Role of the core energy in the vortex Nernst effect

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We present an analytical study of diamagnetism and transport in a film with superconducting phase fluctuations, formulated in terms of vortex dynamics within the Debye-Hückel approximation. We find that the diamagnetic and Nernst signals decay strongly with temperature in a manner that is dictated by the vortex core energy. Using the theory to interpret Nernst measurements of underdoped $La_{2-x}Sr_xCuO_4$ above the critical temperature regime, we obtain a considerably better fit to the data than a fit based on Gaussian order-parameter fluctuations. Our results indicate that the core energy in this system scales roughly with the critical temperature and is significantly smaller than expected from BCS theory. Furthermore, it is necessary to assume that the vortex mobility is much larger than the Bardeen-Stephen value in order to reconcile conductivity measurements with the same vortex picture. Therefore, either the Nernst signal is not due to fluctuating vortices, or vortices in underdoped $La_{2-x}Sr_xCuO_4$ have highly unconventional properties.

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I. INTRODUCTION

Over the past decade, the Nernst effect has become a widely used tool in the study of strongly correlated electronic systems. The Nernst signal, $e_N = E_y/(-\partial_x T)$, defined by the ratio between a measured electric field E_y and a transverse applied temperature gradient $\partial_x T$ in an electrically isolated system subjected to an external magnetic field, H_z, is typically very small in nonmagnetic normal metals. Conversely, a much stronger effect may arise in the flux-flow regime of superconductors, due to the transverse electric fields induced by the motion of vortices down the temperature gradient. Consequently, the observation of a large Nernst signal in the pseudogap state of the cuprates [1–4] has been taken as evidence that these systems support vortexlike superconducting fluctuations over a wide temperature range above their critical temperature, T_c. However, others have attributed the large Nernst signal to the response of quasiparticles in a symmetry-broken state competing with superconductivity [5–7].

Despite its appealing nature, the vortex-based picture has not been previously justified by an analytical treatment. However, several studies have calculated the Nernst signal arising from superconducting order-parameter fluctuations. The contribution of BCS Gaussian fluctuations to the thermoelectric response of the normal state near T_c was obtained in Refs. [8,9]. This result was subsequently extended to a wider range of temperatures and magnetic fields [10–12], as well as to scenarios beyond that of BCS fluctuations [13–15]. Experimentally, good agreement with the Gaussian theory was found in amorphous $Nb_{0.15}Si_{0.85}$ films [16] and in overdoped, but not underdoped, cuprates [8] (see, however, Ref. [17]).

A different approach, more pertinent to the present study, was taken by Podolsky *et al.* [18], who built upon the premise [19] that in underdoped cuprates, superconductivity is destroyed at T_c by strong phase fluctuations, whereas pairing correlations survive up to a considerably higher scale T_p. Ignoring superconducting amplitude fluctuations, the authors calculated the Nernst signal in a stochastic two-dimensional (2D) XY model via numerical simulations and a high-temperature expansion. In addition, they devised a simulation method to calculate the thermoelectric response based on vortex dynamics [20].

In this paper, we aim to bridge the aforementioned theoretical gap and present an analytical study of diamagnetism and transport in an extreme type-II superconducting film that is formulated directly in terms of vortices. We focus on temperatures above T_c, where there is a finite density, n_f, of free, unbound vortices. Our approach, which treats the vortex interactions within a Debye-Hückel approximation, is inspired by Ambegaokar *et al.* [21], who considered vortex dynamics in the context of superfluid films. A similar route was taken in the study of the resistive transition of superconducting films by Halperin and Nelson [22].

Our treatment identifies the vortex core energy ϵ_c as an important energy scale that controls the strong temperature dependence of the fluctuation signals. Using the theory, we are able to obtain a fit to the transverse thermoelectric response of underdoped $La_{2-x}Sr_xCuO_4$ (LSCO) that is superior to the one based on Gaussian fluctuations. The available data imply that both ϵ_c and T_c share a similar doping dependence, with $\epsilon_c \approx 4-5T_c$. Such values are significantly lower than the Fermi energy, which is the expected ϵ_c from BCS theory. Moreover, in order to reconcile the vortex picture with conductivity data, one needs to assume that the vortex mobility is much larger than the Bardeen-Stephen value [23]. Thus, unless the strong Nernst and diamagnetic signals in underdoped LSCO are not due to vortices, it appears that the vortex core is unconventional and plays an important role in this system.

The paper is organized as follows. The vortex Hamiltonian and dynamics are introduced in Sec. II, as well as some results based on the Debye-Hückel approximation. In Sec. III, we calculate the equilibrium magnetization currents; in Sec. IV, we calculate the electric conductivity; and in Sec. V, we calculate the thermoelectric transport coefficients. We conclude with a discussion and comparison to Nernst data in Sec. VI. Some details of the calculation are relegated to the Appendixes.

II. VORTEX HAMILTONIAN AND DYNAMICS

A 2D superconductor, at temperatures well below T_p, where the order parameter amplitude is frozen, can be described by an XY-type Hamiltonian density of a phase field θ coupled via its charge, $(2e < 0)$, to an electromagnetic vector potential $\mathbf{A}$, and a constant superfluid density ρ_s:

$$\mathcal{H} = (1+\psi)\left[\frac{\rho_s}{2}\left(\nabla\theta - \frac{2e}{\hbar c}\mathbf{A}\right)^2 + \sum_i \epsilon_c \delta(\mathbf{r}-\mathbf{r}_i)\right]. \quad (1)$$

We assume that only vortices contribute to the otherwise uniform $\nabla\theta$. A vortex i of vorticity $n_i = \pm 1$ at coordinates $\mathbf{r}_i = (x_i, y_i)$ contributes

$$\nabla\theta_i(\mathbf{r}) = n_i \hat{\mathbf{z}} \times \nabla \ln \frac{|\mathbf{r}-\mathbf{r}_i|}{r_0} = n_i \frac{\hat{\mathbf{z}} \times (\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^2}, \quad (2)$$

where r_0 is the vortex core radius and $\hat{\mathbf{z}}$ is a unit vector perpendicular to the plane. The continuum model and vortex configuration, Eqs. (1) and (2), are valid at scales longer than r_0. Thus, a region of radius r_0 around $\mathbf{r}_i$ is implicitly removed from the first term in Eq. (1). Its energy is given by the vortex core energy [24], ϵ_c, which we assume to be constant across the sample. Following Luttinger [25], we have introduced a "gravitational" field $\psi(\mathbf{r})$ in order to study the response of the system to a temperature gradient.

For concreteness, we consider a superconducting strip of infinite extent along the y direction, and of finite width L in the x direction. When needed, a constant transverse temperature gradient is applied via $\psi(\mathbf{r}) = \psi' x$, and a uniform electric field $\mathbf{E} = E_y \hat{\mathbf{y}}$ is applied along the strip. Working in the extreme type-II limit, we assume the presence of a uniform perpendicular magnetic field $B\hat{\mathbf{z}}$, and we choose the gauge $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_E$, where $\mathbf{A}_0 = Bx\hat{\mathbf{y}}$, and $\mathbf{E} = -\partial_t \mathbf{A}_E / c$. By symmetry, the average (over the vortices' positions) phase gradient $\langle \nabla\theta \rangle$ is directed along the strip and is independent of the y coordinate.

We approach the model given by Eq. (1) within a mean-field Debye-Hückel approximation, in which correlations between vortices are ignored. This is possible at temperatures higher than the Beresinskii-Kosterlitz-Thouless (BKT) transition temperature T_{BKT}, for length scales longer that the Debye-Hückel screening length r_s, where vortex interactions are screened by thermally excited vortices. The effective description at such scales is still given by Eq. (1), provided that ρ_s and ϵ_c assume renormalized values, which include contributions from the superflow at shorter distances [26]. Consequently, these parameters become temperature-dependent. The dynamics is introduced into the model by assuming that the probability $P_i(\mathbf{r}_i, t)$ to find the ith vortex at position $\mathbf{r}_i$ and time t obeys a mean-field Fokker-Planck equation (see Appendix B for more details). The corresponding probability current density for vortex i is given by [27]

$$\mathbf{J}^i(\mathbf{r}_i, t) = -\mu P_i(\mathbf{r}_i, t) \langle \nabla_i H \rangle_i - \mu T \nabla_i P_i(\mathbf{r}_i, t), \quad (3)$$

where $H = \int d^2 r\, \mathcal{H}$, μ is the vortex mobility, T is the temperature (here and throughout, $k_B = 1$), ∇_i is the gradient with respect to $\mathbf{r}_i$, and $\langle \cdots \rangle_i$ denotes an average over the position of all vortices besides $\mathbf{r}_i$. Near equilibrium, this reproduces the mean-field Debye-Hückel theory, provided one ignores fluctuations by taking $\langle (\nabla\theta)^2 \rangle \approx (\langle \nabla\theta \rangle)^2$. The residual effect of fluctuations is accounted for by renormalizing ρ_s and ϵ_c [26].

For convenience, we define the mean field $u(x) \equiv \langle \partial_y \theta \rangle / 2\pi$ and $a(x) \equiv A_y / \phi_0$, where $\phi_0 = \pi\hbar c / e$ is the flux quantum. Using these definitions, we show in Appendix B that the x component of the probability current density of vortex i is given by

$$J_x^i(x) = \mu P_i(x)[4\pi^2 \rho_s n_i (1+\psi)(u-a) - \epsilon_c \partial_x \psi] - \mu T \partial_x P_i(x). \quad (4)$$

Similarly, the average vorticity current density along x is

$$J_x^v(x) = \sum_i n_i J_x^i(x) = 4\pi^2 \rho_s \mu n_f (1+\psi)(u-a) - \mu\epsilon_c \partial_x \psi \partial_x u - \mu T \partial_x^2 u, \quad (5)$$

where $\partial_x u(x) = n(x) = \sum_i n_i P_i(x)$ is the mean vorticity, whose bulk value, as shown below, is set by B, and $n_f(x) = \sum_i P_i(x)$ is the density of free vortices. Within the equilibrium Debye-Hückel approximation, it is possible to show (see Appendix A) that

$$n_f \simeq \sqrt{4 r_0^{-4} e^{-2\epsilon_c/T} + n^2}, \quad (6)$$

which establishes a strong dependence of n_f on T for small B. The average y component of the electric current density $\mathbf{J}^e = -c\langle \delta\mathcal{H}/\delta\mathbf{A} \rangle$ is given by

$$J_y^e = \frac{4\pi^2 \rho_s c}{\phi_0}(1+\psi)(u-a). \quad (7)$$

Thus, the first term in Eq. (4) is just the vortex drift in response to the Magnus force it experiences in an electric current J_y^e. Note that *all* free vortices, and not only those responsible for the excess vorticity, contribute to the vorticity current, Eq. (5), via their response to the Magnus force. As a result, the strong temperature dependence of n_f is also reflected in the transport coefficients.

III. EQUILIBRIUM MAGNETIZATION

In equilibrium, $\psi = 0$, $E_y = 0$, and we must have $J_x^v = 0$. We therefore need to find $u_0(x)$, which solves the following equation:

$$4\pi^2 \rho_s n_f (u_0 - \bar{n}x) - T \partial_x^2 u_0 = 0, \quad (8)$$

with $\bar{n}$ defined such that $a(x) = Bx/\phi_0 = \bar{n}x$. We solve this equation, for small B, by choosing boundary conditions in which the vorticity $n(x) = \partial_x u(x)$ vanishes at $x = 0$ and $x = L$. In terms of the Debye-Hückel screening length, $r_s^{-2} = 4\pi^2 \rho_s n_f / T$, we find

$$u_0(x) = \bar{n}\left[x + r_s \frac{e^{-x/r_s} - e^{-(L-x)/r_s}}{1 + e^{-L/r_s}}\right]. \quad (9)$$

The deviation of u_0 from $\bar{n}x$ near the edge leads, according to Eq. (7), to edge currents. Their integral gives rise to an average magnetization density

$$M_z = \frac{1}{c\mathcal{A}} \int dy \int_0^L dx\, x J_y^e \simeq -\frac{TB}{\phi_0^2 n_f}, \quad (10)$$

where $\mathcal{A}$ is the area of the strip. Here, and in the following, we ignore corrections of order $O(r_s/L)$. Similar expressions to Eq. (10) were obtained in several previous studies [22,29,30].

IV. ELECTRIC CONDUCTIVITY

To study the linear response of the system to a weak perturbing field, $E_y(\omega)e^{-i\omega t}$, we need to obtain the dynamics of $u(x,t)$. By employing translational invariance in the y direction, one can show (see Appendix C) that

$$\frac{\partial u}{\partial t} = -J_x^v. \tag{11}$$

This is a local version of the equation used in Refs. [21,22]. Solving it using Eq. (5), we find in the bulk $u(x,t) = \bar{n}x + u(\omega)e^{-i\omega t}$, where

$$u(\omega) = \frac{1}{1 - i\omega\tau}\frac{cE_y(\omega)}{i\omega\phi_0}, \tag{12}$$

and where we have introduced the relaxation time $1/\tau = 4\pi^2\rho_s\mu n_f$. Equation (7) then implies an electric conductivity

$$\sigma_s(\omega) = \frac{4e^2}{h}\frac{1}{h\mu n_f}\frac{1}{1 - i\omega\tau}. \tag{13}$$

This result is identical to the conductivity obtained by Halperin and Nelson [22] for temperatures above T_c.

V. THE THERMOELECTRIC COEFFICIENTS

For systems with particle-hole symmetry or when superconducting fluctuations dominate, the Nernst signal is given by $e_N = \rho\alpha_{xy} = -\rho\alpha_{yx}$, where α_{yx} is defined by $J_y^e = \alpha_{yx}(-\partial_x T)$ [4]. Luttinger has shown [25] that α_{yx} can be deduced from the response to a "gravitational" field ψ according to the relation $J_y^e = T\alpha_{yx}(-\partial_x\psi)$. Thus, we solve Eq. (11) in the presence of $\psi(x,t) = \psi'(\omega)xe^{-i\omega t}$. By writing $u(x,t) = u_0(x) + u(\omega)e^{-i\omega t}$, where $u_0(x)$ is the equilibrium solution of Eq. (8), we find that to first order in $\psi'(\omega)$,

$$\begin{aligned}\bar{u}(\omega) &= \frac{1}{L}\int_0^L dx\, u(x,\omega)\\ &= \frac{-M_z\phi_0 n_f + \epsilon_c\bar{n}}{1 - i\omega\tau}\frac{\psi'(\omega)}{4\pi^2\rho_s n_f}.\end{aligned} \tag{14}$$

Equation (7) leads then to the average electric current density

$$\begin{aligned}\overline{J_y^e}(\omega) &= \frac{1}{\mathcal{A}}\int dy\int_0^L dx\, J_y^e(x,\omega)\\ &\simeq \frac{-M_z\phi_0 n_f + \epsilon_c\bar{n}}{1 - i\omega\tau}\frac{c\psi'(\omega)}{n_f\phi_0} + cM_z\psi'(\omega).\end{aligned} \tag{15}$$

The response of $u(x,\omega)$ is given by the first term above. An additional contribution, of opposite sign, comes from magnetization currents near the edges. Contrary to some previous studies [8,18], where this additional contribution had to be subtracted [31], in our treatment its opposite effect is explicitly included in the second term. In the dc limit, $\omega \to 0$, we therefore obtain

$$\alpha_{yx} = -\frac{2ek_B}{h}\frac{B}{n_f\phi_0}\frac{\epsilon_c}{k_B T} = \frac{\epsilon_c}{T}\frac{cM_z}{T}. \tag{16}$$

This result should be compared with the constant ratio between α_{yx} and cM_z/T, which was found for high temperatures in Refs. [8,18,20].

Next, we consider the linear-response ratio $\tilde{\alpha}_{xy}$ between an applied electric field and a transverse heat current density, $J_x^Q = \tilde{\alpha}_{xy}E_y$. We deduce $\mathbf{J}^Q$, which in our model equals the energy current density, from the conservation equation $\partial_t\mathcal{H} + \nabla\cdot\mathbf{J}^Q = \mathbf{J}^e\cdot\mathbf{E}$. Its source term originates from the explicit time dependence of $\mathcal{H}$ via $\mathbf{A}$. The result

$$\mathbf{J}^Q = -\rho_s\left\langle\frac{\partial\theta}{\partial t}\left(\nabla\theta - \frac{2e}{\hbar c}\mathbf{A}\right)\right\rangle + \sum_i \epsilon_c\,\mathbf{J}^i \tag{17}$$

is consistent with the form used by Ussishkin *et al.* [8], once modified to include the energy current associated with the vortex cores. If we additionally assume that the long superconducting strip is periodic in the y direction, then the x component of the first term in Eq. (17) must vanish by symmetry, and we find that Onsager's relation $\tilde{\alpha}_{xy}(B) = T\alpha_{yx}(-B)$ is obeyed.

VI. DISCUSSION

Often (see Refs. [1,4] and references therein), a phenomenological quantity called the vortex transport entropy, s_ϕ, is invoked in order to relate the temperature gradient to the thermal force acting on a vortex, i.e., $\mathbf{f} = -s_\phi\nabla T$. Based on Eq. (4) and Luttinger [25], we identify $s_\phi = \epsilon_c/T$. For low temperatures at which there are no thermally excited vortices and the flux-flow resistivity is the dominant form of damping, one can show by neglecting vortex interactions [4] that $\alpha_{yx} = -cs_\phi/\phi_0$. When taken together with the above identification of s_ϕ, this result is consistent with Eq. (16), since at low temperatures $\bar{n}_f\phi_0 = B$.

As the temperature is raised through $T_{\rm BKT}$, the density of free vortices, n_f, rapidly increases. Our results, Eqs. (6), (10), and (16), indicate that both M_z and α_{yx} should exhibit a consequent strong reduction with temperature, much faster than the $1/T\,\ln(T/T_c)$ decay expected from Gaussian fluctuations [8,11,12]. To look for such behavior in the cuprates, we compare Eq. (16) divided by the LSCO layer separation, $d = 6.5$ Å, $\alpha_{yx}^{3D} = \alpha_{yx}/d$, with underdoped LSCO data. According to Eq. (6), n_f is determined by the renormalized vortex core energy ϵ_c, which reflects fluctuations at distances below r_s and is temperature-dependent. For weak magnetic fields and in the critical regime above $T_{\rm BKT}$, this renormalization leads to $n_F \sim \exp(-b/\sqrt{T - T_{\rm BKT}})$ [26], while at high temperatures $n_F \sim \exp[-\epsilon_c/(T - \tilde{b})]$ [26,29]. Here b and $\tilde{b}$ are constants and ϵ_c is the bare core energy. The lack of detailed knowledge about the full temperature dependence of ϵ_c allows for considerable freedom in the fitting procedure. To constrain the fit, and since we are only interested in a rough estimate of ϵ_c, we choose to consider a constant ϵ_c and also set $\phi_0/2\pi r_0^2 = 50$ T [18]. Furthermore, we concentrate on the limit $B \to 0$ and temperatures sufficiently above T_c, where the renormalization effects are expected to be small, but low enough so that vortices are distinct objects, i.e., $r_0^2 n_f \ll 1$. Figure 1 depicts the measured $B \to 0$ limit of $-\alpha_{yx}^{3D}/B$ for LSCO samples with $x\,(T_c) = 0.07\,(11\text{ K}), 0.10\,(27.5\text{ K})$, and $0.12\,(29\text{ K})$. The solid color lines are the theoretical fits in the temperature window, $1.1T_c < T \lesssim 3T_c$, with a constant ϵ_c as the only free fitting parameter. From these curves, we find $\epsilon_c \approx 58$, 114, and 143 K for the different doping levels.

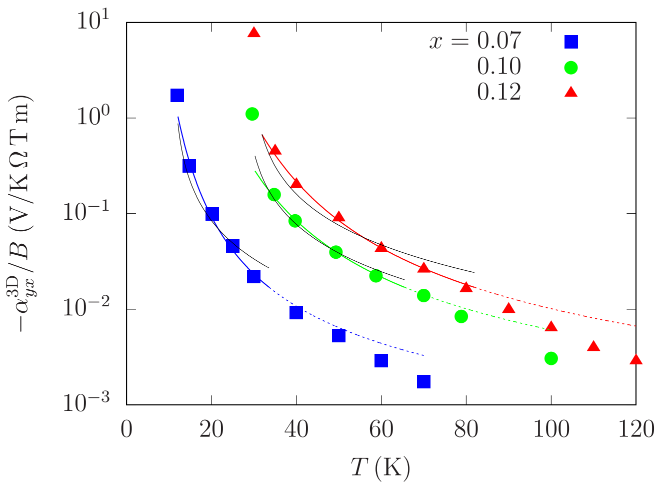

FIG. 1. (Color online) $-\lim_{B\to 0}\alpha^{3D}_{yx}/B = (\nu - \nu_n)$ of underdoped $La_{2-x}Sr_xCuO_4$, where ν is the Nernst coefficient, ν_n is a subtracted background due to quasiparticles, and ρ is the in-plane resistivity. The data for $x = 0.07, 0.10$ were extracted from Refs. [2,3,32], and for $x = 0.12$ from Ref. [18]. The data were fitted to Eqs. (16) (solid color curves). In the regime indicated by the dashed curves, $r_0^2 n_f > 0.35$, and the theory is not expected to be applicable. The solid black curves depict the best fit to the Gaussian fluctuations theory [11,12].

Comparable but somewhat larger values, $\epsilon_c \approx 8T_c$, were found by analyzing penetration depth measurements in underdoped $Y_{1-x}Ca_xBa_2Cu_3O_{7-\delta}$ bilayer films [33]. For comparison, we also include the best fit to the data based on the theory of Gaussian fluctuations [11,12]. Clearly, the data exhibit a faster decay than the Gaussian theory above the critical region around T_c. In addition, we fitted the data to the high-T result $\alpha_{yx} \propto T^{-4}$ of the stochastic XY model [18]. We obtained a good fit for $x = 0.12$, but we found an overestimation of the data in the range $1.1T_c < T < 2T_c$ ($3T_c$) for $x = 0.10$ (0.07).

The Nernst effect onset temperature, T_{onset}, is defined as the temperature for which the Nernst coefficient $\nu = e_N/B$ goes below a threshold value, typically around $\nu = 4\,\text{nV/K T}$. Such levels can be reached using Eq. (16) only if one takes $r_0^2 n_f \sim 1$. This, however, is beyond the validity of our theory. Indeed, we find that the experimental data begin to deviate from the theoretical curves at temperatures where $r_0^2 n_f > 0.35$, indicated by dashed lines in Fig. 1. Thus, although our theory agrees with the Nernst measurements up to $T \approx 3\,T_c$, it cannot account for T_{onset}, which is probably controlled by a combination of lattice effects [18] and amplitude fluctuations [8].

The Nernst signal in the cuprate pseudogap regime exhibits a maximum as a function of the magnetic field, which shifts to higher fields with increasing temperature [4,17]. While we do not have a theory for the maximum, we note that Eqs. (6) and (16) imply a crossover, set by the condition $B/\phi_0 \sim n_f(T, B = 0)$, from a linear-$B$ dependence of α_{yx} at weak fields toward saturation at higher fields. Across this scale, magnetic field-induced vortices dominate, screening is reduced, and correlation effects are enhanced, leading potentially to the suppression of α_{yx}.

In conclusion, we showed that within the vortex picture of phase fluctuating superconductors, ϵ_c plays an essential role in the thermoelectric response. The vortex core energy was also found to be important in determining T_c of layered superconductors [34]. Uncovering the role played by ϵ_c in other phenomena may help in identifying the physics underlying the different temperature scales observed in the cuprates. Equally pertinent is gaining an understanding of the factors that determine ϵ_c itself. Here we briefly mention the need for a model of "cheap vortices," in which vortices support a state close in energy to the superconducting phase [35–38]. It seems to us that the checkerboard state observed around vortex cores [39] is a natural candidate.

Nevertheless, if the Nernst signal in underdoped cuprates is, in fact, due to thermally excited vortices, one must also understand why experiments do not show signatures of fluctuation-enhanced conductivity over a similar temperature range. More specifically, if the vortex mobility is given by the Bardeen-Stephen result [23], $\mu \approx 8\pi e^2 r_0^2/h^2\sigma_n$, then Eq. (13) gives a fluctuation contribution $\sigma_s = \sigma_n/2\pi r_0^2 n_f$, where σ_n is the normal state conductivity. This would imply, using our estimate $\epsilon_c \approx 4 - 5T_c$, from fitting the LSCO Nernst data, and Eq. (6), that $\sigma_s > \sigma_n$ for $T < 2T_c$, in contradiction to experiments. To avoid such a contradiction within our model, we must therefore assume that μ is much larger than the Bardeen-Stephen value, thereby reducing σ_s while not affecting M_z and α_{yx}. A similar conclusion regarding μ was reached based on THz time-domain spectroscopy in LSCO [40]. The above discussion further indicates that understanding the vortex core in the cuprates may call for physics beyond standard BCS theory.

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APPENDIX A: DEBYE-HÜCKEL APPROXIMATION IN EQUILIBRIUM

At high temperatures, it is possible to study the vortex Hamiltonian within the Debye-Hückel approximation. This approximation is best formulated using a variational mean-field approach in which the density matrix is factored into a product of local probabilities, $\rho_{\mathbf{r}}$, of the local vorticity $n_{\mathbf{r}} = 0, \pm 1$,

$$\rho = \prod_{\mathbf{r}} \rho_{\mathbf{r}}(n_{\mathbf{r}}), \tag{A1}$$

with the effect that the entropy is given by

$$S = -\text{Tr}\rho \ln \rho = -\sum_{\mathbf{r}} \sum_{n_{\mathbf{r}}} \rho_{\mathbf{r}}(n_{\mathbf{r}}) \ln \rho_{\mathbf{r}}(n_{\mathbf{r}}). \tag{A2}$$

In addition, one approximates the average Hamiltonian by

$$\langle H \rangle \approx \frac{1}{2}\rho_s \int d^2r \left(\langle \nabla\theta \rangle - \frac{2e}{\hbar c}\mathbf{A} \right)^2 + \epsilon_c \sum_{\mathbf{r}} \langle |n_{\mathbf{r}}| \rangle , \tag{A3}$$

while ignoring the contribution coming from fluctuations in $\nabla\theta$,

$$\langle H_{\text{fluc}}\rangle = \frac{1}{2}\rho_s \int d^2r[\langle(\nabla\theta)^2\rangle - \langle\nabla\theta\rangle^2]. \tag{A4}$$

The average phase gradient $\langle\nabla\theta\rangle$ is given by

$$\langle\nabla\theta(\mathbf{r})\rangle = \overline{\nabla\theta} + \sum_{\mathbf{r}'}\langle n_{\mathbf{r}'}\rangle \frac{\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^2}, \tag{A5}$$

where $\overline{\nabla\theta}$ is the uniform part of $\nabla\theta(\mathbf{r})$, which does not rise from vortices, and $\langle n_{\mathbf{r}}\rangle = \sum_{n_{\mathbf{r}}}\rho_{\mathbf{r}}(n_{\mathbf{r}})n_{\mathbf{r}}$.

The density matrix, $\rho_{\mathbf{r}}(n_{\mathbf{r}})$, itself is determined by minimizing the free energy $F = \langle H\rangle - TS$, subject to the constraint $\sum_{n_{\mathbf{r}}}\rho_{\mathbf{r}}(n_{\mathbf{r}}) = 1$. This amounts to solving

$$\frac{\partial F}{\partial\rho_{\mathbf{r}}(n_{\mathbf{r}})} = \varphi(\mathbf{r})n_{\mathbf{r}} + \epsilon_c|n_{\mathbf{r}}| + T\ln\rho_{\mathbf{r}}(n_{\mathbf{r}}) = \alpha, \tag{A6}$$

where

$$\varphi(\mathbf{r}) = \rho_s\int d^2r'\left[\langle\nabla'\theta(\mathbf{r}')\rangle - \frac{2e}{\hbar c}\mathbf{A}(\mathbf{r}')\right]\cdot\frac{\hat{\mathbf{z}}\times(\mathbf{r}'-\mathbf{r})}{(\mathbf{r}'-\mathbf{r})^2}. \tag{A7}$$

As a result, we find

$$\rho_{\mathbf{r}}(n_{\mathbf{r}}) = \frac{1}{z_{\mathbf{r}}}e^{-\beta\epsilon_c|n_{\mathbf{r}}|-\beta\varphi(\mathbf{r})n_{\mathbf{r}}}, \tag{A8}$$

with $\beta = 1/T$ ($k_B = 1$ is used throughout), and

$$z_{\mathbf{r}} = 1 + e^{-\beta\epsilon_c}2\cosh\beta\varphi(\mathbf{r}). \tag{A9}$$

For small $e^{-\beta\epsilon_c}$ this yields

$$\langle|n_{\mathbf{r}}|\rangle \approx e^{-\beta\epsilon_c}2\cosh\beta\varphi(\mathbf{r}) \tag{A10}$$

and

$$\langle n_{\mathbf{r}}\rangle \approx -e^{-\beta\epsilon_c}2\sinh\beta\varphi(\mathbf{r}). \tag{A11}$$

Eliminating φ gives

$$\langle|n_{\mathbf{r}}|\rangle = \sqrt{4e^{-2\beta\epsilon_c} + \langle n_{\mathbf{r}}\rangle^2}, \tag{A12}$$

which, after dividing through by r_0^2, results in Eq. (6).

APPENDIX B: VORTEX DYNAMICS

1. Mean-field Fokker-Planck equation

To formulate the dynamics of the vortices in our model, we assume that the number of vortices is the same as in equilibrium, and that their vorticity is fixed. The events of vortex-antivortex creation and annihilation are important for nonlinear response at T_c, but they have a negligible effect on linear response and are therefore ignored. Thus, it is possible to formulate vortex dynamics using a Fokker-Planck equation for the positions of all vortices, $\{\mathbf{r}_i\}$, each with a given vorticity $\{n_i = \pm 1\}$:

$$\frac{\partial P(\{\mathbf{r}_i\},t)}{\partial t} = \mu\sum_i\left[\nabla_i\cdot[P(\{\mathbf{r}_i\},t)\nabla_i H] + T\nabla_i^2 P(\{\mathbf{r}_i\},t)\right], \tag{B1}$$

where μ is the vortex mobility and ∇_i is the gradient with respect to $\mathbf{r}_i$. This is a complicated equation to solve, but it can be treated approximately, in a manner similar to the Debye-Hückel approximation in equilibrium, by factoring the probability density into a product of single vortex probabilities,

$$P(\{\mathbf{r}_i\},t) = \prod_i P_i(\mathbf{r}_i,t). \tag{B2}$$

Integrating the left side of Eq. (B1) over the positions of all vortices aside from the position of the ith gives

$$\begin{aligned}\prod_{j\neq i}\int d^2r_j\frac{\partial P(\{\mathbf{r}_i\})}{\partial t} &= \prod_{j\neq i}\int d^2r_j\sum_k\prod_{l\neq k}P_l(\mathbf{r}_l,t)\frac{\partial P_k(\mathbf{r}_k,t)}{\partial t}\\ &= P_i(\mathbf{r}_i,t)\sum_{k\neq i}\prod_{j\neq i,k}\left(\int d^2r_j P_j(\mathbf{r}_j,t)\right)\int d^2r_k\frac{\partial P_k(\mathbf{r}_k)}{\partial t} + \frac{\partial P_i(\mathbf{r}_i,t)}{\partial t}\prod_{j\neq i}\left(\int d^2r_j P_j(\mathbf{r}_j,t)\right)\\ &= \frac{\partial P_i(\mathbf{r}_i,t)}{\partial t},\end{aligned} \tag{B3}$$

where we demand that the single vortex probabilities are normalized,

$$\int d^2r_j P_j(\mathbf{r}_j,t) = 1. \tag{B4}$$

Performing the same integral on the right side of the Fokker-Planck equation gives

$$\begin{aligned}\frac{\partial P_i(\mathbf{r}_i,t)}{\partial t} &= \prod_{j\neq i}\int d^2r_j\mu\sum_k\nabla_k\cdot[P(\{\mathbf{r}_k\},t)\nabla_k H(\{\mathbf{r}_k\}) + T\nabla_k P(\{\mathbf{r}_k\})]\\ &= P_i(\mathbf{r}_i,t)\mu\sum_{k\neq i}\int d^2r_k\nabla_k\cdot[P_k(\mathbf{r}_k,t)\langle\nabla_k H\rangle_{ik} + T\nabla_k P_k(\mathbf{r}_k)] + \mu\nabla_i\cdot[P_i(\mathbf{r}_i,t)\langle\nabla_i H\rangle_i + T\nabla_i P_i(\mathbf{r}_i)],\end{aligned} \tag{B5}$$

where

$$\langle \nabla_i H \rangle_i = \prod_{j\neq i} \left(\int d^2 r_j P_j(\mathbf{r}_j,t) \right) \nabla_i H \quad \text{(B6)}$$

and

$$\langle \nabla_k H \rangle_{ik} = \prod_{j\neq i,k} \left(\int d^2 r_j P_j(\mathbf{r}_i,t) \right) \nabla_k H. \quad \text{(B7)}$$

$\langle \nabla_k H \rangle_{ik}$ is similar to $\langle \nabla_k H \rangle_k$ except for an interaction term H_{ik} between vortex k and vortex i:

$$\langle \nabla_k H \rangle_{ik} = \langle \nabla_k H \rangle_k - \langle \nabla_k H_{ik} \rangle_k + \nabla_k H_{ik}. \quad \text{(B8)}$$

Substituting Eq. (B8) into Eq. (B5), we find that the single-vortex Fokker-Planck equation is

$$\frac{\partial P_i(\mathbf{r}_i)}{\partial t} = \mu \nabla_i \cdot [P_i(\mathbf{r}_i,t) \langle \nabla_i H \rangle_i + T \nabla_i P_i(\mathbf{r}_i)], \quad \text{(B9)}$$

provided that

$$\sum_{k\neq i} \int d^2 r_k \nabla_k \cdot [P_k(\mathbf{r}_k,t)(\nabla_k H_{ik} - \langle \nabla_k H_{ik} \rangle_k)] = 0. \quad \text{(B10)}$$

This can be shown to be the case on our strip, where there is translational invariance in the y direction.

2. Derivation of the vorticity current

Various average quantities can be calculated using the single-vortex probability density

$$P_i(\mathbf{r},t) = \langle \delta(\mathbf{r} - \mathbf{r}_i(t)) \rangle . \quad \text{(B11)}$$

Specifically, the vorticity can be written as

$$\begin{aligned} \partial_x u(x,t) &= n(x,t) \\ &= \sum_i \langle n_i \delta(\mathbf{r} - \mathbf{r}_i(t)) \rangle \\ &= \sum_i n_i P_i(x,t), \end{aligned} \quad \text{(B12)}$$

and the free vortex density is

$$n_f(x,t) = \sum_i \langle \delta(\mathbf{r} - \mathbf{r}_i(t)) \rangle = \sum_i P_i(x,t), \quad \text{(B13)}$$

where we have used the translation invariance in the y direction.

Furthermore, the probability current density,

$$\mathbf{J}_i(\mathbf{r},t) = \langle \delta(\mathbf{r} - \mathbf{r}_i(t)) \dot{\mathbf{r}}_i(t) \rangle , \quad \text{(B14)}$$

can be related to $P_i(\mathbf{r},t)$ by interpreting the single-vortex Fokker-Planck equation (B9) as a probability conservation condition, from which it is evident that

$$\mathbf{J}_i(\mathbf{r}_i,t) = -\mu P_i(\mathbf{r}_i,t) \langle \nabla_i H \rangle_i - \mu T \nabla_i P_i(\mathbf{r}_i,t). \quad \text{(B15)}$$

This can be used to calculate the vorticity current density

$$\begin{aligned} J_x^v(x,t) &= \sum_i \langle n_i \delta(\mathbf{r} - \mathbf{r}_i(t)) \dot{x}_i \rangle \\ &= \sum_i n_i J_{i,x}(x,t), \end{aligned} \quad \text{(B16)}$$

which is of particular interest to us. Ignoring the same fluctuation term in $\langle H \rangle$ as in Eq. (A3), we find

$$\begin{aligned} \frac{\partial \langle H \rangle_i}{\partial x_i} &= \frac{\partial}{\partial x_i} \frac{\delta \langle H \rangle}{\delta P_i(\mathbf{r}_i)} \\ &\approx n_i \rho_s \frac{\partial}{\partial x_i} \int d^2 r' [1 + \psi(x')] \left[\langle \nabla \theta(\mathbf{r}') \rangle - \frac{2e}{\hbar c} \mathbf{A}(\mathbf{r}') \right] \cdot \frac{\hat{\mathbf{z}} \times (\mathbf{r}' - \mathbf{r}_i)}{(\mathbf{r}' - \mathbf{r}_i)^2} + \epsilon_c \frac{\partial}{\partial x_i} \psi(\mathbf{r}_i) \\ &= n_i \rho_s \frac{\partial}{\partial x_i} \int dx'\, 2\pi [1 + \psi(x')][u(x') - a(x')] \int dy' \frac{x' - x_i}{(x' - x_i)^2 + (y' - y_i)^2} + \epsilon_c \frac{\partial}{\partial x_i} \psi(x_i) \\ &= n_i \rho_s \frac{\partial}{\partial x_i} \int dx'\, 2\pi [1 + \psi(x')][u(x') - a(x')]\, \pi \,\mathrm{sgn}(x' - x_i) + \epsilon_c \frac{\partial}{\partial x_i} \psi(x_i) \\ &= -n_i 4\pi^2 \rho_s [1 + \psi(x_i)][u(x_i) - a(x_i)] + \epsilon_c \frac{\partial}{\partial x_i} \psi(x_i). \end{aligned} \quad \text{(B17)}$$

Therefore, the vorticity current density is

$$\begin{aligned} J_x^v(x,t) &= \sum_i n_i J_{i,x}(x,t) \\ &= \sum_i n_i \left[-\mu P_i(x_i,t) \frac{\partial \langle H \rangle_i}{\partial x_i} - \mu T \frac{\partial P_i(x_i,t)}{\partial x_i} \right]_{x_i = x} \\ &= \sum_i n_i \left[\mu P_i(x_i,t) n_i 4\pi^2 \rho_s [1 + \psi(x_i)][u(x_i) - a(x_i)] - \mu P_i(x_i,t) \epsilon_c \frac{\partial}{\partial x_i} \psi(x_i) - \mu T \frac{\partial P_i(x_i,t)}{\partial x_i} \right]_{x_i = x} \\ &= \sum_i [4\pi^2 \rho_s \mu P_i(x,t)[1 + \psi(x)][u(x) - a(x)] - \mu \epsilon_c \partial_x \psi(x) n_i P_i(x,t) - \mu T n_i \partial_x P_i(x,t)], \end{aligned} \quad \text{(B18)}$$

which, with the help of Eqs. (B12) and (B13), finally gives Eq. (5).

APPENDIX C: DYNAMICAL EQUATION FOR U

To study the linear response of the system to weak, time-dependent, perturbing fields $\mathbf{E}$ and $\nabla\psi$, we must obtain the dynamics of the field $u(x,t)$. Using the definition of u, we obtain

$$\frac{\partial u}{\partial t} = \frac{1}{2\pi}\left\langle \sum_i \dot{x}_i \frac{\partial}{\partial x_i}\partial_y\theta \right\rangle, \tag{C1}$$

whose integral over y,

$$\begin{aligned}\int dy\,\frac{\partial u}{\partial t} &= \frac{1}{2\pi}\left\langle \sum_i \dot{x}_i \frac{\partial}{\partial x_i}\int dy\,\partial_y\theta \right\rangle \\ &= \frac{1}{2\pi}\left\langle \sum_i \dot{x}_i \frac{\partial}{\partial x_i} n_i\pi\,\mathrm{sgn}(x-x_i) \right\rangle \\ &= -\left\langle \sum_i \dot{x}_i n_i \delta(x-x_i) \right\rangle \\ &= -\int dy\, J_x^v,\end{aligned} \tag{C2}$$

implies, by translational invariance along y, Eq. (11).

[1] For a review of earlier results, see R. P. Huebener, Supercond. Sci. Technol. **8**, 189 (1995).
[2] Z. A. Xu, N. P. Ong, Y. Wang, T. Kakeshita, and S. Uchida, Nature (London) **406**, 486 (2000).
[3] Y. Wang, Z. A. Xu, T. Kakeshita, S. Uchida, S. Ono, Y. Ando, and N. P. Ong, Phys. Rev. B **64**, 224519 (2001).
[4] Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B **73**, 024510 (2006).
[5] O. Cyr-Choinière, R. Daou, F. Lalibertè, D. LeBoeuf, N. Doiron-Leyraud, J. Chang, J.-Q. Yan, J.-G. Cheng, J.-S. Zhou, J. B. Goodenough, S. Pyon, T. Takayama, H. Takagi, Y. Tanaka, and L. Taillefer, Nature (London) **458**, 743 (2009).
[6] R. Daou, J. Chang, D. LeBoeuf, O. Cyr-Choinière, F. Lalibertè, N. Doiron-Leyraud, B. J. Ramshaw, R. Liang, D. A. Bonn, W. N. Hardy, and L. Taillefer, Nature (London) **463**, 519 (2010).
[7] A. Hackl, M. Vojta, and S. Sachdev, Phys. Rev. B **81**, 045102 (2010).
[8] I. Ussishkin, S. L. Sondhi, and D. A. Huse, Phys. Rev. Lett. **89**, 287001 (2002).
[9] I. Ussishkin, Phys. Rev. B **68**, 024517 (2003).
[10] K. Michaeli and A. M. Finkel'stein, Phys. Rev. B **80**, 214516 (2009).
[11] K. Michaeli and A. M. Finkel'stein, Europhys. Lett. **86**, 27007 (2009).
[12] M. N. Serbyn, M. A. Skvortsov, A. A. Varlamov, and V. Galitski, Phys. Rev. Lett. **102**, 067001 (2009).
[13] S. Mukerjee and D. A. Huse, Phys. Rev. B **70**, 014506 (2004).
[14] S. Tan and K. Levin, Phys. Rev. B **69**, 064510 (2004).
[15] A. Levchenko, M. R. Norman, and A. A. Varlamov, Phys. Rev. B **83**, 020506(R) (2011).
[16] A. Pourret, H. Aubin, J. Lesueur, C. A. Marrache-Kikuchi, L. Bergé, L. Dumoulin, and K. Behnia, Nat. Phys. **2**, 683 (2006).
[17] J. Chang, N. Doiron-Leyraud, O. Cyr-Choinière, G. Grissonnanche, F. Lalibertè, H. Hassinger, J. Ph. Reid, R. Daou, S. Pyon, T. Takayama, H. Takagi, and L. Taillefer, Nat. Phys. **8**, 751 (2012).
[18] D. Podolsky, S. Raghu, and A. Vishwanath, Phys. Rev. Lett. **99**, 117004 (2007).
[19] V. J. Emery and S. A. Kivelson, Nature (London) **374**, 434 (1995).
[20] S. Raghu, D. Podolsky, A. Vishwanath, and D. A. Huse, Phys. Rev. B **78**, 184520 (2008).
[21] V. Ambegaokar, B. I. Halperin, D. R. Nelson, and E. D. Siggia, Phys. Rev. B **21**, 1806 (1980).
[22] B. I. Halperin and D. R. Nelson, J. Low Temp. Phys. **36**, 599 (1979).
[23] J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965).
[24] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).
[25] J. M. Luttinger, Phys. Rev. **135**, A1505 (1964).
[26] P. Minnhagen, Rev. Mod. Phys. **59**, 1001 (1987).
[27] The effect of a temperature gradient on a vortex enters this Fokker-Planck equation via the "gravitational" field ψ in the Hamiltonian. Alternatively, it is possible to introduce a position-dependent temperature, $T(\mathbf{r})$, directly into the Fokker-Plank equation, in a manner that depends on the underlying microscopic dynamics. Assuming a vortex behaves like a Brownian particle, the corresponding probability current density [28] is $\mathbf{J}_i(\mathbf{r}_i,t) = -\mu P_i(\mathbf{r}_i,t)\,\langle\nabla_i H\rangle_i - \mu\nabla\,[T(\mathbf{r}_i)P(\mathbf{r}_i,t)]$. This approach reproduces the same results we get by calculating the response to ψ, assuming that the density of free vortices is at local equilibrium, $n_f(\mathbf{r}) \sim \exp[-\epsilon_c/T(\mathbf{r})]$.
[28] N. G. van Kampen, J. Phys. Chem. Solids **49**, 673 (1988).
[29] L. Benfatto, C. Castellani, and T. Giamarchi, Phys. Rev. Lett. **99**, 207002 (2007).
[30] V. Oganesyan, D. A. Huse, and S. L. Sondhi, Phys. Rev. B **73**, 094503 (2006).
[31] N. R. Cooper, B. I. Halperin, and I. M. Ruzin, Phys. Rev. B **55**, 2344 (1997).
[32] Y. Ando, S. Komiya, K. Segawa, S. Ono, and Y. Kurita, Phys. Rev. Lett. **93**, 267001 (2004).
[33] L. Benfatto, C. Castellani, and T. Giamarchi, Phys. Rev. B **77**, 100506(R) (2008).
[34] L. Benfatto, C. Castellani, and T. Giamarchi, Phys. Rev. Lett. **98**, 117008 (2007).
[35] P. A. Lee, N. Nagaosa, and X. Wen, Rev. Mod. Phys. **78**, 17 (2006).
[36] D. P. Arovas, A. J. Berlinsky, C. Kallin, and S-C. Zhang, Phys. Rev. Lett. **79**, 2871 (1997).
[37] L. B. Ioffe and A. J. Millis, Phys. Rev. B **66**, 094513 (2002).
[38] C. Honerkamp and P. A. Lee, Phys. Rev. Lett. **92**, 177002 (2004).
[39] J. E. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, H. Eisaki, S. Uchida, and J. C. Davis, Science **295**, 466 (2002).
[40] L. S. Bilbro, R. Valdés Aguilar, G. Logvenov, I. Bozovic, and N. P. Armitage, Phys. Rev. B **84**, 100511(R) (2011).