptibility has temperature dependence of type $1/T$ (Curie type) or $1/\sqrt{T}$. This indicates that the destruction of the mesoscopic Stoner instability by uniaxial anisotropy is not abrupt. The magnetic field suppresses the temperature dependence of χ_{zz} making spins aligned along the field.

For the case of the Ising exchange interaction we study the effect of single-particle level fluctuations on χ_{zz} in detail. The temperature dependence of χ_{zz} appears only due to level fluctuations. We showed that at low temperatures and for $\delta - J_z \ll \delta$ (where fluctuations are strong), the statistical properties of the longitudinal spin susceptibility are determined by the statistics of the extrema of a Gaussian process with a drift. This random process resembles locally a fractional Brownian motion. We rigorously prove that in this regime of strong fluctuations, all moments of zero-field static longitudinal spin susceptibility χ_{zz} are finite for $J_z < \delta$ and temperatures $T \gg \delta$. This means that the Stoner instability is not shifted by the level fluctuations away from its average position at $J_z = \delta$. Also, our results imply that randomness in the single-particle levels does not lead to a transition at finite $T \gg \delta$ between a paramagnetic and a ferromagnetic phase. We expect that these conclusions hold also for temperatures $T \lesssim \delta$. However, we cannot argue it within our approach; a separate (perhaps numerical) analysis is needed. We found that the magnetic field suppresses the effect of level fluctuations on the average longitudinal spin susceptibility. Interestingly, the dependence of $\overline{\chi}_{zz}$ on b is nonmonotonous with a minimum. We extended the analysis of the effect of strong level fluctuations to the case of Heisenberg exchange. We demonstrated that in this case the very same conclusions as for the Ising exchange hold.

For equidistant single-particle levels, we computed the temperature and magnetic field dependence of the imaginary part of the transverse spin susceptibility $\mathrm{Im}\,\chi_\perp(\omega)$. We found that it always has a maximum and a minimum whose positions tend to zero frequency with the decrease of anisotropy. The height of the maximum and the depth of the minimum increase with the decrease of anisotropy.

For the Ising exchange, we took into account the effect of single-particle level fluctuations on $\mathrm{Im}\,\chi_\perp(\omega)$. We argued that all moments of the dynamic transverse spin susceptibility $\chi_\perp(\omega)$ do not diverge for $J_z < \delta$. We found that at $\delta - J_z \ll \delta$ the positions of the extrema of $\mathrm{Im}\,\overline{\chi}_\perp(\omega)$ have a $\sqrt{T}$-type dependence at high temperatures and become independent of T at low temperatures (in the regime of strong level fluctuations). Interestingly, the level fluctuations do not change the minimal and maximal values of $\mathrm{Im}\,\overline{\chi}_\perp(\omega)$ significantly.

Our results, in principle, can be checked in quantum dots and nanoparticles made of materials close to the Stoner instability, such as Co impurities in a Pd or Pt host, Fe or Mn dissolved in various transition-metal alloys, Ni impurities in a Pd host, and Co in Fe grains, as well as nearly ferromagnetic rare-earth materials [43]. However, to test our most interesting results on spin susceptibility in the regime of strong level fluctuations, one needs to explore the regime $(\delta - J_z)/\delta \ll 1/(\pi^2\beta)$. The closest material to the Stoner instability we are aware of, YFe_2Zn_{20} [44], has the exchange interaction $J \approx 0.94\delta$ which is near the border of the regime with strong level fluctuations at low temperatures.

ACKNOWLEDGMENTS

We acknowledge useful discussions with Y. Fyodorov, Y. Gefen, A. Ioselevich, A. Shnirman, and M. Skvortsov. The research was funded in part by RFBR Grant No. 14-02-00333, the Council for Grant of the President of Russian Federation (Grant No. MK-4337.2013.2), Dynasty Foundation, RAS Programs “Quantum mesoscopics and disordered systems,” “Quantum physics of condensed matter” and “Fundamentals of nanotechnology and nanomaterials,” and by Russian Ministry of Education and Science.

APPENDIX A: DERIVATION OF $Z(b)$ USING THE WEI-NORMAN-KOLOKOLOV TRANSFORMATION

In this Appendix, we present a derivation of the partition function for the Hamiltonian (1). For simplicity, we consider the case of zero magnetic field. We use the notation of Ref. [12]. We start from the Hamiltonian $H_0 + H_S$. Then, the corresponding partition function can be written as $Z_J = \mathrm{Tr}\langle\exp(-\beta H_S)\rangle$, where H_S is given by Eq. (5) and $\langle\ldots\rangle$ denotes the averaging over all many-particle states with the weight $\exp(-\beta H_0)$. To get rid of terms of the fourth order in electron operators in the exponent H_S we apply the Hubbard-Stratonovich transformation

$$e^{it[J_\perp(S_x^2+S_y^2)+J_zS_z^2]} = \lim_{N\to\infty}\int\left[\prod_{n=1}^{N} d\boldsymbol{\theta}_n\right]\prod_\alpha \mathcal{T}e^{it\boldsymbol{\theta}_n\mathbf{s}_\alpha/N}\exp\left[-\frac{i\Delta}{4}\sum_{n=1}^{N}\left(\frac{\theta_{x,n}^2+\theta_{y,n}^2}{J_\perp}+\frac{\theta_{z,n}^2}{J_z}\right)\right], \tag{A1}$$

where $\Delta = t/N$. Here and further, we omit the normalization factors. We restore the correct normalization factor (depending on $T, J_\perp$, and J_z) in the final result. To calculate the time-ordered exponent ($\mathcal{T}$) of noncommuting operators, it is useful to apply the Wei-Norman-Kolokolov transformation [45,46] allowing us to rewrite the $\mathcal{T}$ exponent as a product of usual exponents:

$$\prod_\alpha \mathcal{T}e^{it\boldsymbol{\theta}_n\mathbf{s}_\alpha/N} = e^{ps_\alpha^{-p}\kappa_{p,N}^p}\exp\left(is_\alpha^z\Delta\sum_{n=1}^{N}\rho_{p,n}\right)\exp\left(is_\alpha^p\Delta\sum_{n=1}^{N}\kappa_{p,n}^{-p}\prod_{j=1}^{n}e^{-ip\Delta\rho_{p,j}}\right), \tag{A2}$$

where $p = \pm$ and $s_\alpha^p = s_\alpha^x + ips_\alpha^y$. We employ the initial condition $\kappa_{p,1}^p = 0$. The new variables ρ_p, κ_p^p, and κ_p^{-p} are related to the variables $\boldsymbol{\theta}$ as follows:

$$\begin{aligned}\frac{\theta_{x,n} - ip\theta_{y,n}}{2} &= \kappa_{p,n}^{-p},\ \theta_{z,n} = \rho_{p,n} - \kappa_{p,n}^{-p}\left(\kappa_{p,n}^p + \kappa_{p,n-1}^p\right),\\ \frac{\theta_{x,n} + ip\theta_{y,n}}{2} &= \frac{\kappa_{p,n}^p - \kappa_{p,n-1}^p}{ip\Delta} + \frac{\rho_{p,n}\left(\kappa_{p,n}^p + \kappa_{p,n-1}^p\right)}{2} - \frac{\left(\kappa_{p,n}^p + \kappa_{p,n-1}^p\right)^2}{4}\kappa_{p,n}^{-p}.\end{aligned} \tag{A3}$$

The vector variables $\boldsymbol{\theta}_n$ are real but the transformation (A3) assumes that the contour of integration in Eq. (A2) has been rotated. In order to preserve the number of variables, we impose the following constraints on the new variables: $\rho_{p,n} = -\rho_{p,n}^*$ and $\kappa_{p,n}^+ = (\kappa_{p,n}^-)^*$. We mention that the transformation (A3) assumes such a discretization of time that the quantity $(\kappa_{p,N}^p + \kappa_{p,N-1}^p)/2$ corresponds to $\kappa_p^p(t)$ in the continuous limit. In general, there are a lot of discrete representations of $\kappa_p^p(t)$, e.g., of the form $\nu\kappa_{p,N}^p + (1-\nu)\kappa_{p,N-1}^p$ with $0 \leqslant \nu \leqslant 1$. However, the choice of the symmetric one (with $\nu = \frac{1}{2}$) is optimal since it allows us to work within the first order in Δ in Eq. (A1). We note that the Jacobian of the transformation (A3) is equal to $\exp(ip\Delta\sum_{n=1}^{N}\rho_{p,n}/2)$.

Having in mind the further usage of the results, we rewrite $\exp(-\beta H)$ as the product $\exp(-it_+ H)\exp(it_- H)$ with $t_+ - t_- = -i\beta$. Now, rewriting two exponents in terms of two sets of new variables, we obtain

$$Z_J = \prod_{p=\pm}\Bigg\{\lim_{N_p\to\infty}\prod_{n_p=1}^{N_p}\int d\kappa^p_{p,n_p} d\kappa^{-p}_{p,n_p} d\rho_{p,n_p} e^{\frac{ip\Delta}{2}\rho_{p,n_p} - \frac{ip\Delta}{4J_z}\rho^2_{p,n_p} - \frac{1}{J_\perp}\kappa^{-p}_{p,n_p}(\kappa^p_{p,n_p} - \kappa^p_{p,n_p-1}) - \frac{ip\Delta\varkappa}{2J_\perp}\rho_{p,n_p}\kappa^{-p}_{p,n_p}(\kappa^p_{p,n_p}+\kappa^p_{p,n_p-1})}$$
$$\times\, e^{\frac{ip\Delta\varkappa}{4J_\perp}[\kappa^{-p}_{p,n_p}(\kappa^p_{p,n_p}+\kappa^p_{p,n_p-1})]^2}\Bigg\}\prod_\alpha \mathrm{Tr}\prod_{p=\pm}\left[e^{-ipt_p\epsilon_\alpha n_\alpha} e^{ps_\alpha^{-p}\kappa^p_{p,N_p}} e^{is^z_\alpha\Delta\sum_{n=1}^{N_p}\rho_{p,n}} e^{is^p_\alpha\Delta\sum_{n=1}^{N_p}\kappa^{-p}_{p,n}\exp(-ip\Delta\sum_{j=1}^n\rho_{p,j})}\right], \tag{A4}$$

where $\varkappa = 1 - J_\perp/J_z$ and $\Delta = t_p/N_p$. Let us introduce a set of auxiliary variables η_{p,n_p} to get rid of terms of the fourth order in κ_p's:

$$e^{\frac{ip\Delta\varkappa}{4J_\perp}[\kappa^{-p}_{p,n_p}(\kappa^p_{p,n_p}+\kappa^p_{p,n_p-1})]^2} = \int d\eta_{p,n_p} e^{\frac{ip\Delta\varkappa}{4J_\perp}\eta^2_{p,n_p}} e^{-\frac{ip\Delta\varkappa}{2J_\perp}[\eta_{p,n_p}\kappa^{-p}_{p,n_p}(\kappa^p_{p,n_p}+\kappa^p_{p,n_p-1})]}. \tag{A5}$$

To proceed with the evaluation of Z_J we need to calculate the following integrals over κ_p's:

$$\prod_{n_p=1}^{N_p}\int d\kappa^p_{p,n_p} d\kappa^{-p}_{p,n_p}\exp\left(-\frac{1}{J_\perp}\kappa^{-p}_{p,n_p}\left(\kappa^p_{p,n_p} - \kappa^p_{p,n_p-1}\right)\right)\exp\left(-\frac{ip\Delta\varkappa}{2J_\perp}(\rho_{p,n_p} - \eta_{p,n_p})\kappa^{-p}_{p,n_p}\left(\kappa^p_{p,n_p} + \kappa^p_{p,n_p-1}\right)\right). \tag{A6}$$

Following Ref. [46], we introduce the new variables

$$\kappa^{-p}_{p,n_p} = \chi^{-p}_{p,n_p} e^{\alpha_{p,n_p}}, \quad \kappa^p_{p,n_p} = \chi^p_{p,n_p} e^{\beta_{p,n_p}}, \tag{A7}$$

where

$$\beta_{p,n_p} = -ip\Delta\varkappa \textstyle\sum_{n=1}^{n_p}(\rho_{p,n} - \eta_{p,n}), \quad \alpha_{p,n_p} = -\beta_{p,n_p} - \dfrac{ip\Delta\varkappa}{2}(\rho_{p,n_p} - \eta_{p,n_p}). \tag{A8}$$

Such choice of α_{p,n_p} and β_{p,n_p} allows us to get rid of terms of the third order (second order in χ's and first order in ρ) in Eq. (A6) within the first order in Δ. The last term in the right-hand side of the second equation in (A8) determines the Jacobian $\mathcal{J}_p$ of the transformation (A7), $\mathcal{J}_p = \exp[-ip\Delta\varkappa(\rho_{p,n_p} - \eta_{p,n_p})/2]$. We emphasize that it can be missed in the continuous representation.

Evaluating the single-particle trace Tr in the expression (A4) explicitly, one can obtain (the limit $N_p \to \infty$ is assumed)

$$Z_J = \prod_{p=\pm}\left\{\prod_{n_p=1}^{N_p}\int d\chi^p_{p,n_p} d\chi^{-p}_{p,n_p} d\rho_{p,n_p} d\eta_{p,n_p}\, e^{\frac{ip\Delta}{2}[(1-\varkappa)\rho_{p,n_p}+\varkappa\eta_{p,n_p}]} e^{-\frac{ip\Delta}{4J_z}[\rho^2_{p,n_p}+\frac{\varkappa}{1-\varkappa}\eta^2_{p,n_p}]} e^{-\frac{1}{J_\perp}\chi^{-p}_{p,n_p}(\chi^p_{p,n_p}-\chi^p_{p,n_p-1})}\right\}$$
$$\times\prod_\alpha\Bigg\{1 + e^{-2i\epsilon_\alpha(t_+-t_-)} + 2e^{-i\epsilon_\alpha(t_+-t_-)}\cos\left(\frac{\Delta}{2}\sum_{p=\pm}\sum_{n_p=1}^{N_p}\rho_{p,n_p}\right) + \prod_{p=\pm} e^{-ip\epsilon_\alpha t_p}\exp\left(\frac{ip\Delta}{2}\sum_{n_p=1}^{N_p}\rho_{p,n_p}\right)$$
$$\times\left(p\chi^p_{p,N_p} e^{-ip\Delta\varkappa\sum_{n_p=1}^{N_p}(\rho_{p,n_p}-\eta_{p,n_p})} + i\Delta\sum_{n_{-p}=1}^{N_{-p}}\chi^p_{-p,n_{-p}} e^{-ip\Delta\varkappa\sum_{n=1}^{n_{-p}}(\rho_{-p,n}-\eta_{-p,n})} e^{ip\Delta\sum_{n=1}^{n_{-p}}\rho_{-p,n}}\right)\Bigg\}. \tag{A9}$$

Now, the integration over variables χ_{p,n_p} can be performed (see details in Appendix B of Ref. [12]). Then, we find

$$Z_J = \prod_{p=\pm}\left\{\prod_{n_p=1}^{N_p}\int d\rho_{p,n_p} d\eta_{p,n_p}\, e^{\frac{ip\Delta}{2}[(1-\varkappa)\rho_{p,n_p}+\varkappa\eta_{p,n_p}]} e^{-\frac{ip\Delta}{4J_z}[\rho^2_{p,n_p}+\frac{\varkappa}{1-\varkappa}\eta^2_{p,n_p}]}\right\}\prod_\alpha\left(\oint_{|z_\alpha|=1}\frac{i dz_\alpha}{2\pi z_\alpha^2}\right) e^{-w}$$
$$\times\exp\left\{-2v\cos\left[\frac{\Delta}{2}\sum_{p=\pm}\sum_{n_p=1}^{N_p}\rho_{p,n_p}\right]\right\}\int_0^\infty dy\, e^{-y}\exp\Bigg\{-iJ_\perp vy\left(\prod_{p=\pm} e^{i\frac{p\Delta}{2}\sum_{n_p=1}^{N_p}\rho_{p,n_p}}\right)$$
$$\times\left(\sum_{p=\pm} p\, e^{-ip\Delta\varkappa\sum_{n_p=1}^{N_p}(\rho_{p,n_p}-\eta_{p,n_p})}\Delta\sum_{n_p=1}^{N_p} e^{-ip\Delta\sum_{n=1}^{n_p}[(1-\varkappa)\rho_{p,n}+\varkappa\eta_{p,n}]}\right)\Bigg\}. \tag{A10}$$

Here, we introduce the notation

$$w = \sum_\alpha z_\alpha(1 + e^{-2i\epsilon_\alpha(t_+-t_-)}), \quad v = \sum_\alpha z_\alpha e^{-i\epsilon_\alpha(t_+-t_-)}. \tag{A11}$$

Let us introduce new variables to make the expression (A10) more standard:

$$\xi_p(t) = ip \int_0^t dt'[(1-\varkappa)\rho_p(t') + \varkappa\eta_p(t')] + \xi_p(0). \tag{A12}$$

Here, we switch to continuous representation. We obtain

$$Z_J = \prod_\alpha \left(\oint_{|z_\alpha|=1} \frac{idz_\alpha}{2\pi z_\alpha^2} \right) \int_0^\infty dy\, e^{-y-w} \prod_{p=\pm} \left\{ \int \mathcal{D}[\xi_p,\eta_p] e^{\frac{1}{2}[\xi_p(t_p)-\xi_p(0)]} e^{-\frac{ip}{4J_z}\int_0^{t_p} dt [\frac{(ip\dot{\xi}_p+\varkappa\eta_p)^2}{(1-\varkappa)^2} + \frac{\varkappa\eta_p^2}{1-\varkappa}]} \right\}$$
$$\times e^{-2v\cosh\{\frac{1}{2(1-\varkappa)}\sum_{p=\pm} p[\xi_p(t_p)-\xi_p(0)-ip\varkappa\int_0^{t_p} dt'\eta_p(t')]\}}$$
$$\times \exp\left\{ -iJ_\perp vy \left(\prod_{p=\pm} e^{\frac{1}{2(1-\varkappa)}[\xi_p(t_p)-\xi_p(0)-ip\varkappa\int_0^{t_p} dt\eta_p(t)]} \right) \left(\sum_{p=\pm} p\, e^{\frac{1}{1-\varkappa}[\xi_p(0)-\varkappa\xi_p(t_p)+ip\varkappa\int_0^{t_p} dt\eta_p(t)]} \int_0^{t_p} dt\, e^{-\xi_p(t)} \right) \right\}. \tag{A13}$$

There is some freedom in choosing the initial conditions for field variables $\xi_p(t)$. It is convenient to choose them such that the following relations hold:

$$\sum_{p=\pm} \xi_p(t_p) + 2\ln(4vy) = 0, \tag{A14}$$

$$\sum_{p=\pm} p\left[\xi_p(0) - \varkappa\xi_p(t_p) + ip\varkappa \int_0^{t_p} dt\eta_p(t) \right] = 0. \tag{A15}$$

Then, Eq. (A13) can be rewritten as

$$Z_J = \prod_\alpha \left(\oint_{|z_\alpha|=1} \frac{idz_\alpha}{2\pi z_\alpha^2} \right) \int_0^\infty dy\, e^{-y-w} \prod_{p=\pm} \left\{ \int \mathcal{D}[\xi_p,\eta_p] \int_{-\infty}^\infty dx\, e^{\frac{1}{2}[\xi_p(t_p)-\xi_p(0)]} e^{-\frac{ip}{4J_z}\int_0^{t_p} dt [\frac{(ip\dot{\xi}_p+\varkappa\eta_p)^2}{(1-\varkappa)^2} + \frac{\varkappa\eta_p^2}{1-\varkappa}]} \right.$$
$$\left. \times e^{\frac{ixp}{1-\varkappa}[\xi_p(0)-\varkappa\xi_p(t_p)+ip\varkappa\int_0^{t_p} dt\eta_p(t)]} \right\} e^{-2v\cosh\frac{\xi_+(t_+)-\xi_-(t_-)}{2}} e^{-\frac{iJ_\perp}{4}\sum_{p=\pm} p\int_0^{t_p} dt\, e^{-\xi_p(t)}} \delta\left(\sum_{p=\pm} \xi_p(t_p) + 2\ln(4vy) \right). \tag{A16}$$

Integrating over the variables η_p we find

$$Z_J = \prod_\alpha \left(\oint_{|z_\alpha|=1} \frac{idz_\alpha}{2\pi z_\alpha^2} \right) \int_{-\infty}^\infty dx \prod_{p=\pm} \left\{ e^{-iJ_z\varkappa x^2 pt_p} \int \mathcal{D}[\xi_p]\, e^{ip\int_0^{t_p} dt\mathcal{L}_p} e^{-(1-2ipx)\xi_p(0)/2} \right\}$$
$$\times \int_0^\infty \frac{dy}{4yv} e^{-y-w} \exp\left(-2v\cosh\left[\frac{1}{2}\sum_{p=\pm} p\xi_p(t_p) \right] \right) \delta\left(\sum_{p=\pm} \xi_p(t_p) + 2\ln(4vy) \right). \tag{A17}$$

The functional integral (A17) is of Feynman-Kac type with the Lagrangian

$$\mathcal{L}_p = \frac{1}{4J_\perp}\dot{\xi}_p^2 - \frac{J_\perp}{4}e^{-\xi_p}. \tag{A18}$$

Then, the calculation of the partition function can be reduced to an evaluation of two matrix elements:

$$Z_J = \prod_\alpha \left(\oint_{|z_\alpha|=1} \frac{idz_\alpha}{2\pi z_\alpha^2} \right) \int_{-\infty}^\infty dx\, e^{-iJ_z\varkappa x^2(t_+-t_-)} \int_0^\infty \frac{dy}{4yv} e^{-y-w} \prod_{p=\pm} \left\{ \int d\xi_p d\xi_p'\, e^{-(1-2ipx)\xi_p'/2} \right\}$$
$$\times\delta\left(\sum_{p=\pm} \xi_p + 2\ln(4vy) \right) e^{-2v\cosh[(\xi_+-\xi_-)/2]} \langle \xi_+|e^{-i\mathcal{H}_J t_+}|\xi_+'\rangle \langle \xi_-'|e^{i\mathcal{H}_J t_-}|\xi_-\rangle. \tag{A19}$$

Here, the one-dimensional quantum mechanical Hamiltonian

$$\mathcal{H}_J = -J_\perp \frac{\partial^2}{\partial\xi^2} + \frac{J_\perp}{4}e^{-\xi}. \tag{A20}$$

Its eigenfunctions are given by the modified Bessel functions $K_{2i\nu}$ where ν is a real number:

$$\langle \xi|\nu\rangle = \frac{2}{\pi}\sqrt{\nu\sinh(2\pi\nu)}K_{2i\nu}(e^{-\xi/2}). \tag{A21}$$

The eigenvalues of $\mathcal{H}_J$ are equal to $J\nu^2$: $\mathcal{H}_J|\nu\rangle = J\nu^2|\nu\rangle$. After integration over y we obtain

$$Z_J = \frac{4}{\pi^2}\prod_\alpha\left(\oint_{|z_\alpha|=1}\frac{idz_\alpha}{2\pi z_\alpha^2}\right)\int_{-\infty}^{\infty}dx\, e^{-iJ_z\varkappa x^2(t_+-t_-)}\int_0^\infty\frac{d\nu}{\nu}\,\nu\sinh(2\pi\nu)K_{2i\nu}(2\upsilon)e^{-w}$$
$$\times\prod_{p=\pm}\left\{\int d\xi_p d\xi_p'\; e^{-(1-2ipx)\xi_p'/2}K_{2i\nu}(e^{-\xi_p/2})\right\}\langle\xi_+|e^{-i\mathcal{H}_Jt_+}|\xi_+'\rangle\langle\xi_-'|e^{i\mathcal{H}_Jt_-}|\xi_-\rangle. \tag{A22}$$

Here, we use the following result (see formula 6.794.11 on p. 794 of Ref. [47]):

$$\int_0^\infty d\nu\,\nu\sinh(2\pi\nu)K_{2i\nu}(2\upsilon)K_{2i\nu}(e^{-\xi_+/2})K_{2i\nu}(e^{-\xi_-/2}) = \frac{\pi^2}{16}\exp\left(-\frac{1}{4\upsilon}e^{-\frac{\xi_++\xi_-}{2}} - 2\upsilon\cosh\frac{\xi_+-\xi_-}{2}\right). \tag{A23}$$

Integration over ξ_p can be now easily performed, and we obtain

$$Z_J = \frac{32}{\pi^2}\prod_\alpha\left(\oint_{|z_\alpha|=1}\frac{idz_\alpha}{2\pi z_\alpha^2}\right)\int_{-\infty}^{\infty}dx\, e^{-iJ_z\varkappa x^2(t_+-t_-)}\int_0^\infty\frac{d\nu}{\nu}\nu\sinh(2\pi\nu)K_{2i\nu}(2\upsilon)e^{-w}e^{-iJ_\perp\nu^2(t_+-t_-)}\int d\eta_+d\eta_-$$
$$\times e^{-2\eta_++4ix\eta_-}K_{2i\nu}(e^{-\eta_+-\eta_-})K_{2i\nu}(e^{-\eta_++\eta_-}). \tag{A24}$$

Using the identity (see formula 6.521.3 on p. 658 of Ref. [47])

$$\int_0^\infty dx\,xK_\nu(ax)K_\nu(bx) = \frac{\pi(ab)^{-\nu}(a^{2\nu}-b^{2\nu})}{2\sin(\pi\nu)(a^2-b^2)}, \tag{A25}$$

we can perform the integration over η_+. With the help of the integral representation of the modified Bessel function

$$K_\nu(x) = \int_0^\infty dh\, e^{-x\cosh h+\nu h}, \tag{A26}$$

we integrate over ν. Finally, integration over x yields

$$Z_J = \frac{2e^{-\beta J_\perp/2}}{\pi\beta\sqrt{J_\perp(J_z-J_\perp)}}\int_{-\infty}^{\infty}dh\,\sinh h\prod_{\alpha,\sigma}(1+e^{-\epsilon_\alpha\beta+h\sigma})e^{-\frac{h^2}{J_\perp\beta}}\int_{-\infty}^{\infty}d\eta_-\, e^{-\frac{4\eta_-^2}{\varkappa J_\perp\beta}}\frac{\sinh\frac{4\eta_-h}{J_\perp\beta}}{\sinh(2\eta_-)}. \tag{A27}$$

Here, we restored the correct numerical factor using the normalization condition $Z_J = 1$ at $\epsilon_\alpha\to+\infty$.

In order to derive the partition function for the Hamiltonian (1) from Eq. (A27), one needs to make the substitution $\epsilon_\alpha\to\epsilon_\alpha+i\phi_0T$ and to integrate over the variable ϕ_0:

$$Z = \sum_{k\in\mathbb{Z}}e^{-\beta E_c(k-N_0)^2}\int_{-\pi}^{\pi}\frac{d\phi_0}{2\pi}e^{i\phi_0k}Z_J. \tag{A28}$$

Then, we obtain Eq. (11) with $b=0$.

It is easy to obtain the partition function with nonzero magnetic field. The field shifts the z projection of the total spin in the evolution operator (A1): $S_z\to S_z+\frac{B}{2J_z}$. This shift affects only the boundary conditions on ξ in (A15).

APPENDIX B: ASYMPTOTIC RESULTS FOR THE FUNCTIONS $F_1(x,y)$ AND $F_2(x,y)$

At $y\ll\min\{1,1/\sqrt{x}\}$, the value of the integral in Eq. (23) is determined by the region $||t|-xy/2|\sim1$. Thus, one can expand sinh in the denominator into a series in $yt\sim y^2x\ll1$ and obtain

$$F_1(x,y) = \frac{\sqrt{\pi}}{y}\,\mathrm{erfi}(xy/2) - \frac{xy^2}{12}\exp(x^2y^2/4). \tag{B1}$$

Here, we performed the expansion to the second order in yt, having the further calculation for the spin susceptibility in mind.

At $1/\sqrt{x}\ll y\ll1$, the argument of the sinh in the denominator is large and one can make the following replacement: $\sinh yt\sim\mathrm{sgn}(t)\exp(y|t|)/2$. Then, we find

$$F_1(x,y) = 2\exp[(x-1)^2y^2/4]. \tag{B2}$$

The same simplification for $\sinh yt$ can be used in the limit $y\gg1$ and $x\geqslant1$. Then, we obtain

$$F_1(x,y) = e^{(x-1)^2y^2/4}\{1+\mathrm{erf}[(x-1)y/2]\}. \tag{B3}$$

At $y\gg1$ and $x\ll1$ the relevant region of integration in Eq. (23) is determined by the denominator. Thus, one can omit e^{-t^2}, expand $\sinh(xyt)$ in the numerator, and find

$$F_1(x,y) = \frac{\pi^{3/2}x}{2y}. \tag{B4}$$

The only relevant case for our values of x and y in Eq. (38) is the case with $y\ll1/\sqrt{x}$. In this regime, the denominator can be substituted by yt and the region of integration can be extended to infinity. Then, the following result can be obtained from (B1) by replacement $y\to iy$:

$$F_2(x,y) = \frac{\sqrt{\pi}}{y}\,\mathrm{erf}(xy/2) + \frac{xy^2}{12}\exp(-x^2y^2/4). \tag{B5}$$

APPENDIX C: DERIVATION OF EQS. (31) AND (95)

In this Appendix, we present brief arguments as to why Eqs. (31) and (95) are correct. Since Eq. (31) can be obtained from Eq. (95), we consider only the latter. We start from the following exact expression:

$$Z_{n_\uparrow} Z_{n_\downarrow} = \int_{-\pi}^{\pi} \frac{d\theta_1 d\theta_2}{(2\pi)^2} e^{-i\phi n - i\theta m} \prod_{\sigma=\pm} e^{-\beta\Omega_0(iT\phi + i\sigma T\theta/2)}, \quad \text{(C1)}$$

where $\theta_{1,2} = \phi \pm \theta/2$. As usual, at $\delta \ll T$ the integral over ϕ can be performed in the saddle-point approximation. This yields

$$Z_{n_\uparrow} Z_{n_\downarrow} \approx \sqrt{\frac{\beta\delta}{4\pi}} e^{-\beta\mu_n n - 2\beta\Omega_0(\mu_n)} \mathcal{X}_m(\beta\delta), \quad \text{(C2)}$$

where

$$\mathcal{X}_m(x) = \int_{-\pi}^{\pi} \frac{d\theta}{\pi} e^{-2mi\theta} e^{-\theta^2/x - V(i\theta)}. \quad \text{(C3)}$$

The function $\mathcal{X}_m(x)$ can be rewritten as

$$\mathcal{X}_m(x) = \int_{-\infty}^{\infty} \frac{d\theta}{\pi} e^{-\theta^2/x - V(i\theta)} \cos 2m\theta + \mathcal{R}_m(x). \quad \text{(C4)}$$

Now, we bound

$$\mathcal{R}_m(x) = 2\int_{\pi}^{\infty} \frac{d\theta}{\pi} e^{-\theta^2/x - V(i\theta)} \cos 2m\theta \quad \text{(C5)}$$

from above. Using that random function $V(i\theta)$ depends, in fact, on $\sin^2(\theta/2)$ [cf. Eq. (D3)], we obtain the following set of inequalities:

$$\begin{aligned} |\mathcal{R}_m(x)| &\leqslant 4\int_0^{2\pi} \frac{d\theta}{2\pi} \sum_{l=0}^{\infty} e^{-(\theta+\pi+2\pi l)^2/x} e^{-V(i\theta)} \\ &\leqslant 4\sum_{l=0}^{\infty} e^{-\pi^2(2l+1)^2/x} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-V(i\theta)} \\ &\leqslant 4\sum_{l=1}^{\infty} e^{-\pi^2 l/x} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-V(i\theta)} \\ &\leqslant \frac{4}{e^{\pi^2/x} - 1} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-V(i\theta)}. \end{aligned} \quad \text{(C6)}$$

Hence, we demonstrate that $|\mathcal{R}_m(x)| \leqslant O(e^{-\pi^2/x})$ is independent of m. Therefore, we can write $\mathcal{X}_m(x)$ at $x \ll 1$ as follows:

$$\mathcal{X}_m(x) \approx \int_{-\infty}^{\infty} \frac{d\theta}{\pi} e^{-\theta^2/x - V(i\theta)} \cos 2m\theta. \quad \text{(C7)}$$

APPENDIX D: CORRELATION FUNCTION $\overline{V(h_1)V(h_2)}$

In this Appendix, we present a brief derivation of Eq. (42). The correlation function of the single-particle density of states is given by [34]

$$\langle \delta\nu_0(E)\delta\nu_0(E+\omega)\rangle = \frac{1}{\delta^2}\left[\delta\left(\frac{\omega}{\delta}\right) - R\left(\frac{\pi\omega}{\delta}\right)\right]. \quad \text{(D1)}$$

Here, the function $R(x)$ depends on the statistics of the ensemble of single-particle energies. Using Eq. (D1), the identity $\int_{-\infty}^{\infty} R(x)dx = \pi$ and the definition of $V(h)$ we obtain

$$\begin{aligned} \overline{V(h_1)V(h_2)} = T^2 \int_{-\infty}^{\infty} \frac{dE\, d\omega}{\delta^2} R\left(\frac{\pi T\omega}{\delta}\right) [g(E,h_1)g(E,h_2) \\ - g(E+\omega/2,h_1)g(E-\omega/2,h_2)], \end{aligned} \quad \text{(D2)}$$

where

$$g(E,h) = \ln\left[1 + \frac{\sinh^2\left(\frac{h}{2}\right)}{\cosh^2\left(\frac{E}{2}\right)}\right]. \quad \text{(D3)}$$

The function $g(E,h)$ has the following Fourier transform with respect to variable E:

$$\begin{aligned} g(t,h) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{iEt} g(E,h) = \frac{1}{2\pi t} \operatorname{Im} \int_{-\infty}^{\infty} dE\, e^{iEt} \\ \times \tanh\frac{E}{2} \frac{\sinh^2(h/2)}{\sinh^2(h/2) + \cosh^2(E/2)}. \end{aligned} \quad \text{(D4)}$$

Since the function $g(E,h)$ is even in E, the function $g(t,h)$ is even in t. The function under the integral in the right-hand side of Eq. (D4) has poles at $E = \pi(2n+1)i, \pm h + \pi(2m+1)i$ where n and m are integers. Computation of the residues yields

$$\begin{aligned} g(t,h) &= \frac{1}{2\pi t} \operatorname{Im} 4\pi i \sum_{n\geqslant 0} e^{-\pi(2n+1)t} \left(1 - \frac{1}{2}e^{-iht} - \frac{1}{2}e^{iht}\right) \\ &= \frac{1 - \cos(ht)}{t \sinh(\pi t)}. \end{aligned} \quad \text{(D5)}$$

Substitution into Eq. (D2) leads to

$$\begin{aligned} \overline{V(h_1)V(h_2)} = 2\pi T^2 \int_{-\infty}^{\infty} \frac{dt\, d\omega}{\delta^2} R\left(\frac{\pi T\omega}{\delta}\right) g(t,h_1) \\ \times g(t,h_2)[1 - e^{-i\omega t}]. \end{aligned} \quad \text{(D6)}$$

At $x \gg 1$ the function $R(x)$ has the following asymptotic behavior:

$$R(x) = \frac{1}{\boldsymbol{\beta} x^2}, \quad x \gg 1. \quad \text{(D7)}$$

Recall that $\boldsymbol{\beta} = 1$ for the orthogonal Wigner-Dyson ensemble, $\boldsymbol{\beta} = 2$ for the unitary Wigner-Dyson ensemble, and $\boldsymbol{\beta} = 4$ for the symplectic Wigner-Dyson ensemble. Then, at $\max\{|h|, T/\delta\} \gg 1$ we find

$$\begin{aligned} \overline{V(h_1)V(h_2)} &= \frac{4}{\boldsymbol{\beta}} \int_0^{\infty} dt\, \frac{[1-\cos(h_1 t)][1-\cos(h_2 t)]}{t \sinh^2(\pi t)} \\ &= \sum_{\sigma=\pm} L(h_1 + \sigma h_2) - 2L(h_1) - 2L(h_2), \end{aligned} \quad \text{(D8)}$$

where

$$L(h) = \frac{2}{\boldsymbol{\beta}} \int_0^{\infty} dt\, \frac{\cos(ht) - 1 + h^2t^2/2}{t \sinh^2(\pi t)} \quad \text{(D9)}$$

is even in h. Next, for $h > 0$,

$$L'(h) = \frac{2}{\beta}\int_0^\infty dt\, \frac{ht - \sin(ht)}{\sinh^2(\pi t)} = \frac{8}{\beta}\int_0^\infty dt \sum_{n=1}^{\infty} n[ht - \sin(ht)]e^{-2\pi n t} = \frac{2h}{\pi^2\beta}\left[\mathrm{Re}\,\psi\left(1 + \frac{ih}{2\pi}\right) - \psi(1)\right]. \quad \text{(D10)}$$

This is the Eq. (42) of the paper. Using the well-known asymptotic expressions for the Euler digamma function $\psi(x)$ at small and large values of its argument, one arrives at Eq. (43).

APPENDIX E: FOURTH-ORDER PERTURBATION THEORY FOR $\overline{\chi}_{zz}$ IN THE CASE OF THE ISING EXCHANGE

In this Appendix, we present the derivation of the perturbative results (46) and (49) for $b = 0$. In addition, we compute the next order in L for the correction to $\overline{\chi}_{zz}$.

We start from the expansion of the average $\ln Z_S$ to the fourth order in V:

$$\overline{\ln Z_S} = \frac{1}{2}\ln\frac{\bar{J}_z}{J_z} - \frac{1}{2}F_2 - \frac{1}{2}F_{1,1} - \frac{1}{24}F_4 - \frac{1}{8}F_{2,2} - \frac{1}{6}F_{3,1} - \frac{1}{2}F_{2,1,1} - \frac{1}{4}F_{1,1,1,1} + O(V^6). \quad \text{(E1)}$$

Here, we introduced

$$F_{k_1,\dots,k_q} = (-1)^q \int_{-\infty}^{\infty} \frac{dh_1\dots dh_q}{\pi^{q/2}} \exp\left(\sum_{j=1}^{q} h_j^2\right) \times \overline{V^{k_1}(h_1)\dots V^{k_q}(h_q)}. \quad \text{(E2)}$$

1. Second order in V

The contribution of the second order in V is given by F_2 and $F_{1,1}$. We find

$$F_2 + F_{1,1} = 2\int_0^\infty \frac{dh}{\sqrt{\pi}}\, e^{-h^2}[2L(h\sqrt{2y}) - L(2h\sqrt{y})]. \quad \text{(E3)}$$

Here, we remind $y = \beta\bar{J}_z$. It is instructive to compare the second-order contribution (E3) with the second-order contribution to the variance of $\ln Z_S$:

$$\overline{(\ln Z_S - \overline{\ln Z_S})^2} = F_{1,1} = 4\int_0^\infty \frac{dh}{\sqrt{\pi}}\, e^{-h^2} \times [L(h\sqrt{2y}) - 2L(h\sqrt{y})]. \quad \text{(E4)}$$

In the regime $T \gg \bar{J}_z$, the arguments of L in the right-hand side of Eqs. (E3) and (E4) are small. Using the asymptotic expression for $L(h)$ at $|h| \ll 1$, we obtain

$$F_2 + F_{1,1} = -\frac{3\zeta(3)}{4\pi^4\beta}\frac{\bar{J}_z^2}{T^2}, \quad F_{1,1} = \frac{3\zeta(3)}{8\pi^4\beta}\frac{\bar{J}_z^2}{T^2}. \quad \text{(E5)}$$

The result (E5) for $F_2 + F_{1,1}$ is translated into Eq. (46) of the paper. From Eq. (E5), we find that

$$\frac{\overline{(\chi_{zz} - \overline{\chi}_{zz})^2}}{\overline{\chi}_{zz}^2} \propto \frac{\bar{J}_z^2}{\pi^2\beta T^2} \ll 1, \quad T \gg \bar{J}_z. \quad \text{(E6)}$$

At low temperatures $T \ll \bar{J}_z$, the asymptotic expression of $L(h)$ for $|h| \gg 1$ must be used in Eq. (E3). We find

$$F_2 + F_{1,1} = -\frac{\ln 2}{\pi^2\beta}\frac{\bar{J}_z}{T}, \quad F_{1,1} = \frac{\ln 2}{\pi^2\beta}\frac{\bar{J}_z}{T}. \quad \text{(E7)}$$

From Eq. (E5), it follows that

$$\frac{\overline{(\chi_{zz} - \overline{\chi}_{zz})^2}}{\overline{\chi}_{zz}^2} \propto \frac{\bar{J}_z}{\pi^2\beta T} \ll 1, \quad \frac{\bar{J}_z}{\pi^2\beta} \ll T \ll \bar{J}_z. \quad \text{(E8)}$$

In view of the result (E8) we can expect that $\ln Z_S$ has a normal distribution with mean $[\ln(\bar{J}_z/J_z) - F_2 - F_{1,1}]/2$ and variance $F_{1,1}$ in the regime $\bar{J}_z/(\pi^2\beta) \ll T \ll \bar{J}_z$. For $T = 3\delta$ and $J_z/\delta = 0.97$, the complementary cumulative distribution function for the normal distribution and the complementary cumulative distribution function obtained numerically for the process $V(h)$ are compared in Fig. 6. We note that for $T = 3\delta$ and $J_z/\delta = 0.94$, numerical integration of Eqs. (E3) and (E4) yields $F_2 + F_{1,1} \approx -0.07$ and $F_{1,1} \approx 0.05$. These values are still different from the asymptotic estimates (E7).

2. Fourth order in V

In the regime $T \gg \bar{J}_z$, the fourth-order contributions are proportional to $(J_z/T)^4$ and therefore negligible. For low temperatures $T \ll \bar{J}_z$, the contributions of the fourth order in V are listed in the following:

$$F_4 = -3\int_{-\infty}^{\infty} \frac{dh}{\sqrt{\pi}}\, e^{-h^2}[\overline{V^2(h\sqrt{y})}]^2 = -36\ln^2 2\, z^4, \quad \text{(E9)}$$

$$F_{2,2} = \left[\int_{-\infty}^{\infty} \frac{dh}{\sqrt{\pi}}\, e^{-h^2}\overline{V^2(h\sqrt{y})}\right]^2 + 2\int_{-\infty}^{\infty} \frac{dh_1 dh_2}{\pi}\, e^{-h_1^2-h_2^2}[\overline{V(h_1\sqrt{y})V(h_2\sqrt{y})}]^2 = (4\ln^2 2 + 8b_{2,2})\, z^4, \quad \text{(E10)}$$

$$b_{2,2} = \frac{1}{2}\int_0^{2\pi} \frac{d\phi}{2\pi}(\overline{v(\cos\phi)v(\sin\phi)})^2 \approx 0.35, \quad \text{(E11)}$$

$$F_{3,1} = 3\int_{-\infty}^{\infty} \frac{dh_1 dh_2}{\pi}\, e^{-h_1^2-h_2^2}\overline{V(h_1\sqrt{y})V(h_2\sqrt{y})} \times \overline{V^2(h_2\sqrt{y})} = 12\ln^2 2\, z^4, \quad \text{(E12)}$$

$$F_{2,1,1} = -\int_{-\infty}^{\infty} \frac{dh_1 dh_2 dh_3}{\pi^{3/2}}\, e^{-h_1^2-h_2^2-h_3^2}\, \overline{V(h_1\sqrt{y})V(h_2\sqrt{y})} \times [\overline{V^2(h_3\sqrt{y})} + 2\overline{V(h_1\sqrt{y})V(h_3\sqrt{y})}] = -(2\ln^2 2 + 2b_{2,1,1})\, z^4, \quad \text{(E13)}$$

$$b_{2,1,1} = \frac{15}{4}\int_0^{2\pi}\frac{d\phi}{4\pi}\int_0^{\pi} d\theta\ \sin^3\theta\ \overline{v(\cos\phi)v(\sin\phi)} \times \overline{v(\cos\theta)v(\sin\theta\cos\phi)} \approx 0.79, \tag{E14}$$

$$F_{1,1,1,1} = 3\left[\int_{-\infty}^{\infty}\frac{dh}{\sqrt{\pi}}\, e^{-h^2}\overline{V^2(h\sqrt{y})}\right]^2 = 3\ln^2 2\, z^4. \tag{E15}$$

Here, we recall that $z^2 = \bar{J}_z/(\pi^2\boldsymbol{\beta}T)$. Summing up, for $T \ll \bar{J}_z$ we obtain

$$\overline{\ln Z_S} = \frac{1}{2}\ln\frac{\bar{J}_z}{J_z} + \frac{\ln 2}{2\pi^2\boldsymbol{\beta}}\frac{\bar{J}_z}{T} + \frac{a_2}{4}\left(\frac{\bar{J}_z}{\pi^2\boldsymbol{\beta}T}\right)^2, \tag{E16}$$

where

$$a_2 = -3\ln^2 2 - 4b_{2,2} + 4b_{2,1,1} \approx 0.29. \tag{E17}$$

Using Eq. (E16) and the definition of the spin susceptibility, we obtain for $b = 0$

$$\overline{\chi}_{zz} = \frac{1}{2(\delta - J_z)}\left[1 + \frac{\bar{J}_z\ln 2}{\pi^2\boldsymbol{\beta}T} + a_2\left(\frac{\bar{J}_z}{\pi^2\boldsymbol{\beta}T}\right)^2\right]. \tag{E18}$$

[1] Y. Alhassid, Rev. Mod. Phys. **72**, 895 (2000).
[2] W. G. van der Wiel, S. De Franceschi, J. M. Elzerman, T. Fujisawa, S. Tarucha, and L. P. Kouwenhoven, Rev. Mod. Phys. **75**, 1 (2002).
[3] I. Aleiner, P. Brouwer, and L. Glazman, Phys. Rep. **358**, 309 (2002).
[4] R. Hanson, L. P. Kouwenhoven, J. R. Petta, S. Tarucha, and L. M. K. Vandersypen, Rev. Mod. Phys. **79**, 1217 (2007).
[5] D. Ullmo, Rep. Prog. Phys. **71**, 026001 (2008).
[6] I. L. Kurland, I. L. Aleiner, and B. L. Altshuler, Phys. Rev. B **62**, 14886 (2000).
[7] R. I. Shekhter, Zh. Eksp. Teor. Fiz. **63**, 1410 (1973) [Sov. Phys.–JETP **36**, 474 (1973)]; I. O. Kulik and R. I. Shekhter, Zh. Eksp. Teor. Fiz. **68**, 623 (1975) [Sov. Phys.–JETP **41**, 308 (1975)]; E. Ben-Jacob and Y. Gefen, Phys. Lett. A **108**, 289 (1985).
[8] A. Kamenev and Y. Gefen, Phys. Rev. B **54**, 5428 (1996).
[9] N. Sedlmayr, I. V. Yurkevich, and I. V. Lerner, Europhys. Lett. **76**, 109 (2006).
[10] M. N. Kiselev and Y. Gefen, Phys. Rev. Lett. **96**, 066805 (2006).
[11] I. S. Burmistrov, Y. Gefen, and M. N. Kiselev, JETP Lett. **92**, 179 (2010).
[12] I. S. Burmistrov, Y. Gefen, and M. N. Kiselev, Phys. Rev. B **85**, 155311 (2012).
[13] B. Sothmann, J. König, and Y. Gefen, Phys. Rev. Lett. **108**, 166603 (2012).
[14] J. A. Folk, C. M. Marcus, R. Berkovits, I. L. Kurland, I. L. Aleiner, and B. L. Altshuler, Phys. Scr. T **90**, 26 (2001); G. Usaj and H. U. Baranger, Phys. Rev. B **67**, 121308 (2003); Y. Alhassid and T. Rupp, Phys. Rev. Lett. **91**, 056801 (2003); Y. Alhassid, T. Rupp, A. Kaminski, and L. I. Glazman, Phys. Rev. B **69**, 115331 (2004).
[15] Y. Alhassid and T. Rupp, arXiv:cond-mat/0312691.
[16] H. E. Türeci and Y. Alhassid, Phys. Rev. B **74**, 165333 (2006); G. Murthy, *ibid.* **77**, 073309 (2008); O. Zelyak and G. Murthy, *ibid.* **80**, 205310 (2009).
[17] I. L. Aleiner and V. I. Fal'ko, Phys. Rev. Lett. **87**, 256801 (2001).
[18] B. Nissan-Cohen, Y. Gefen, M. N. Kiselev, and I. V. Lerner, Phys. Rev. B **84**, 075307 (2011).
[19] S. Guéron, M. M. Deshmukh, E. B. Myers, and D. C. Ralph, Phys. Rev. Lett. **83**, 4148 (1999); M. M. Deshmukh, S. Kleff, S. Guéron, E. Bonet, A. N. Pasupathy, J. von Delft, and D. C. Ralph, *ibid.* **87**, 226801 (2001).
[20] C. M. Canali and A. H. MacDonald, Phys. Rev. Lett. **85**, 5623 (2000); S. Kleff, J. von Delft, M. M. Deshmukh, and D. C. Ralph, Phys. Rev. B **64**, 220401 (2001); S. Kleff and J. von Delft, *ibid.* **65**, 214421 (2002).
[21] A. Cehovin, C. M. Canali, and A. H. MacDonald, Phys. Rev. B **66**, 094430 (2002); G. Usaj and H. U. Baranger, Europhys. Lett. **72**, 110 (2005).
[22] P. W. Brouwer and D. A. Gorokhov, Phys. Rev. Lett. **95**, 017202 (2005).
[23] M. Misiorny, M. Hell, and M. R. Wegewijs, Nat. Phys. **9**, 801 (2013).
[24] A. M. Finkel'stein, in *Soviet Scientific Reviews*, edited by I. M. Khalatnikov, Vol. 14 (Harwood Academic, London, 1990).
[25] A. V. Andreev and A. Kamenev, Phys. Rev. Lett. **81**, 3199 (1998).
[26] X. Yang, Ch. Nayak, Phys. Rev. B **72**, 052405 (2005).
[27] M. Schechter, Phys. Rev. B **70**, 024521 (2004); Zu-Jian Ying, M. Cuoco, C. Noce, and Huan-Qiang Zhou, *ibid.* **74**, 012503 (2006); **74**, 214506 (2006); S. Schmidt, Y. Alhassid, and K. van Houcke, Europhys. Lett. **80**, 47004 (2007); S. Schmidt and Y. Alhassid, Phys. Rev. Lett. **101**, 207003 (2008); K. Van Houcke, Y. Alhassid, S. Schmidt, and S. M. A. Rombouts, arXiv:1011.5421; Y. Alhassid, K. N. Nesterov, and S. Schmidt, Phys. Scr. T **151**, 014047 (2012); K. N. Nesterov and Y. Alhassid, Phys. Rev. B **87**, 014515 (2013).
[28] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, Phys. Rev. Lett. **78**, 2803 (1997).
[29] Ya. M. Blanter, A. D. Mirlin, and B. A. Muzykantskii, Phys. Rev. Lett. **78**, 2449 (1997).
[30] D. S. Lyubshin, A. U. Sharafutdinov, and I. S. Burmistrov, Phys. Rev. B **89**, 201304(R) (2014).
[31] D. Huertas-Hernando and Y. Alhassid, Phys. Rev. B **75**, 153312 (2007).
[32] A. Saha, Y. Gefen, I. S. Burmistrov, A. Shnirman, and A. Altland, Ann. Phys. (NY) **327**, 2543 (2012).
[33] K. B. Efetov and A. Tschersich, Phys. Rev. B **67**, 174205 (2003).
[34] M. L. Mehta, *Random Matrices* (Academic, Boston, 1991).
[35] H. D. Politzer, Phys. Rev. B **40**, 11917 (1989).
[36] T. Bojdecki, L. G. Gorostiza, and A. Talarczyk, Potential Anal. **28**, 71 (2008).
[37] R. J. Adler, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes* (Institute of Mathematical Statistics, Hayward, California, 1990).

[38] S. Asmussen and H. Albrecher, *Ruin Probabilities* (World Scientific, Singapore, 2010).
[39] J. Hüsler and V. Piterbarg, Stoch. Proc. Appl. **83**, 257 (1999).
[40] A. I. Larkin, Zh. Eksp. Teor. Fiz. **58**, 1466 (1970) [Sov. Phys.–JETP **31**, 784 (1970)].
[41] Y. Imry and Sh.-K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[42] G. D. Mahan, *Many-Particle Physics* (Kluwer Academic, New York, 2000).
[43] L. D. Graham and D. S. Schreiber, J. Appl. Phys. **39**, 963 (1968); L. Shen, D. S. Schreiber, and A. J. Arko, Phys. Rev. **179**, 512 (1969); P. Gambardella, S. Rusponi, M. Veronese, S. S. Dhesi, C. Grazioli, A. Dallmeyer, I. Cabria, R. Zeller, P. H. Dederichs, K. Kern, C. Carbone, and H. Brune, Science **300**, 1130 (2003); A. M. Clogston, B. T. Matthias, M. Peter, H. J. Williams, E. Corenzwit, and R. C. Sherwood, Phys. Rev. **125**, 541 (1962); D. Shaltiel, J. H. Wrenick, H. J. Williams, and M. Peter, *ibid.* **135**, A1346 (1964); J. W. Loram and K. A. Mirza, J. Phys. F: Met. Phys. **15**, 2213 (1985); J. W. Loram, K. A. Mirza, and Z. Chen, *ibid.* **16**, 233 (1986); G. Mpourmpakis, G. E. Froudakis, A. N. Andriotis, and M. Menon, Phys. Rev. B **72**, 104417 (2005).
[44] S. Jia, S. L. Bud'ko, G. D. Samolyuk, and P. C. Canfield, Nat. Phys. **3**, 334 (2007).
[45] J. Wei and E. Norman, J. Math. Phys. **4**, 575 (1963).
[46] I. V. Kolokolov, Phys. Lett. A **114**, 99 (1986); Ann. Phys. (NY) **202**, 165 (1990); M. Chertkov and I. V. Kolokolov, Phys. Rev. B **51**, 3974 (1995); Zh. Eksp. Teor. Fiz. **106**, 1525 (1994) [Sov. Phys.–JETP **79**, 824 (1994)]; for a review, see I. V. Kolokolov, Int. J. Mod. Phys. B **10**, 2189 (1996).
[47] I. S. Gradsteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, San Diego, 2000).