Marginal Anderson localization and many-body delocalization

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We consider d-dimensional systems which are localized in the absence of interactions, but whose single-particle localization length diverges near a discrete set of (single-particle) energies, with critical exponent ν. This class includes disordered systems with intrinsic or symmetry protected topological bands, such as disordered integer quantum Hall insulators. We show that such marginally localized systems exhibit anomalous properties intermediate between localized and extended, including vanishing dc conductivity but subdiffusive dynamics, and fractal entanglement (an entanglement entropy with a scaling intermediate between area and volume law). We investigate the stability of marginal localization in the presence of interactions, and argue that arbitrarily weak short-range interactions trigger delocalization for partially filled bands at nonzero energy density if $\nu \geqslant 1/d$. We use the Harris-Chayes bound $\nu \geqslant 2/d$ to conclude that marginal localization is generically unstable in the presence of interactions. Our results suggest the impossibility of stabilizing quantized Hall conductance at nonzero energy density.

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I. INTRODUCTION

Edge states of topological systems give rise to a striking set of coherent quantum phenomena. For example, chiral edge states of two-dimensional quantum Hall systems lead to precisely quantized Hall conductance at zero temperature. However in thermal systems with nonzero temperature, these edge-state properties are washed out due to thermal excitations that communicate between opposite edges of the system. Such deleterious thermal effects can sometimes be mitigated by careful cooling and isolation. Nevertheless, it is interesting and potentially useful to ask whether one can obtain systems with topological edge state properties that are immune to thermal washing out, circumventing the need for cooling.

An intriguing possibility arises from the study of many-body localization (MBL), a phenomenon whereby well-isolated quantum systems fail to thermalize due to the localization of excitations by strong disorder [1–11]. In MBL systems, disorder can protect quantum coherence against such thermal degradation by localizing the offending excitations. In this fashion certain coherent quantum phenomena including symmetry breaking and topological order may survive in MBL systems at nonzero energy density (the appropriate analog of finite temperature in nonthermal systems) [12–17].

This naturally raises the question of whether topological edge states can be protected by localization of the bulk states. Here, we face a new complication not present in previous examples of localization protected quantum order: topological bands in fermionic systems cannot be completely localized by disorder. Rather, the nontrivial band topology guarantees the existence of an extended single-particle (SP) bulk orbital at some critical SP energy E_c [18] (or more generally a discrete set of such extended orbitals). SP orbitals with energy E near E_c exhibit diverging SP localization length [19]:

$$\xi(E) \sim \frac{1}{|E - E_c|^{\nu}}, \tag{1}$$

where ν is an exponent whose value depends on the particular system under consideration (see Fig. 1). We dub such systems "marginally localized." Examples include disordered integer quantum Hall systems [20,21] and chiral superconductors, intrinsically topological superconductors [22], and symmetry protected topological phases (e.g., topological insulators) with symmetry-preserving disorder [23]. Here, the diverging localization length accompanies each change in the topological edge state structure (e.g., the Chern number changing quantum Hall plateau to plateau transitions). The presence of at least one extended state is guaranteed by the topological obstruction to constructing localized Wannier orbitals in a topological band [20,23]. In the vicinity of the critical energy, the SP localization length diverges as in Eq. (1). A diverging SP localization length can occur in nontopological contexts, such as 1D XX spin chains with random bond disorder (or equivalently, noninteracting fermion chains with random hopping). Such marginally localized systems have properties intermediate between localized and extended systems, as we shall discuss in Sec. II.

A central issue is whether marginal localization persists away from the noninteracting limit. In fully localized systems [$\xi(E) \leqslant \xi_0$ for all E] there is strong evidence that localization survives the addition of weak interactions even at nonzero energy density [5–7]. In contrast, interactions are expected to delocalize a system whose single-particle (SP) spectrum contains a mobility edge, as a generic excited state will contain a finite density of occupied extended orbitals that thermalize among themselves and then act as a bath for the localized orbitals. In the presence of a nonzero coupling to a bath, the many-body wave functions become thermal, and localization is destroyed, even though signatures of proximity to localization may survive in the spectral functions of local operators [24,25]. Even if extended states only arise in a narrow energy range of width W, the results of [26] suggest that this narrow band of extended states will be sufficient to destroy localization, for any nonzero W. However, for marginally localized phases, extended states arise only at a single energy E_c. In the language

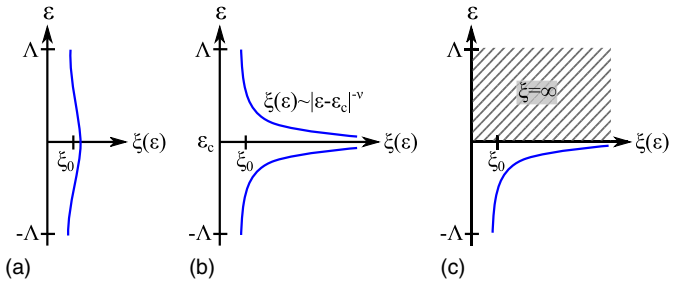

FIG. 1. (Color online) Schematic dependence of SP localization length ξ on single-particle energy E for a fully localized system (a), marginally localized system (b), and extended system with a mobility edge (c).

of [26], these extended states constitute a bath with vanishing bandwidth, and it is not *a priori* clear whether this will be sufficient to destroy localization upon inclusion of interactions.

In order to discuss the stability of marginal localization in the presence of interactions, we must first settle on a definition of marginal localization. It has been pointed out that fully many-body localized systems are locally integrable in the sense that all eigenstates can be completely specified by a set of exponentially well localized conserved, commuting quantities or "integrals of motion" [10,27–29]. For noninteracting systems the integrals of motion are simply the occupation numbers of single-particle orbitals. In an MBL phase, these integrals of motion are dressed by interactions but (apart from a exponentially small fraction of rare resonances [10]) retain their localized character. All of the striking properties of the fully many-body localized phase can be understood as following from the existence of this extensive number of exponentially localized integrals of motion, and the existence of exponentially well localized integrals of motion can thus be viewed as the diagnostic for full MBL.

The emergent integrability in marginally localized systems is rather different from that in fully MBL systems. This is easiest to see in the noninteracting limit, when the integrals of motion are simply the occupation numbers of the SP orbitals (as noted in [28]). It follows straightforwardly from (1) that the fraction of integrals of motion that will *not* be localized on a length scale R will go to zero as $R \to \infty$ only as a power-law function of R (unlike [10,28], where this fraction goes to zero as an exponential function of R). Thus, if we use the [10,28] criterion for localization, then even the noninteracting system is not localized. Clearly, we need a more robust criterion. Inspired by [10,27–29], we take the defining property of marginally localized systems to be the existence of a (nearly) complete set of algebraically well localized integrals of motion. Here, algebraically well localized means that the fraction of integrals of motion with appreciable support on scale larger than R decays as a power law in R (unlike traditional MBL systems where it decays exponentially in R, and also unlike a delocalized system, where the integrals of motion are not localized), and the caveat "nearly" allows for a possible set of nonpercolating resonances. We can then ask whether marginal localization (defined in terms of algebraically well localized integrals of motion) can survive in the presence of interactions.

We emphasize that we have defined "algebraically well localized" in terms of the fraction of integrals of motion that are well localized on a length scale R, and not in terms of how the support of individual integrals of motion falls off with distance. Indeed, in the noninteracting limit it is clearly the case that individual integrals of motion have only exponential tails in real space, but the localization lengths controlling these exponential tails are continuously varying and unbounded, such that the fraction of integrals of motion that are not well localized on a given length scale R decays only algebraically with R. This constitutes marginal localization in the sense that we have defined it above.

In this paper, we give analytic arguments showing that marginal localization is impossible for arbitrarily weak interaction strength if $\nu d > 1$, where d is the spatial dimension. The breakdown of localization originates from a proliferation of multiparticle resonances between well-separated single-particle orbitals. Notably, such breakdown occurs in disordered integer quantum Hall systems for which $\nu \approx 2.3$ [21,30]. Moreover, in the generic case of systems that are governed by the Harris-Chayes bound, $\nu d \geqslant 2$, and which have a nonzero density of states (e.g., integer quantum Hall systems), our arguments preclude the existence of many-body localization at finite energy density in partially filled bands where the localization length diverges at a discrete set of energies. Our results can be extended to all ten Altland-Zirnbauer symmetry classes.

We note that the effect of short-range interactions in quantum Hall systems has been previously discussed in Refs. [31,32]. These works are complementary to our analysis, in that they focus on systems where the Fermi level is exactly at the critical energy, whereas we focus on noncritical systems. Reference [32] shows that the finite-temperature dc conductivity of the critical system is a discontinuous function of the interaction strength, being zero for the noninteracting system and nonzero for a system with infinitesimally weak interactions. Our results establish that the finite-temperature dc conductivity is a similar discontinuous function of interaction strength even away from criticality, and tends to a nonzero constant in the weak-interaction limit. However, this nonzero constant is an exponential function of temperature, unlike critical systems, where it is a power-law function of temperature [32].

Our arguments do not exclude many-body localization in systems that evade the Harris bound due to symmetry reasons, and have $\nu < 1/d$. Additionally, the Harris-Chayes bound applies to the critical exponent for the *mean* localization length, while our argument involves the critical exponent of the *typical* localization length, so there remains a possibility of marginal localization in systems where the mean and typical localization lengths diverge with very different exponents.

This paper is structured as follows. In Section II we review the properties of marginally localized phases in the absence of interactions. In Sec. III we introduce the model of interest and the basic approach. In Sec. IV we derive a condition for the perturbative instability of marginal localization. The analysis in Sec. IV initially assumes that the wave functions are uniform on length scales less than a localization length, and decay exponentially on longer length scales. This is an oversimplification, since the wave functions close to criticality

have multifractal character on length scales short compared to the localization length. In Sec. IV H we take this multifractality into account and demonstrate that it does not alter our essential results. In Sec. V we discuss the consequences of our results and use the Harris-Chayes argument [33,34] to rule out stability of marginal localization in a wide class of models, including the integer quantum Hall effect. We also discuss the possibility of perturbatively stable marginal localization in systems that exploit certain loopholes in our argument, which we discuss.

II. PROPERTIES OF MARGINALLY LOCALIZED PHASES

Before analyzing the effects of interactions, we begin by reviewing the properties of marginally localized phases in the absence of interactions. In this and all that follows, we consider the case of only a single critical energy ε_c and shift our SP energy scale such that $\varepsilon_c = 0$. Moreover, we suppose that the SP density of states is regular and nonzero in the vicinity of ε_c. We work at partial filling in the band at a nonzero energy density (i.e., we are not discussing ground states or low-lying excited states). In the quantum Hall context, this amounts to working at a nonzero temperature, but a temperature that is still much less than the cyclotron frequency (allowing us to focus on a single Landau subband).

A. Subdiffusive relaxation

Marginally localized states show anomalous subdiffusive dynamics, as is known from previous studies of dynamics in disordered quantum Hall systems [35–37]. These results are reproduced by the following simple argument. Consider the relaxation of a wave packet Ψ initially localized in the vicinity of $\mathbf{R}(t=0)=0$. The wave packet can be written in the form $|\Psi(t=0)\rangle = \sum_\alpha w_\alpha |\psi_\alpha\rangle$, where $|\psi_\alpha\rangle$ are the single-particle states with energy E_α and amplitude $\varphi_\alpha(\mathbf{r})$ at a position $\mathbf{r}$. Meanwhile, w_α are appropriate complex amplitudes that produce a localized wave packet. The wave functions φ_α are all localized, with energy dependent localization lengths $\xi(E_\alpha)$ as in Eq. (1). This wave packet has the time evolution $|\Psi(t)\rangle = \sum_\alpha w_\alpha |\psi_\alpha\rangle e^{-iE_\alpha t}$. We can now straightforwardly evaluate the spatial spreading of the wave packet by examining

$$\begin{aligned} R^2(t) &= \int d^d r \sum_{\alpha,\beta} e^{i(E_\alpha - E_\beta)t} w_\alpha^* w_\beta \varphi_\alpha^*(r) r^2 \varphi_\beta(r) \\ &\approx \int d^d r \sum_{\alpha} |w_\alpha|^2 \varphi_\alpha^*(r) r^2 \varphi_\beta(r) \\ &\approx \sum_\alpha |w_\alpha|^2 \sum_{\beta:|E_{\alpha\beta}|<1/t} \min\left(D_\alpha t, \xi_\alpha^2\right), \end{aligned}$$

where the second line is valid at long times when the phases between states with $|E_{\alpha\beta}| \equiv |E_\alpha - E_\beta| \gg 1/t$ have effectively randomized, and in going to the third line we have assumed that the spreading in each energy interval is diffusive up to the localization length at that energy, and have allowed for the possibility that the diffusion "constant" varies with energy. The diffusion constant at an energy E may be estimated by setting the Thouless energy $D(E)/\xi(E)^2$ equal to the level spacing $\xi^{-d}(E)$ at that energy, giving $D(E) \sim \xi^{2-d}$. The long-time dynamics will turn out to be controlled by states near the critical energy E_c. In dimensions $d \neq 2$ the singular behavior of ξ near the critical energy will translate into a singular behavior of $D(E)$. However, in two dimensions (the case relevant for quantum Hall), the singularity in ξ will not translate into a singularity in D, and thus we may replace $D(E)$ by a constant value D_0, the diffusion constant for the states near the critical energy. Moreover, the detailed initial structure of the wave packet is irrelevant for the long-time asymptotics, which are captured by simply taking $|w_\alpha|$ to be roughly equal amplitude for all energies E_α within a bandwidth Λ of E_c. Under these assumptions, the spreading of the wave packet follows:

$$R^2(t) = \int_0^{E(t)} \frac{dE}{\Lambda} D_0 t + \int_{E(t)}^{\Lambda} \frac{dE}{\Lambda} \xi^2(E). \tag{2}$$

In the fourth line we have defined the integration limit $E(t)$ through $\xi^2[E(t)] \approx \xi_0^2 (\frac{E(t)}{\Lambda})^{-2\nu} = D_0 t$, i.e., $E(t) = \Lambda (\frac{D_0 t}{\xi_0^2})^{-1/2\nu}$. The behavior is qualitatively different for $\nu < 1/2$ and for $\nu > 1/2$. However, provided the Harris-Chayes argument [33,34] applies (see discussion in Appendix A), $\nu \geqslant 1$, and only the latter case is relevant. Thus, the dominant contribution comes from low-energy states with diverging $\xi(E)$, i.e., from the first term in Eq. (2) and from the lower end with $E \gtrsim E(t)$ in the second term of Eq. (2), and we obtain

$$R^2(t) \sim t^\alpha, \quad \alpha = 1 - \frac{1}{2\nu}. \tag{3}$$

Thus the dynamics are characterized by the dynamical exponent $\alpha = 1 - \frac{1}{2\nu}$ intermediate between diffusive ($\alpha = 1$) and localized ($\alpha = 0$) scaling, and taking the numerical value $\alpha \approx 0.78$ in the case of integer quantum Hall. We note too that while the above derivation was for the noninteracting system, a similar analysis could be performed also for the interacting system, if and only if there were an extensive number of algebraically well localized integrals of motion. Indeed, one could take the φ to be the wave functions of such algebraically well localized integrals of motion (instead of the wave functions of the SP orbitals) and the analysis would proceed unchanged. If the fraction of integrals of motion not localized on the length scale R scales as $\sim R^{-1/\tilde{\nu}}$, then the relaxation will follow (3) with $\alpha = 1 - 1/2\tilde{\nu}$. Thus, the existence of algebraically well localized integrals of motion implies a subdiffusive relaxation, as well as a scale-dependent conductivity (following subsection).

B. Scale-dependent conductivity

The sub diffusive spreading Eq. (3) can be interpreted as a length-scale-dependent diffusion "constant": $D(L) \sim L^{-2/(2\nu-1)}$. Using the usual Einstein relation between diffusion and conductivity, these considerations predict scale-dependent dc conductivity

$$\sigma(L) \sim L^{-2/(2\nu-1)}, \tag{4}$$

which vanishes in the the thermodynamic limit. Again we see scaling intermediate between the constant conductivity $\sigma_{\text{ex}} \sim$ constant of an extended system and the exponential scaling $\sigma_{\text{loc}}(L) \sim e^{-L/\xi}$ of a localized system.

C. Fractal entanglement scaling

Entanglement scaling provides insight into the character of many-body eigenstates, and provides a conceptually powerful means to distinguish extended and localized systems. The entanglement entropy of a spatial subregion of linear size R for an excited energy eigenstate in a localized system comes only from states within a localization length ξ_0 of the boundary and exhibits boundary-law scaling $S(R) \sim \xi_0 R^{d-1}$. Here d is the number of spatial dimensions. In extended systems, such boundary law scaling occurs only for the ground states, whereas excited states typically display volume law scaling $S(R) \sim R^d$. In this section we analyze the spatial scaling of entanglement entropy in excited states of a marginally localized system, which has not been considered in previous studies.

For a marginally localized system, a fraction $f(R) \sim R^{-1/\nu}$ states are extended across the subregion of size R [i.e., have $\xi(\varepsilon) > R$], and make a volume-law type contribution to entanglement $f(R)R^d \sim R^{d-1/\nu}$. For $\nu > 1$, this dominates the area-law contribution from well-localized states leading to a scaling

$$S(R) \sim R^{d-1/\nu} \tag{5}$$

intermediate between volume law and area law, which we dub "fractal" entanglement scaling as it scales like the boundary of a fractal subregion of a localized phase with fractional dimension $d - 1/\nu$.

Marginally localized systems exhibit fractal entanglement scaling so long as $\nu > 1$. For $\nu < 1$, the entanglement entropy becomes effectively area law. When applicable, a Harris-type criterion (see Appendix A) $\nu \geqslant 2/d$ ensures that the fraction of modes which are extended on the length scale R dominate the entanglement entropy in all systems with $d < 2$. Integer quantum Hall systems have $\nu \approx 2.3$, and will exhibit fractal entanglement scaling. However Harris-Chayes arguments do not rule out that some marginally localized systems exhibit conventional boundary-law scaling in spatial dimensions $d \geqslant 2$. The arguments laid out above are once again not particular to the noninteracting problem, but will generalize also to the interacting problem, iff there exists an extensive number of algebraically well-localized integrals of motion. If the fraction of integrals of motion not localized on the length scale R scales as $R^{-1/\tilde{\nu}}$, then the entanglement entropy will scale as $S(R) \sim R^{d-1/\tilde{\nu}}$, for $\tilde{\nu} > 1$.

Thus, we see that in dynamics, response properties, and the scaling of entanglement entropy, marginally localized noninteracting systems display properties that are intermediate between traditional localized systems and delocalized systems. Next, we investigate whether this intermediate behavior survives in the presence of interactions.

III. MODEL AND APPROACH

The Hamiltonian of interest takes the form $\hat{H} = \hat{H}_0 + \hat{V}$, where H_0 is the Hamiltonian of the disordered noninteracting fermion system (for example Landau levels with disorder). This noninteracting Hamiltonian has single-particle eigenmodes of energy E_α created by operators $\psi_\alpha^\dagger = \int d^d r \varphi_\alpha^*(r) c^\dagger(r)$, where $c^\dagger(r)$ creates an electron at position r. Meanwhile, the interaction term takes the form

$$\hat{V} = V \sum_{\alpha,\beta,\gamma,\delta} \lambda_{\alpha\beta\gamma\delta} \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\gamma \psi_\delta. \tag{6}$$

This represents a short-range interaction of strength V, where $\lambda_{\alpha\beta\gamma\delta} = \int d^d r \varphi_\alpha^*(r)\varphi_\beta^*(r)\varphi_\gamma(r)\varphi_\delta(r)$ are the matrix elements of the interaction in the eigenbasis of H_0. We note that the indices $\alpha,\beta,\gamma,\delta$ label not only the spatial structure of the wave function but also the spin state of the fermions. We note too that while modeling the interaction as a delta function is adequate for the physically relevant problem of multicomponent fermions (e.g., fermions with a spin degeneracy), for the theoretically interesting toy problem of a single species of spinless fermion, a pure delta function interaction has no effect, because of Pauli exclusion. In this case one should work with an interaction with nonzero but finite range, and must work with properly antisymmetrized wave functions [31,38]. This introduces several subtleties, which do not materially affect our argument. For a full discussion of these issues, we refer the reader to Appendix B.

In the integer quantum Hall system, there are an infinite number of critical energies. For simplicity, we consider a simplified problem [illustrated in Fig. 1(b)], where the SP localization length diverges only at one SP energy, E_c, and choose our zero of energy to set $E_c \equiv 0$. The localization lengths for energies close to zero then diverge as $\xi(E) \sim |E|^{-\nu}$, such that at any length scale L, a nonzero fraction $\sim L^{-1/\nu}$ of the states appear extended. We take the SP density of states to be a nonzero constant in the vicinity of $E = 0$. For generality we will keep the spatial dimensionality d and the localization-length exponent ν arbitrary; for example $d = 2$ and $\nu \approx 2.3$ for the disordered integer quantum Hall plateau [21].

We now ask whether inclusion of the interaction causes a breakdown of localization. We begin by noting that if we attempt to construct the many-body eigenstates perturbatively in weak interactions, then perturbation theory for the eigenstates will break down already at $O(V)$ for states involving SP orbitals sufficiently close to the mobility edge, irrespective of the value of ν. However, the fact that the eigenstates cannot be perturbatively constructed does not imply that localization itself is unstable; to address the stability of localization we need a more sophisticated diagnostic.

Motivated by [27–29] and [10] we have conjectured that that defining property of marginal localization is the existence of an extensive number of algebraically well localized integrals of motion. We have also argued (in Sec. II) that the existence of such algebraically well localized integrals of motion implies the characteristic properties of marginally localized systems, such as subdiffusive relaxation, scale-dependent conductivity, and an entanglement entropy which is intermediate between area law and volume law. We wish to investigate whether this picture of marginal localization can be stable in the presence of interactions. To this end, we first introduce some notation. We use $|\Phi_E\rangle$ to denote a (many-body) eigenstate of H_0 at an energy E, labeled by a particular set of marginally localized integrals of motion, in the sense described above. We use $|\Psi_E\rangle$ to denote a (many-body) eigenstate of $\hat{H}_0 + \hat{V}$ at an energy E. We now introduce the $\hat{T}$ matrix of the problem by $\hat{V}|\Psi_E\rangle = \hat{T}|\Phi_E\rangle$.

The matrix elements of $\hat{T}$ between SP states with energy E and spatial separation R gives the effective matrix element for long-range hopping. It is well known that the effective hopping matrix element must fall off with distance faster than $1/R^d$, else localization is impossible [1,39]. Similarly, the matrix elements of $\hat{T}$ between two-particle states with the particles separated by a distance R gives the effective matrix element for a long-range two-body interaction, the matrix elements between three-particle states give the effective matrix element for a long-range three-body interaction, etc. Again, there are conditions on how fast these matrix elements must fall off with R in order for localization to be stable [2,40,41]. Our main strategy in the present work will be to determine how the matrix elements of $\hat{T}$ fall off with distance R between the particles involved. Hence, we will obtain a condition on ν for breakdown of localization.

The $\hat{T}$ matrix is related to $\hat{V}$ by the Dyson equation [42]

$$\hat{T} = \hat{V} + \hat{V}\frac{1}{E - \hat{H}_0 + i0}\hat{T}. \tag{7}$$

This expression fully specifies the $\hat{T}$ matrix for states at a particular energy E. We note that even though $\hat{V}$ is short range, $\hat{T}$ can have long-range matrix elements because $\hat{H}_0$ has eigenstates with arbitrarily long range matrix elements. Within perturbation theory in small $\hat{V}$, we can approximate the $\hat{T}$ matrix by the Born series

$$\hat{T} = \hat{V} + \hat{V}\hat{G}_0(E)\hat{V} + \hat{V}\hat{G}_0(E)\hat{V}\hat{G}_0(E)\hat{V} + \cdots,$$
$$\hat{G}_0(E) = \frac{1}{E - \hat{H}_0 + i0}. \tag{8}$$

This perturbative expression can be represented in standard diagrammatic perturbation theory, where G_0 is represented by a (directed) line, and V is represented by a vertex with two lines going in and two coming out. The diagrammatic representation of the $\hat{T}$ matrix contains only connected diagrams.

IV. DELOCALIZATION IS INEVITABLE IF $d\nu > 1$

We use $T_{A,B}$ to denote the matrix element of $\hat{T}$ between incoming state A and outgoing state B. If $T_{A,B}$ exceeds the level spacing between states A and B then these two states are "in resonance," and the occupation number of state A cannot be an integral of motion. However, any time two states are in resonance, we can define new states A' and B' which are orthogonal linear combinations of A and B, such that $T_{A',B'} = 0$. The occupation number of A' and B' can then be an integral of motion (at least if the states are not resonant with any third state C). However, the new integrals of motion will have support wherever A and B had support. We see therefore that the spatial structure of resonances is crucial. If resonances only occur between states that have support at nearby points in real space, then we may be able to define new integrals of motion which are still localized, if the local resonances do not percolate [39]. In contrast, if resonances occur on all length scales, then the integrals of motion will not be localized (even algebraically), and so the system will not be able to support marginal MBL.

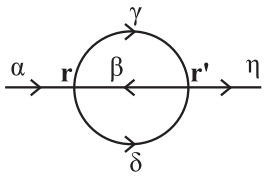

FIG. 2. Diagrammatic representation of mediated hopping, to lowest order.

The question we are interested in asking is thus the following: if A is an eigenstate of H_0 with localization length ξ_A, then what is the probability that A is "in resonance" with another state B a distance R apart, where $R \gg \max(\xi_A,\xi_B)$ [the situation when $R < \max(\xi_A,\xi_B)$ constitutes a local resonance, which just redefines the localized integrals of motion]. If the probability of having a resonance at a distance greater than R goes to zero as $R \to \infty$, then we will be able to define algebraically localized integrals of motion with arbitrary precision, in the sense that all but a fraction ϵ of the integrals of motion will be localized on a length scale $R_c(\epsilon)$, where $R_c(\epsilon)$ is set by the condition that the probability of having a resonance at a length scale greater than R_c is equal to ϵ. In contrast, if the probability of having a resonance at a length scale greater than R does not go to zero as $R \to \infty$, then we will not be able to define algebraically localized integrals of motion with arbitrary precision, and the system cannot support marginal MBL.

We thus want to calculate how the matrix elements of $\hat{T}$ scale with distance. We recall that the $\hat{T}$ matrix is given by the Born series (8). The first-order term $\hat{V}$ is purely short range. Long-range terms first appear at order (V^2). We now examine how the matrix elements of $\hat{T}$ fall off with distance at $O(V^2)$.

A. Hopping resonances

We begin by evaluating the matrix elements of $\hat{T}$ between well-localized SP orbitals ψ_α and ψ_η at a separation $R \gg \max(\xi_\alpha,\xi_\eta)$. This can be viewed as the effective matrix element for hopping at a range R. The mediated hopping process at order V^2 is illustrated in Fig. 2. Thus, we have

$$T_{\alpha,\eta}(R) = \sum_{\beta\gamma\delta}\frac{V^2\lambda_{\alpha\beta\gamma\delta}\lambda_{\gamma\delta\beta\eta}}{(E_\alpha + E_\beta) - (E_\gamma + E_\delta)} \approx \frac{V^2}{\xi_\alpha^{d/2}\xi_\eta^{d/2}}\sum_{\beta\gamma\delta}\frac{\exp\left[-R\left(\frac{1}{\xi_\beta}+\frac{1}{\xi_\gamma}+\frac{1}{\xi_\delta}\right)\right]}{\left(\xi_\beta^d\xi_\gamma^d\xi_\delta^d\right)(E_\alpha + E_\beta - E_\gamma - E_\delta)}\exp(-i\phi_{\beta\gamma\delta})$$
$$= V^2\left|E_\alpha^{\nu d/2}E_\eta^{\nu d/2}\right|\sum_{\beta\gamma\delta}\Pi_{\beta\gamma\delta}(R), \tag{9}$$

where the sum ranges over intermediate states, (β,γ,δ). Since orbitals α,η are well localized we have replaced $\lambda_{\alpha\beta\gamma\delta} \approx \varphi^*_\alpha(0)\varphi^*_\beta(0)\varphi_\gamma(0)\varphi_\delta(0)$, and $\lambda_{\alpha\beta\gamma\delta} \approx \varphi^*_\lambda(R)\varphi^*_\delta(R)\varphi_\beta(R)\varphi_\eta(R)$. We approximate each of the wave functions by the simplistic localized form $|\varphi_\alpha(r)| \approx \frac{1}{\xi_\alpha^{d/2}} e^{-|r-r_\alpha|}$ (and similarly for $\beta,\lambda,\ldots$). This assumption ignores the multifractal character of the SP orbitals with ε close to ε_c, but is expected to reproduce the typical behavior. The multifractal character of the wave functions will be taken into account in Sec. V. For convenience we have defined

$$\Pi_{\beta\gamma\delta}(R) = |E_\beta E_\gamma E_\delta|^{dv} \frac{\exp[-R(|E_\beta|^v + |E_\gamma|^v + |E_\delta|^v)]}{(E_\alpha + E_\beta - E_\gamma - E_\delta)} \times \exp(-i\phi_{\beta\gamma\delta}). \tag{10}$$

Each matrix element contains an overall phase, $\phi_{\beta\gamma\delta}$, and the phases are essentially uncorrelated for different choices of $\beta\gamma\delta$. Hence the sum over intermediate states accumulates random amplitude and phase characteristic of a random walk in the complex plane of characteristic step size $\approx |\Pi_{\beta,\gamma,\delta}(R)|$. The typical rms modulus of the effective hopping is then

$$|T_{\alpha,\eta}(R)| = V^2 |E_\alpha E_\eta|^{vd/2} \sqrt{\sum_{\beta\gamma\delta\beta'\gamma'\delta'} \Pi^*_{\beta\gamma\delta}(R)\Pi_{\beta'\gamma'\delta'}(R)},$$
$$= V^2 |E_\alpha E_\eta|^{vd/2} \sqrt{\sum_{\beta\gamma\delta} |\Pi_{\beta\gamma\delta}(R)|^2}, \tag{11}$$

where we have assumed that the "off diagonal" terms come with random phases and cancel out.

We can trade the sum over discrete states for a continuous integral by identifying the appropriate density of states for intermediate energies $E_{\alpha,\beta,\delta}$. Recall that the localization length scales as $|E|^{-\nu}$. Thus there are $\approx E^{-d\nu} dE$ states with appreciable overlap with states α and η in an energy interval $[E, E+dE]$. This implies that we should take $\sum_\beta \to \int dE_\beta |E_\beta^{-\nu d}|$. With the above rule for translating sums to integrals, we get

$$\langle |T_{\alpha,\eta}(R)| \rangle = V^2 |E_\alpha E_\eta|^{vd/2} \sqrt{\int dE_\beta dE_\gamma dE_\delta |E_\beta E_\gamma E_\delta|^{dv} \frac{\exp\left[-2R\left(E_\beta^v + E_\gamma^v + E_\delta^v\right)\right]}{(E_\alpha + E_\beta - E_\gamma - E_\delta)^2}}. \tag{12}$$

The exponential scaling of the wave-function amplitudes provides a cutoff, restricting us to states with $|E|_{\beta,\gamma,\delta} < R^{-1/\nu}$. Meanwhile, if $E_{\beta,\gamma,\delta}$ are all close to zero (which must be the case given the exponential cutoff) then the denominator is $\approx E_\alpha^2$, so we obtain

$$\frac{\langle |T_{\alpha,\eta}(R)| \rangle}{V^2} = \left|E_\alpha^{vd-1}\right| \sqrt{2\int_0^{R^{-1/\nu}} dE_\beta dE_\gamma dE_\delta |E_\beta E_\gamma E_\delta|^{dv}}$$
$$= |E_\alpha|^{dv-1} R^{-\frac{3}{2}(d+1/\nu)}. \tag{13}$$

This matrix element should be compared to the level spacing between the states ψ_α and ψ_η. If the matrix element is greater than the level spacing, then the two states are in resonance; i.e., a particle originally in state ψ_α can readily hop to ψ_η. If we are to have any hope of localization, then the probability of having a resonance at a length scale R must go to zero as $R \to \infty$.

We recall that the effective matrix elements for long-range hopping [at $O(V^2)$] fall off as $R^{-\frac{3}{2}(d+1/\nu)}$. The probability that we have resonances at a distance $R > R_c$ is then

$$P_{\rm res}(R > R_c) = \int_{R_c}^{\infty} R^{d-1} dR \frac{|T_{\alpha,\eta}(R)|}{\delta}, \tag{14}$$

where $\delta \sim E_\alpha^{\nu d}$ is the typical level spacing at energy E_α. Thus, we conclude that the probability of hopping resonances at a distance R behaves as

$$P_{\rm res}(R > R_c) \sim V^2 |E_\alpha|^{-1} R_c^{-(d/2+3/(2\nu)}. \tag{15}$$

For any nonzero ϵ and any nonzero E_α there is an R_c such that $P_{\rm res}(R > R_c) < \epsilon$. Thus, the effective long-range hopping falls off sufficiently rapidly that the probability of having long-range hopping resonances vanishes, and thus the effective long-range hopping does not present an obstacle to the construction of marginally localized integrals of motion.

B. Flip-flop resonances

Next, we consider the matrix elements of $\hat{T}$ between two-particle states, where the particles in question are separated by a distance R. Such processes may be represented diagrammatically by processes of the form shown in Fig. 3, and may be interpreted as "flip flop" processes which can transport energy over long distances [43]. The effective matrix element for this sort of "flip flop" process scales as

$$T_{\alpha\mu,\beta\eta}(R) = \sum_{\gamma\delta} \frac{\lambda_{\alpha\beta\gamma\delta}\lambda_{\gamma\delta\mu\eta}}{(E_\alpha + E_\beta) - (E_\gamma + E_\delta)} \tag{16}$$

$$= \frac{V^2}{(\xi_\alpha\xi_\beta\xi_\mu\xi_\eta)^{d/2}} \sum_{\gamma\delta} \frac{\exp\left[-R\left(\frac{1}{\xi_\gamma} + \frac{1}{\xi_\delta}\right)\right]}{(\xi_\gamma^d \xi_\delta^d)[(E_\alpha - E_\beta) - (E_\gamma - E_\delta)]} \times \exp(-i\phi_{\gamma\delta}), \tag{17}$$

where again $\phi_{\gamma\delta}$ is a random phase. The arguments proceed exactly as for the hopping resonances (except that there is one

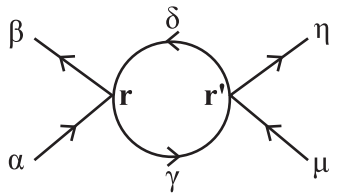

FIG. 3. Diagram illustrating flip-flop process.

fewer internal variable to sum over), and lead to the result

$$\langle |T_{\alpha\mu,\beta\eta}(R)| \rangle = V^2 \frac{|E_\alpha E_\beta E_\mu E_\eta|^{d\nu/2}}{|E_\alpha - E_\beta|} R^{-(d+1/\nu)}. \quad (18)$$

Since the strongest contributions will come from states α,β,μ,η with similar energies, we take $E_{\gamma,\delta,\beta} \approx E_\alpha$ in the numerator, and replace the energy denominator, $|E_\alpha - E_\beta|$, by minimum level spacing for states of size $\xi(E_\alpha) \approx E_\alpha^{\nu d}$, producing

$$\langle |T_{\alpha\mu,\beta\eta}(R)| \rangle = V^2 E_\alpha^{\nu d} R^{-(d+1/\nu)}. \quad (19)$$

We now ask what is the probability that a *particular* typical transition has a resonant pair within a distance R. The typical SP level spacing at an energy E_α is $\sim \xi_\alpha^{-d} \sim E_\alpha^{\nu d}$. The number of transitions which could be paired with the transition of interest goes as R^d. Thus, the minimum energy change for flip flops involving a particular typical transition goes as $E_\alpha^{\nu d} R^{-d}$. Meanwhile, the matrix element falls off as $V^2 E_\alpha^{\nu d} R^{-(d+1/\nu)}$. The probability that a particular transition has a resonant partner on the length scale R is thus $\sim V^2 R^{-1/\nu}$, which goes to zero as $R \to \infty$ for any ν. Thus, the probability of having a flip-flop resonance will drop below ϵ on length scales $R > R_c$, where $R_c = (V^2/\epsilon)^\nu$. It is instructive to contrast with the usual MBL case, Fig. 1(a), where the SP localization length is finite at all energies. This situation may be modeled by taking $\nu \to 0$, so that all states (except the exact zero energy state) are localized with localization length of order unity. Taking this limit predicts that for the traditional MBL problem, the length scale R_c beyond which the probability of flip-flop resonances drops below ϵ should grow slower than any power law of ϵ. Meanwhile, the situation with a mobility edge can be modeled by taking $\nu \to \infty$, such that all states with $E < 1$ are delocalized. In this case there will be no finite length scale R_c beyond which the probability of flip-flop resonances drops below V^2, indicating instability of localization to interactions at finite energy density in partially filled bands with a SP mobility edge.

C. Flip-flop assisted hopping

Finally, we consider the matrix elements of $\hat{T}$ between three-particle states. This process is represented diagrammatically by Fig. 4, and may be interpreted as a "flip-flop assisted hopping," whereby a particle hops from $\mathbf{r_1}$ to $\mathbf{r_2}$ by triggering SP transitions at both $\mathbf{r_1}$ and $\mathbf{r_2}$. We are interested as before in determining how this matrix element scales with $R = |\mathbf{r_1} - \mathbf{r_2}|$. This process has matrix element

$$T_{\alpha\beta\mu,\gamma\eta\kappa}(R) = \sum_\delta \frac{\lambda_{\alpha\beta\gamma\delta}\lambda_{\delta\mu\eta\kappa}}{(E_\alpha + E_\beta) - (E_\gamma + E_\delta)} \quad (20)$$

$$= \frac{V^2}{(\xi_\alpha\xi_\beta\xi_\gamma\xi_\mu\xi_\eta\xi_\kappa)^{d/2}} \sum_\delta \frac{\exp\left[-\frac{R}{\xi_\delta}\right]}{\xi_\delta^d[(E_\alpha - E_\beta) - (E_\gamma - E_\delta)]} \times \exp(-i\phi_\delta), \quad (21)$$

where again ϕ_δ is a random phase. The arguments proceed exactly as for the flip-flop resonances (except that there is one fewer internal variable to sum over), and lead to the result

$$\langle |T_{\alpha\beta\mu,\gamma\eta\kappa}(R)| \rangle = V^2 \frac{|E_\alpha E_\beta E_\gamma E_\mu E_\eta E_\kappa|^{d\nu/2}}{|E_\alpha - E_\beta|} R^{-\frac{1}{2}(d+1/\nu)}. \quad (22)$$

Since the strongest contributions will come from states α,β,μ,η with similar energies, we take $E_{\gamma,\delta,\beta} \approx E_\alpha$ in the numerator, and replace the energy denominator, $|E_\alpha - E_\beta|$, by minimum level spacing for states of size $\xi(E_\alpha) \approx E_\alpha^{\nu d}$, producing

$$\langle |T_{\alpha\beta\mu,\gamma\eta\kappa}(R)| \rangle = V^2 E_\alpha^{2\nu d} R^{-\frac{1}{2}(d+1/\nu)}. \quad (23)$$

There are $\sim R^d$ sites within a distance R of ψ_α to which a particle could hop via this "flip-flop assisted" hopping process. Thus, the minimum level spacing for such a process falls off as R^{-d}. Meanwhile, the matrix element for this process falls off only as $R^{-\frac{1}{2}(d+1/\nu)}$. Thus, the probability that there is no "assisted hopping" resonance at a length scale R vanishes at long distances as a power law of R for $\nu d > 1$ (for $\nu d = 1$ it vanishes logarithmically with R). Thus, for $\nu d \geqslant 1$, a particle can always find a (distant) resonant site to hop to, with some assistance from flip flops. This establishes that localization is impossible in the presence of interactions if

$$d\nu \geqslant 1. \quad (24)$$

This is the main result of this section, and concludes our discussion of the matrix elements of $\hat{T}$ at $O(V^2)$ [we note that $\hat{T}$ has no matrix elements between four-particle states at $O(V^2)$ because $\hat{T}$ must be represented by a connected diagram]. Next, we discuss whether we can get a more stringent condition for breakdown of localization at higher orders in V.

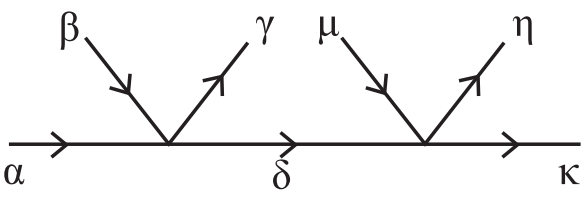

FIG. 4. The flip-flop assisted hopping process.

D. Higher orders in V

We now discuss whether a more stringent condition on breakdown of localization could be obtained by going to higher order in V and considering diagrams, e.g., of the sort illustrated in Fig. 5 (an explicit discussion of how to calculate the matrix elements represented by these diagrams is provided in Appendix C). We will conclude that the answer is no, and that (24) is the criterion for perturbative breakdown of marginal localization. We begin by noting that the diagrammatic representation of the Born series (8) contains vertices, internal lines (which begin and end on a vertex), and external lines (which are connected to vertices only at one end). At $O(V^n)$ in perturbation theory, we will have n vertices, m internal lines, and p external lines, where $2m + p = 4n$. Since only connected diagrams contribute to the $\hat{T}$ matrix, every

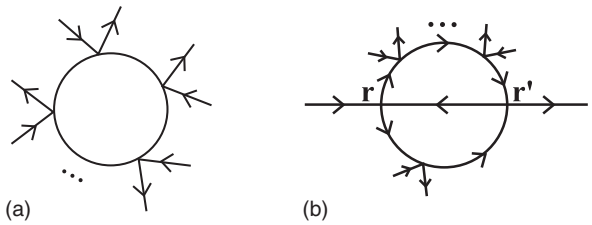

FIG. 5. Some high-order diagrams contributing to $\hat{T}$.

vertex must be connected to at least one internal line, and all but two of the vertices must be connected to at least two internal lines. This gives the constraint $m \geqslant (n-1)$.

Now, the R scaling depends only on the number of internal lines m. After summing over internal variables in a rms sense, we find that the R scaling has the form

$$T(R) \sim \sqrt{\left[\int^{R^{-1/\nu}} dE E^{-\nu d}(E^{\nu d})^2\right]^m}, \tag{25}$$

where the first factor $E^{-\nu d}$ comes about when we change the sum over states to an integral, and the second factor $(E^{\nu d})^2$ comes from the normalization factors associated with the internal lines. This gives

$$T(R) \sim R^{-\frac{m}{2}(d+1/\nu)}. \tag{26}$$

Now, if we evaluate the matrix element of $\hat{T}$ that connects a particle at a given position to particles at $\tilde{n}$ other positions within a ball of radius R (i.e., if we look at diagrams containing $n \geqslant \tilde{n}+1$ vertices that are connected to at least one external line), then the minimum level spacing for such processes will fall off as $\sim R^{-\tilde{n}d}$. If this level spacing falls off faster than the corresponding matrix element, then the probability of having resonances will approach one at large R, so that localization will be unstable. The condition for delocalization thus takes the form

$$\tilde{n}d > \frac{m}{2}\left(d+\frac{1}{\nu}\right). \tag{27}$$

Thus, to maximize our chances of delocalization, we should maximize $\tilde{n}$ and should minimize m, subject to the constraints we have already identified $\tilde{n}+1 \leqslant n$ and $m \geqslant n-1$. Substituting $\tilde{n} = n-1$ and $m = n-1$ into the above formula, we find that the condition for delocalization becomes

$$(n-1)d > \frac{n-1}{2}\left(d+\frac{1}{\nu}\right) \Rightarrow d\nu > 1. \tag{28}$$

Thus, the tightest constraint that we could possibly get within perturbation theory is the constraint (24) coming from the flip-flop mediated hopping process; i.e., delocalization occurs if $d\nu > 1$. Meanwhile, for $d\nu < 1$, the probability of resonances vanishes at large R at all orders in perturbation theory, so that marginal localization appears to be perturbatively stable. For $d\nu = 1$ the probability of having a resonance, calculated in the manner of (13), grows logarithmically with R, suggesting that the system should be delocalized (this is also consistent with results on noninteracting models with critically long range hopping [39]).

E. Systems with a vanishing critical density of states

The result (24) is slightly modified if the density of states vanishes as E^{Υ} in the vicinity of the critical energy. In this case, the integrals over energy come with a factor of E^{Υ}, and the matrix elements with m internal lines are of order

$$T(R) \sim R^{-\frac{m}{2}[d+(1+\Upsilon)/\nu]}. \tag{29}$$

In this case similar arguments establish that delocalization occurs if

$$\nu d \geqslant 1+\Upsilon. \tag{30}$$

The situation where the density of states diverges at the critical energy can also be treated within the above formalism, by taking $\Upsilon < 0$.

We note that the possible values for ν and Υ have been recently studied in Ref. [44]. In none of the ten Altland-Zirnbauer symmetry classes is the condition (30) violated; thus, fermionic topological bands at nonzero temperature always delocalize in the presence of arbitrarily weak interactions. However, in symmetry classes C and $C1$, the condition (30) is saturated, and $\nu d = 1+\Upsilon$. In these symmetry classes, the T matrix will diverge only as a logarithmic function of length scale, and correspondingly the length scale and time scale for delocalization will be *exponential* functions of the bare interaction strength. This is to be contrasted with the integer quantum Hall case, where these scales are power-law functions of the bare interaction strength (see Sec. IV G). A specific example of such a marginally localized system, which is only marginally unstable to interactions, is the spin quantum Hall effect in $d = 2$, which has $\nu = 4/7$ and $\Upsilon = 1/7$ (Ref. [44]).

F. An intuitive explanation for these results

We now draw attention to a very simple and intuitive explanation for the delocalization criterion (24). On any length scale R, there are states with energies $|E| < R^{-1/\nu}$ that look effectively extended. Meanwhile, there are $\xi^d(E)$ states at an energy E that have support at a particular position $\mathbf{r}$. Thus, the number of states to which we can couple with a local interaction which are extended on the scale R is

$$N(R) \sim \int_0^{R^{-1/\nu}} dE E^{\Upsilon}\xi^d(E) = \int_0^{R^{-1/\nu}} dE E^{\Upsilon-\nu d}. \tag{31}$$

If $\nu d < 1+\Upsilon$ then the "bath" of extended states to which we can locally couple contains a finite number of states (and the number of states in the bath goes at zero as the length scale $R \to \infty$). In this case, localization can be perturbatively stable. In contrast, if $\nu d > 1+\Upsilon$, then the above integral diverges near $E = 0$, such that the "bath" to which we can locally couple contains an infinite number of extended states at any length scale R. In this case, the algebraically delocalized states near the critical energy form a good bath, and are able to thermalize the system by mediating long-range interactions that fall off sufficiently slowly with R, as we have discussed.

G. Characteristic length and time scales for delocalization

We now estimate the length scale and time scale for the resonances that give rise to a breakdown of localization.

Recall that the T matrix elements between states separated by a distance R at order $n+1$ in perturbation theory scale as $V^{n+1}R^{-n(d+1/\nu)/2}$, while the level spacing scales as R^{-nd}. The length scale $R_c(n)$ on which resonances appear at order $n+1$ in perturbation theory is the length scale at which the matrix elements and the level spacings become comparable; i.e., $V^{n+1}R_c^{-n(d+1/\nu)/2} = R_c^{-nd}$. This happens at a critical length scale R_c, where

$$R_c(n) \sim V^{-\frac{2\nu}{d\nu-1}\frac{n+1}{n}}. \tag{32}$$

By inspection, the smallest R_c comes from $n \to \infty$ for which

$$R_c \sim V^{-2\nu/(d\nu-1)}. \tag{33}$$

The scaling of R_c in Eq. (33) controls, for example, the length scale at which the longitudinal conductivity ceases to follow (4) and saturates instead to a nonzero constant. Remarkably, even though breakdown of perturbation theory occurs already at second order in perturbation theory, the shortest length scale on which resonances appear is controlled by high orders of perturbation theory.

We can also estimate the time scale on which breakdown of localization occurs. Processes at order $n+1$ cause a breakdown on length scales $R_c(n)$, given by (32). The matrix element for these processes is of order $V^{n+1}R_c(n)^{-n(d+1/\nu)/2} = R_c(n)^{-nd}$, and the level spacing is $R_c(n)^{-nd}$, so that the density of final states is $R_c(n)^d$. A straightforward application of the golden rule implies that the time scale on which breakdown of localization happens due to processes at $n+1$ order in perturbation theory is

$$t_c(n) \sim R_c(n)^{nd} = V^{-\frac{2\nu d(n+1)}{\nu d-1}}. \tag{34}$$

Thus we see that the shortest time scale on which breakdown occurs is controlled by $n=1$ (i.e., by processes at second order in perturbation theory), and is of order

$$t_c \sim V^{-\frac{4\nu d}{\nu d-1}}. \tag{35}$$

For the case of integer quantum Hall, with $\nu = 2.3$ and $d=2$, $R_c \sim V^{-1.3}$ and $t_c \sim V^{-2.5}$. We note that the crossover time scales R_c and t_c both diverge as $V \to 0$, recovering marginal localization in the noninteracting limit ($V=0$). We note also that in Altland-Zirnbauer symmetry classes C and $C1$, where the condition [Eq. (30)] is saturated and the system is only marginally unstable to interactions, the critical length scale and time scale will depend exponentially on the bare interaction, instead of the power-law dependence discussed above (see Sec. IV E for details).

So far we have suppressed temperature-dependent occupation factors in estimating the T-matrix amplitudes. The breakdown of marginal localization happens (for a closed quantum system) because the "internal heat bath" presented by the states with diverging localization length allows the system to thermalize. At low temperature, the coupling to this internal bath is suppressed by a factor of $\exp(-\Delta/T)$, where Δ is the detuning of the chemical potential from the critical energy; i.e., the crossover scales R_c and t_c will be enhanced by an "Arrhenius" factor $\sim\exp(c\Delta/T)$, where c is a positive numerical prefactor that depends on ν and d. Thus, the crossover scales will also diverge in the zero-temperature limit, so that the zero-temperature system will indeed be localized, as long as the Fermi level is not at the critical energy. We contrast the exponential dependence on temperature with the power-law dependence expected for a system at criticality [32].

The emergence of a new length scale and time scale in the interacting problem suggests that the scaling behavior near the zero-temperature quantum critical point [21] may be richer for the interacting system than for the noninteracting system. In particular, the presence of an additional interaction-related length scale R_c may help explain why the experimentally observed longitudinal conductance is not temperature independent at the critical point [21]—the longitudinal conductance could depend on the ratio of R_c to the thermal length $\hbar/kT$. Additionally, if one examines the scaling of, e.g., the longitudinal conductance with temperature at a nonzero detuning Δ from the critical point, then the coupling to the "internal" heat bath will provide a mechanism for a thermally activated scaling of the conductance, with an activation gap proportional to the detuning from the critical point. Of course, any real experimental system will also be coupled to an "external" heat bath (e.g., phonons), which will allow for variable range hopping at any nonzero temperature, and this variable range hopping contribution to the conductance will dominate over the "activated" contribution at the lowest temperatures. However, if one observed a regime where the low-temperature longitudinal conductance was activated, with an activation gap proportional to the detuning from criticality, then this may be tentatively identified as evidence for the "delocalization due to the internal bath" discussed in this paper. An additional complication is that we have considered short-range interactions, while in real quantum Hall systems there are also Coulomb interactions (unless screened by a nearby metallic gate). The inclusion of long-range Coulomb interactions may introduce additional corrections to scaling that are not captured by the present theory.

H. Multifractality

Thus far we have assumed that the wave functions were uniform on length scales short compared to ξ. In fact, this is not the case, because close to criticality the wave functions exhibit multifractal behavior on length scales less than ξ. It is known [45,46] that for wave functions close to criticality, the disorder-averaged two-point correlations fall off as $\langle\psi_\alpha^*(0)\psi_\beta(0)\psi_\beta^*(r)\psi_\alpha(r)\rangle \sim \frac{1}{\xi^{2d}}(\xi/r)^\eta$ for $r<\xi$, where $\eta>0$ is a multifractal exponent. This suggests that the wave functions actually have higher amplitudes on scales $r \ll \xi$ than would be expected based on the "plain vanilla" model we were working with thus far. This will tend to *enhance* the matrix elements of the $\hat{T}$ matrix over and above the estimates above. Thus, the condition $\nu d > 1+\Upsilon$ may be viewed as a lower bound condition for delocalization. If we take into account the effects of multifractality in a simple-minded way by attaching a factor of $[\xi_{\rm int}(E)/R]^{\eta/2}$ to each internal line, then we will conclude that the contribution to the $\hat{T}$ matrix from a diagram with m internal lines (after rms summation over internal variables) will be

$$T(R) \sim \sqrt{\left[\int^{R^{-1/\nu}} dE\, E^{\Upsilon-\nu d}(E^{\nu d})^2\left(\frac{E^{-\nu}}{R}\right)^\eta\right]^m}$$
$$\sim R^{-\frac{m}{2}\left(d+\frac{1+\Upsilon}{\nu}\right)}, \tag{36}$$

which is independent of η; i.e., if we take multifractality into account in this simple-minded way then the threshold for delocalization is unchanged, and is $\nu d \geqslant 1 + \Upsilon$. A more sophisticated treatment of multifractality may reduce the critical value of ν somewhat, but (as we have discussed), multifractality can only make it *easier* to get delocalization.

V. DISCUSSION AND CONCLUSION

We have discussed marginally localized systems, which for partially filled bands at nonzero energy density are characterized in the noninteracting limit by subdiffusive relaxation (3), a scale-dependent dc conductivity that vanishes in the thermodynamic limit (4), and an entanglement entropy of the excited eigenstates that is intermediate between area law and volume law (5). We have shown that such systems are generically unstable to interactions, since interactions trigger a breakdown of localization for any $\nu > 1/d$. We further note that, for most systems, general arguments ([33,34] and Appendix A) place a lower bound, $\nu \geqslant 2/d$, on the localization-length exponent. Thus, we have shown that marginally localization is generically unstable to interactions, for partially filled bands at non-zero energy density.

We therefore predict that in a marginally localized system such as integer quantum Hall, the inclusion of arbitrarily weak short-range interactions will cause the entanglement entropy in the excited many-body eigenstates to become volume law, instead of the fractal scaling shown in (5). We also predict that in two-dimensional weakly interacting marginally localized systems, the scale-dependent conductivity at nonzero energy density will follow (4) on small length scales, but will saturate to a nonzero constant on large length scales. Similarly, relaxation in such marginally localized systems should be subdiffusive according to (3) for short times, but should cross over to diffusive relaxation on long time scales. We have estimated the crossover length scales and time scales, and have also discussed in Sec. IV G the interaction-induced corrections to scaling in the vicinity of the zero-temperature quantum critical point. These ideas may be directly tested in experiments in cold-atomic systems and possibly also in solid-state materials for which the electron-phonon interactions are negligible, so that the electronic system well approximates an isolated quantum system. We speculate in particular that disordered graphene in the quantum Hall regime may provide a playground to test these ideas, due to its weak electron-phonon coupling and isolation from any substrate in suspended devices.

While our arguments were for short-range interactions, the question of whether MBL can survive with long-range interactions has also been recently studied [41]. It was claimed there that full MBL is perturbatively stable against weak interactions as long as the interactions fall off with distance faster than $1/r^p$, where p is a critical power that depends on the model in question. For interactions that fall off faster than $1/r^p$, our arguments can be straightforwardly generalized to demonstrate that marginal localization is perturbatively unstable. The situation with truly long range interactions (which fall off slower than $1/r^p$) is beyond the scope of the present work.

We now discuss the prospects for numerical tests of these ideas. Unfortunately, these are rather poor since there are no existing numerical techniques suitable for simulating the highly entangled 2D or 3D systems relevant for the predicted many-body delocalization of a marginally Anderson localized phase. The prospects for experimental tests may however be brighter. Experimental quantum Hall type systems with tunable interaction strength may shortly be available using cold atoms in artificial gauge fields [47], and would provide a natural testing ground for the ideas presented in this paper.

We note that there are three classes of short-range interacting system that may evade our arguments. If the divergence in the localization length is accompanied by a density of states (DOS) which vanishes as E^{Υ}, our argument only predicts delocalization for $d\nu > 1 + \Upsilon$. Given a sufficiently rapid vanishing of the DOS at the critical energy, marginal localization could be perturbatively stable. However, the arguments presented in Ref. [44] suggest that the density of states does not vanish sufficiently rapidly to stabilize localization in any of the Altland-Zirnbauer symmetry classes. Additionally, in systems where the critical energy is pinned (e.g., by symmetry) to a particular value and is locally insensitive to disorder, the Harris criterion does not apply and we can have $\nu < 2/d$ (see [48] and Appendix A). A particular example of this is models of particle-hole symmetric localization in one dimension [22], where the divergence in the localization length is only a logarithmic function of energy. In this case our analysis would suggest that marginal localization could be perturbatively stable. Finally, our arguments establish that marginal localization is unstable for $\nu > 1/d$, where ν is the critical exponent for the localization length of *typical* states. However, the Harris-Chayes criteria only constrain the critical exponent for the *mean* localization length $\nu_{mean} \geqslant 2/d$. For infinite randomness critical points, $\nu_{typical}$ and ν_{mean} can be markedly different [49]. Thus, it is conceivable that there may be systems with infinite randomness critical points where $\nu_{mean} \geqslant 2/d$ (satisfying Harris-Chayes), but $\nu_{typical} < 1/d$, so that marginal localization could be perturbatively stable. Investigations of whether such marginally localized systems can be stable to interactions would be a fruitful topic for future work.

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APPENDIX A: HARRIS-CHAYES BOUNDS FOR LOCALIZATION-LENGTH EXPONENT

The generating function for two-body Green's functions with fixed energy ε of a noninteracting quantum system in d spatial dimensions can be written in terms of a partition function with a d-dimensional action:

$$Z(\varepsilon) = \int D[\bar{\psi},\psi] e^{i\bar{\psi}(\varepsilon + i0^+ - H)\psi}, \tag{A1}$$

where H is the single-particle Hamiltonian including random disorder. A diverging SP localization length corresponds to a critical value of ε_c for which $Z(\varepsilon)$ exhibits a phase transition. Harris's argument, which we briefly review below, can then be applied to this d-dimensional theory to constrain the correlation length exponent ν.

Within a region of size L, the disorder potential gives a random shift to the average value of ε within the region of order $\delta(L) \sim L^{-d/2}$. Then, in order to determine the average value of $\varepsilon - \varepsilon_c$ to accuracy δ, one needs to look at length scale $L \sim \delta^{-2/d}$. On the other hand, if the transition occurs inside the single-particle band, one should be able to determine on which side of the transition the system lies by examining a subsystem with size L much bigger than the correlation length $\xi(\varepsilon) \sim |\varepsilon - \varepsilon_c|^{-\nu}$. Together these considerations require that the random shift in average ε in a subsystem of size $\xi(\varepsilon)$ goes to zero no slower than than the distance to the critical point $\lim_{\varepsilon \to \varepsilon_c} \delta(\xi(\varepsilon))/|\varepsilon - \varepsilon_c| = 0$, or

$$d\nu \geqslant 2, \tag{A2}$$

the celebrated Harris criterion.

This argument applies only if the critical energy can locally be shifted by disorder [48]. In situations where the extended states are pinned at a particular energy independent of disorder, perhaps by a symmetry, then the Harris bound does not apply, and the system can have $\nu d < 2$. An example of this situation is the case of particle-hole symmetric localization [22], where the delocalized states sit exactly at $E = 0$ because of particle-hole symmetry independent of the strength of the (random hopping) disorder. In this case, the divergence of the localization length is only a logarithmic function of the energy (i.e., slower than any power law), and our argument suggests that marginal localization could be at least perturbatively stable.

APPENDIX B: SPINLESS FERMIONS AND SHORT-RANGE INTERACTIONS

In this Appendix we discuss the proper definition of $\lambda_{\alpha\beta\gamma\delta}$ [Eq. (6)] for a system containing a single species of spinless fermion. In such a system, delta function interactions have no effect, because of Pauli exclusion, and one must work instead with a density-density interaction of nonzero but finite range. For example, one could include in the Hamiltonian a term of the form

$$\int_{\mathbf{r},\mathbf{R}} V(r)\psi^\dagger_{\mathbf{R}}\psi^\dagger_{\mathbf{R}+\mathbf{r}}\psi_{\mathbf{R}+\mathbf{r}}\psi_{\mathbf{R}},$$

where $\psi^\dagger_{\mathbf{R}}$ creates a fermion at position $\mathbf{R}$. We can then define $\lambda_{\alpha\beta\gamma\delta}$ as the matrix element of this interaction term between the properly antisymmetrized incoming state $\frac{1}{\sqrt{2}}(|\alpha\rangle|\beta\rangle - |\beta\rangle|\alpha\rangle)$ and the properly antisymmetrized outgoing state $\frac{1}{\sqrt{2}}(|\gamma\rangle|\delta\rangle - |\delta\rangle|\gamma\rangle)$. This yields

$$\begin{aligned}\lambda_{\alpha\beta\gamma\delta} = \frac{1}{2}\int_{\mathbf{r},\mathbf{R}} V(r)[&\varphi^*_\alpha(\mathbf{R})\varphi^*_\beta(\mathbf{R}+\mathbf{r})\varphi_\gamma(\mathbf{R})\varphi_\delta(\mathbf{R}+\mathbf{r}) \\ &- \varphi^*_\alpha(\mathbf{R})\varphi^*_\beta(\mathbf{R}+\mathbf{r})\varphi_\delta(\mathbf{R})\varphi_\gamma(\mathbf{R}+\mathbf{r}) \\ &+ \varphi^*_\beta(\mathbf{R})\varphi^*_\alpha(\mathbf{R}+\mathbf{r})\varphi_\delta(\mathbf{R})\varphi_\gamma(\mathbf{R}+\mathbf{r}) \\ &- \varphi^*_\beta(\mathbf{R})\varphi^*_\alpha(\mathbf{R}+\mathbf{r})\varphi_\gamma(\mathbf{R})\varphi_\delta(\mathbf{R}+\mathbf{r})].\end{aligned} \tag{B1}$$

When $r = 0$ the integrand vanishes identically; this is one way to see that delta function interactions cannot have any effect in a system of spinless fermions. We therefore consider instead an interaction with finite but nonzero range a. We assume that $a \ll \xi_{\alpha,\beta,\gamma,\delta}$. With this interaction, the portion of the integrand enclosed in square brackets [...] can be Taylor expanded in small r. We can straightforwardly see that Pauli exclusion suppresses $\lambda_{\alpha\beta\gamma\delta}$ by powers of $\frac{a}{\min(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\delta)}$. When all four states $\alpha,\beta,\gamma,\delta$ are near criticality, the matrix element λ is strongly suppressed. In Ref. [31] this physics shows up in the irrelevance of the interaction. One might therefore worry that our analysis must be modified for spinless fermions, to allow for a λ that vanishes at the critical energy. However, the processes that dominate delocalization (Fig. 4) have only one of the four "legs" near criticality, while the other three "legs" are well away from the critical energy. As a result, λ is just suppressed by a constant, noncritical factor. In the language of Ref. [31], the noncritical energies of the external legs in Fig. 4 "cut off" the RG flow of the interaction. As a result, $\lambda_{\alpha\beta\gamma\delta}$ does not acquire a singular dependence on the energy for the processes that dominate delocalization, and the subtleties associated with antisymmetrization do not materially alter the argument, even for a single species of spinless fermions.

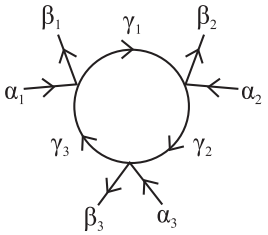

FIG. 6. A diagram of the class shown in Fig. 5(a), which appears at third order in perturbation theory, and is used to illustrate the calculation of high-order diagrams.

APPENDIX C: CALCULATING HIGH-ORDER DIAGRAMS

In this Appendix we illustrate how to calculate high-order diagrams. For specificity, we consider the particular diagram shown in Fig. 6, which appears at third order in perturbation theory, and is in the class of diagrams illustrated in Fig. 5(a). Other high-order diagrams may be calculated analogously. This diagram represents a matrix element between three-particle states. Explicitly, we have

$$T_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3} = \sum_{\gamma_1\gamma_2\gamma_3} \frac{\lambda_{\alpha_1\gamma_3\beta_1\gamma_1}}{(E_{\alpha_1}+E_{\gamma_3})-(E_{\beta_1}+E_{\gamma_1})} \frac{\lambda_{\alpha_2\gamma_1\beta_2\gamma_2}}{(E_{\alpha_2}+E_{\gamma_1})-(E_{\beta_2}+E_{\gamma_2})} \frac{\lambda_{\alpha_3\gamma_2\beta_3\gamma_3}}{(E_{\alpha_3}+E_{\gamma_2})-(E_{\beta_3}+E_{\gamma_3})} \tag{C1}$$

$$= \frac{V^3}{\left(\xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3}\xi_{\beta_1}\xi_{\beta_2}\xi_{\beta_3}\right)^{d/2}} \sum_{\gamma_1\gamma_2\gamma_3} \frac{\exp\left(-i\phi_{\gamma_3\gamma_1}\right)}{\left(\xi_{\gamma_3}\xi_{\gamma_1}\right)^{d/2}\left[\left(E_{\alpha_1}+E_{\gamma_3}\right)-\left(E_{\beta_1}+E_{\gamma_1}\right)\right]}$$
$$\times \frac{\exp\left(-i\phi_{\gamma_1\gamma_2}\right)}{\left(\xi_{\gamma_1}\xi_{\gamma_2}\right)^{d/2}\left[\left(E_{\alpha_2}+E_{\gamma_1}\right)-\left(E_{\beta_2}+E_{\gamma_2}\right)\right]} \frac{\exp\left(-i\phi_{\gamma_2\gamma_3}\right)}{\left(\xi_{\gamma_2}\xi_{\gamma_3}\right)^{d/2}\left[\left(E_{\alpha_3}+E_{\gamma_2}\right)-\left(E_{\beta_3}+E_{\gamma_3}\right)\right]}. \quad \text{(C2)}$$

We now repeat the sequence of manipulations performed in Sec. IV. Performing the sum in a rms sense, we obtain

$$\langle |T_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3}|(R)\rangle \sim \frac{V^3}{\left(\xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3}\xi_{\beta_1}\xi_{\beta_2}\xi_{\beta_3}\right)^{d/2}} \sqrt{\sum_{\gamma_1\gamma_2\gamma_3} \frac{1}{\left(\xi_{\gamma_1}\xi_{\gamma_2}\xi_{\gamma_3}\right)^{d}}} \times \left[\frac{1}{E_{\alpha_1}-E_{\beta_1}} \frac{1}{E_{\alpha_2}-E_{\beta_2}} \frac{1}{E_{\alpha_3}-E_{\beta_3}}\right], \quad \text{(C3)}$$

where we have assumed that the E_γ are much smaller than the E_α, E_β, because they are the energies of internal states that must be close to the critical energy $E_c = 0$ in order to connect vertices at large separations R. Now the above equation contains two types of localization length. There are the localization lengths associated with the "external" legs $\xi_{\alpha,\beta} \approx \xi_{\text{ext}}$, which can be short, and which (crucially) are independent of R, and there are the "internal" localization lengths $\xi_\gamma \approx \xi_{\text{int}}$, which are restricted to be greater than or equal to R if the same state γ is to have support at two positions differing by a distance of order R. The energy denominators $(E_{\alpha_j} - E_{\beta_j})$ are typically of order ξ_{ext}^{-d} since that is the level spacing for states with localization length ξ_{ext}. Trading the sum over states for an integral, we get

$$\langle |T|\rangle \sim \frac{V^n}{\xi_{\text{ext}}^{\text{pd}/2}} \left(\int^{R^{-1/\nu}} dE_{\text{int}} \xi_{\text{int}}^{-d}\right)^{m/2} \xi_{\text{ext}}^{\text{md}}, \quad \text{(C4)}$$

where $n = 3$ is the number of vertices, $m = 3$ is the number of internal lines, and $p = 6$ is the number of external lines for the diagram currently under consideration (although the expression is more general). Evaluating the integral, using $\xi_{\text{int}} \sim E_{\text{int}}^{-\nu}$, we get

$$\langle |T|\rangle \sim V^n R^{-m(d+1/\nu)/2}, \quad \text{(C5)}$$

dropping the constant factors of ξ_{ext}, which are independent of V and R. For the diagram presently under consideration, with $n = 3$ and $m = 3$, we have $\langle |T|\rangle \sim V^3 R^{-3(d+1/\nu)/2}$. However, the analysis also applies more generally to any diagram with n vertices, m internal lines, and p external lines, and underpins the discussion in Sec. IV D.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] L. Fleishman and P. W. Anderson, Phys. Rev. B **21**, 2366 (1980).
[3] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, Phys. Rev. Lett. **78**, 2803 (1997).
[4] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. **95**, 206603 (2005).
[5] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Ann. Phys. **321**, 1126 (2006).
[6] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[7] V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).
[8] M. Znidaric, T. Prosen, and P. Prelovsek, Phys. Rev. B **77**, 064426 (2008).
[9] R. Vosk and E. Altman, Phys. Rev. Lett **110**, 067204 (2013).
[10] J. Z. Imbrie, arXiv:1403.7837.
[11] R. Nandkishore and David A. Huse, arXiv:1404.0686 and references contained therein.
[12] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, Phys. Rev. B **88**, 014206 (2013).
[13] D. Pekker, G. Refael, E. Altman, E. Demler, and V. Oganesyan, Phys. Rev. X **4**, 011052 (2014).
[14] R. Vosk and E. Altman, Phys. Rev. Lett. **112**, 217204 (2014).
[15] B. Bauer and C. Nayak, J. Stat. Mech. (2013) P09005.
[16] Y. Bahri, R. Vosk, E. Altman, and A. Vishwanath, arXiv:1307.4092.
[17] A. Chandran, V. Khemani, C. R. Laumann, and S. L. Sondhi, Phys. Rev. B **89**, 144201 (2014).
[18] B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).
[19] We note that we are interested in how the localization length of a *typical* state diverges with energy. For infinite-randomness critical points, the typical and mean localization lengths may well scale differently. In such a cast, ν in our arguments should be understood as $\nu_{typical}$.
[20] D. J. Thouless, J. Phys. C: Solid State Phys. **17**, L325 (1984).
[21] B. Huckenstein, Rev. Mod. Phys. **67**, 357 (1995) and references contained therein; S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, *ibid.* **69**, 315 (1997) and references contained therein.
[22] O. Motrunich, K. Damle, and D. A. Huse, Phys. Rev. B **63**, 224204 (2001).
[23] A. A. Soluyanov and D. Vanderbilt, Phys. Rev. B **83**, 035108 (2011).
[24] R. Nandkishore, S. Gopalakrishnan, and D. A. Huse, Phys. Rev. B **90**, 064203 (2014).
[25] S. Johri, R. Nandkishore, and R. N. Bhatt, arXiv:1405.5515.
[26] S. Gopalakrishnan and R. Nandkishore, arXiv:1405.1036.
[27] M. Serbyn, Z. Papic, and D. A. Abanin, Phys. Rev. Lett. **111**, 127201 (2013).
[28] D. A. Huse and V. Oganesyan, arXiv:1305.4915; D. A. Huse, R. Nandkishore, and V. Oganesyan, arXiv:1408.4297.

[29] A. Chandran, I. H. Kim, G. Vidal, and D. A. Abanin, arXiv:1407.8480.
[30] J. T. Chalker and P. D. Coddington, J. Phys. C **21**, 2665 (1988).
[31] D.-H. Lee and Z. Wang, Phys. Rev. Lett. **76**, 4014 (1996).
[32] Z. Wang, M. P. A. Fisher, S. M. Girvin, and J. T. Chalker, Phys. Rev. B **61**, 8326 (2000).
[33] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[34] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).
[35] J. Sinova, V. Meden, and S. M. Girvin, Phys. Rev. B **62**, 2008 (2000).
[36] N. Sandler, H. R. Maei, and J. Kondev, Phys. Rev. B **68**, 205315 (2003); **70**, 045309 (2004).
[37] J. E. Moore, Nucl. Phys. B **661**, 514 (2003).
[38] I. A. Gruzberg, A. D. Mirlin, and M. R. Zirnbauer, Phys. Rev. B **87**, 125144 (2013).
[39] L. S. Levitov, Phys. Rev. Lett. **64**, 547 (1990); Ann. Phys. (Leipzig) **7**, 697 (1999).
[40] A. L. Burin, arXiv:cond-mat/0611387.
[41] N. Y. Yao, C. R. Laumann, S. Gopalakrishnan, M. Knap, M. Muller, E. A. Demler, and M. D. Lukin, arXiv:1311.7151.
[42] N. R. Cooper, Lecture notes on quantum mechanics, Cambridge University, 2007 (unpublished).
[43] Permuting the arrows in the diagram also gives a "long-range pair hopping" process whereby two particles at energies $\pm E$ and position $\mathbf{r_1}$ hop to two states at energy $\approx \pm E$ and position $\mathbf{r_2}$ with $|\mathbf{r_2} - \mathbf{r_1}| = R$. However, this matrix element scales with R in the same way as the flip-flop process, and so we do not discuss it separately.
[44] F. J. Wegner, arXiv:1003.0787.
[45] J. T. Chalker, Physica A **167**, 253 (1990).
[46] F. Evers and A. D. Mirlin, Rev. Mod. Phys. **80**, 1355 (2008).
[47] J. Dalibard, F. Gerbier, G. Juzeliunas, and P. Ohberg, Rev. Mod. Phys. **83**, 1523 (2011).
[48] R. R. P. Singh and M. E. Fisher, Phys. Rev. Lett. **60**, 548 (1988).
[49] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).