

## Movement of dislocations dressed with $^3\text{He}$ impurities in $^4\text{He}$ crystals

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Solid  $^4\text{He}$  is a unique example of crystal where dislocations may move at macroscopic speeds with impurities attached to them. In  $^4\text{He}$  crystals, the only impurities are  $^3\text{He}$  atoms, whose concentration can be reduced to zero and measured down to the ppt ( $10^{-12}$ ) level. We present measurements of the mobility of dislocations dressed with  $^3\text{He}$  impurities as a function of the crystal purity. They show that the damping of dislocation motion is proportional to the concentration of  $^3\text{He}$  bound to these dislocations. It has allowed us to measure the  $^3\text{He}$  binding energy  $E_B$  to dislocations without any ambiguity. Our results solve the controversy concerning  $E_B$ : We confirm our previously measured value  $0.7 \pm 0.1$  K, and we demonstrate that it cannot be 0.2 or 0.4 K as estimated by other authors. Finally, we present a simple model for the damping magnitude, where dissipation is due to the emission by moving  $^3\text{He}$  impurities of transverse waves along the dislocation lines.

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Solid  $^4\text{He}$  is an interesting system for the study of dislocation effects, which strongly influence the mechanical properties of crystals [1,2]. This is because of the extraordinary purity of solid  $^4\text{He}$ , where the only impurity is  $^3\text{He}$  whose concentration can be lowered to zero [3,4] and accurately measured down to the ppt level ( $10^{-12}$ ). Dislocation motion is highly sensitive to the presence of tiny amounts of impurities. In  $^4\text{He}$  crystals, large quantum fluctuations have at least two important consequences. First,  $^3\text{He}$  atoms move freely at velocities of order 1 cm/s through the lattice [5,6]. They can also bind to dislocations [7,8], in which case there is a thermodynamic equilibrium between the bound  $^3\text{He}$  atoms in the potential well of dislocation cores and a gas of free  $^3\text{He}$  quasiparticles in the bulk crystal. By binding to dislocations,  $^3\text{He}$  atoms damp the dislocation motion, which is free otherwise, even at low temperature, because of a second consequence of quantum fluctuations: They reduce the classical “Peierls barriers” against dislocation motion to a negligible value, possibly to zero as proposed by Haziot *et al.* [4,9]. Recent work [4,6,9–11] has demonstrated that the mechanical properties of hcp  $^4\text{He}$  crystals can be understood by modeling their dislocations as elastic strings that can move back and forth at kHz frequencies between pinning points [1] under the action of an ac-driving strain. Around 0.2 K, the mobility is very large without measurable dissipation. It induces a “giant plasticity,” a spectacular phenomenon due to the vanishing of the shear modulus in certain directions. At higher temperature, the dislocation motion is damped by scattering with thermal phonons while at lower temperature it is damped or pinned by  $^3\text{He}$  binding.

By studying oriented  $^4\text{He}$  single crystals, we have demonstrated that dislocations glide parallel to the basal planes of the hcp structure. Thanks to a detailed understanding of the scattering by thermal phonons, which is well described by the “fluttering mechanism” [1,12,13], we could measure the density  $\Lambda$  of dislocation length per unit volume. It varies between  $10^4$  cm $^{-2}$  and  $10^6$  cm $^{-2}$  depending on growth conditions. We could also measure the “network length”  $L_N$  between network nodes, which are strong pinning points where dislocations intersect: It is well represented by a wide distribution around an average value of order 100  $\mu\text{m}$ .

From a study of the shear modulus as a function of driving strain amplitude and frequency, Haziot *et al.* [6] discovered an original phenomenon. In usual crystals, dislocations zig zag between impurities which are fixed and form quenched disorder.  $^3\text{He}$  atoms are mobile in  $^4\text{He}$  crystals but still, Haziot *et al.* have found that at large amplitude, large frequency, and low temperature, the dislocations are really pinned by  $^3\text{He}$  which cannot move fast enough. However, at amplitude and frequency small enough that the dislocation speed is less than 45  $\mu\text{m/s}$ ,  $^3\text{He}$  move bound to them, so that the motion is damped but not pinned. Haziot *et al.* [6] realized that there are two different regimes: pinning or damping. The damping regime was not understood. The present study clarifies it.

In previous work [6,8,11,14], it was assumed that the damping was proportional to the concentration  $x_d$  of  $^3\text{He}$  bound to the dislocation and their binding energy  $E_B$  was found equal to  $0.7 \pm 0.1$  K. However, Kim *et al.* [15] and Iwasa [16] made the estimates  $E_B = 0.4$  and 0.2 K, respectively. If instead of being proportional to the concentration  $x_d$ , the damping had been proportional to  $x_d^2$ , our previous measurements would imply  $E_B = 0.35$  K, in better agreement with Iwasa and Kim *et al.* As explained below, our new series of measurements is consistent with a damping proportional to  $x_d$ , not with  $x_d^2$ . The present experiment solves the running controversy [17]. Furthermore, we present a simple model at the end of

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this Rapid Communication, which qualitatively explains the magnitude of the measured damping.

To describe the dislocation dynamics and its consequences on mechanical properties, we follow Granato's approach [1] and describe them as elastic strings moving between pinning points in response to an oscillating stress with no effect of the lattice potential. This dislocation motion results in a strain  $\epsilon_{\text{dis}}$  that adds to the strain  $\epsilon_{\text{el}}$  only due to the elastic deformation of the lattice [18]. The additional strain  $\epsilon_{\text{dis}}$  causes the shear modulus  $\mu$  to decrease from the intrinsic value  $\mu_{\text{el}}$ :

$$\mu = \frac{\mu_{\text{el}}}{1 + \epsilon_{\text{dis}}/\epsilon_{\text{el}}}. \quad (1)$$

The magnitude of the softening  $(\mu_{\text{el}} - \mu)/\mu_{\text{el}}$  can approach 1 if the dislocations move freely, but it is reduced by the damping of dislocation motion, which is due, in the case of  $^4\text{He}$ , to bound  $^3\text{He}$  impurities or thermal phonons. The equation of motion of the dislocation is

$$A\xi'' + B\dot{\xi} - C\frac{\partial^2\xi}{\partial y^2} = b\sigma, \quad (2)$$

where  $\xi(y,t)$  is the dislocation displacement as a function of time  $t$  and position  $y$  between its pinning points,  $A = \pi\rho b^2 = 8.2 \times 10^{-17}$  kg/m is the dislocation's effective mass per unit length in a material with density  $\rho = 191$  kg/m<sup>3</sup>,  $b = 0.367$  nm the Burgers vector amplitude,  $B$  is the damping force per unit length, and  $\sigma$  the applied stress. For an edge dislocation, the line tension  $C$  is given by  $C = [\mu_{\text{el}}b^2 \ln(R/r)]/[4\pi(1-\nu)] \approx 2.3 \times 10^{-12}$  N, where  $\nu = 0.33$  is Poisson's ratio of the material in an isotropic approximation,  $R \approx L_N \approx 100$   $\mu\text{m}$  is an average distance between dislocations, and  $r \approx 1$  nm is the core radius [19].

We define a relaxation time,

$$\tau = BL^2/\pi^2C, \quad (3)$$

where the length  $L$  equals  $L_N$  in the absence of pinning by  $^3\text{He}$ , and we have [11]

$$\frac{\epsilon_{\text{dis}}}{\epsilon_{\text{el}}} = \alpha\Lambda L_N^2 \frac{1 - i\omega\tau}{1 + (\omega\tau)^2}, \quad (4)$$

where  $\omega/2\pi$  is the driving frequency, and  $\alpha = 32(1-\nu)/\pi^4 \ln(R/r) \approx 0.019$ . Equations (1) and (4) show that as  $B$  increases,  $\mu$  increases to its intrinsic value  $\mu_{\text{el}}$  and the dissipation  $Q^{-1} = \text{Im}[\mu]/\text{Re}[\mu]$  passes through a maximum at  $\omega\tau = \sqrt{1 + \alpha\Lambda L_N^2}$  [11]. We use Eqs. (1) and (4) along with measurements in the phonon damping regime to determine  $\Lambda$  and  $L_N$  for a given crystal [10], and then use Eqs. (1) and (4) along with measurements on the same crystal in the  $^3\text{He}$  damping regime to determine the  $^3\text{He}$  damping coefficient.

Our crystal growth and shear modulus measurement techniques are explained in previous publications [4,11]. Inside the measurement cell, two piezoelectric shear plates face each other with a separation of 0.7 mm, forming a narrow gap that can be filled with an oriented  $^4\text{He}$  crystal (Fig. 1 inset). The orientation of single crystals is determined from photographs of growth shapes. Applying a voltage  $V$  to one transducer produces a shear strain  $\epsilon = Vd_{15}$  in the  $^4\text{He}$  crystal, where  $d_{15} = 0.095$  nm/V below 1K [4]. The resulting stress  $\sigma$  in the  $^4\text{He}$  is measured with the opposite transducer. The shear modulus is then given by  $\mu = \sigma/\epsilon$ . For the present study, we

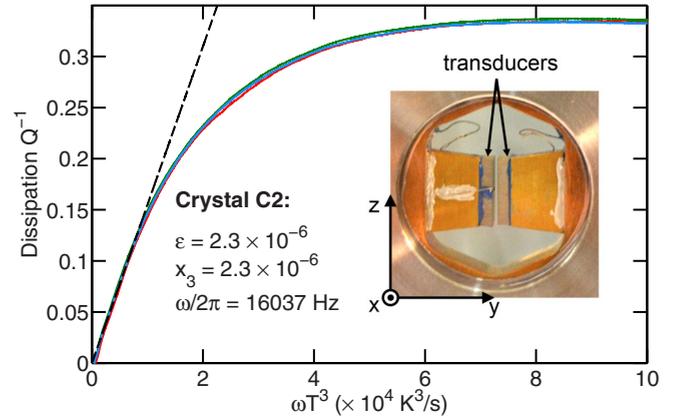


FIG. 1. (Color online) (Solid curves) Dissipation measurement on cooling C2 ( $^3\text{He}$  concentration  $x_3 = 2.3 \times 10^{-8}$ ) at  $\omega/2\pi = 16037$  Hz and driving strain  $\epsilon = 2.3 \times 10^{-6}$ . The three different colors correspond to measurements on three different days, which demonstrate excellent reproducibility. The dashed line shows the linear variation of  $Q^{-1}$  in the low  $\omega T^3$  limit, which is predicted by Eq. (6) and allows a determination of the average network length  $L_N = 158$   $\mu\text{m}$  and dislocation density  $\Lambda = 4.1 \times 10^5$  cm<sup>-2</sup>. (Inset) The measurement cell (see text).

have used three different helium samples whose  $^3\text{He}$  purity was carefully measured as  $2.5 \times 10^{-8}$  (commercial  $^4\text{He}$  from Air Liquide, probably originating from Qatar where the isotopic impurity concentration is much lower than in Texas [20]),  $3.8 \times 10^{-7}$  and  $2.3 \times 10^{-6}$ , respectively. Single crystals were grown at 1.4 K so that any small liquid droplets that could trap  $^3\text{He}$  solidified as the crystal was cooled below 1 K. In order to verify that the latter was true, we compared with polycrystals grown by cooling the cell at constant volume, starting typically from liquid at 67 bar and 2.8 K, ending with a cell completely full of solid at 37 bar and 2 K. As shown by Figs. 3 and 4, results obtained with single crystals are perfectly compatible with results obtained with polycrystals. We used the method of Maris and Balibar [21] to relate the shear modulus of polycrystals to the elastic coefficients  $c_{ij}$  of single crystals.

We used high drive, high frequency measurements to determine  $L_N$  and  $\Lambda$  for all crystals, as in [10]. Let us consider the example of crystal C2. Figure 1 shows the dissipation measured on cooling C2 at a driving frequency  $\omega/2\pi = 16037$  Hz and an rms driving strain  $\epsilon = 2.3 \times 10^{-6}$ . Measurements made on three different days demonstrate their high degree of reproducibility. The high  $\epsilon$  prevented  $^3\text{He}$  atoms from binding to the dislocations, so that all the dissipation was due to phonon scattering. In this regime, the damping coefficient is  $B_{\text{ph}} = 14.4k_B^3 T^3/\pi^2 h^2 c^3 \approx 2.1 \times 10^{-8} T^3$  Pa s where  $c$  is the sound speed [12,13], and the relaxation time is  $\tau_{\text{ph}} = B_{\text{ph}} L_N^2/\pi^2 C$ . Substituting  $\tau_{\text{ph}}$  for  $\tau$  in Eq. (4) and using Eq. (1) yields, at low temperatures where the phonon damping is small, a maximum softening,

$$\frac{\Delta\mu}{\mu_{\text{el}}} \equiv \frac{\mu_{\text{el}} - \mu}{\mu_{\text{el}}} = \frac{\alpha\Lambda L_N^2}{1 + \alpha\Lambda L_N^2}, \quad (5)$$

TABLE I. Results obtained with different types of crystals.

		Y3	Z5	E1	F1	C1	C2
$x_3$	( $\times 10^{-6}$ )	0.025	0.025	0.025	0.38	2.3	2.3
$L_N$	( $\mu\text{m}$ )	73	96	68	81	189	169
$\Lambda$	( $\times 10^5 \text{ cm}^{-2}$ )	7.6	7.9	15	37	4.7	3.6
$E_B$	(K)	0.65	0.67	0.60	0.71	0.65	0.73

and a dissipation,

$$Q^{-1} = \frac{\Delta\mu}{\mu_{\text{el}}} \omega\tau_{\text{ph}} = \frac{\Delta\mu}{\mu_{\text{el}}} \frac{14.4k_B^3 L_N^2}{\pi^4 \hbar^2 c^3 C} \omega T^3. \quad (6)$$

The dashed line in Fig. 1 shows that  $Q^{-1}$  varies linearly in the low  $\omega T^3$  limit, as predicted by Eq. (6). The nonlinear behavior at higher  $\omega T^3$  was explained in Ref. [11]. We determined  $L_N = 158 \mu\text{m}$  and  $\Lambda = 4.1 \times 10^5 / \text{cm}^2$  by using Eqs. (5) and (6) after measuring a maximum softening  $\Delta\mu/\mu_{\text{el}} = 0.664$  and a low temperature slope  $Q^{-1}/\omega T^3 = 1.55 \times 10^{-5} \text{ s/K}^3$ . We used the same method for all crystals and we obtained respective values  $2.5 \times 10^{-6}$ ,  $5 \times 10^{-6}$ ,  $2.5 \times 10^{-5}$ ,  $1.75 \times 10^{-5}$ ,  $2.4 \times 10^{-6}$ , and  $5 \times 10^{-6}$  for crystals Y3, Z5, C1, and C2, and polycrystals E1 and F1 (see Table I).

After having characterized the dislocation network, we determined the  $^3\text{He}$  damping coefficient by using a low drive and a low frequency, so that the dislocation speed  $v$  is less than  $45 \mu\text{m/s}$ . Figure 2 shows the temperature dependence of  $\mu$  and  $Q^{-1}$  obtained when cooling crystal C1. The shear modulus increases monotonically as  $^3\text{He}$  atoms bind to the dislocations and damp their motion. Because the measurements in Fig. 2 are at low frequency, damping by thermal phonons is now negligible and  $B = B_3$  is only due to  $^3\text{He}$  bound to the dislocations. At the lowest temperatures,  $\mu = \mu_{\text{el}}$ . Near the midpoint of the temperature variation of  $\mu$  at each frequency,  $Q^{-1}$  reaches a maximum at a temperature  $T_p$ .

Figure 3 shows six sets of data corresponding to six different crystals with various concentrations  $x_3$ . For each crystal, the relaxation time is obtained from  $\tau = (1/\omega)\sqrt{1 + \alpha\Lambda L_N^2}$  at the peak dissipation temperature  $T_p$  and the damping coefficient  $B_3$  from Eq. (3). Let us now assume that  $B_3$  scales with some power  $n$  of the  $^3\text{He}$  concentration on the dislocation  $x_d = x_3 \exp(E_B/k_B T)$ . This may be written as  $B_3 = B_0 x_3^n \exp(nE_B/k_B T)$ . Figure 4 shows a graph of the prefactor  $B_0 x_3^n$ , which is obtained by extrapolating to  $1/T = 0$  on Fig. 3, as a function of  $x_3$ . Our results are well compatible with  $n = 1$ , not with  $n = 2$ . It justifies the use of Eq. (2) with a damping force proportional to the dislocation velocity. In 2010, Syshchenko *et al.* [14] made frequency measurements on polycrystals where  $x_3$  could be  $2 \times 10^{-7}$ . Assuming that their values for  $\lambda$  and  $L_N$  were the same as for F1 that was grown in similar conditions, we found that the present results are consistent with their preliminary results.

This is an important result which solves the running controversy on the interpretation of our measurement of slopes in Arrhenius plots like the one of Fig. 3. These slopes measure the binding energy  $E_B$  of  $^3\text{He}$  atoms to dislocations, not any function of  $E_B$  like  $2E_B$ . Here we found the respective values 0.65, 0.67, 0.65, 0.73, 0.6, and 0.71 K for crystals Y3, Z5, C1, C2, E1, and F1. The small scatter in these values

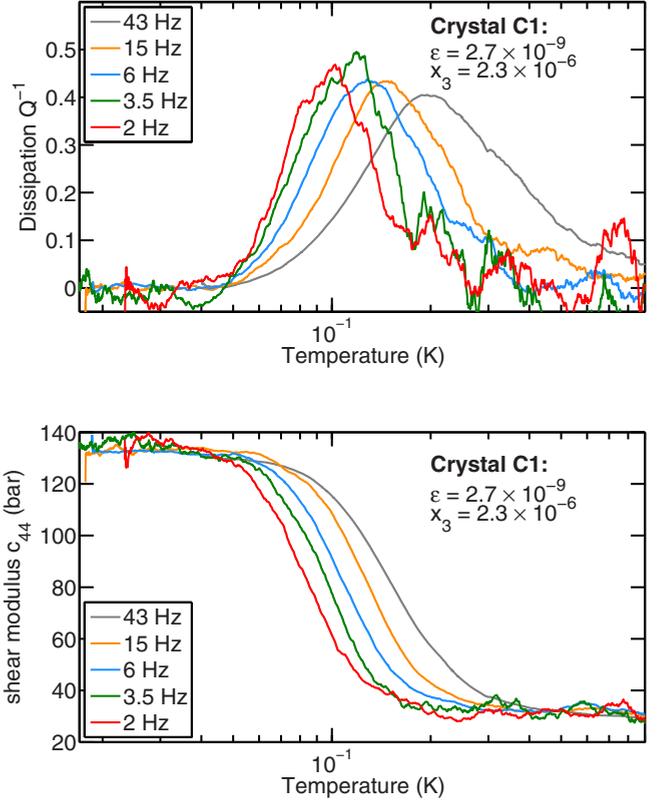


FIG. 2. (Color online) The dissipation and shear modulus of crystal C1 whose  $^3\text{He}$  concentration was  $2.3 \times 10^{-6}$ . Measurements made on cooling at a low rms driving strain of  $2.7 \times 10^{-9}$  and low frequency, where the low temperature damping is due to the binding of  $^3\text{He}$  impurities only.

perfectly agrees with the distribution around 0.7 K found by Fefferman *et al.* in Ref. [11], also with preliminary results by Syshchenko *et al.* [14]. A numerical calculation could show if

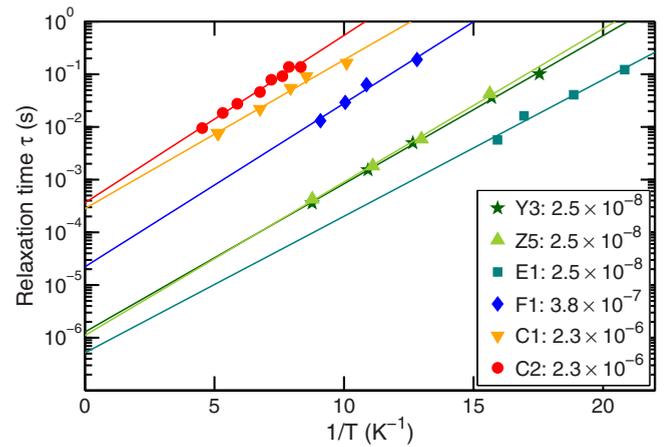


FIG. 3. (Color online) A semilog plot of the relaxation time vs inverse temperature. Different symbols correspond to six different crystals with various  $^3\text{He}$  concentrations. The measurements were made at low driving strain  $\epsilon = 2.7 \times 10^{-9}$  and at frequencies in the range 1–100 Hz. This plot is used to determine the binding energy of  $^3\text{He}$  atoms to dislocations (see text).

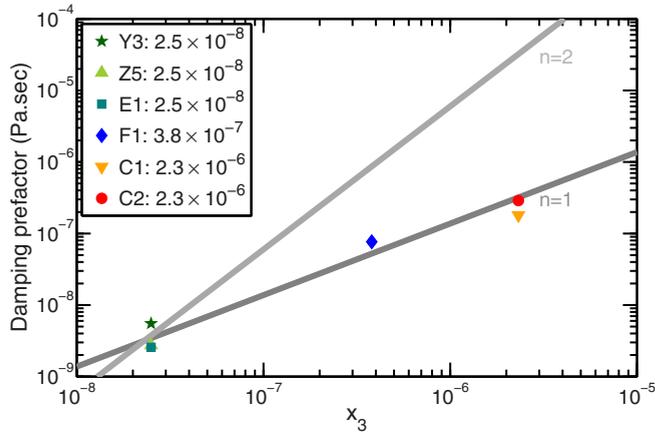


FIG. 4. (Color online) The dependence on  ${}^3\text{He}$  concentration  $x_3$  of the prefactor of the exponential in the expression of the damping coefficient  $B_3$  (see text).  $x_3$  is measured within 5% in two different mass spectrometers. The agreement with  $n = 1$  demonstrates that the damping is proportional to  $x_3$  (i.e.,  $n = 1$ ), not to  $x_3^2$  (i.e.,  $n = 2$ ).

the particular value of this binding energy is consistent with the predicted splitting of dislocations [22]. Our results contradict the assumptions by Iwasa [16] or Kim *et al.* [15]. As for the magnitude  $B_3$  of the damping due to bound  ${}^3\text{He}$ , Fig. 4 shows that it is equal to  $B_0 x_3$  with  $B_0 = 0.15$  Pa.s.

Let us now present a simple model for the magnitude of the damping  $B_0$ . The dislocation line moves in the direction  $x$  at an average velocity  $v$ . Bound  ${}^3\text{He}$  atoms move at the same average velocity  $v$  but they probably lag behind, so that, in the moving frame of the dislocation, their position  $x(t)$  oscillates back and forth:

$$x(t) = -a_1 - a_1 \cos(2\pi vt/a), \quad (7)$$

where  $a = 0.3$  nm is the periodicity of the lattice in the  $x$  direction. This oscillatory motion emits waves of amplitude  $a_1$ , which should be of order the core radius  $r = 1$  nm. These

waves are emitted in both directions away from the  ${}^3\text{He}$  atom and they should decay into heat via some interaction with bulk phonons. Suppose that the velocity of transverse waves along dislocations is  $c_d$ , the energy of two such waves represents a dissipation per  ${}^3\text{He}$  atom and per unit time  $\dot{q} = 4\pi^2 A(a_1 v/a)^2 c_d \approx 400 A v^2 c_d$ , where  $A = 8.2 \times 10^{-17}$  kg/m is the mass already mentioned in Eq. (2). Note that the dissipation  $\dot{q}$  has the right proportionality to  $v^2$  corresponding to a force linear in  $v$  as assumed in Eq. (2) and verified in our experiments. Since the concentration of  ${}^3\text{He}$  atoms is  $x_d = x_3 \exp(E_B/k_B T)$ , and assuming that the density of  ${}^4\text{He}$  atoms is  $1/a$  along the line, we find

$$B_0 = 400 A c_d / a. \quad (8)$$

Taking  $c_d$  equal to the transverse sound speed 250 m/s, we obtain  $B_0 = 0.027$  Pa.s. This is about 6 times smaller than the experimental value 0.15 Pa.s, but the simple model above contains several rough approximations. The splitting of dislocations into partials [22] should be considered in a more accurate estimate of the speed of such transverse waves. The estimated dissipation could be increased if  ${}^3\text{He}$  motion is periodic but not sinusoidal, also if the motion of  ${}^3\text{He}$  directly emits phonons. Our model is a simple starting point for more precise future calculations. One may like to verify that the energy loss per  ${}^3\text{He}$  atom and per lattice period is smaller than the binding energy  $E_B = 0.7$  K. Even at the highest velocity  $v = 45$   $\mu\text{m/s}$  of the damping regime, this energy loss is  $B_0 v a^2 \approx 0.05$  K only.

In summary, we have shown that, when dislocations move dressed with  ${}^3\text{He}$  atoms, the damping of their motion is proportional to the density of  ${}^3\text{He}$  attached to them. Our work solves the controversy on the value of the binding energy. We propose that the damping is due to the emission of transverse waves along them.

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