Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory

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Symmetry-protected topological (SPT) phases are gapped short-range-entangled quantum phases with a symmetry G, which can all be smoothly connected to the trivial product states if we break the symmetry. It has been shown that a large class of *interacting* bosonic SPT phases can be systematically described by group cohomology theory. In this paper, we introduce a (special) group supercohomology theory which is a generalization of the standard group cohomology theory. We show that a large class of *short-range interacting* fermionic SPT phases can be described by the group supercohomology theory. Using the data of supercocycles, we can obtain the ideal ground state wave function for the corresponding fermionic SPT phase. We can also obtain the bulk Hamiltonian that realizes the SPT phase, as well as the anomalous (i.e., non-onsite) symmetry for the boundary effective Hamiltonian. The anomalous symmetry on the boundary implies that the *symmetric* boundary must be gapless for (1+1)-dimensional [(1+1)D] boundary, and must be gapless or topologically ordered beyond (1+1)D. As an application of this general result, we construct a new SPT phase in three dimensions, for interacting fermionic superconductors with coplanar spin order (which have $T^2 = 1$ time-reversal Z_2^T and fermion-number-parity Z_2^f symmetries described by a full symmetry group $Z_2^T \times Z_2^f$). Such a fermionic SPT state can neither be realized by free fermions nor by interacting bosons (formed by fermion pairs), and thus are not included in the K-theory classification for free fermions or group cohomology description for interacting bosons. We also construct three interacting fermionic SPT phases in two dimensions (2D) with a full symmetry group $Z_2 \times Z_2^f$. Those 2D fermionic SPT phases all have central-charge $c = 1$ gapless edge excitations, if the symmetry is not broken.

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I. INTRODUCTION

A. Short- and long-range entangled states

Recently, it was realized that highly entangled quantum states can give rise to new kind of quantum phases beyond Landau symmetry breaking [1–3], which include topologically ordered phases [4–6], and symmetry-protected topological (SPT) phases [7,8] (see Fig. 1). The topologically ordered phases contain long-range entanglement [9] as revealed by topological entanglement entropy [10,11], and cannot be transformed to product states via local unitary (LU) transformations [12–14]. Fractional quantum Hall states [15,16], chiral spin liquids [17,18], Z_2 spin liquids [19–21], non-Abelian fractional quantum Hall states [22–25], etc., are examples of topologically ordered phases. The mathematical foundation of topological orders is closely related to tensor category theory [9,12,26,27] and simple current algebra [22,28]. Using this point of view, we have developed a systematic and quantitative theory for topological orders with gappable edge for (2+1)-dimensional [(2+1)D] interacting boson and fermion systems [9,12,27]. Also, for (2+1)D topological orders with only Abelian statistics, we find that we can use integer K-matrices to describe them [29–32].

The SPT states are short-range entangled (SRE) states with symmetry (i.e., they do not break the symmetry of the Hamiltonian), which can be transformed to product states via LU transformations that break the symmetry. However, nontrivial SPT states cannot be transformed to product states via the LU transformations that preserve the symmetry, and different SPT states cannot be transformed to each other via the LU transformations that preserve the symmetry. The one-dimensional (1D) Haldane phase for spin-1 chain [7,8,33,34] and topological insulators [35–40] are nontrivial examples of SPT phases. Some examples of two-dimensional (2D) SPT phases protected by translation and some other symmetries were discussed in Refs. [41–43].

It turns out that there is no gapped bosonic LRE state in 1D [13], so all 1D gapped bosonic states are either symmetry breaking states or SPT states. This realization led to a complete classification of all (1+1)D gapped bosonic quantum phases [42–44].

In Refs. [45,46], the result for 1D SPT phase is generalized to any dimensions: *For gapped bosonic systems in d_{sp}-spatial dimensions with an onsite symmetry G, we can construct distinct SPT phases that do not break the symmetry G from the distinct elements in $\mathcal{H}^{d_{sp}+1}[G,U_T(1)]$, the $(1+d_{sp})$-cohomology class of the symmetry group G with G-module $U_T(1)$ as coefficient.* Note that the above result does not require the translation symmetry. The results for some simple onsite symmetry groups are summarized in Table I. In particular, the interacting bosonic topological insulators/superconductors are included in the table.

B. Definition of fermionic SPT phases

In Ref. [27], the LU transformations for fermionic systems were introduced, which allow us to define and study gapped quantum liquid phases and topological orders in fermionic systems. In particular, we developed a fermionic tensor category theory to classify intrinsic fermionic topological orders with gappable edge, as defined through the fermionic LU transformations without any symmetry. In this paper, we

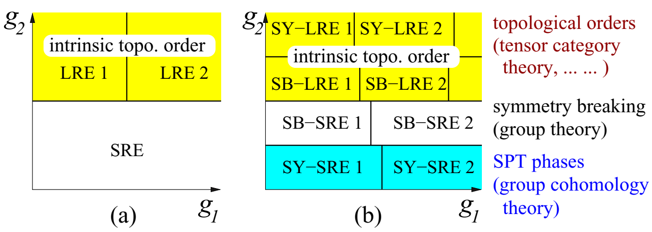


FIG. 1. (Color online) (a) The possible gapped phases for a class of Hamiltonians $H(g_1,g_2)$ without any symmetry. (b) The possible gapped phases for the class of Hamiltonians $H_{\text{symm}}(g_1,g_2)$ with a symmetry. The yellow regions in (a) and (b) represent the phases with long-range entanglement. Each phase is labeled by its entanglement properties and symmetry breaking properties. SRE stands for short-range entanglement, LRE for long-range entanglement, SB for symmetry breaking, SY for no symmetry breaking. SB-SRE phases are the Landau symmetry breaking phases, which are understood by introducing group theory. The SY-SRE phases are the SPT phases, which can be understood by introducing group cohomology theory.

are going to use the similar line of thinking to study fermionic gapped liquid phases with symmetries. To begin, we will study the simplest kind of them: *short-range-entangled fermionic phases with symmetries*. Those phases are called fermionic SPT phases:

(1) They are $T=0$ gapped phases of fermionic Hamiltonians with certain symmetries.

(2) Those phases do not break any symmetry of the Hamiltonian.

(3) Different SPT phases cannot be connected without phase transition if we deform the Hamiltonian while preserving the symmetry of the Hamiltonian.

(4) All the SPT phases can be connected to the trivial product states without phase transition through the deformation the Hamiltonian if we allow to break the symmetry of the Hamiltonian.

TABLE I. (Color online) Bosonic SPT phases (from group cohomology theory) in d_{sp}-spatial dimensions protected by some simple symmetries (represented by symmetry groups G). Here, $\mathbb{Z}_1$ means that our construction only gives rise to the trivial phase. $\mathbb{Z}_n$ means that the constructed nontrivial SPT phases plus the trivial phase are labeled by the elements in $\mathbb{Z}_n$. Z_2^T represents time-reversal symmetry, $U(1)$ represents $U(1)$ symmetry, Z_n represents cyclic symmetry, etc. Also, (m,n) is the greatest common divisor of m and n. The red rows are for bosonic topological insulators and the blue rows bosonic topological superconductors.

Symm. G	$d_{sp}=0$	$d_{sp}=1$	$d_{sp}=2$	$d_{sp}=3$
$U(1)\rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1)\times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)\times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_m\times Z_n$	$\mathbb{Z}_m\times\mathbb{Z}_n$	$\mathbb{Z}_{(m,n)}$	$\mathbb{Z}_m\times\mathbb{Z}_n\times\mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(m,n)}^2$
$Z_n\times Z_2^T$	$\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_2\times\mathbb{Z}_{(2,n)}$	$\mathbb{Z}_{(2,n)}^2$	$\mathbb{Z}_2\times\mathbb{Z}_{(2,n)}^2$

[The term "connected" can also mean connected via fermionic LU transformations (with or without symmetry) defined in Ref. [27].]

Note that in 0 spatial dimension, the trivial product states can have even or odd numbers of fermions. So the trivial product states in 0D can belong to two different phases. In higher dimensions, all the trivial product states belong to one phase (if there is no symmetry). This is why when there is no symmetry, there are two fermionic SPT phases in 0D and only one trivial fermionic SPT phase in higher dimensions.

C. Group supercohomology construction of fermionic SPT states

After introducing the concept of fermionic SPT phase, we find that we can generalize our construction of bosonic SPT orders [45,46] to fermion systems, by generalizing the group cohomology theory [46,47] to group supercohomology theory. This allows us to systematically construct a large class of fermionic SPT phases for interacting fermions in any dimensions.

In the group cohomology theory for bosonic SPT states, we find that the different topological terms in bosonic nonlinear σ models in *discrete* space-time are described by different cocycles in group cohomology theory (see Sec. III for a detailed review). The different types of topological terms lead to different bosonic SPT phases, and thus different cocycles describe different bosonic SPT phases. The idea behind our approach in this paper is similar: we show that different topological terms in fermionic nonlinear σ models in *discrete* space-time are described by different cocycles in group supercohomology theory. The different types of fermionic topological terms lead to different fermionic SPT phases, and thus different supercocycles describe different fermionic SPT phases.

So far, our construction only applies for a certain type of symmetries where the fermions form a 1D representation of the symmetry group. It cannot handle the situation where fermions do not form a 1D representation of the symmetry group in the fixed-point wave functions. For this reason, we call the current formulation of group supercohomology theory a special group supercohomology theory. On the other hand, the current version of group supercohomology theory can indeed systematically generate a large class of fermionic SPT phases, and many of those examples are totally new since they can neither be constructed from free fermions nor from interacting bosons (that correspond to bound states of fermion pair).

D. A summary of main results

The constructed fermionic SPT phases for some simple symmetry groups are given in Table II, which lists the special group supercohomology class $\mathscr{H}^{d+1}[G_f,U(1)]$. The rows correspond to different symmetries for the fermion systems. We note that, in literature, when we describe the symmetry of a fermion system, sometimes we include the fermion-number-parity transformation $P_f=(-)^N$ in the symmetry group, and sometimes we do not. In this paper and in Table II, we always use the full symmetry group G_f which includes the fermion-number-parity transformation to describe the symmetry of

TABLE II. (Color online) Fermionic SPT phases in d_{sp}-spatial dimensions constructed using group supercohomology for some simple symmetries (represented by the full symmetry groups G_f). The red symmetry groups can be realized by electron systems. Here $\mathbb{Z}_1$ means that our construction only gives rise to the trivial phase. $\mathbb{Z}_n$ means that the constructed nontrivial SPT phases plus the trivial phase are labeled by the elements in $\mathbb{Z}_n$. Note that fermionic SPT phases always include the bosonic SPT phases with the corresponding bosonic symmetry group $G_b \equiv G_f/Z_2^f$ as special cases. The blue entries are complete constructions which become classifications. Z_2^T represents time-reversal symmetry, $U(1)$ represents $U(1)$ symmetry, etc. As a comparison, the results for noninteracting fermionic gapped/SPT phases [48–50], as well as the interacting symmetric phases in 1D [44,51–53], are also listed. Note that the symmetric interacting and noninteracting fermionic gapped phases can be the SPT phases or intrinsically topologically ordered phases. This is why the lists for gapped phases and for SPT phases are different. $2\mathbb{Z}$ means that the phases are labeled by even integers. Note that all the phases listed above respect the symmetry G_f.

Interacting fermionic SPT phases					
$G_f \backslash d_{sp}$	0	1	2	3	Example
"none" $= Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	Superconductor
$Z_2 \times Z_2^f$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_1$	
$Z_2^T \times Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	Supercond. with coplanar spin order
$Z_{2k+1} \times Z_2^f$	$\mathbb{Z}_{4k+2}$	$\mathbb{Z}_1$	$\mathbb{Z}_{2k+1}$	$\mathbb{Z}_1$	
$Z_{2k} \times Z_2^f$	$\mathbb{Z}_{2k} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{4k}$	$\mathbb{Z}_1$	
Interacting fermionic gapped symmetric phases					
$G_f \backslash d_{sp}$	0	1	2	3	Example
"none" $= Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	?	?	Superconductor
$Z_2 \times Z_2^f$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	?	?	
$Z_2^T \times Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	?	?	Supercond. with coplanar spin order
$Z_{2k+1} \times Z_2^f$	$\mathbb{Z}_{4k+2}$	$\mathbb{Z}_2$	?	?	
$Z_{2k} \times Z_2^f$	$\mathbb{Z}_{2k} \times \mathbb{Z}_2$	$\mathbb{Z}_4$	?	?	
Noninteracting fermionic SPT phases					
$G_f \backslash d_{sp}$	0	1	2	3	Example
"none" $= Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	Superconductor
$Z_2 \times Z_2^f$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$	
$Z_2^T \times Z_2^f$	$\mathbb{Z}_2$	$2\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	Supercond. with coplanar spin order
Noninteracting fermionic gapped phases					
$G_f \backslash d_{sp}$	0	1	2	3	Example
"none" $= Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$	Superconductor
$Z_2 \times Z_2^f$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	$\mathbb{Z}^2$	$\mathbb{Z}_1$	
$Z_2^T \times Z_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	Supercond. with coplanar spin order

fermion systems [48]. So, the full symmetry group of a fermion system with no symmetry is $G_f = Z_2^f$ generated by P_f. The bosonic symmetry group G_b is given by $G_b = G_f/Z_2^f$, which correspond to the physical symmetry of the fermion system that can be broken. In fact, the full symmetry group is a projective symmetry group discussed in Ref. [54], which is a Z_2^f extension of the physical symmetry group G_b.

The columns of Table II correspond to different spatial dimensions. In 0D and 1D, our results reproduce the exact results obtained from previous studies [44,51–53]. The results for 2D and 3D are new.

Each entry indicates the number of nontrivial phases plus one trivial phase. For example, $\mathbb{Z}_2$ means that there is one nontrivial SPT phase labeled by 1 and one trivial phase labeled by 0. Also, say, $\mathbb{Z}_8$ means that there are seven nontrivial SPT phases and one trivial phase. For each nontrivial fermionic SPT phase, we can construct the ideal ground state wave function and the ideal Hamiltonian that realizes the SPT phase, using group supercohomology theory.

When $G_f = G_b \times Z_2^f$, $\mathscr{H}^d[G_f,U(1)]$ can be calculated from the following short exact sequence:

$$0 \to \mathcal{H}^d[G_b,U_T(1)]/\,\Gamma \to \mathscr{H}^d[G_f,U_T(1)] \to B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2) \to 0. \quad (1)$$

In the above, $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ is a subgroup of $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, which is formed by elements n_{d-1} that satisfy $Sq^2(n_{d-1}) = 0$ in $\mathcal{H}^{d+1}[G_b,U(1)_T]$, where Sq^2 is the Steenrod square $Sq^2 w : \mathcal{H}^{d-1}(G_b,\mathbb{Z}_2) \to \mathcal{H}^{d+1}(G_b,\mathbb{Z}_2) \subset \mathcal{H}^{d+1}[G_b,U(1)_T]$. Also, Γ is a subgroup of $\mathcal{H}^d[G_b,U(1)_T]$ that is generated by $Sq^2(n_{d-2}) = f_d$, where $n_{d-2} \in \mathcal{H}^{d-2}(G_b,\mathbb{Z}_2)$ and f_d are viewed as elements of $\mathcal{H}^d[G_b,U_T(1)]$.

E. Boundary of fermionic SPT states

Topologically ordered states and SPT states have short-ranged correlations for any local operators. So, it is impossible to probe and distinguish the different topological states by bulk linear response measurements. However, for chiral topological order (such as integer and fractional quantum Hall states) [55,56] and free fermion SPT states (such as topological insulators/superconductors [35–40]), their boundary states are gapless if the symmetry is not broken. In this case, we can use boundary linear response measurements to probe those gapless excitations, which allow us to indirectly measure the bulk topological phases.

Thus, it is very natural to ask the following: Can we use the gapless boundary states to probe the SPT order in interacting fermionic SPT states? It turns out that, in the presence of interaction, the situation is much more complicated. In fact, the situation is already complicated even for noninteracting cases. It is well known that even trivial band insulator can have gapless boundary states. So it is incorrect to say that topological insulators/superconductors are characterized by gapless boundary states. The situation gets worse for interacting cases: regardless if the bulk is a trivial or nontrivial bosonic/fermionic SPT phase, the interacting boundary can be symmetry breaking, gapless, topological, etc. In fact, there can be infinite many different boundary phases for every fixed (3+1)D bulk phase. So, in order to use the boundary states to characterize the bulk SPT order, we must identify the common features among *all* those infinite many different boundary phases of the same bulk. Only those common features characterize the bulk SPT order. This is a highly nontrivial task and has not been studied carefully in literature.

In this paper, just like the bosonic case [45,46], we propose the boundary anomalous symmetry (i.e., the boundary non-onsite symmetry) as one of the common features that characterize the bulk SPT order. The standard global symmetry transformation (i.e., onsite symmetry transformation) in the bulk $\hat{U}(g), g \in G$ has the following tensor product decomposition:

$$\hat{U}(g) = \prod_i \hat{U}_i(g), \tag{2}$$

where $\hat{U}_i(g)$ acts on single site i. However, although the boundary of a SPT state has the same symmetry as the bulk, the symmetry transformation on the boundary cannot be onsite if the bulk SPT order is nontrivial:

$$\hat{U}_{\text{bndry}}(g) = w(\{g_i\},g) \prod_i \hat{U}_i(g). \tag{3}$$

Here, g_i labels the effective degrees of freedom on the boundary site i, and the $U(1)$ phase factor $w(\{g_i\},g)$ makes the boundary symmetry transformation $\hat{U}_{\text{bndry}}(g)$ non-onsite or anomalous. In fact, the $U(1)$ phase factor $w(\{g_i\},g)$ can be constructed from the supercocycle in $\mathscr{H}^d[G_f,U_T(1)]$ that describes the bulk SPT state (for details, see Appendix G 4).

Since all the effective boundary Hamiltonians satisfy

$$\hat{U}^\dagger_{\text{bndry}}(g) H_{\text{bndry}} \hat{U}_{\text{bndry}}(g) = H_{\text{bndry}}, \tag{4}$$

and all the possible boundary types are described by the ground states of the above boundary Hamiltonians, many low energy properties of boundary state are determined by the anomalous (i.e., non-onsite) symmetry $\hat{U}_{\text{bndry}}(g)$. For example, for (1+1)D boundary, an anomalous symmetry makes the boundary state gapless if the symmetry is not broken [45]. For (2+1)D boundary and beyond, an anomalous symmetry makes the symmetric boundary state gapless or topologically ordered.

The above results can also be understood from space-time path integral point of view. Our discrete fermionic topological nonlinear σ model, when defined on a space-time with boundary, can be viewed as a "nonlocal" boundary effective Lagrangian, which is a fermionic and discrete generalization of the bosonic continuous Wess-Zumino-Witten (WZW) term [57,58]. As a result of this "nonlocal" boundary effective Lagrangian, the action of symmetry transformation on the low energy boundary degrees of freedom must be non-onsite, and, we believe, the boundary excitations of a nontrivial SPT phase are gapless or topologically ordered if the symmetry is not broken.

F. Structure of the paper

In the rest of this paper, we will first compare the results in Table II for a few interacting and free fermion systems. This will give us some physical understanding of Table II. We then briefly review the topological bosonic nonlinear σ model on discretized space-time, which leads to the group cohomology theory for the bosonic SPT states. We start our development of group supercohomoloy theory for fermionic SPT phases by carefully defining fermionic path integral for the fermionic nonlinear σ model on discrete space-time. Next we discuss the conditions under which the fermionic path integral becomes a fixed-point theory under the coarse-graining transformation of the space-time complex. Such a fixed-point theory is a fermionic topological nonlinear σ model. The fixed-point path integral describes a fermionic SPT phase. We then construct the ground state wave function from the fixed-point path integral, as well as the exact solvable Hamiltonian that realizes the SPT states. In the Appendices, we develop a group supercohomology theory, and calculate the Table II from the group supercohomology theory.

II. PHYSICAL PICTURES OF SOME GENERIC RESULTS

Before describing how to obtain the generic results (1) and Table II, in this section, we will compare some of our results with known results for 1D interacting systems and 2D/3D noninteracting systems. The comparison will give us a physical understanding for some of the interacting fermionic SPT phases.

A. Fermion systems with symmetry $G_f = Z_2^T \times Z_2^f$

First, from Table II, we find that interacting fermion systems with $T^2 = 1$ time-reversal symmetry (or the full symmetry group $G_f = Z_2^T \times Z_2^f$) can have three nontrivial fermionic SPT phases in $d_{sp} = 1$ spatial dimension and one nontrivial fermionic SPT phase in $d_{sp} = 3$ spatial dimensions. We would like to compare such results with those for free fermion systems.

1. 0D case

For 0D free or interacting fermion systems with $T^2 = 1$ time-reversal symmetry Z_2^T, the possible symmetric gapped phases are the 1D representations of $Z_2^T \times Z_2^f$. Since the time-reversal transformation T is antiunitary, Z_2^T has only one trivial 1D representation. Z_2^f has two 1D representations. Thus, 0D fermion systems with $T^2 = 1$ time-reversal symmetry are classified by $\mathbb{Z}_2$ corresponding to even fermion states and odd fermion states.

2. 1D case

For 1D *free* fermion systems with $T^2 = 1$ time-reversal symmetry Z_2^T, the possible gapped phases are classified by $\mathbb{Z}$ [49,50]. For the phase labeled by $n \in \mathbb{Z}$, it has n Majorana zero modes at one end of the chain [59]. Those boundary states form a representation of n Majorana fermion operators $\eta_1, \dots, \eta_n$ which all transform in the same way under the time-reversal transformation T: $\eta_a \to \eta_a$ for all a or $\eta_a \to -\eta_a$ for all a. (If, say η_1 and η_2 transform differently under T: $\eta_1 \to \eta_1$ and $\eta_2 \to -\eta_2$, $i\eta_1\eta_2$ will be invariant under T and such a term will gap out the η_1 and η_2 modes.) Some of those 1D gapped states have intrinsic topological orders. Only those labeled by even integers (which have even numbers of Majorana boundary zero modes) become the trivial phase after we break the time-reversal symmetry Z_2^T. Thus, the free fermion SPT phases are labeled by even integers $2\mathbb{Z}$.

For interacting 1D systems with $T^2 = 1$ time-reversal symmetry, there are eight possible gapped phases that do not break the time-reversal symmetry [44,51–53]. Four of them have intrinsic topological orders (which break the fermion-number-parity symmetry in the bosonized model) [44] and the other four are fermionic SPT phases given in Table II (three of them are nontrivial fermionic SPT phases and the fourth one is the trivial phase).

3. 3D case

For 3D fermion systems with $T^2 = 1$ time-reversal symmetry, there is no nontrivial gapped phase for noninteracting fermions [49,50] but, in contrast, there is (at least) one nontrivial SPT phase for strongly interacting fermions. Such a nontrivial 3D fermionic SPT phase cannot even be viewed as a nontrivial bosonic SPT state formed by bounded fermion pairs. To see this point, we note that bounded fermion pairs behave like a boson system with Z_2^T time-reversal symmetry, which has one nontrivial bosonic SPT state (see Table I). Such a state cannot be smoothly deformed into bosonic product state without breaking the Z_2^T symmetry within the space of *many-boson* states. But, it can be smoothly deformed into bosonic product state without breaking the Z_2^T symmetry within the space of *many-fermion* states. Therefore, the discovered nontrivial fermionic SPT phase with $T^2 = 1$ time-reversal symmetry is totally new.

B. Fermion systems with symmetry $G_f = Z_2 \times Z_2^f$

We also find that fermion systems with symmetry group $G_b = Z_2$ (or the full symmetry group $G_f = Z_2 \times Z_2^f$) can have one nontrivial fermionic SPT phase in $d_{sp} = 1$ spatial dimension and three nontrivial fermionic SPT phases in $d_{sp} = 2$ spatial dimensions. (One of the nontrivial 3D fermionic SPT phases is actually a bosonic SPT phase while the other two are intrinsically fermionic.)

1. Free fermion systems

To compare the above result with the known free fermion result, let us consider free fermion systems with symmetry group $G_b = Z_2$ (or the full symmetry group $G_f = Z_2 \times Z_2^f$), which contain two kinds of fermions: one carries the Z_2 charge 0 and the other carries Z_2 charge 1. For such free fermion systems, their gapped phases are classified by [48]

$$\begin{array}{lcccccccc} d_{sp}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7, \\ \text{gapped phases:} & \mathbb{Z}_2^2 & \mathbb{Z}_2^2 & \mathbb{Z}^2 & 0 & 0 & 0 & \mathbb{Z}^2 & 0. \end{array} \tag{5}$$

The four $d_{sp} = 0$ phases correspond to the ground state with even or odd Z_2-charge-0 fermions and even or odd Z_2-charge-1 fermions. The four $d_{sp} = 1$ phases correspond to the phases where the Z_2-charge-0 fermions are in the trivial or nontrivial phases of Majorana chain [59] and the Z_2-charge-1 fermions are in the trivial or nontrivial phases of Majorana chain. A $d_{sp} = 2$ phase labeled by two integers $(m,n) \in \mathbb{Z}^2$ corresponds to the phase where the Z_2-charge-0 fermions form m $(p_x + ip_y)$ states with m right-moving Majorana chiral modes and the Z_2-charge-1 fermions form n $(p_x + ip_y)$ states with n right-moving Majorana chiral modes. [If m and/or n are negative, the fermions then form the corresponding number of $(p_x - ip_y)$ states with the corresponding number of left-moving Majorana chiral modes.]

Some of the above noninteracting gapped phases have intrinsic fermionic topological orders. (Those phases are stable and have intrinsic fermionic topological orders even after we turn on interactions.) So, only a subset of them are noninteracting fermionic SPT phases:

$$\begin{array}{lcccccccc} d_{sp}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7, \\ \text{SPT phases:} & \mathbb{Z}_2^2 & \mathbb{Z}_2 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0. \end{array} \tag{6}$$

The four $d_{sp} = 0$ phases correspond to the ground states with even or odd numbers of fermions and 0 or 1 Z_2 charges. The two $d_{sp} = 1$ phases correspond to the phases where the Z_2-charge-0 fermions and the Z_2-charge-1 fermions are both in the trivial or nontrivial phases of Majorana chain. The $d_{sp} = 2$ phase labeled by one integer $n \in \mathbb{Z}$ corresponds to the phase where the Z_2-charge-0 fermions have n right-moving Majorana chiral modes and the Z_2-charge-1 fermions have n left-moving Majorana chiral modes. We see that in 0D and 1D, the noninteracting fermionic SPT phases are the same as the interacting fermionic SPT phases.

2. 2D case

However, in two dimensions, the noninteracting fermionic SPT phases are quite different from the interacting ones. In this paper, we are able to construct three nontrivial interacting fermionic SPT phases. Despite very different phase diagram, it appears that the above three nontrivial interacting fermionic SPT phases in 2D can be realized by free fermion systems.

In fact, it appears that there are seven nontrivial interacting 2D fermionic SPT phases protected by Z_2 symmetry [60]. All the seven nontrivial SPT phases can be represented by free fermion SPT phases. This suggests that our current construction is incomplete since we only obtain four fermionic SPT phases (including the trivial one) with the $G_f = Z_2 \times Z_2^f$ symmetry. One possible reason of the incompleteness may be due to our limiting requirement that fermions only form 1D representations of G_f.

One way to understand why there can only be seven nontrivial interacting 2D fermionic SPT phases protected by Z_2 symmetry is using the idea of duality between intrinsic topological orders and SPT orders discovered recently [61]. The key observation in such a duality map is that for any SPT orders associated with a (discrete) global symmetry G, we can always promote the global symmetry to a local (gauge) symmetry. For different SPT phases, the corresponding promoted (discrete) gauge models describe different intrinsic topological orders. For fermionic SPT phases protected by Z_2 symmetry, we can let Z_2-charge-1 fermion couple to a Z_2 gauge field. In this way, different intrinsic topological ordered phases can be characterized by different statistics of the Z_2 flux. According to Kitaev's classification [62] for different types of vortices in superconductors,[1] we know that there are seven nontrivial cases. Thus, we see that although in free 2D fermionic systems SPT phases protected by Z_2 symmetry are classified by an integer (Chern number), the interactions dramatically change the classifications.

Following Kitaev's idea [62], we can have a very simple way to understand the seven nontrivial types of vortices by

[1]In a superconductor, the $U(1)$ symmetry is broken down to Z_2 symmetry, hence the vortex of a superconductor can be regarded as Z_2 flux.

counting the number of Majorana modes in the vertex core. In the corresponding free fermion model, the number of Majorana modes n corresponds to the Chern number of the Z_2-charge-1 free fermion. The seven nontrivial SPT phases are described by $n = 1,2,\ldots,7$.

We see that an interacting 2D fermionic SPT state with Z_2 symmetry ($G_f = Z_2 \times Z_2^f$) is characterized by having n right-moving Majorana chiral modes for the Z_2-charge-0 fermions and n left-moving Majorana chiral modes for the Z_2-charge-1 fermions, where $n \in \mathbb{Z}_8$. Such kinds of edge states can be realized by free fermions. Thus the interacting 2D fermionic SPT states with Z_2 symmetry can be realized by free fermions. When $n =$ even, the 2D fermionic SPT states with Z_2 symmetry can be realized by 2D topological insulators with fermion-number conservation, S^z spin rotation symmetry, and time-reversal symmetry. The Z_2 symmetry transformation corresponds to $\pi/2$ charge rotations and π spin rotation

$$U_{Z_2} = e^{i\pi N_F/2} e^{i\pi S^z}, \tag{7}$$

where N_F is the total fermion number and S^z is the total S^z spin. Such a state is stable even if we break the fermion-number conservation, S^z spin rotation symmetry, and time-reversal symmetry, as long as we keep the above Z_2 symmetry.

Now, let us try to understand why four of the nontrivial fermionic SPT phases protected by the Z_2 symmetry require fermions to form high dimensional representations of $G_f = Z_2 \times Z_2^f$. It is easy to see that when $n =$ even, the free fermion models are described by $n/2$ Z_2-charge-1 complex fermions per site and these fermions form 1D representations of $G_f = Z_2 \times Z_2^f$. However, the situations are more complicated when $n =$ odd. For example, when $n = 1$, the free fermion Hamiltonian describes one Z_2-charge-0 Majorana fermion and one Z_2-charge-1 Majorana fermion per site, labeled as $\gamma_{i,\uparrow}$ and $\gamma_{i,\downarrow}$. Under the Z_2 action, $\gamma_{i,\uparrow}$ does not change while $\gamma_{i,\downarrow}$ changes to $-\gamma_{i,\downarrow}$. Thus the symmetry operation U_{Z_2} can be constructed as $U_{Z_2} = i^{(N-1)N/2} \prod_{i=1}^{N} \gamma_{i,\uparrow}$, where N is the number of sites. It is easy to check that $U_{Z_2}\gamma_{i,\uparrow}U_{Z_2}^\dagger = \gamma_{i,\uparrow}$, $U_{Z_2}\gamma_{i,\downarrow}U_{Z_2}^\dagger = -\gamma_{i,\downarrow}$ and $U_{Z_2}^2 = 1$. However, the symmetry operator U_{Z_2} can be regarded as the Z_2 symmetry transformation only when $N =$ even, in that case U_{Z_2} contains an even number of fermion operators. But in our construction, the Z_2 symmetry can be realized as bosonic unitary transformation regardless $N =$ even or odd. Thus, we can only construct four fermionic SPT phases that are labeled by even n.

Another example requiring fermions to carry high dimensional representations is the well known time-reversal symmetry with $T^2 = P_f$. We believe that the principle/framework developed in this paper should be applicable for all these interesting cases and the results will be presented in our future work.

III. PATH INTEGRAL APPROACH TO BOSONIC SPT PHASES

In this paper, we are going develop a group supercohomology theory, trying to describe the SPT phase for interacting fermions. Our approach is motivated by the group cohomology theory for bosonic SPT phases, based on the path integral approach. In this section, we will briefly review such path integral approach for the group cohomology description of bosonic SPT phases. Those who are familar with bosonic SPT theory can go directly to the next section.

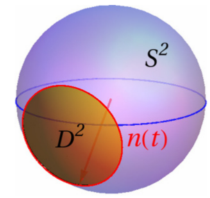


FIG. 2. (Color online) If we extend $\boldsymbol{n}(t)$ that traces out a loop to $\boldsymbol{n}(t,\xi)$ that covers the shaded disk, then the WZW term $\int_{D^2} dt\, d\xi\ \boldsymbol{n}(t,\xi)\cdot[\partial_t \boldsymbol{n}(t,\xi)\times\partial_\xi \boldsymbol{n}(t,\xi)]$ corresponds to the area of the disk.

Here we are going to use the Haldane phase of spin-1 chain as an example. It has been pointed out that the Haldane phase (a nontrivial 1D SPT phase) is described by a 2π-quantized topological term in continuous nonlinear σ model [63]. However, such kinds of 2π-quantized topological terms cannot describe more general 1D SPT phases. We argue that to describe SPT phases correctly, we must generalize the 2π-quantized topological terms to discrete space-time. The generalized 2π-quantized topological terms turn out to be nothing but the cocycles of group cohomology (see Refs. [45,46] for more details.)

A. Path integral approach to a single spin

Before considering a spin-1 chain, let us first consider a (0+1)D nonlinear σ model that describes a single spin, whose imaginary-time action is given by

$$\oint dt \frac{[\partial_t \boldsymbol{n}(t)]^2}{2g} + is\int_{D^2} dt\, d\xi\ \boldsymbol{n}(t,\xi)\cdot[\partial_t \boldsymbol{n}(t,\xi)\times\partial_\xi \boldsymbol{n}(t,\xi)], \tag{8}$$

where $\boldsymbol{n}(t)$ is an unit $3d$ vector and we have assumed that the time direction forms a circle. The second term is the Wess-Zumino-Witten (WZW) term [57,58]. We note that the WZW term cannot be calculated from the field $\boldsymbol{n}(t)$ on the time circle. We have to extend $\boldsymbol{n}(t)$ to a disk D^2 bounded by the time circle: $\boldsymbol{n}(t)\to\boldsymbol{n}(t,\xi)$ (see Fig. 2). Then the WZW term can be calculated from $\boldsymbol{n}(t,\xi)$. When $2s$ is an integer, WZW terms from different extensions only differ by a multiple of $2i\pi$. So e^{-S} is determined by $\boldsymbol{n}(t)$ and is independent of how we extend $\boldsymbol{n}(t)$ to the disk D^2.

The ground states of the above nonlinear σ model have $(2s+1)$-fold degeneracy, which form the spin-s representation of $SO(3)$. The energy gap above the ground state approaches to infinity as $g\to\infty$. Thus a pure WZW term describes a pure spin-s spin.

B. Path integral approach to a spin-1 chain

To obtain the action for the $SO(3)$ symmetric antiferromagnetic spin-1 chain, we can assume that the spins $\boldsymbol{S}_i$ are described by a smooth unit vector field $\boldsymbol{n}(x,t)$: $\boldsymbol{S}_i = (-)^i \boldsymbol{n}(ia,t)$ [see Fig. 3(b)]. Putting the above single-spin action for different spins together, we obtain the following (1+1)D

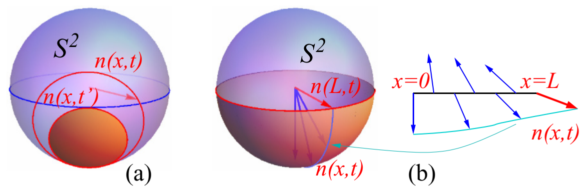


FIG. 3. (Color online) (a) The topological term W describes the number of times that $\boldsymbol{n}(x,t)$ wraps around the sphere (as we change t). (b) On an open chain $x \in [0,L]$, the topological term W in the (1+1)D bulk becomes the WZW term for the end spin $\boldsymbol{n}_L(t) = \boldsymbol{n}(L,t)$ (where the end spin at $x = 0$ is held fixed).

nonlinear σ model [64]:

$$S = \int dx\, dt \frac{1}{2g}[\partial \boldsymbol{n}(x,t)]^2 + i\theta W, \quad \theta = 2\pi \tag{9}$$

where $W = (4\pi)^{-1} \int dt\, dx\, \boldsymbol{n}(t,x) \cdot [\partial_t \boldsymbol{n}(t,x) \times \partial_x \boldsymbol{n}(t,x)]$ and $i\theta W$ is the topological term [64]. If the space-time manifold has no boundary, then $e^{-i\theta W} = 1$ when $\theta = 0 \bmod 2\pi$. We will call such a topological term a 2π-quantized topological term. The above nonlinear σ model describes a gapped phase with short-range correlation and the $SO(3)$ symmetry, which is the Haldane phase [33]. In the low energy limit, g flows to infinity and the fixed-point action contains only the 2π-quantized topological term. Such a nonlinear σ model will be called topological nonlinear σ model.

It appears that the 2π-quantized topological term has no contribution to the path integral and can be dropped. In fact, the 2π-quantized topological term has physical effects and cannot be dropped. On an open chain, the 2π-quantized topological term $2\pi i W$ becomes a WZW term for the boundary spin $\boldsymbol{n}_L(t) \equiv \boldsymbol{n}(x = L,t)$ (see Fig. 3) [63]. The motion of $\boldsymbol{n}_L$ is described by Eq. (8) with $s = \frac{1}{2}$. So, the Haldane phase of spin-1 chain has a spin-$\frac{1}{2}$ boundary spin at each chain end [63,65,66]!

We see that the Haldane phase is described by a fixed-point action which is a topological nonlinear σ model containing only the 2π-quantized topological term. *The nontrivialness of the Haldane phase is encoded in the nontrivially quantized topological term* [50,63].

C. Topological term on discrete space-time

From the above example, one might guess that various SPT phases can be described by various topological nonlinear σ models, and thus by various 2π-quantized topological terms. But, such a guess is not correct.

This is because the fixed-point action (the topological nonlinear σ model) describes a short-range-correlated state. Since the renormalized cutoff length scale of the fixed-point action is always larger than the correlation length, the field $\boldsymbol{n}(x,t)$ fluctuates strongly even at the cutoff length scale. Thus, the fixed-point action has no continuum limit, and must be defined on discrete space-time. On the other hand, in our fixed-point action, $\boldsymbol{n}(x,t)$ is assumed to be a continuous field in space-time. The very existence of the continuum 2π-quantized topological term depends on the nontrivial mapping classes from the *continuous* space-time manifold T^2 to the *continuous* target space S^2. It is not self-consistent to use such a continuum topological term to describe the fixed-point action for the Haldane phase.

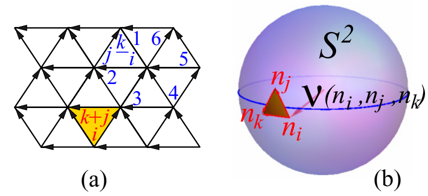


FIG. 4. (Color online) (a) A branched triangularization of space-time. Each edge has an orientation and the orientations on the three edges of any triangle do not form a loop. The orientations on the edges give rise to a natural order of the three vertices of a triangle (i,j,k) where the first vertex i of a triangle has two outgoing edges on the triangle and the last vertex k of a triangle has two incoming edges on the triangle. $s(i,j,k) = \pm 1$ depending on the orientation of $i \to j \to k$ to be clockwise or anticlockwise. (b) The phase factor $\nu(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k)$ depends on the image of a space-time triangle on the sphere S^2.

As a result, the continuum 2π-quantized topological terms fail to properly describe bosonic SPT phases. For example, different possible continuum 2π-quantized topological terms in Eq. (9) are labeled by integers, while the integer spin chain has only two gapped phases protected by spin rotation symmetry: all even-integer topological terms give rise to the trivial phase and all odd-integer topological terms give rise to the Haldane phase. Also, nontrivial SPT phases may exist even when there are no continuum 2π-quantized topological terms (such as when the symmetry G is discrete).

However, the general idea of using fixed-point actions to describe SPT phases is still correct. But, to use 2π-quantized topological terms to describe bosonic SPT phases, we need to generalize them to discrete space-time. In the following, we will show that this indeed can be done, using the (1+1)D model (9) as an example.

A discrete (1+1)D space-time is given by a branched triangularization [46,67] (see Fig. 4). Since $S = \int dx\, dt L$, on triangularized space-time, we can rewrite

$$e^{-S} = \prod \nu^{s(i,j,k)}(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k),$$
$$\nu^{s(i,j,k)}(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k) = e^{-\int_\triangle dx\, dt\, L} \in U(1), \tag{10}$$

where $\int_\triangle dx\, dt\, L$ is the action on a single triangle. We see that, on discrete space-time, the action and the path integral are described by a 3-variable function $\nu(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k)$, which is called action amplitude. The $SO(3)$ symmetry requires that

$$\nu(g\boldsymbol{n}_i,g\boldsymbol{n}_j,g\boldsymbol{n}_k) = \nu(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k), \quad g \in SO(3). \tag{11}$$

In order to use the action amplitude $\nu^{s(i,j,k)}(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k)$ to describe a 2π-quantized topological term, we must have $\prod \nu^{s(i,j,k)}(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k) = 1$ on any sphere. This can be satisfied iff $\prod \nu^{s(i,j,k)}(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k) = 1$ on a tetrahedron: the simplest discrete sphere [see Fig. 5(a)]

$$\frac{\nu(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)\nu(\boldsymbol{n}_0,\boldsymbol{n}_1,\boldsymbol{n}_3)}{\nu(\boldsymbol{n}_0,\boldsymbol{n}_2,\boldsymbol{n}_3)\nu(\boldsymbol{n}_0,\boldsymbol{n}_1,\boldsymbol{n}_2)} = 1. \tag{12}$$

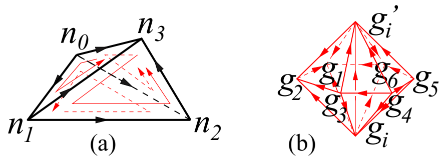


FIG. 5. (Color online) A tetrahedron: the simplest discrete sphere. $\prod \nu^{s(i,j,k)}(\boldsymbol{n}_i,\boldsymbol{n}_j,\boldsymbol{n}_k) = 1$ on the tetrahedron becomes Eq. (12). Note that $s(1,2,3) = s(0,1,3) = 1$ and $s(0,2,3) = s(0,1,2) = -1$. (b) The total action amplitude of the topological nonlinear σ model on the complex gives rise to the phase factor in Eq. (17).

(Another way to define topological term on discretized space-time can be found in Ref. [69].)

A $\nu(\boldsymbol{n}_0,\boldsymbol{n}_1,\boldsymbol{n}_2)$ that satisfies Eqs. (11) and (12) is called a 2-cocycle. If $\nu(\boldsymbol{n}_0,\boldsymbol{n}_1,\boldsymbol{n}_2)$ is a 2-cocycle, then

$$\nu'(\boldsymbol{n}_0,\boldsymbol{n}_1,\boldsymbol{n}_2) = \nu(\boldsymbol{n}_0,\boldsymbol{n}_1,\boldsymbol{n}_2)\frac{\mu(\boldsymbol{n}_1,\boldsymbol{n}_2)\mu(\boldsymbol{n}_0,\boldsymbol{n}_1)}{\mu(\boldsymbol{n}_0,\boldsymbol{n}_2)} \tag{13}$$

is also a 2-cocycle, for any $\mu(\boldsymbol{n}_0,\boldsymbol{n}_1)$ satisfying $\mu(g\boldsymbol{n}_0,g\boldsymbol{n}_1) = \mu(\boldsymbol{n}_0,\boldsymbol{n}_1)$, $g \in SO(3)$. Since $\nu(\boldsymbol{n}_0,\boldsymbol{n}_2,\boldsymbol{n}_3)$ and $\nu'(\boldsymbol{n}_0,\boldsymbol{n}_2,\boldsymbol{n}_3)$ can continuously deform into each other, they correspond to the same kind of 2π-quantized topological term. So, we say that $\nu(\boldsymbol{n}_0,\boldsymbol{n}_2,\boldsymbol{n}_3)$ and $\nu'(\boldsymbol{n}_0,\boldsymbol{n}_2,\boldsymbol{n}_3)$ are equivalent. The equivalent classes of the 2-cocycles $\nu(\boldsymbol{n}_0,\boldsymbol{n}_2,\boldsymbol{n}_3)$ give us $\mathcal{H}^2[S^2,U(1)]$: the 2-cohomology group of sphere S^2 with $U(1)$ coefficient. $\mathcal{H}^2[S^2,U(1)]$ classifies the 2π-quantized topological term in *discrete* space-time and with S^2 as the target space.

D. Maximum symmetric space and group cohomology classes

Does $\mathcal{H}^2[S^2,U(1)]$ classify the SPT phases with $SO(3)$ symmetry? The answer is no. We know that S^2 is just one of the symmetric spaces of $SO(3)$. To classify the SPT phases, we need to replace the target space S^2 by the maximal symmetric space, which is the group itself $SO(3)$ (see Ref. [46] for more discussions). So, we need to consider discrete nonlinear σ model described by $\nu(g_i,g_j,g_k)$, $g_i,g_j,g_k \in SO(3)$. Now, the 2-cocycle conditions become

$$\nu(gg_i,gg_j,gg_k) = \nu(g_i,g_j,g_k) \in U(1),$$
$$\frac{\nu(g_1,g_2,g_3)\nu(g_0,g_1,g_3)}{\nu(g_0,g_2,g_3)\nu(g_0,g_1,g_2)} = 1, \tag{14}$$

which defines a "group cohomology" $\mathcal{H}^2[SO(3),U(1)]$. It classifies the 2π-quantized topological term for the maximal symmetric space. It also classifies the SPT phases with $SO(3)$ symmetry in (1+1)D.

The above 2-cocycle condition can be generalized to any group G, including discrete groups! We conclude that $\mathcal{H}^2[G,U(1)]$ classifies the SPT phases with onsite symmetry G in (1+1)D. The above discussion can also be generalized to any dimensions by replacing the 2-cocycle condition on $\nu(g_0,g_1,g_2)$ with d-cocycle condition on $\nu(g_0,g_1,\dots,g_d)$. For example, the functions $\nu(g_0,\dots,g_3)$, $g_i \in G$, satisfying the following conditions

$$\nu(gg_0,gg_1,gg_2,gg_3) = \nu(g_0,g_1,g_2,g_3) \in U(1),$$
$$\frac{\nu(g_1,g_2,g_3,g_4)\nu(g_0,g_1,g_3,g_4)\nu(g_0,g_1,g_2,g_3)}{\nu(g_0,g_2,g_3,g_4)\nu(g_0,g_1,g_2,g_4)} = 1, \tag{15}$$

are 3-cocycles which form $\mathcal{H}^3[G,U(1)]$. Using $\nu(g_0,\dots,g_{d_{sp}+1}) \in \mathcal{H}^{d_{sp}+1}[G,U(1)]$ as the action amplitude on a simplex in $(d_{sp}+1)$D space-time, we can construct the path integral of the topological nonlinear σ model, which describes the corresponding SPT phase. This way, we show that the SPT phases with symmetry G in d spatial dimensions are described by the elements in $\mathcal{H}^{d_{sp}+1}[G,U(1)]$.

In conclusion, a quantized topological term in the path integral of a $(d_{sp}+1)$D system can be written in terms of the $(d_{sp}+1)$-cocycles by (1) discretizing the $(d_{sp}+1)$D space-time with branched triangularization (a branching structure [46,67] will induce a local order of vertices on each simplex); (2) putting group element labeled degrees of freedom onto the vertices; (3) assigning action amplitude to each simplex with the corresponding cocycle. The path integral then takes the form

$$Z = |G|^{-N_v} \sum_{\{g_i\}} \prod_{\{ij\dots k\}} \nu_{d_{sp}+1}^{s_{ij\dots k}}(g_i,g_j,\dots,g_k), \tag{16}$$

where $s_{ij\dots k} = \pm 1$ depending on the orientation of the simplex $ij\dots k$. Similar to the $(1+1)$D case, it can be shown that the path integral is symmetric under symmetries in group G, the action amplitude $e^{-S(\{g_i\})} = \prod_{\{ij\dots k\}} \nu_{d_{sp}+1}^{s_{ij\dots k}}(g_i,g_j,\dots,g_k)$ is in a fixed-point form and is quantized to 1 on a closed manifold.

On a space with boundary, the 2π-quantized topological term obtained from a nontrivial cocycle $\nu(g_0,\dots,g_{d_{sp}+1})$ gives rise to a discretized analog of the WZW term in the low energy effective theory for boundary excitations. We believe that such a discretized WZW term will make the boundary excitations gapless if the symmetry is not broken. In $(2+1)$D, we can indeed prove rigorously that a nontrivial SPT state must have gapless edge modes if the symmetry is not broken [45]. In addition, it has been further pointed out that by "gauging" the global symmetry [61], each SPT phase can be identified by the braiding statistics of the corresponding gauge flux, which is known to be classified by $\mathcal{H}^3[G,U(1)]$ in $(2+1)$D [68].

E. Ground state wave function and Hamiltonian for a bosonic SPT phase

From each element in $\mathcal{H}^{d_{sp}+1}[G,U(1)]$ we can also construct the d_{sp}-dimensional ground state wave function for the corresponding SPT phase. In 2D, we can start with a triangle lattice model where the physical states on site i are given by $|g_i\rangle$, $g_i \in G$ [see Fig. 4(a)]. The ground state wave function can be obtained by viewing the 2D lattice (which is a torus) as the surface of a solid torus. The evaluation of the path integral of the topological nonlinear σ model {which is given by $\nu_3(g_0,g_1,g_2,g_3)$, an element in $\mathcal{H}^3[G,U(1)]$} on the solid torus gives rise to the following ground state wave function: $\Phi(\{g_i\}) = \prod_\triangle \nu_3(1,g_i,g_j,g_k)\prod_\bigtriangledown \nu_3^{-1}(1,g_i,g_j,g_k)$, where $\prod_\triangle$ and $\prod_\bigtriangledown$ multiply over all up and down triangles, and the order of ijk is clockwise for up triangles and anticlockwise for down triangles [see Fig. 4(a)]. To construct exactly solvable Hamiltonian H that realizes the above wave function as the ground state, we start with an exactly solvable Hamiltonian $H_0 = -\sum_i |\phi_i\rangle\langle\phi_i|$, $|\phi_i\rangle = \sum_{g_i\in G}|g_i\rangle$, whose ground state is $\Phi_0(\{g_i\}) = 1$. Then, using the local unitary transformation $U = \prod_\triangle \nu_3(1,g_i,g_j,g_k)\prod_\bigtriangledown \nu_3^{-1}(1,g_i,g_j,g_k)$, we find that $\Phi = U\Phi_0$ and $H = \sum_i H_i$, where $H_i = U|\phi_i\rangle\langle\phi_i|U^\dagger$. H_i acts on a seven-spin cluster labeled by i, 1–6 in blue in Fig. 4(a) [see Fig. 5(b)]:

$$H_i|g_i,g_1g_2g_3g_4g_5g_6\rangle = \sum_{g_i'} |g_i',g_1g_2g_3g_4g_5g_6\rangle \times \frac{\nu_3(g_4,g_5,g_i,g_i')\nu_3(g_5,g_i,g_i',g_6)\nu_3(g_i,g_i',g_6,g_1)}{\nu_3(g_i,g_i',g_2,g_1)\nu_3(g_3,g_i,g_i',g_2)\nu_3(g_4,g_3,g_i,g_i')}. \quad (17)$$

We see that H has a short-ranged interaction and has the symmetry G: $|\{g_i\}\rangle \to |\{gg_i\}\rangle$, $g \in G$ [45].

IV. GRASSMANN TENSOR NETWORK AND FERMIONIC PATH INTEGRAL

After having some physical understanding of the fermionic SPT phases protected by $Z_2^T \times Z_2^f$ or $Z_2 \times Z_2^f$ symmetries and after some understanding of the path integral approach to the bosonic SPT phases, in the following, we will discuss a generic construction that allows us to obtain fermionic SPT phases protected by more general symmetries in any dimensions.

As discussed above, one way to obtain a systematic construction of bosonic SPT phases is through a systematic construction of *topological* bosonic path integral [46] using a tensor network representation of the path integral [7,70]. Here, we will use a similar approach to obtain a systematic construction of fermionic SPT phases through *topological* fermionic path integral using a Grassmann tensor network [27,71] representation of the fermionic path integral. The Grassmann tensor network and the associated fermion path integral are defined on a discretized space-time. So first, let us describe the structure of a discretized space-time.

A. Discretized space-time

A d-dimensional discretized space-time is a d-dimensional complex Σ_d formed by many d-dimensional simplexes $\Sigma_{[i_0...i_d]}$. We will use $i,j,\dots$ to label the vertices of complex Σ_d. $\Sigma_{[i_0...i_d]}$ is the simplex with vertices $i_0,\dots,i_d$.

However, in order to define Grassmann tensor network (and the associated fermion path integral) on the discretized space-time Σ_d, the space-time complex must have a so-called branching structure [46,67]. A branching structure is a choice of orientation of each edge in the d-dimensional complex so that there is no oriented loop on any triangle (see Fig. 6).

The branching structure induces a *local order* of the vertices on each d-simplex. The first vertex of a d-simplex is the vertex with d outgoing edges, and the second vertex is the vertex with $d-1$ outgoing edges and 1 incoming edge, etc. So, the simplex in Fig. 6(a) has the following vertex ordering: 0,1,2,3.

The branching structure also gives the simplex (and its subsimplexes) an orientation. Figure 6 illustrates two 3-simplexes with opposite orientations. The red arrows indicate the orientations of the 2-simplexes which are the subsimplexes of the 3-simplexes. The black arrows on the edges indicate the orientations of the 1-simplexes.

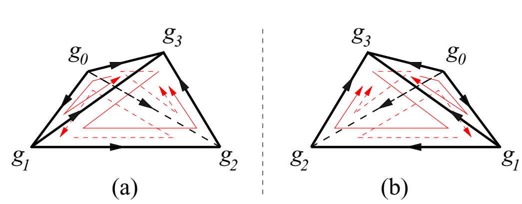


FIG. 6. (Color online) Two branched simplexes with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

B. Physical variables on discretized space-time

To define a path integral on a discretized space-time (i.e., on a complex Σ_d with a branching structure), we associate each vertex i (a 0-simplex) in Σ_d with a variable g_i. So, g_i is a local physical dynamical variable of our system. The allowed values of g_i form a space which is denoted as G_b. At the moment, we treat G_b as an arbitrary space. Later, we will assume that G_b is the space of a group.

We also associate each $(d-1)$-simplex $(i_0\dots\hat{i}_i\dots i_d)$ of a d-simplex $[i_0\dots i_d]$ with a Grassmann number. Here, the $(d-1)$-simplex $(i_0\dots\hat{i}_i\dots i_d)$ is a subsimplex of the d-simplex $[i_0\dots i_d]$ that does not contain the vertex i_i. Relative to the orientation of the d-simplex $[i_0\dots i_d]$, the $(d-1)$-simplex $(i_0\dots\hat{i}_i\dots i_d)$ can have a "+" or "−" orientation (see Fig. 6). We associate $(d-1)$-simplex $(i_0\dots\hat{i}_i\dots i_{d-1})$ with a Grassmann number $\theta_{(i_0\dots\hat{i}_i\dots i_d)}$ if the $(d-1)$-simplex has a "+" orientation and with a Grassmann number $\bar{\theta}_{(i_0\dots\hat{i}_i\dots i_d)}$ if the $(d-1)$-simplex has a "−" orientation. However, the Grassmann number $\theta_{(i\dots k)}$ (or $\bar{\theta}_{(i\dots k)}$) may or may not present on the $(d-1)$-simplex $(i\dots k)$. So, on each $(d-1)$-simplex $(i\dots k)$ we also have a local physical dynamical variable $n_{i\dots k}=0,1$ or $\bar{n}_{i\dots k}=0,1$, indicating whether $\theta_{(i\dots k)}$ or $\bar{\theta}_{(i\dots k)}$ is present or not. So, each $(d-1)$-simplex $(i\dots k)$ is really associated with a dynamical Grassmann variable $\theta_{(i\dots k)}^{n_{i\dots k}}$ or $\bar{\theta}_{(i\dots k)}^{\bar{n}_{i\dots k}}$.

We see that each $(d-1)$-simplex $(i_0\dots i_{d-1})$ on the surface of the d-complex Σ_d is associated with two types of Grassmann variables $\theta_{(i_0\dots i_{d-1})}^{n_{i_0\dots i_{d-1}}}$ and $\bar{\theta}_{(i_0\dots i_{d-1})}^{\bar{n}_{i_0\dots i_{d-1}}}$, each belongs to the d-simplex on each side of the $(d-1)$-simplex $(i_0\dots i_{d-1})$ (see Fig. 7). We also see that each d-simplex has its own set of Grassmann variables that do not overlap with the Grassmann variables of other simplexes.

C. A constraint on physical variables

In general, the physical dynamical variables g_i, $n_{i\dots k}$, and $\bar{n}_{i\dots k}$ are independent. We can formulate fermionic topological theory and obtain fermionic topological phases by treating g_i,

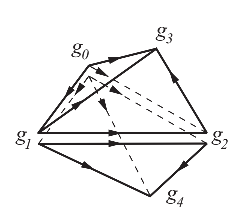


FIG. 7. The 2-simplex (012) is associated with $\bar{\theta}_{(012)}$ which belongs to the top 3-simplex [0123]. The 2-simplex (012) is also associated with $\theta_{(012)}$ which belongs to the bottom 3-simplex [0124]. A rank-4 tensor $\mathcal{V}_3$ on the top tetrahedron is given by $\mathcal{V}_3^+(g_0,g_1,g_2,g_3)$ and on the lower tetrahedron by $\mathcal{V}_3^-(g_0,g_1,g_2,g_4)$.

$n_{i...k}$, and $\bar{n}_{i...k}$ as independent variables. This is essentially the approach used in Ref. [27] in (2+1)D, and we have obtained a system of nonlinear algebraic equations, whose solutions describe various 2D fermionic topological phases. However, the nonlinear algebraic equations are very hard to solve, despite they only describe 2D fermionic topological orders. As a result, we have obtained very few fermionic solutions.

In this paper, we are going to study a simpler and less general problem by putting a constraint between the physical dynamical variables g_i and $n_{i...k}$ ($\bar{n}_{i...k}$):

$$n_{i...k} = n(g_i,\ldots,g_k), \quad \bar{n}_{i...k} = n(g_i,\ldots,g_k). \tag{18}$$

That is, when $n_{i...k} \neq n(g_i,\ldots,g_k)$ or $\bar{n}_{i...k} \neq n(g_i,\ldots,g_k)$, there will be a huge energy cost. So at low energies, the system always stays in the subspace that satisfies the above constraint. In this case, $n_{i...k}$ and $\bar{n}_{i...k}$ are determined from g_i's.

It turns out that the constraint indeed simplifies the mathematics a lot, which allows us to systematically describe fermionic topological phases in any dimensions. The resulting nonlinear algebraic equations also have nice structures which can be solved more easily. So we can also find many solutions in any dimensions and hence obtain many new examples of topological phases in any dimensions. However, the constraint limits us to describe only SPT phases. To describe phases with intrinsic topological orders, we have to use more general unconstrained formalism, which is discussed in Ref. [27]. In this paper, we will only consider the simple constraint cases.

D. Grassmann tensor on a single d-dimensional simplex

In the low energy constraint subspace, we can associate each d-simplex of "+" orientation with a Grassmann rank-$(1+d)$ tensor $\mathcal{V}_d^+$, and each d-simplex of "−" orientation with a different Grassmann rank-$(1+d)$ tensor $\mathcal{V}_d^-$.

Let us first assume that a d-simplex $[0\ldots d]$ has a "+" orientation, and discuss the structure of the Grassmann tensor $\mathcal{V}_d^+$. The Grassmann tensor $\mathcal{V}_d^+$ is a map from space $G_b^{1+d} \to M_f$:

$$\mathcal{V}_d^+(g_0,g_1,\ldots,g_d) \in M_f, \quad g_i \in G_b. \tag{19}$$

The order of the variables $g_0,g_1,\ldots,g_d$ in $\mathcal{V}_d$ is the same as the order of the branching structure: vertex-0 < vertex-1 < $\cdots$ < vertex-d.

Here, an element in M_f is a complex number times an even number of Grassmann numbers. Thus $\mathcal{V}_d^+(g_0,g_1,\ldots,g_d)$ has a form

$$\begin{aligned}\mathcal{V}_d^+&(g_0,g_1,\ldots,g_d)\\ &= \nu_d^+(g_0,g_1,g_2,g_3,g_4,\ldots,g_d)\\ &\quad\times\theta_{(1234...d)}^{n_{d-1}(g_1,g_2,g_3,g_4,\ldots,g_d)}\theta_{(0134...d)}^{n_{d-1}(g_0,g_1,g_3,g_4,\ldots,g_d)}\cdots\\ &\quad\times\bar{\theta}_{(0234...d)}^{n_{d-1}(g_0,g_2,g_3,g_4,\ldots,g_d)}\bar{\theta}_{(0124...d)}^{n_{d-1}(g_0,g_1,g_2,g_4\ldots,g_d)}\cdots,\end{aligned} \tag{20}$$

where $\nu_d^+(g_0,g_1,\ldots,g_d)$ is a pure phase factor and $\theta_{(ij...k)}$ or $\bar{\theta}_{(ij...k)}$ is the Grassmann number associated with $(d-1)$-dimensional subsimplex on Σ_d^0, $(ij\ldots k)$. The constraint is implemented explicitly by writing, say, $\theta_{(1234...d)}^{n_{1234...d}}$ as $\theta_{(1234...d)}^{n_{d-1}(g_1,g_2,g_3,g_4,\ldots,g_d)}$.

Note that the vertices on the branched simplex Σ_d^0 have a natural order: vertex-0 < vertex-1 < $\cdots$ < vertex-d. This leads to the ordering of the Grassmann numbers: the first Grassmann number is associated with the subsimplex that does not contain the vertex-0, the second Grassmann number is associated with the subsimplex that does not contain the vertex-2, etc. Then it is followed by the Grassmann number associated with the subsimplex that does not contain the vertex-1, the Grassmann number associated with the subsimplex that does not contain the vertex-3, etc.

The above example is for simplexes with a "+" orientation. For branched simplex Σ_d^0 with a "−" orientation, we have

$$\begin{aligned}\mathcal{V}_d^-&(g_0,g_1,\ldots,g_d)\\ &= \nu_d^-(g_0,g_1,g_2,g_3,g_4,\ldots,g_d)\\ &\quad\times\cdots\theta_{(0124...d)}^{n_{d-1}(g_0,g_1,g_2,g_4,\ldots,g_d)}\theta_{(0234...d)}^{n_{d-1}(g_0,g_2,g_3,g_4,\ldots,g_d)}\\ &\quad\times\cdots\bar{\theta}_{(0134...d)}^{n_{d-1}(g_0,g_1,g_3,g_4,\ldots,g_d)}\bar{\theta}_{(1234...d)}^{n_{d-1}(g_1,g_2,g_3,g_4,\ldots,g_d)},\end{aligned} \tag{21}$$

where the order of the Grassmann numbers is reversed and θ's are switched with $\bar{\theta}$'s.

The number of the Grassmann numbers on the $(d-1)$-dimensional simplex $(ij\ldots k)$ is given by

$$n_{d-1}(g_i,g_j,\ldots,g_k) = 0,1, \tag{22}$$

which depends on d variables $g_i,g_j,\ldots,g_k$, and must satisfy

$$\sum_{i=0}^{d} n_{d-1}(g_0,\ldots,\hat{g}_i,\ldots,g_d) = \text{ even}, \tag{23}$$

so that the Grassmann rank-$(1+d)$ tensor always has an even number of Grassmann numbers. Here, the sequence $g_0,\ldots,\hat{g}_i,\ldots,g_d$ is the sequence $g_0,\ldots,g_d$ with g_i removed.

All the solutions of Eq. (23) can be obtained using an integer function $m_{d-2}(g_1,g_2,\ldots,g_{d-1}) = 0,1$ with $d-1$ variables [while $n_{d-1}(g_1,g_2,\ldots,g_d)$ is an integer function of d variables]:

$$n_{d-1}(g_1,g_2,\ldots,g_d) = \sum_{i=1}^{d} m_{d-2}(g_1,\ldots,\hat{g}_i,\ldots,g_d) \bmod 2. \tag{24}$$

For example, a rank-1 tensor has a form

$$\mathcal{V}_0(g_0) = \nu_0(g_0) \tag{25}$$

which contains no Grassmann number. A rank-2 tensor has a form

$$\mathcal{V}_1^+(g_0,g_1) = \nu_1^+(g_0,g_1)\theta_{(1)}^{n_0(g_1)}\bar{\theta}_{(0)}^{n_0(g_0)}, \tag{26}$$

where $n_0(g)$ has only two consistent choices

$$n_0(g) = 0, \quad \forall\, g \in G_b \quad \text{and} \quad n_0(g) = 1, \quad \forall\, g \in G_b. \tag{27}$$

As another example, a rank-4 tensor $\mathcal{V}_3^+$ for the tetrahedron in Fig. 6(a) is given by

$$\begin{aligned}\mathcal{V}_3^+(g_0,g_1,g_2,g_3) &= \nu_3^+(g_0,g_1,g_2,g_3)\theta_{(123)}^{n_2(g_1,g_2,g_3)}\\ &\quad\times\theta_{(013)}^{n_2(g_0,g_1,g_3)}\bar{\theta}_{(023)}^{n_2(g_0,g_2,g_3)}\bar{\theta}_{(012)}^{n_2(g_0,g_1,g_2)}.\end{aligned} \tag{28}$$

Note that the four triangles of the tetrahedron have different orientations: the triangles (123) and (013) point outwards and the triangles (023) and (012) point inwards. We have used θ and $\bar\theta$ for triangles with different orientations.

For the tetrahedron in Fig. 6(b), which has an opposite orientation to that of the tetrahedron in Fig. 6(a), we have a rank-4 tensor $\mathcal{V}_3^-$ which is given by

$$\mathcal{V}_3^-(g_0,g_1,g_2,g_3) = \nu_3^-(g_0,g_1,g_2,g_3)\theta_{(012)}^{n_2(g_0,g_1,g_2)}\theta_{(023)}^{n_2(g_0,g_2,g_3)} \times \bar\theta_{(013)}^{n_2(g_0,g_1,g_3)}\bar\theta_{(123)}^{n_2(g_1,g_2,g_3)}. \tag{29}$$

Note the different order of the Grassmann numbers $\theta_{(012)}^{n_2(g_0,g_1,g_2)}\theta_{(023)}^{n_2(g_0,g_2,g_3)}\bar\theta_{(013)}^{n_2(g_0,g_1,g_3)}\bar\theta_{(123)}^{n_2(g_1,g_2,g_3)}$.

E. Evaluation of the Grassmann tensors on a complex

Let Σ_d be a d-complex with a branching structure which is formed by several simplexes $[ab\dots c]$. As discussed in the last section, a simplex is associated with one of the two Grassmann tensors $\mathcal{V}_d^\pm$. Let us use $\int_{\text{in}(\Sigma_d)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)}$ to represent the evaluation of rank-$(1+d)$ Grassmann tensors $\mathcal{V}_d^\pm$ on the d-complex Σ_d:

$$\begin{aligned}\int_{\text{in}(\Sigma_d)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)} &\equiv \int\prod_{(ij\dots k)} d\theta_{(ij\dots k)}^{n_{d-1}(g_i,g_j,\dots,g_k)} d\bar\theta_{(ij\dots k)}^{n_{d-1}(g_i,g_j,\dots,g_k)}\\ &\quad\times\prod_{\{xy\dots z\}}(-)^{m_{d-2}(g_x,g_y,\dots,g_z)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)}(g_a,g_b,\dots,g_c),\end{aligned} \tag{30}$$

where $\prod_{[ab\dots c]}$ multiplies over all the d-dimensional simplexes $[ab\dots c]$ in Σ_d and $\mathcal{V}_d^{s(a,b,\dots,c)}(g_a,g_b,\dots,g_c)$ is a Grassmann rank-$(1+d)$ tensor associated with the simplex $[ab\dots c]$. $s(a,b,\dots,c)=\pm$ depending on the orientation of the d-dimensional simplex $[ab\dots c]$. Also, $\prod_{(ij\dots k)}$ multiplies over all the interior $(d-1)$-dimensional simplexes, $(ij\dots k)$, of the complex Σ_d [i.e., those $(d-1)$-dimensional simplexes that are not on the surface of Σ_d]. $\prod_{\{xy\dots z\}}$ multiplies over all the interior $(d-2)$-dimensional simplexes [i.e., those $(d-2)$-dimensional simplexes that are not on the surface of Σ_d], $\{xy\dots z\}$, of the complex Σ_d. We note that, when Σ_d is a single simplex $[ab\dots c]$, $\int_{\text{in}(\Sigma_d)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)}$ is given by $\mathcal{V}_d^{s(a,b,\dots,c)}(g_a,g_b,\dots,g_c)$.

We see that in the formal notation $\int_{\text{in}(\Sigma_d)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)}$, $\int_{\text{in}(\Sigma_d)}$ represents a Grassmann integral over the Grassmann numbers on all the interior $(d-1)$-simplexes in the d-complex Σ_d. So in(Σ_d) in $\int_{\text{in}(\Sigma_d)}$ actually represents the collection of all those interior $(d-1)$-simplexes in the d-complex Σ_d. The Grassmann numbers on the $(d-1)$-simplexes on the surface of the d-complex Σ_d are not integrated over. In fact $\int_{\text{in}(\Sigma_d)}$ has the following explicit form:

$$\int_{\text{in}(\Sigma_d)} = \int\prod_{(ij\dots k)} d\theta_{(ij\dots k)}^{n_{d-1}(g_i,g_j,\dots,g_k)} d\bar\theta_{(ij\dots k)}^{n_{d-1}(g_i,g_j,\dots,g_k)} \times\prod_{\{xy\dots z\}}(-)^{m_{d-2}(g_x,g_y,\dots,g_z)}. \tag{31}$$

We note that $d\theta$ always appears in front of $d\bar\theta$. We also note that the integration measure contains a nontrivial sign factor $(-)^{m_{d-2}(g_x,g_y,\dots,g_z)}$ on the interior $(d-2)$ simplexes $\{xy\dots z\}$.

The sign factor $\prod_{\{xy\dots z\}}(-)^{m_{d-2}(g_x,g_y,\dots,g_z)}$ is included to help us to define topological fermionic path integral later. Choosing orientation dependent tensors $\mathcal{V}_d^\pm$ also help us to define topological fermionic path integral. Adding the sign factor $\prod_{\{xy\dots z\}}(-)^{m_{d-2}(g_x,g_y,\dots,g_z)}$ and choosing orientation dependent tensors $\mathcal{V}_d^\pm$ appear to be very unnatural. However, they are two extremely important features of our approach. We cannot obtain topological fermionic path integral without these two features. Realizing these two features is one of a few breakthroughs that allows us to obtain topological fermionic path integral on discrete space-time. The two features are related to our choice of the ordering convention of θ and $\bar\theta$ in the Grassmann tensors $\mathcal{V}_d^\pm$, and the ordering convention of $d\theta$ and $d\bar\theta$ in the integration measure.

Now let us consider two d-complexes Σ_d^1 and Σ_d^2, which do not overlap but may share part of their surfaces. Let Σ_d be the union of the two d-complexes. From our definition of the Grassmann integral, we find that the Grassmann integral on Σ_d can be expressed as

$$\int_{\text{in}(\Sigma_d)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)} = \int_{\Sigma_d^1\cap\Sigma_d^2}\left(\int_{\text{in}(\Sigma_d^1)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)}\int_{\text{in}(\Sigma_d^2)}\prod_{[ab\dots c]}\mathcal{V}_d^{s(a,b,\dots,c)}\right), \tag{32}$$

where $\Sigma_d^1\cap\Sigma_d^2$ is the intersection of the two complexes, which contains only $(d-1)$-simplexes on the shared surface of the two complexes Σ_d^1 and Σ_d^2. More precisely,

$$\int_{\Sigma_d^1\cap\Sigma_d^2} = \int\prod_{(ij\dots k)} d\theta_{(ij\dots k)}^{n_{d-1}(g_i,g_j,\dots,g_k)} d\bar\theta_{(ij\dots k)}^{n_{d-1}(g_i,g_j,\dots,g_k)} \times\prod_{\{xy\dots z\}}(-)^{m_{d-2}(g_x,g_y,\dots,g_z)}, \tag{33}$$

where $\prod_{(ij\dots k)}$ is a product over all the $(d-1)$-simplexes in $\Sigma_d^1\cap\Sigma_d^2$ and $\prod_{\{xy\dots z\}}$ is a product over all the interior $(d-2)$-simplexes in $\Sigma_d^1\cap\Sigma_d^2$ [note that $\Sigma_d^1\cap\Sigma_d^2$ is a $(d-1)$-complex]. Equation (32) describes the process to glue the two complexes Σ_d^1 and Σ_d^2 together.

F. An example for evaluation of Grassmann tensor network

Let us consider an example to help us to understand the complicated expression (30). When Σ is formed by two tetrahedrons as in Fig. 7, the top tetrahedron with a "+" orientation is associated with a rank-4 tensor $\mathcal{V}_3^+$:

$$\mathcal{V}_3^+(g_0,g_1,g_2,g_3) = \nu_3^+(g_0,g_1,g_2,g_3)\theta_{(123)}^{n_2(g_1,g_2,g_3)}\theta_{(013)}^{n_2(g_0,g_1,g_3)} \times\bar\theta_{(023)}^{n_2(g_0,g_2,g_3)}\bar\theta_{(012)}^{n_2(g_0,g_1,g_2)}. \tag{34}$$

The lower tetrahedron with a "−" orientation is associated with a rank-4 tensor $\mathcal{V}_3^-$:

$$\mathcal{V}_3^-(g_0,g_1,g_2,g_4) = \nu_3^-(g_0,g_1,g_2,g_4)\theta_{(012)}^{n_2(g_0,g_1,g_2)}\theta_{(024)}^{n_2(g_0,g_2,g_4)} \times \bar\theta_{(014)}^{n_2(g_0,g_1,g_4)}\bar\theta_{(124)}^{n_2(g_1,g_2,g_4)}. \quad (35)$$

Note that $\mathcal{V}_3^+(g_0,g_1,g_2,g_3)$ and $\mathcal{V}_3^-(g_0,g_1,g_2,g_4)$ have independent Grassmann numbers $\theta_{(012)}$ and $\bar\theta_{(012)}$ on the shared face (012). $\bar\theta_{(012)}$ belongs to $\mathcal{V}_3^-(g_0,g_1,g_2,g_4)$ and $\theta_{(012)}$ belongs to $\mathcal{V}_3^+(g_0,g_1,g_2,g_3)$. Since $\text{in}(\Sigma) = (012)$, the evaluation of $\mathcal{V}_3^\pm$ on the complex Σ is given by

$$\int_{(012)} \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_2,g_4) = \int d\theta_{(012)}^{n_2(g_0,g_1,g_2)} d\bar\theta_{(012)}^{n_2(g_0,g_1,g_2)} \times \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_2,g_4) \quad (36)$$

which is a special case of Eq. (30).

G. Fermionic path integral

Given a closed space-time manifold M_{ST}, with a time direction, its triangularization Σ is a complex with no boundary. The time direction gives rise to a local order of the vertices. So, the complex Σ has a branching structure.

Since Σ has no boundary, the evaluation of Grassmann tensors $\mathcal{V}_d^\pm$ on it gives rise to a complex number:

$$e^{-S} = \int_{\text{in}(\Sigma)} \prod_{[ab...c]} \mathcal{V}_d^{s(a,b,...,c)}. \quad (37)$$

Such a complex number can be viewed as the action amplitude in the imaginary-time path integral of a fermionic system. The different choices of the complex functions $\nu_d^\pm(g_0,\dots,g_d)$ and the integer functions $m_{d-2}(g_0,\dots,g_{d-2})$ correspond to different choices of Lagrangian of the fermionic system. The partition function of the imaginary-time path integral is given by

$$Z = \sum_{\{g_i\}} \int_{\text{in}(\Sigma)} \prod_{[ab...c]} \mathcal{V}_d^{s(a,b,...,c)}, \quad (38)$$

where each vertex of the complex Σ is associated with a variable g_i. We note that

> the fermionic partition function is determined from two $(1+d)$-variable complex functions: $\nu_d^+(g_i,g_j,\dots)$, $\nu_d^-(g_i,g_j,\dots)$, and one $(d-1)$-variable integer function $m_{d-2}(g_i,g_j,\dots)$.

V. TOPOLOGICALLY INVARIANT GRASSMANN TENSOR NETWORK AND FERMIONIC TOPOLOGICAL NONLINEAR σ MODEL

Now we would like to study the low energy effective theory of a gapped fermion system. The fixed-point low energy effective action amplitude of a gapped system must describe a topological quantum field theory. So the fixed-point action amplitude must be invariant under a renormalization group (RG) flow which generates coarse-graining transformation of the space-time complex Σ. In fact, such a fixed-point requirement under RG flow completely fixes the form of the fixed-point action amplitude. The fixed-point action amplitude, in turn, describes the possible fermionic SPT phases.

The detailed RG flow steps depend on the dimensions of the space-time. We will discuss different space-time dimensions and the corresponding fixed-point action amplitude separately. Here, we will only give a brief discussion. A more detailed discussion will be given in Appendix A.

A. (1+1)D case

We first consider fermion systems in $(1+1)$ space-time dimension. Its imaginary-time path integral is given by

$$Z = \sum_{\{g_i\}} \int_{\text{in}(\Sigma_2)} \prod_{[abc]} \mathcal{V}_2^{s(a,b,c)}, \quad (39)$$

where Σ_2 is the (1+1)D space-time complex with a branching structure (see Fig. 8), and $\mathcal{V}_2^\pm$ are rank-3 Grassmann tensors.

The first type of the RG flow step that changes the space-time complex is described by Fig. 8(a). If the Grassmann tensors $\mathcal{V}_2^\pm$ describe a fixed-point action amplitude, their evaluation on the two complexes in Fig. 8(a) should be the same. This leads to the following condition:

$$\nu_2(g_0,g_1,g_3)\nu_2(g_1,g_2,g_3) = \nu_2(g_0,g_1,g_2)\nu_2(g_0,g_2,g_3), \quad (40)$$

where ν_2 is a complex phase related to $\nu_2^\pm$ and $m_0(g)$ through

$$\nu_2^+(g_0,g_1,g_2) = \nu_2(g_0,g_1,g_2),$$
$$\nu_2^-(g_0,g_1,g_2) = (-)^{m_0(g_1)}/\nu_2(g_0,g_1,g_2). \quad (41)$$

Note that the above condition is obtained for a particular vertex ordering (0,1,2,3) (a particular branching structure) as described in Fig. 8(a). Different *valid* branching structures, in general, lead to other conditions on $\nu_2(g_0,g_1,g_2)$. It turns out that all those conditions are equivalent to Eq. (40). For

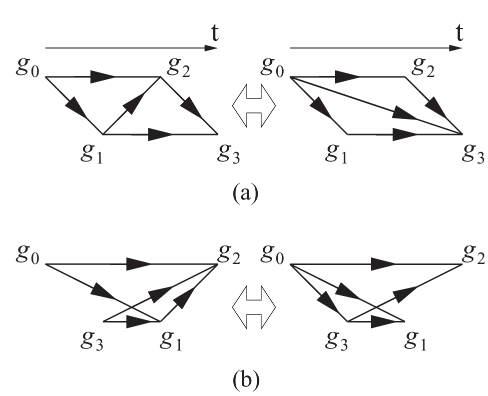


FIG. 8. (a) Consider two triangles which are a part of a space-time complex Σ with a branching structure. The first type of the RG flow step changes the two triangles to other two triangles. The branching structure leads to the following vertex ordering: (0,1,2,3). (b) A different space-time complex Σ which corresponds to a different ordered vertices (0,3,1,2). The corresponding complex does have a valid branching structure because some triangles overlap.

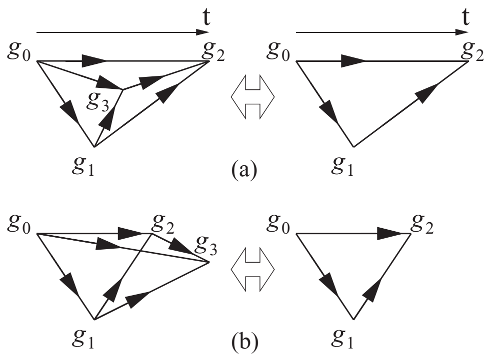


FIG. 9. (a) Consider three triangles which are a part of a space-time complex Σ with a branching structure. The second type of the RG flow step changes the three triangles to one triangle. The branching structure leads to the following vertex ordering: (0,1,3,2). (b) The vertex order (0,1,2,3) does not correspond to a valid branching structure.

example, for vertex ordering (1,3,0,2), we obtain

$$\nu_2(g_1,g_0,g_2)\nu_2(g_1,g_3,g_0) = \nu_2(g_1,g_3,g_2)\nu_2(g_3,g_0,g_2), \quad (42)$$

which is the same as Eq. (40) after replacing $g_1 \to g_0$, $g_3 \to g_1$, $g_0 \to g_2$, and $g_2 \to g_3$.

Some vertex orders do not correspond to valid branching structure due to the overlap of the simplexes, and are not considered. One of the invalid orderings is described by Fig. 8(b) which has a vertex order (0,3,1,2). The invalid orderings can give rise to conditions on $\nu_2(g_0,g_1,g_2)$ that are *not* equivalent to Eq. (40).

The second type of the RG flow step that changes the space-time complex is described by Fig. 9. Different *valid* branching structures [see Fig. 9(a) for one example] lead to different conditions on $\nu_2(g_0,g_1,g_2)$ [which is defined in Eq. (41)]. It turns out that all those conditions are equivalent to Eq. (40). [Some vertex orderings do not correspond to valid branching structures. One of the invalid orderings is described by Fig. 9(b).]

Since $\nu_2(g_0,g_1,g_2)$ is related to $\nu_2^\pm(g_0,g_1,g_2)$ and $m_0(g)$ through Eq. (41), Eq. (40) is actually a condition on $\nu_2^\pm(g_0,g_1,g_2)$ and $m_0(g)$ that determines the fermion path integral. The fermion path integral described by $\nu_2^\pm(g_0,g_1,g_2)$ and $m_0(g)$ that satisfies Eq. (40) is a topological fermion path integral which is a fixed-point theory. We would like to point out that Eq. (40) is the standard 2-cocycle condition of group cohomology [46,47].

B. (2+1)D case

We next consider fermion systems in $(2+1)$ space-time dimension described by imaginary-time path integral

$$Z = \sum_{\{g_i\}} \int_{\text{in}(\Sigma_3)} \prod_{[abcd]} \mathcal{V}_3^{s(a,b,c,d)}, \quad (43)$$

where Σ_3 is the (2+1)D space-time complex with a branching structure and $\mathcal{V}_3^\pm$ are rank-4 Grassmann tensors.

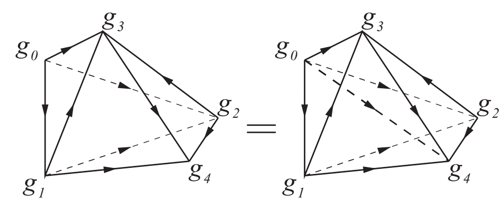


FIG. 10. (a) A 3D complex formed by two 3-simplexes (g_0,g_1,g_2,g_3) and (g_1,g_2,g_3,g_4). The two 3-simplexes share a 2-simplex (g_1,g_2,g_3). (b) A 3D complex formed by three 3-simplexes (g_0,g_2,g_3,g_4), (g_0,g_1,g_3,g_4), and (g_0,g_1,g_2,g_4).

The first type of the RG flow step that changes the space-time complex is described by Fig. 10. In order for the Grassmann tensors $\mathcal{V}_3^\pm$ to describe a fixed-point action amplitude, their evaluations on the two complexes in Fig. 10 should be the same. This leads to the following condition:

$$\begin{aligned}&\nu_3(g_1,g_2,g_3,g_4)\nu_3(g_0,g_1,g_3,g_4)\nu_3(g_0,g_1,g_2,g_3)\\&\quad = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)}\nu_3(g_0,g_2,g_3,g_4)\nu_3(g_0,g_1,g_2,g_4),\end{aligned} \quad (44)$$

where ν_3 is a complex phase and is given by

$$\begin{aligned}\nu_3^+(g_0,g_1,g_2,g_3) &= (-)^{m_1(g_0,g_2)}\nu_3(g_0,g_1,g_2,g_3),\\ \nu_3^-(g_0,g_1,g_2,g_3) &= (-)^{m_1(g_1,g_3)}/\nu_3(g_0,g_1,g_2,g_3).\end{aligned} \quad (45)$$

Different vertex orders give rise to different branching structures. It turns out that all the different *valid* branching structures give rise to the same condition (44).

The second type of the RG flow step that changes the space-time complex is described by Fig. 11. If the Grassmann tensors $\mathcal{V}_3^\pm$ describe a fixed-point action amplitude, their evaluations on the two complexes in Fig. 11 should also be the same. This leads to the same condition (44).

C. (3+1)D case

Last, we consider fermion systems in $(3+1)$ space-time dimension, described by imaginary-time path integral

$$Z = \sum_{\{g_i\}} \int_{\text{in}(\Sigma_4)} \prod_{[abcde]} \mathcal{V}_4^{s(a,b,c,d,e)}, \quad (46)$$

where Σ_4 is the (3+1)D space-time complex with a branching structure and $\mathcal{V}_4^\pm$ are rank-5 Grassmann tensors.

The first type of the RG flow step that changes the space-time complex is described by Fig. 12. The second type

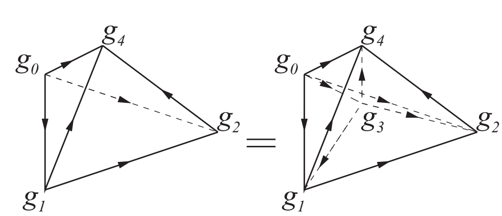
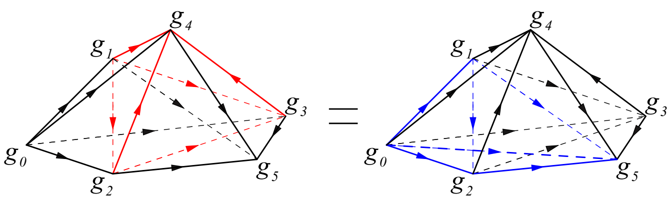
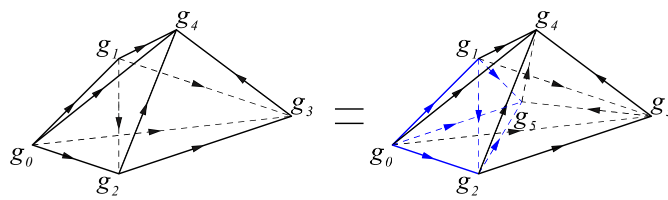


FIG. 11. (a) A 3D complex formed by one 3-simplex (g_0,g_1,g_2,g_4). (b) A 3D complex formed by four 3-simplexes (g_1,g_2,g_3,g_4), (g_0,g_2,g_3,g_4), (g_0,g_1,g_3,g_4), and (g_0,g_1,g_2,g_3).



(a) (b)

FIG. 12. (Color online) (a) A 4D complex formed by two 4-simplexes (g_0,g_1,g_2,g_3,g_4) and (g_1,g_2,g_3,g_4,g_5). The two 4-simplexes share a 3-simplex (g_1,g_2,g_3,g_4) (color red). (b) A 4D complex formed by four 4-simplexes (g_0,g_2,g_3,g_4,g_5), (g_0,g_1,g_3,g_4,g_5), (g_0,g_1,g_2,g_4,g_5), and (g_0,g_1,g_2,g_3,g_5). The two simplexes (g_0,g_1,g_2,g_4,g_5), and (g_0,g_1,g_2,g_3,g_5) share a 3-simplex (g_0,g_1,g_2,g_5) (color blue).



(a) (b)

FIG. 13. (Color online) (a) A 4D complex formed by one 4-simplex (g_0,g_1,g_2,g_3,g_4). (b) A 4D complex formed by five 4-simplexes (g_1,g_2,g_3,g_5,g_4), (g_0,g_2,g_3,g_5,g_4), (g_0,g_1,g_3,g_5,g_4), (g_0,g_1,g_2,g_5,g_4), and (g_0,g_1,g_2,g_3,g_5). The two simplexes (g_0,g_1,g_2,g_5,g_4), and (g_0,g_1,g_2,g_3,g_5) share a 3-simplex (g_0,g_1,g_2,g_5) (color blue).

of the RG flow step that changes the space-time complex is described by Fig. 13. If the Grassmann tensor $\mathcal{V}_4$ describes a fixed-point action amplitude, its evaluation on the two complexes in Figs. 12 or 13 should be the same. This leads to the following condition:

$$\begin{aligned}&\nu_4(g_1,g_2,g_3,g_4,g_5)\nu_4(g_0,g_1,g_3,g_4,g_5)\nu_4(g_0,g_1,g_2,g_3,g_5)\\&\quad=(-)^{n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5)}\\&\qquad\times\nu_4(g_0,g_2,g_3,g_4,g_5)\nu_4(g_0,g_1,g_2,g_4,g_5)\nu_4(g_0,g_1,g_2,g_3,g_4),\end{aligned}\tag{47}$$

where ν_4 is a complex phase and is given by

$$\begin{aligned}\nu_4^+(g_0,g_1,g_2,g_3,g_4)&=(-)^{m_2(g_0,g_1,g_3)+m_2(g_1,g_3,g_4)+m_2(g_1,g_2,g_3)}\nu_4(g_0,g_1,g_2,g_3,g_4),\\\nu_4^-(g_0,g_1,g_2,g_3,g_4)&=(-)^{m_2(g_0,g_2,g_4)}/\nu_4(g_0,g_1,g_2,g_3,g_4).\end{aligned}\tag{48}$$

The sign factors in the above relations between $\nu_4^\pm$ and ν_4, and the sign factor in Eq. (47) are very strange. Obtaining those strange sign factors is another breakthrough that allows us to obtain topological fermionic path integral beyond (1+1) dimensions. It appears that those sign factors are related to a deep mathematical structure called Steenrod squares [72,73] Different vertex orders give rise to different branching structures. It turns out that all the valid branching structures give rise to the same condition (47).

D. Fixed-point action on a closed complex

We know that the fixed-point theory of a bosonic SPT phase is a discrete bosonic topological nonlinear σ model [46]. The bosonic topological nonlinear σ models are characterized by action amplitudes that are equal to 1 on any closed complex. The action amplitude being equal to 1 on any closed complex is the key reason why the models describe bosonic SPT phases without intrinsic topological orders.

In the above, we have developed a fixed-point theory of fermionic SPT phases, and defined discrete fermionic topological nonlinear σ model. A discrete fermionic topological nonlinear σ model is described by a pair of functions $[\nu_d(g_0,\dots,g_d),m_{d-2}(g_0,\dots,g_{d-2})]$, that satisfy the conditions (40), (44), or (47). According to the branching moves discussed above, the evaluation on any closed manifold is reduced to an equivalent evaluation

$$\int \mathcal{V}_d^+\mathcal{V}_d^-,\tag{49}$$

where the symbol $\int$ means integrating out all the Grassmann variables on the corresponding $d-1$ simplexes [see Eq. (31)]. We note that the sign factor $\prod_{\{x,y,\dots,z\}}(-)^{m_{d-2}(g_x,g_y,\dots,g_z)}$ should be included in the fermionic path integral. In Appendix B, we will show that $\int \mathcal{V}_d^+\mathcal{V}_d^-$ is always equal to 1.

E. Symmetry and stability of the fixed-point action amplitude

We have seen that a fermionic system can be described by a fermionic path integral on the discretized space-time. In d-dimensional space-time, the fermionic path integral is determined by two $(d+1)$-variable complex functions $\nu_d^\pm(g_0,\dots,g_d)$ and one $(d-1)$-variable integer function $m_{d-2}(g_0,\dots,g_{d-2})$. In the last few sections, we have shown that if $\nu_d^\pm(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ satisfy some conditions [Eqs. (40), (44), and (47)], the fermionic path integral determined from $\nu_d^\pm(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ is actually a fixed-point theory under the coarse-graining transformation of the space-time complex.

A fixed-point theory can be used to describe a phase if the fixed-point theory is stable. To see whether a fixed-point theory is stable or not, we perturb the fixed-point action amplitude $\nu_d^\pm(g_0,\dots,g_d)$ by a small amount $\nu_d^\pm(g_0,\dots,g_d)\to\nu_d^\pm(g_0,\dots,g_d)+\delta\nu_d^\pm(g_0,\dots,g_d)$. If $\nu_d^\pm(g_0,\dots,g_d)+\delta\nu_d^\pm(g_0,\dots,g_d)$ flows back to $\nu_d^\pm(g_0,\dots,g_d)$ under the coarse-graining transformation of space-time complex, the fixed-point theory is stable.

From this point of view, our constructed fixed-point theories described by $\nu_d^\pm(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ are not stable. If we add a perturbation of form $\delta\nu_d^{\pm}(g_0,\dots,g_d)=\epsilon\prod_{i=0}^d \delta_{g_i,\bar g}$ to increase the weight of the $\bar g$ state in the action amplitude, we believe that the action amplitude will flow to the simple form $\nu_d^{\pm}(g_0,\dots,g_d)=\prod_{i=0}^d \delta_{g_i,\bar g}$ where only the $\bar g$ state has a nonzero weight.

However, if we require the path integral to have a symmetry, then our constructed fixed-point theories can be stable. To impose a symmetry, we assume that our fermion system has full symmetry group G_f. Since G_f contains a normal subgroup Z_2^f, we can view G_f as a fiber bundle with fiber Z_2^f and base space $G_b = G_f/Z_2^f$. Thus, there is natural projection map $p:\ G_f \to G_b$.

In general, g_i in $\nu_d^{\pm}(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ can be a group element in G_f. Here, we will assume that $\nu_d^{\pm}(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ satisfy

$$\begin{aligned}\nu_d^{\pm}(g_0,\dots,g_d)&=\nu_d^{\pm}(\tilde g_0,\dots,\tilde g_d),\\ m_{d-2}(g_0,\dots,g_{d-2})&=m_{d-2}(\tilde g_0,\dots,\tilde g_{d-2}),\\ g_i\in G_f,\quad \tilde g_i&=p(g_i)\in G_b.\end{aligned}\tag{50}$$

Therefore, we can view g_i in $\nu_d^{\pm}(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ as a group element in G_b.

From Sec. IV C, we see that different degrees of freedom are attached to the $(d-1)$-simplexes for different choices of $g_i,\dots,g_k$ in G_b. The degrees of freedom on a $(d-1)$-simplex form a 1D representation u_{d-1}^g of the *full* symmetry group $g\in G_f$, which may depend on $g_i,\dots,g_k$:

$$\begin{aligned}u_{d-1}^{g_1g_2}(g_i,g_j,\dots,g_k)&=u_{d-1}^{g_1}(g_i,g_j,\dots,g_k)\\ &\quad\times u_{d-1}^{g_2}(g_i,g_j,\dots,g_k),\\ u_{d-1}^{g}[p(g'g_i),\dots,p(g'g_k)]&=u_{d-1}^{g}(g_i,\dots,g_k),\end{aligned}\tag{51}$$

where p is the projection map $G_f\to G_b$. Here we would like to stress that, in $u_{d-1}^g(g_i,g_j,\dots,g_k)$, $g\in G_f$ and $g_i,g_j,\dots\in G_b$. We also require the variables with bar and without bar to carry opposite quantum number (i.e., their 1D representations are inverse of each other). In this case, the symmetry of the path integral (i.e., the invariance of the action amplitude) can be implemented by requiring

$$\begin{aligned}\theta_{(ij\dots k)}&\to\left(u_{d-1}^g\right)^{-1}(g_i,g_j,\dots,g_k)\theta_{(ij\dots k)},\\ \bar\theta_{(ij\dots k)}&\to u_{d-1}^g(g_i,g_j,\dots,g_k)\bar\theta_{(ij\dots k)},\end{aligned}\tag{52}$$

and

$$\begin{aligned}&[\nu_d^{\pm}(gg_0,\dots,gg_d)]^{s(g)}\\ &\quad=\nu_d^{\pm}(g_0,\dots,g_d)\prod_i\left(u_{d-1}^g\right)^{(-)^i}d(g_0,\dots,\hat g_i,\dots,g_d),\\ &m_{d-2}(gg_0,\dots,gg_{d-2})=m_{d-2}(g_0,\dots,g_{d-2}),\quad g\in G_f\end{aligned}\tag{53}$$

where $s(g)=-1$ if g contains the antiunitary time-reversal transformation and $s(g)=1$ otherwise. If we view $\mathcal{V}_d^{\pm}(g_0,\dots,g_d))$ as a whole object, it is invariant under the symmetry action g:

$$\mathcal{V}_d^{\pm}(gg_0,\dots,gg_d)^{s(g)}=\mathcal{V}_d^{\pm}(g_0,\dots,g_d).\tag{54}$$

Indeed, the above symmetric $\nu_d^{\pm}(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ describe a fermion system with G_f symmetry. If $\nu_d^{\pm}(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ further satisfy Eqs. (40), (44), or (47), we believe they even describe a fixed-point theory that is stable against perturbations which do not break the symmetry. But, further consideration suggests that such a symmetry condition is too strong. The fermionic systems that are described by the symmetric $\nu_d^{\pm}(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$ appear to be essentially bosonic. At least, their description is the same as the description of bosonic SPT phases.

So, in this paper, we will use a weaker symmetry condition. We only require $\nu_d(g_0,\dots,g_d)$ and $n_{d-1}(g_0,\dots,g_{d-1})$ [which are certain combinations of $\nu_d^{\pm}(g_0,\dots,g_d)$ and $m_{d-2}(g_0,\dots,g_{d-2})$] to be symmetric:

$$\begin{aligned}&\nu_d^{s(g)}(gg_0,\dots,gg_d)\\ &\quad=\nu_d(g_0,\dots,g_d)\prod_i\left(u_{d-1}^g\right)^{(-)^i}(g_0,\dots,\hat g_i,\dots,g_d),\\ &n_{d-1}(gg_0,\dots,gg_{d-1})=n_{d-1}(g_0,\dots,g_{d-1}),\quad g\in G_f.\end{aligned}\tag{55}$$

Since the fermion fields always change sign under the fermion-number-parity transformation $P_f\in G_f$, the 1D representations satisfy

$$u_{d-1}^{P_f}(g_0,\dots,g_{d-1})=(-)^{n_{d-1}(g_0,\dots,g_{d-1})}.\tag{56}$$

Note that we only require $n_{d-1}(g_0,\dots,g_{d-1})$ to be symmetric. In general, $m_{d-2}(g_0,\dots,g_{d-2})$ is not symmetric $m_{d-2}(gg_0,\dots,gg_{d-2})\neq m_{d-2}(g_0,\dots,g_{d-2})$. Although the fermionic path integral appears not to be symmetric under the weaker symmetry condition, it turns out that the symmetry breaking is only a boundary effect and the symmetry can be restored by adding an additional boundary action to the space-time path integral. In fact, we will show that the ground state wave function and the Hamiltonian obtained from the path integral with the weaker symmetry condition are indeed symmetric. So, a fermion path integral determined by $\nu_d(g_0,\dots,g_d)$ and $n_{d-1}(g_0,\dots,g_{d-1})$ that satisfies Eq. (55) as well as Eqs. (40), (44), or (47) does describe a symmetric stable fixed-point theory. Such $\nu_d(g_0,\dots,g_d)$, $n_{d-1}(g_0,\dots,g_{d-1})$, and $u_{d-1}^g(g_0,\dots,g_{d-1})$ describe a fermionic topological phase with symmetry.

Let us introduce

$$\begin{aligned}f_1(g_0,g_1)&=0;\\ f_2(g_0,g_1,g_2)&=0;\\ f_3(g_0,g_1,\dots,g_3)&=0;\\ f_4(g_0,g_1,\dots,g_4)&=n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4);\\ f_5(g_0,g_1,\dots,g_5)&=n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)\\ &\quad+n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)\\ &\quad+n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5).\end{aligned}\tag{57}$$

Using f_d, we can define a mapping from $(d+1)$-variable functions $\nu_d(g_0,\dots,g_d)$ to $(d+2)$-variable functions $(\delta\nu_d)(g_0,\dots,g_{d+1})$:

$$\begin{aligned}&(\delta\nu_d)(g_0,\dots,g_{d+1})\\ &\quad\equiv(-)^{f_{d+1}(g_0,\dots,g_{d+1})}\prod_{i=0}^{d+1}\nu_d^{(-)^i}(g_0,\dots,\hat g_i,\dots,g_{d+1}).\end{aligned}\tag{58}$$

Then using $(\delta\nu_d)$, we can rewrite the conditions (40), (44), and (47), in a uniform way:

$$(\delta\nu_d)(g_0,\ldots,g_{d+1}) = 1. \tag{59}$$

Here, $[\nu_d(g_0,\ldots,g_d),\ n_{d-1}(g_0,\ldots,g_{d-1}),\ u^g_{d-1}(g_0,\ldots,g_{d-1})]$, that satisfies Eqs. (55), (23), and (59), will be called a d-cocycle. The space of d-cocycle is denoted as $\mathscr{Z}^d[G_f,U_T(1)]$. The fermionic path integral obtained from a d-cocycle $(\nu_d,n_{d-1},u^g_{d-1})$ [via $\nu^\pm_d(g_0,\ldots,g_d)$ and $m_{d-2}(g_0,\ldots,g_{d-2})$] will be called a fermionic topological nonlinear σ model.

In Appendix C, we study the fermionic cocycles (ν_{d+1},n_d,u^g_d) systematically. In particular, we will study the equivalence relation between them. This will lead to the notion of group supercohomology class $\mathscr{H}^{d+1}[G_f,U_T(1)]$.

Similar to the bosonic case, we argue that each element in group supercohomology class $\mathscr{H}^{d_{sp}+1}[G_f,U_T(1)]$ corresponds to a fermionic topological nonlinear σ model defined via the fermion path integral in imaginary time (see Sec. IV G). The fermion path integral over a $(d_{sp}+1)$-dimensional complex Σ will give rise to a ground state on its d_{sp} boundary. Such a ground state represents the corresponding fermionic SPT state. So, each element in group supercohomology class $\mathscr{H}^{d_{sp}+1}[G_f,U_T(1)]$ corresponds to a fermionic SPT state.

In Appendix G 1, we will show the elements in $\mathscr{H}^{d_{sp}+1}[G_f,U_T(1)]$ correspond to distinct fermionic SPT phases. We will calculate $\mathscr{H}^{d_{sp}+1}[G_f,U_T(1)]$ for some simple symmetry groups G_f in Appendix F. This allows us to construct several new fermionic SPT phases. The results are summarized in Table II.

VI. GENERAL SCHEME OF CALCULATING THE GROUP SUPERCOHOMOLOGY CLASSES $\mathscr{H}^d[G_f,U_T(1)]$: AN OUTLINE

In this section, we will give a general description on how to calculate the group supercohomology classes. More concrete calculations will be given in Appendix F. From the discussions in Appendix C, we see that to calculate $\mathscr{H}^d[G_f,U_T(1)]$ {which is formed by the equivalent classes of the fermionic cocycles $[\nu_d(g_0,\ldots,g_d),n_{d-1}(g_0,\ldots,g_{d-1}),u^g_{d-1}(g_0,\ldots,g_{d-1})]$}, we need to go through the following steps.

First step. Calculate $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ which gives us different classes of $(d-1)$D-graded structure $n_{d-1}(g_0,\ldots,g_{d-1})$. For each $n_{d-1}(g_0,\ldots,g_{d-1})$, find an allowed $u^g_{d-1}(g_0,\ldots,g_{d-1})$.

Second step. Next we need to find out $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ from $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$. We note that $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ denotes the subset of $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, such that for each graded structure $n_{d-1}(g_0,\ldots,g_{d-1})$ in $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, the corresponding $(-)^{f_{d+1}}$ is a $(d+1)$-coboundary in $\mathcal{B}^{d+1}[G_b,U_T(1)]$. We note that for $n_{d-1}(g_0,\ldots,g_{d-1})\notin B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, the corresponding $(-)^{f_{d+1}}$ will not be a coboundary in $\mathcal{B}^{d+1}[G_b,U_T(1)]$ and the super cocycle condition (59) will not have any solution. Mathematically, such an inconsistency is called as an *obstruction*, and the group $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ is called as the obstruction-free subgroup of $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$. We note that for $d\leqslant 2$, $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)\equiv\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ since $f_{d+1}\equiv 0$ when $d\leqslant 2$.

Third step. We need to calculate $\nu_d(g_0,\ldots,g_d)$ from Eq. (59). For each fixed $n_{d-1}(g_0,\ldots,g_{d-1})\in B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, the solutions ν_d of Eq. (59) have a one-to-one correspondence with the standard group cocycles, provided that $u^g_{d-1}(g_0,\ldots,g_{d-1})=1$ for $g\in G_b$. We see that, for each element in $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, there is a class of solutions. For two different elements in $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, their classes of solutions have a one-to-one correspondence. Therefore, we have an exact sequence

$$\mathscr{H}^d[G_f,U_T(1)]\to B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)\to 0. \tag{60}$$

Fourth step. From the equivalence relation of supercohomology class defined in Appendix C, we see that, for each fixed element $n_{d-1}(g_0,\ldots,g_{d-1})$ in $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, the equivalent classes of the supercocycle solutions can be labeled by $\mathcal{H}^d[G_b,U_T(1)]$. However, the labeling is not one to one since in obtaining $\mathcal{H}^d[G_b,U_T(1)]$, we only used the equivalence relation for the standard group cohomology class. The labeling is one to one only when $(-)^{f_d}$ happen to be a coboundary in $\mathcal{B}^d[G_b,U_T(1)]$. But, when $(-)^{f_d}$ is not a coboundary in $\mathcal{B}^d[G_b,U_T(1)]$, we need to consider the more general equivalence relation including all possible shifts $(-)^{f_d}$, where f_d is generated by $n_{d-2}(g_0,\ldots,g_{d-2})\in\mathcal{H}^{d-2}(G_b,\mathbb{Z}_2)$, e.g., see Eq. (57) for the precise definition for f_4 and f_5 in terms of $n_2(g_0,g_1,g_2)$ and $n_3(g_0,g_1,g_2,g_3)$. So, the group cohomology description of bosonic SPT phases will collapse into a smaller quotient group $\mathcal{H}^d_{\text{rigid}}[G_b,U_T(1)]=\mathcal{H}^d[G_b,U_T(1)]/\Gamma$, where Γ is a subgroup of $\mathcal{H}^d[G_b,U_T(1)]$ generated by $(-)^{f_d}$. Physically, such a result implies that when we embed interacting bosons systems into interacting fermion systems by viewing bosons as tightly bounded fermion pairs, sometimes, a nontrival bosonic SPT state, described by a cocyle ν_d in $\mathcal{H}^d[G_b,U_T(1)]$, may correspond to a trivial fermionic SPT state since ν_d corresponds to the trivial element in $\mathcal{H}^d_{\text{rigid}}[G_b,U_T(1)]$. We call $\mathcal{H}^d_{\text{rigid}}[G_b,U_T(1)]$ as a *rigid* center, which is a normal subgroup of the standard group cohomology class $\mathcal{H}^d[G_b,U_T(1)]$. Fortunately, such an additional complication only happens when $d\geqslant 4$ since $f_d\equiv 0$ for all $d<4$, and we have $\mathcal{H}^d_{\text{rigid}}[G_b,U_T(1)]\equiv\mathcal{H}^d[G_b,U_T(1)]$ for all $d<4$. We see that, for each element (such as the trivial element) in $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, the class of solutions is described by $\mathcal{H}^d_{\text{rigid}}[G_b,U_T(1)]$. This leads to the following short exact sequence:

$$\begin{aligned}0&\to\mathcal{H}^d_{\text{rigid}}[G_b,U_T(1)]\to\mathscr{H}^d[G_f,U_T(1)]\\&\to B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)\to 0,\end{aligned} \tag{61}$$

which completely determines $\mathscr{H}^d[G_f,U_T(1)]$. Roughly speaking,

$$\mathscr{H}^d[G_f,U_T(1)]=\mathcal{H}^d_{\text{rigid}}[G_b,U_T(1)]\times B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2). \tag{62}$$

We see that, through the above four steps and combining with results from the standard group cohomology, we can calculate the group supercohomology classes $\mathscr{H}^d[G_f,U_T(1)]$. In Appendix D, we will further prove the group structure of the supercohomology classes $\mathscr{H}^d[G_f,U_T(1)]$. In the following, we summarize how to describe a minimal set of fermionic SPT phases by using (special) group supercohomology class in 0, 1, 2, and 3 spatial dimensions with arbitrary $G_b=G_f/Z_2^f$.

TABLE III. Computing (special) group supercohomology class by using short exact sequence.

d_{sp}	Short exact sequence
0	$0 \to \mathcal{H}^1[G_b,U_T(1)] \to \mathscr{H}^1[G_f,U_T(1)] \to \mathbb{Z}_2 \to 0$
1	$0 \to \mathcal{H}^2[G_b,U_T(1)] \to \mathscr{H}^2[G_f,U_T(1)] \to \mathcal{H}^1(G_b,\mathbb{Z}_2) \to 0$
2	$0 \to \mathcal{H}^3[G_b,U_T(1)] \to \mathscr{H}^3[G_f,U_T(1)] \to B\mathcal{H}^2(G_b,\mathbb{Z}_2) \to 0$
3	$0 \to \mathcal{H}^4_{\text{rigid}}[G_b,U_T(1)] \to \mathscr{H}^4[G_f,U_T(1)] \to B\mathcal{H}^3(G_b,\mathbb{Z}_2) \to 0$

In $d_{sp}=0$ spatial dimension, the elements in $\mathscr{H}^1[G_f,U_T(1)]$ can always have the trivial graded structure $n_0(g_0)=0$, or the nontrivial graded structure $n_0(g_0)=1$. The corresponding fermionic gapped states can have even or odd numbers of fermions. So, we can have two different fermionic SPT phases in $d_{sp}=0$ spatial dimension even without symmetry. In $d_{sp}=1$ spatial dimension, $\mathscr{H}^2[G_f,U_T(1)]$ is just an extension of the graded structure $\mathcal{H}^1(G_b,\mathbb{Z}_2)$ by the standard group cohomology class $\mathcal{H}^2[G_b,U_T(1)]$. In $d_{sp}=2$ spatial dimensions, $\mathscr{H}^3[G_f,U_T(1)]$ is just an extension of the graded structure $B\mathcal{H}^2(G_b,\mathbb{Z}_2)$ by the standard group cohomology class $\mathcal{H}^3[G_b,U_T(1)]$. In $d_{sp}=3$ spatial dimensions, $\mathscr{H}^4[G_f,U_T(1)]$ is just an extension of the graded $B\mathcal{H}^3(G_b,\mathbb{Z}_2)$ by $\mathcal{H}^4_{\text{rigid}}[G_b,U_T(1)]$, the rigid center of the standard group cohomology class $\mathcal{H}^4[G_b,U_T(1)]$ (see Table III).

VII. IDEAL GROUND STATE WAVE FUNCTION

In the following, we will show that we can construct an exactly solvable local fermionic Hamiltonian in d spatial dimensions from each $(d+1)$-cocycle (ν_{d+1},n_d,u_d^g) in $\mathscr{Z}^{d+1}[G_f,U_T(1)]$. The Hamiltonian has a symmetry G_f. The ground state wave function of the constructed Hamiltonian can also be obtained exactly from the $(d+1)$-cocycle. Such a ground state does not break the symmetry G_f and describes a fermionic SPT phase.

We have shown that from each element (ν_{d+1},n_d,u_d^g) of $\mathscr{Z}^{d+1}[G_f,U_T(1)]$, we can define a fermionic topological nonlinear σ model in $(d+1)$ space-time dimensions. The action amplitude $\mathcal{V}^{\pm}_{d+1}$ of the model [obtained from (ν_{d+1},n_d,u_d^g)] is a fixed-point action amplitude under the coarse-graining transformation of the space-time complex. The fermionic path integral is supposed to give us a quantum ground state. We claim that such a quantum ground state is a fermionic SPT phase described by (ν_{d+1},n_d,u_d^g). In the section, we will construct the ground state wave function, in (2+1)D as an example.

A. Construction of 2D wave function

We will assume that our 2D system lives on a triangular lattice which forms a 2D torus (see Fig. 14). On each lattice site i, we have physical states $|g_i\rangle$ labeled by $g_i \in G_b$. On each triangle (ijk), we have two states: no-fermion state $|0\rangle$ and one-fermion state $|1\rangle$.

The ideal ground state wave function can be obtained by viewing the 2D torus as the surface of a 3D solid torus. The fermionic path integral on the 3D solid torus with the action amplitude $\mathcal{V}^{\pm}_{d+1}$ obtained from the cocycle (ν_{d+1},n_d,u_d^g) should give us the ground state wave function for the corresponding fermionic SPT state. To do the fermionic path integral, we need to divide the 3D solid torus into a 3D complex with

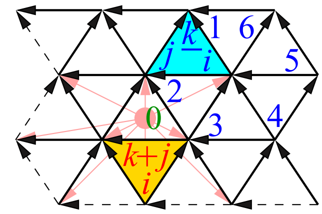


FIG. 14. (Color online) A nine-site triangular lattice on a torus, where each site i has physical states $|g_i\rangle$ labeled by $g_i \in G_b$, and each triangle has fermionic states $|n_{ijk}\rangle$ where $n_{ijk}=0,1$ is the fermion occupation number. The orientations on the edges give rise to a natural order of the three vertices of a triangle (i,j,k) where the first vertex i of a triangle has two outgoing edges on the triangle and the last vertex k of a triangle has two incoming edges on the triangle. The triangular lattice can be viewed as the surface of solid torus. A discretization of the solid torus can be obtained by adding a vertex-0 inside the solid torus. The branching structure of the resulting complex is indicated by the arrows on the edges.

a branching structure. Due to the topological invariance of our constructed action amplitude, the resulting wave function should not depend on how we divide the 3D solid torus into 3D complex. So, we choose a very simple 3D complex which is formed by the triangular lattice on the surface of the 3D solid torus and one additional vertex-0 inside the 3D solid torus. The resulting 3D complex is formed by simplexes $[0ijk]$, where ijk is a triangle on the surface (see Fig. 14). The branching structure of the 3D complex is given by the orientations on the edges. Those orientations for the edges on the surface are given in Fig. 14. For the edges inside the 3D solid torus, the orientation is always pointing from vertex-0 to the vertex on the surface. We note that the simplexes associated with the down triangles in Fig. 14 have a "+" orientation while the simplexes associated with the up triangles have a "−" orientation.

From each 3-cocycle (ν_3,n_2,u_2^g) in $\mathscr{Z}^3[G_f,U_T(1)]$, we can construct a fixed-point action amplitude. For simplexes with "+" orientation, the fixed-point action amplitude is given by

$$\mathcal{V}_3^+(g_0,g_i,g_j,g_k) = \nu_3^+(g_0,g_i,g_j,g_k) \times \theta_{(123)}^{n_2(g_i,g_j,g_k)}\theta_{(013)}^{n_2(g_0,g_i,g_k)}\bar\theta_{(023)}^{n_2(g_0,g_j,g_k)}\bar\theta_{(012)}^{n_2(g_0,g_i,g_j)} \tag{63}$$

and for simplexes with "−" orientation given by

$$\mathcal{V}_3^-(g_0,g_i,g_j,g_k) = \nu_3^-(g_0,g_i,g_j,g_k) \times \theta_{(012)}^{n_2(g_0,g_i,g_j)}\theta_{(023)}^{n_2(g_0,g_j,g_k)}\bar\theta_{(013)}^{n_2(g_0,g_i,g_k)}\bar\theta_{(123)}^{n_2(g_i,g_j,g_k)}, \tag{64}$$

where

$$\begin{aligned}\nu_3^+(g_0,g_i,g_j,g_k) &= (-)^{m_1(g_0,g_j)}\nu_3(g_0,g_i,g_j,g_k),\\ \nu_3^-(g_0,g_i,g_j,g_k) &= (-)^{m_1(g_i,g_k)}/\nu_3(g_0,g_i,g_j,g_k).\end{aligned} \tag{65}$$

Since $n_2(g_0,g_1,g_2)$ is in $\mathcal{H}^2(G,\mathbb{Z}_2)$, we can always write $n_2(g_0,g_1,g_2)$ in terms of $m_1(g_0,g_1)$:

$$\begin{aligned}n_2(g_0,g_1,g_2) &= m_1(g_1,g_2)+m_1(g_0,g_2)+m_1(g_0,g_1) \bmod 2,\\ m_1(g_0,g_1) &= 0,1.\end{aligned} \tag{66}$$

However, in general $m_1(gg_0,gg_1) \neq m_1(g_0,g_1)$, even though $n_2(g_0,g_1,g_2)$ satisfies $n_2(gg_0,gg_1,gg_2)=n_2(g_0,g_1,g_2)$.

Now, the wave function is given by

$$\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\}) = \int \prod_{(0ij)} d\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} d\bar\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} \prod_{\{0i\}} (-)^{m_1(g_0,g_i)} \times \prod_{\triangle} \mathcal{V}_3^-(g_0,g_i,g_j,g_k) \prod_{\bigtriangledown} \mathcal{V}_3^+(g_0,g_i,g_j,g_k), \quad (67)$$

where $\prod_{(0ij)}$ is the product over all links of the triangular lattice, $\prod_{\{0i\}}$ is the product over all sites of the triangular lattice, $\prod_\triangle$ is the product over all up triangles, and $\prod_\bigtriangledown$ is the product over all down triangles. Clearly, the above wave function depends on g_i through ν_3, n_2, and m_1. The wave function also appears to depend on g_0, the variable that we assigned to the vertex-0 inside the solid torus. In fact, the g_0 dependence cancels out, and the wave function is independent of g_0. We can simply set $g_0 = 1$.

If all the m_1 dependence also cancels out, the wave function will have a symmetry described by G_f: $\Psi(\{gg_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\}) = \Psi^{s(g)}(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\})$. But does the m_1 dependence cancel out? Let us only write the m_1 dependence of the wave function:

$$\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\}) = \prod_{\{0i\}} (-)^{m_1(g_0,g_i)} \prod_{\triangle} (-)^{m_1(g_i,g_k)} \prod_{\bigtriangledown} (-)^{m_1(g_0,g_j)} \dots = \prod_{\triangle} (-)^{m_1(g_i,g_k)} \dots, \quad (68)$$

where ijk around the up and down triangles are arranged in a way as illustrated in Fig. 14. We also have used the relation $\prod_{\{0i\}}(-)^{m_1(g_0,g_i)} \prod_\bigtriangledown (-)^{m_1(g_0,g_j)} = 1$. We see that m_1 does not cancel out and the wave function is not symmetric since $m_1(gg_0,gg_1) \neq m_1(g_0,g_1)$.

This is a serious problem, but the symmetry breaking is only on the surface and it can be easily fixed: we simply redefine the wave function by including an extra factor $\prod_{\text{up-left}}(-)^{m_1(g_i,g_k)}$ on the surface:

$$\begin{aligned}\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\}) &= \prod_{\text{up-left}} (-)^{m_1(g_i,g_k)} \int \prod_{(0ij)} d\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} d\bar\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} \\ &\times \prod_{\{0i\}} (-)^{m_1(g_0,g_i)} \prod_\triangle \mathcal{V}_3^-(g_0,g_i,g_j,g_k) \prod_\bigtriangledown \mathcal{V}_3^+(g_0,g_i,g_j,g_k) \\ &= \int \prod_{(0ij)} d\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} d\bar\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} \\ &\times \prod_\triangle \nu_3^{-1}(g_0,g_i,g_j,g_k) \prod_\bigtriangledown \nu_3(g_0,g_i,g_j,g_k) \\ &\times \prod_\triangle \theta_{(0ij)}^{n_2(g_0,g_i,g_j)} \theta_{(0jk)}^{n_2(g_0,g_j,g_k)} \bar\theta_{(0ik)}^{n_2(g_0,g_i,g_k)} \bar\theta_{(ijk)}^{n_2(g_i,g_j,g_k)} \\ &\times \prod_\bigtriangledown \theta_{(ijk)}^{n_2(g_i,g_j,g_k)} \theta_{(0ik)}^{n_2(g_0,g_i,g_k)} \bar\theta_{(0jk)}^{n_2(g_0,g_j,g_k)} \bar\theta_{(0ij)}^{n_2(g_0,g_i,g_j)}, \end{aligned} \quad (69)$$

where $\prod_{\text{up-left}}$ is a product over all the links with the up-left orientation [note that $\prod_{\text{up-left}}(-)^{m_1(g_i,g_k)} = \prod_\triangle (-)^{m_1(g_i,g_k)}$ and see Fig. 14]. The redefined wave function is indeed symmetric, and is the wave function that corresponds to the fermionic SPT state described by the cocycle $(\nu_3,n_2) \in \mathscr{Z}^3[G_f,U_T(1)]$.

We note that the wave function $\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\})$ described above contains Grassmann numbers. Indeed, $\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\})$ can be regarded as the wave function in the fermion coherent state basis. After we expand the wave function in power of the Grassmann numbers, we obtain

$$\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\}) = \sum_{n_{ijk}=0,1} \Phi(\{g_i\},\{n_{ijk}\}) \prod_\triangle \bar\theta_{(ijk)}^{n_{ijk}} \prod_\bigtriangledown \theta_{(ijk)}^{n_{ijk}}. \quad (70)$$

Then, $\Phi(\{g_i\},\{n_{ijk}\})$ is the amplitude of the ground state on the fermion-number basis $\otimes_i |g_i\rangle \otimes_{(ijk)} |n_{ijk}\rangle$, where $|g_i\rangle$ is the state on site i and $|n_{ijk}\rangle$ is the state on triangle (ijk) [where $n_{ijk} = 0,1$ is the fermion occupation number on the triangle (ijk)]. Note that the sign of $\Phi(\{g_i\},\{n_{ijk}\})$ will depend on how the Grassmann numbers are ordered in $\prod_\triangle \bar\theta_{(ijk)}^{n_{ijk}} \prod_\bigtriangledown \theta_{(ijk)}^{n_{ijk}}$. We also note that the wave function vanishes if $n_{ijk} \neq n_2(g_i,g_j,g_k)$.

B. No intrinsic topological orders

In this section, we are going to show that the wave function constructed in the last section $\Phi(\{g_i\},\{n_{ijk}\})$ contains no intrinsic topological orders as a fermion system. In other words, starting with the following pure bosonic direct product state

$$|\Phi_0\rangle = \otimes_i |\phi_i\rangle, \quad |\phi_i\rangle = |G_b|^{-1/2} \sum_{g_i \in G_b} |g_i\rangle, \quad (71)$$

we can obtain the fermionic state $|\Phi\rangle$ constructed in the last section after a fermionic LU transformation defined in Ref. [27].

To show this, we start with an expanded fermionic Hilbert space, where we have four fermionic orbitals within each triangle. We also have bosonic state $|g_i\rangle$ on each vertex (see Fig. 15). In the Grassmann number form, the constructed fermionic

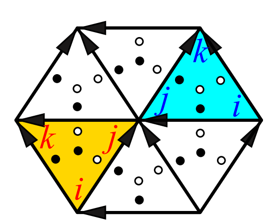


FIG. 15. (Color online) A triangular lattice with four fermionic orbitals in each triangle (ijk). The fermions on the two solid dots in the yellow triangle are described by operators $c_{(ijk)}$ (the center one) and $c_{(0ik)}$ (the side one), and the fermions on the two open dots are described by operators $\bar c_{(0ij)}$ and $\bar c_{(0jk)}$. The fermions on the two open dots in the blue triangle are described by operators $\bar c_{(ijk)}$ (the center one) and $\bar c_{(0ik)}$ (the side one), and the fermions on the two solid dots are described by operators $c_{(0ij)}$ and $c_{(0jk)}$. Each vertex has bosonic states described by $|g_i\rangle$, $g_i \in G_b$.

wave function is given by Eq. (69):

$$\begin{aligned}\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\}) &= \int \prod_{(0ij)} d\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} d\bar\theta_{(0ij)}^{n_2(g_0,g_i,g_j)} \\ &\times \prod_{\triangle} \nu_3^{-1}(g_0,g_i,g_j,g_k) \prod_{\triangledown} \nu_3(g_0,g_i,g_j,g_k) \\ &\times \prod_{\triangle} \theta_{(0ij)}^{n_2(g_0,g_i,g_j)} \theta_{(0jk)}^{n_2(g_0,g_j,g_k)} \bar\theta_{(0ik)}^{n_2(g_0,g_i,g_k)} \bar\theta_{(ijk)}^{n_2(g_i,g_j,g_k)} \\ &\times \prod_{\triangledown} \theta_{(ijk)}^{n_2(g_i,g_j,g_k)} \theta_{(0ik)}^{n_2(g_0,g_i,g_k)} \bar\theta_{(0jk)}^{n_2(g_0,g_j,g_k)} \bar\theta_{(0ij)}^{n_2(g_0,g_i,g_j)}. \end{aligned} \tag{72}$$

We would like to point out that although the above expression contains g_0, the topological invariance of the fermion path integral ensures that the Grassmann wave function on the left-hand side does not depend on g_0.

If we treat θ's and $\bar\theta$'s as the following complex fermion operators

$$\begin{aligned} \theta_{(abc)} &= c_{(abc)}^\dagger, \quad \bar\theta_{(abc)} = \bar c_{(abc)}^\dagger, \\ d\theta_{(abc)} &= c_{(abc)}, \quad d\bar\theta_{(abc)} = \bar c_{(abc)}, \end{aligned} \tag{73}$$

the expression (72) can be viewed as an operator

$$\begin{aligned}\hat U &= \prod_{(0ij)} c_{(0ij)}^{n_2(g_0,g_i,g_j)} \bar c_{(0ij)}^{n_2(g_0,g_i,g_j)} \\ &\times \prod_{\triangle} \nu_3^{-1}(g_0,g_i,g_j,g_k) \prod_{\triangledown} \nu_3(g_0,g_i,g_j,g_k) \\ &\times \prod_{\triangle} c_{(0ij)}^{\dagger n_2(g_0,g_i,g_j)} c_{(0jk)}^{\dagger n_2(g_0,g_j,g_k)} \bar c_{(0ik)}^{\dagger n_2(g_0,g_i,g_k)} \bar c_{(ijk)}^{\dagger n_2(g_i,g_j,g_k)} \\ &\times \prod_{\triangledown} c_{(ijk)}^{\dagger n_2(g_i,g_j,g_k)} c_{(0ik)}^{\dagger n_2(g_0,g_i,g_k)} \bar c_{(0jk)}^{\dagger n_2(g_0,g_j,g_k)} \bar c_{(0ij)}^{\dagger n_2(g_0,g_i,g_j)}. \end{aligned} \tag{74}$$

Again, $\hat U$ is independent of g_0, despite the appearance of g_0 on the right-hand side. Then, the fermionic state constructed in the last section can be obtained from the bosonic product state $|\Phi_0\rangle$:

$$|\Psi\rangle = \hat U|\Phi_0\rangle. \tag{75}$$

Note that $|\Phi_0\rangle$ is a "no-fermion" state satisfying

$$c_{(abc)}|\Phi_0\rangle = 0. \tag{76}$$

Now we would like to point out that $\hat U$ itself is formed by several layers of *fermionic* LU transformations. Since ν_3 is a pure $U(1)$ phase, thus $\prod_\triangle \nu_3^{-1}(g_0,g_i,g_j,g_k)\prod_\triangledown \nu_3(g_0,g_i,g_j,g_k)$ represents layers of bosonic LU transformations. Also, the $\hat U$ has a property that when acting on $|\Phi_0\rangle$, $c^\dagger$ and $\bar c^\dagger$ always act on states with no fermion and c and $\bar c$ always act on states with one fermion. So, those operators map a set of orthogonal states to another set of orthogonal states. In this case, an even number of c's and $\bar c$'s correspond to a fermionic LU transformation. Therefore, the terms in $\hat U$, such as $c_{(ijk)}^{\dagger n_2(g_i,g_j,g_k)} c_{(0ik)}^{\dagger n_2(g_0,g_i,g_k)} \bar c_{(0jk)}^{\dagger n_2(g_0,g_j,g_k)} \bar c_{(0ij)}^{\dagger n_2(g_0,g_i,g_j)}$ and $c_{(0ij)}^{n_2(g_0,g_i,g_j)} \bar c_{(0ij)}^{n_2(g_0,g_i,g_j)}$, all represent fermionic LU transformation as defined in [27]. Note that none of the above transformations change g_i. So, we can show those transformations to be unitary within each fixed set of $\{g_i\}$. Therefore, the state $|\Psi\rangle$ has no fermionic long-range entanglement as defined in [27] (i.e., no fermionic intrinsic topological order). $|\Psi\rangle$ is the fermionic SPT state described by a cocycle (ν_3,n_2,u_2^g).

The fermionic LU transformation (74) that maps the fermionic SPT state to a trivial product state is one of the most important results in this paper. All the properties of the fermionic SPT state as well as the classification of the fermionic SPT states can be described in terms of the fermionic LU transformation. The fact that the fermionic LU transformation is expressed in terms of group supercohomology gives us a systematic understanding of fermionic SPT states in terms of group supercohomology.

VIII. IDEAL HAMILTONIANS THAT REALIZE THE FERMIONIC SPT STATES

After obtaining the wave function $|\Psi\rangle = \hat U|\Phi_0\rangle$ for the fermionic SPT state, it is easy to construct a Hamiltonian H such that $|\Psi\rangle$ is its ground state. We start with a non-negative definite Hermitian operator H_0 that satisfies $H_0|\Phi_0\rangle = 0$. For example, we may choose

$$H_0 = \sum_i (1 - |\phi_i\rangle\langle\phi_i|). \tag{77}$$

The Hamiltonian H can then be obtained as

$$\begin{aligned} H &= VH' + \sum_i H_i, \\ H_i &= \hat U(1 - |\phi_i\rangle\langle\phi_i|)\hat U^\dagger, \\ H' &= \sum_\triangledown [c_{(ijk)}^\dagger c_{(ijk)} - n_2(g_i,g_j,g_k)]^2 \\ &\quad + \sum_\triangle [\bar c_{(ijk)}^\dagger \bar c_{(ijk)} - n_2(g_i,g_j,g_k)]^2. \end{aligned} \tag{78}$$

When V is positive and very large, the H' enforces that the fermion number on each triangle (ijk) is given by $n_2(g_i,g_j,g_k)$. Since $\hat U^\dagger \hat U|\Phi_0\rangle = |\Phi_0\rangle$, one can easily show that H is non-negative definite and $H|\Psi\rangle = H\hat U|\Phi_0\rangle = 0$.

Similar to the bosonic case, H_i acts on site i as well as its six neighbors. However, since there are six more fermionic degrees of freedom in the six triangles surrounding i, H_i also acts on these six triangles. Moreover, when V is positive and very large, the low states are the zero energy subspace of H'. Within such a low energy subspace, all the H_i are Hermitian (unconstrained)commuting projectors satisfying $H_i^2 = H_i$ and $H_iH_j = H_jH_i$. [We note that in the zero energy subspace of H', $H_i^2 = \hat U(1 - |\phi_i\rangle\langle\phi_i|)\hat U^\dagger \hat U(1 - |\phi_i\rangle\langle\phi_i|)\hat U^\dagger = \hat U(1 - |\phi_i\rangle\langle\phi_i|)\hat U^\dagger = H_i$ and $H_iH_j = \hat U(1 - |\phi_i\rangle\langle\phi_i|)(1 - |\phi_j\rangle\langle\phi_j|)\hat U^\dagger = \hat U(1 - |\phi_j\rangle\langle\phi_j|)(1 - |\phi_i\rangle\langle\phi_i|)\hat U^\dagger = H_jH_i$.] Such a nice property (frustration free) makes it exactly solvable, with $\Psi(\{g_i\},\{\theta_{(ijk)}\},\{\bar\theta_{(ijk)}\})$ as its unique ground state.

In Appendix H, we provide an alternative way to construct the parent Hamiltonian from path integral formulism, which is equivalent to the construction here in the infinity V limit.

IX. AN EXAMPLE OF 2D FERMIONIC SPT STATES WITH A Z_2 SYMMETRY

A. Ideal Hamiltonians that realize the fermionic 2D Z_2 SPT states

In this section, we will apply the above general discussion to a particular example: the fermionic 2D Z_2 SPT states. The 3-cocycles that describe the three nontrivial fermionic 2D Z_2 SPT states are given in Sec. IX B. All those SPT phases can be realized on a triangle lattice as described in Fig. 14. Each site has two bosonic states $|g_i\rangle$, $g_i = 0,1$, and each triangle has a fermionic orbital which can occupied $|n_{ijk} = 1\rangle$ or unoccupied $|n_{ijk} = 0\rangle$. The bulk Hamiltonian

$$H = H' + \sum_i H_i \tag{79}$$

that realizes the SPT phase on such a triangle lattice can constructed from the data in the 3-cocycles $[\nu_3(g_0,g_1,g_2,g_3),n_2(g_0,g_1,g_2)]$ as discussed in the above section [see Eq. (78)].

H_i in the Hamiltonian acts on the bosonic states on site i and the six sites 1,2,3,4,5,6 around the site i (see Fig. 14). H_i also acts on the fermionic states on the six triangles $(i21)$, $(3i2)$, $(43i)$, $(45i)$, $(5i6)$, $(i61)$ around the site i. H_i has a property that it does not change g_1, g_2, g_3, g_4, g_5, g_6, but it can change the bosonic state on site i: $|g_i\rangle \to |g_i'\rangle$ and the fermionic states on the triangles (ijk): $|n_{ijk}\rangle \to |n'_{ijk}\rangle$. So, we can express H_i as operator-valued 2×2 matrix $M(g_1,g_2,g_3,g_4,g_5,g_6)$ where the matrix elements are given by

$$\begin{aligned} M_{g_i'g_i}&(g_1,g_2,g_3,g_4,g_5,g_6)\\ &= \langle g_1g_2g_3g_4g_5g_6g_i'|H_i|g_1g_2g_3g_4g_5g_6g_i\rangle. \end{aligned} \tag{80}$$

It turns out that $M(g_1,g_2,g_3,g_4,g_5,g_6)$ are always 2×2 projection matrices

$$\begin{aligned} M&(g_1,g_2,g_3,g_4,g_5,g_6)\\ &= \tfrac{1}{2}v^\dagger(g_1,g_2,g_3,g_4,g_5,g_6)v(g_1,g_2,g_3,g_4,g_5,g_6), \end{aligned} \tag{81}$$

where $v(g_1,g_2,g_3,g_4,g_5,g_6)$ are operator-valued 1×2 matrices that satisfy the Z_2 symmetry condition

$$\begin{aligned} v&(g_1,g_2,g_3,g_4,g_5,g_6)\\ &= v(1-g_1,1-g_2,1-g_3,1-g_4,1-g_5,1-g_6)\sigma^x. \end{aligned} \tag{82}$$

H' in the Hamiltonian is given by

$$H' = U\sum_{(ijk)}[c^\dagger_{(ijk)}c_{(ijk)} - n_2(g_i,g_j,g_k)]^2. \tag{83}$$

It enforces the constraints that the fermion number on each triangle (ijk) is given by $n_2(g_i,g_j,g_k)$ in the ground state. The H_i mentioned above preserves the constraints: $[H_i,H'] = 0$. We see that $n_2(g_i,g_j,g_k)$ describes the fermionic character of the SPT phases. For the SPT phases described by Eq. (88), $n_2(g_i,g_j,g_k) = 0$. So, there is no fermion in those SPT states. Those SPT states are actually bosonic SPT states. For the SPT phases described by Eq. (89), $n_2(g_i,g_j,g_k) \neq 0$. The corresponding SPT states are fermionic SPT states.

We also would like to mention that the constructed Hamiltonian H has an onsite Z_2 symmetry generated by

$$\hat W(g) = \prod_i |g_i\rangle\langle gg_i|, \quad g,g_i \in Z_2. \tag{84}$$

For the trivial phase described by

$$\begin{aligned} &n_2(g_0,g_1,g_2) = 0,\\ &\nu_3(0,1,0,1) = \nu_3(1,0,1,0) = 1, \quad \text{other } \nu_3 = 1, \end{aligned} \tag{85}$$

we find that $v(g_1,g_2,g_3,g_4,g_5,g_6) = (1,-1)$ and $M(g_1,g_2,g_3,g_4,g_5,g_6) = \frac{1}{2}\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix} = H_i$. Note that H_i is a projection operator on the site i. Thus, the ground state of H is a product state $\otimes_i|\phi_i\rangle$, $|\phi_i\rangle \propto |0\rangle + |1\rangle$.

One of the nontrivial SPT phases is described by

$$\begin{aligned} &n_2(g_0,g_1,g_2) = 0,\\ &\nu_3(0,1,0,1) = \nu_3(1,0,1,0) = 1, \quad \text{other } \nu_3 = 1. \end{aligned} \tag{86}$$

It is a bosonic SPT state since $n_2(g_0,g_1,g_2) = 0$. Such a bosonic SPT state can also be viewed as a fermionic SPT state forming tightly bounded fermion pairs. The Hamiltonian for such a bosonic SPT state is given by $v(g_1,g_2,g_3,g_4,g_5,g_6)$ in Table IV. The corresponding $H_i = M(g_1,g_2,g_3,g_4,g_5,g_6)$ are either $\frac{1}{2}\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}$ or $\frac{1}{2}\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}$. H_i are still projection operators. But, now H_i can be two different projection operators depending on the values of g_1,g_2,g_3,g_4,g_5,g_6 on the neighboring sites. Such a bosonic SPT phase is nothing but the Z_2 SPT state studied in [45].

The 3-cocycle

$$\begin{aligned} &n_2(0,1,0) = n_2(1,0,1) = 1, \quad \text{other } n_2 = 0,\\ &\nu_3(0,1,0,1) = \nu_3(1,0,1,0) = i, \quad \text{other } \nu_3 = 1 \end{aligned} \tag{87}$$

describes a nontrivial fermionic SPT state. The Hamiltonian for such a fermionic SPT state is given by $v(g_1,g_2,g_3,g_4,g_5,g_6)$ in Table V. We see that H_i are still projection operators. But, now H_i can be many different projection operators depending on the values of g_1,g_2,g_3,g_4,g_5,g_6 on the neighboring sites. Also, the projection operators mix the bosonic and fermionic states.

B. Edge excitations of 2D fermionic SPT state with Z_2 symmetry

In this section, we will discuss 2D fermionic SPT states with Z_2 symmetry in more detail. In particular, the nontrivial realization of the Z_2 symmetry on the edge states and its protection of gapless edge excitations against interactions.

In Appendix F, we have calculated the group supercohomology classes $\mathcal{H}^3[Z_2\times Z_2^f,U(1)] = \mathbb{Z}_4$. This means that interacting fermion systems with a Z_2 symmetry can have (at least) four different SPT phases: a trivial one plus three nontrivial ones. This result is described by the $Z_2\times Z_2^f$ row and $d_{sp} = 2$ column of Table II.

1. 3-cocycles: Data that characterize the fermionic 2D Z_2 SPT states

The fermionic 2D Z_2 SPT phases are characterized by the data $[\nu_3(g_0,g_1,g_2,g_3),n_2(g_0,g_1,g_2)]$ where $\nu_3(g_0,g_1,g_2,g_3)$

TABLE IV. The 1×2 matrices $v(g_1,g_2,g_3,g_4,g_5,g_6)$ for the bosonic SPT state (86).

$g_1g_2g_3g_4g_5g_6$	$v(g_1,g_2,g_3,g_4,g_5,g_6)$	$g_1g_2g_3g_4g_5g_6$	$v(g_1,g_2,g_3,g_4,g_5,g_6)$
000000	$(1,-1)$	000001	$(1,-1)$
000010	$(1,-1)$	000011	$(-1,-1)$
000100	$(1,-1)$	000101	$(1,-1)$
000110	$(1,1)$	000111	$(-1,1)$
001000	$(1,-1)$	001001	$(1,-1)$
001010	$(1,-1)$	001011	$(-1,-1)$
001100	$(1,1)$	001101	$(1,1)$
001110	$(1,-1)$	001111	$(-1,-1)$
010000	$(1,-1)$	010001	$(1,-1)$
010010	$(1,-1)$	010011	$(-1,-1)$
010100	$(1,-1)$	010101	$(1,-1)$
010110	$(1,1)$	010111	$(-1,1)$
011000	$(-1,-1)$	011001	$(-1,-1)$
011010	$(-1,-1)$	011011	$(1,-1)$
011100	$(-1,1)$	011101	$(-1,1)$
011110	$(-1,-1)$	011111	$(1,-1)$

is a complex function and $n_2(g_0,g_1,g_2)$ an integer function with variables $g_i\in G_b=G_f/Z_2^f=Z_2$. The data $[\nu_3(g_0,g_1,g_2,g_3),n_2(g_0,g_1,g_2)]$ are called a fermionic 3-cocycle, which is an element in $\mathcal{H}^3[Z_2\times Z_2^f,U(1)]$. The first two SPT phases are given by

$$n_2(g_0,g_1,g_2)=0,$$
$$\nu_3(0,1,0,1)=\nu_3(1,0,1,0)=\pm 1,\quad \text{other } \nu_3=1, \tag{88}$$

and the next two SPT phases are given by

$$n_2(0,1,0)=n_2(1,0,1)=1,\quad \text{other } n_2=0,$$
$$\nu_3(0,1,0,1)=\nu_3(1,0,1,0)=\pm i,\quad \text{other } \nu_3=1, \tag{89}$$

where we have assumed that the elements in $G_b=Z_2$ are described by {0,1} with 0 being the identity element. The 3-cocycle $n_2(g_0,g_1,g_2)=0$, $\nu_3(g_0,g_1,g_2,g_3)=1$ corresponds to the trivial fermionic SPT phase.

Using the above 3-cocycle data $[\nu_3(g_0,g_1,g_2,g_3),n_2(g_0,g_1,g_2)]$ we can construct the ideal wave functions that realize the above fermionic SPT phases (see Sec. VII). We can also construct the local Hamiltonian (see Sec. VIII) such that the above ideal wave functions are the exact ground states.

2. *Low energy edge excitation: Their effective symmetry and effective Hamiltonian*

The nontrivial 2D Z_2 SPT states described by the above three 3-cocycles have symmetry-protected gapless edge excitations. The detailed discussions of those gapless edge excitations are presented in Appendices G 3 and G 4. Here, we just present the results.

The low energy edge excitations of the SPT phase can be described by an effective Hamiltonian $H_{\text{eff}}=\sum_{i=1}^{L}H_{\text{eff}}(i)$,

TABLE V. The 1×2 matrices $v(g_1,g_2,g_3,g_4,g_5,g_6)$ for the fermionic SPT state (87).

$g_1g_2g_3g_4g_5g_6$	$v(g_1,g_2,g_3,g_4,g_5,g_6)$	$g_1g_2g_3g_4g_5g_6$	$v(g_1,g_2,g_3,g_4,g_5,g_6)$
000000	$(1,-\bar c_{(5i6)}c_{(3i2)})$	000001	$(c_{(i61)},-c_{(3i2)})$
000010	$(c_{(45i)},-c_{(3i2)})$	000011	$(ic_{(i61)}\bar c_{(5i6)}c_{(45i)},-c_{(3i2)})$
000100	$(1,-\bar c_{(5i6)}c_{(45i)}\bar c_{(43i)}c_{(3i2)})$	000101	$(c_{(i61)},-c_{(45i)}\bar c_{(43i)}c_{(3i2)})$
000110	$(1,i\bar c_{(43i)}c_{(3i2)})$	000111	$(ic_{(i61)}\bar c_{(5i6)},i\bar c_{(43i)}c_{(3i2)})$
001000	$(\bar c_{(43i)},-\bar c_{(5i6)})$	001001	$(c_{(i61)}\bar c_{(43i)},-1)$
001010	$(c_{(45i)}\bar c_{(43i)},-1)$	001011	$(ic_{(i61)}\bar c_{(5i6)}c_{(45i)}\bar c_{(43i)},-1)$
001100	$(1,-i\bar c_{(5i6)}c_{(45i)})$	001101	$(c_{(i61)},-ic_{(45i)})$
001110	$(1,-1)$	001111	$(ic_{(i61)}\bar c_{(5i6)},-1)$
010000	$(\bar c_{(i21)},-\bar c_{(5i6)})$	010001	$(c_{(i61)}\bar c_{(i21)},-1)$
010010	$(-\bar c_{(i21)}c_{(45i)},-1)$	010011	$(ic_{(i61)}\bar c_{(i21)}\bar c_{(5i6)}c_{(45i)},-1)$
010100	$(\bar c_{(i21)},-\bar c_{(5i6)}c_{(45i)}\bar c_{(43i)})$	010101	$(c_{(i61)}\bar c_{(i21)},-c_{(45i)}\bar c_{(43i)})$
010110	$(\bar c_{(i21)},i\bar c_{(43i)})$	010111	$(-ic_{(i61)}\bar c_{(i21)}\bar c_{(5i6)},i\bar c_{(43i)})$
011000	$(-i\bar c_{(i21)}\bar c_{(43i)}c_{(3i2)},-\bar c_{(5i6)})$	011001	$(-ic_{(i61)}\bar c_{(i21)}\bar c_{(43i)}c_{(3i2)},-1)$
011010	$(i\bar c_{(i21)}c_{(45i)}\bar c_{(43i)}c_{(3i2)},-1)$	011011	$(c_{(i61)}\bar c_{(i21)}\bar c_{(5i6)}c_{(45i)}\bar c_{(43i)}c_{(3i2)},-1)$
011100	$(i\bar c_{(i21)}c_{(3i2)},-i\bar c_{(5i6)}c_{(45i)})$	011101	$(ic_{(i61)}\bar c_{(i21)}c_{(3i2)},-ic_{(45i)})$
011110	$(i\bar c_{(i21)}c_{(3i2)},-1)$	011111	$(c_{(i61)}\bar c_{(i21)}\bar c_{(5i6)}c_{(3i2)},-1)$

where L is the number of sites on the edge and the states on each edge site are described by $|g_i\rangle$, $g_i \in G_b$, $i = 1,2,\ldots,L$. We use $|\{g_i\}_{\text{edge}},g_0\rangle$ to denote a low energy edge state (where g_0 takes a fixed value, say $g_0 = 0$). Since the edge is 1D, we can use a purely bosonic model to describe the edge states of 2D fermion system (see Appendix G 3).

For the nontrivial SPT phase described by 3-cocycles $[\nu_3(g_0,g_1,g_2,g_3),n_2(g_0,g_1,g_2)]$, the low energy effective edge Hamiltonian H_{eff} satisfies an unusual symmetry condition (see Appendix G 4):

$$W_{\text{eff}}^\dagger(g)H_{\text{eff}}(i)W_{\text{eff}}(g) = H_{\text{eff}}(i),$$
$$W_{\text{eff}}(g)|\{g_i\}_{\text{edge}},g_0\rangle = -(-)^{n_2(g^{-1}g_0,g_0,g_1)\sum_{i=1}^L[n_2(g_0,g_{i+1},g_i)+1]} \times \prod_i w^*_{i,i+1}|\{gg_i\}_{\text{edge}},g_0\rangle, \tag{90}$$

where

$$w_{i,i+1} = i^{n_2(g^{-1}g_0,g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)+2n_2(g^{-1}g_0,g_0,g_i)} \times \nu_3(g^{-1}g_0,g_0,g_{i+1},g_i). \tag{91}$$

We see that the effective edge symmetry $W_{\text{eff}}(g)$ on the low energy edge states is determined by the cocycle data $[\nu_3(g_0,g_1,g_2,g_3),n_2(g_0,g_1,g_2)]$. Due to the $w_{i,i+1}$ factor, the symmetry is not an onsite symmetry. Such a symmetry can protect the gaplessness of the edge excitations if the symmetry is not spontaneously broken on the edge.

For the trivial SPT phase (85), we find that the symmetry action on the edge states is given by $W_{\text{eff}}(g)|\{g_i\}_{\text{edge}},g_0\rangle = |\{gg_i\}_{\text{edge}},g_0\rangle$ which is an onsite symmetry. Such an onsite symmetry cannot protect the gapless edge excitations: the edge excitations can be gapped without breaking the symmetry.

For the nontrivial bosonic SPT phase (86), we find the symmetry action on the edge states to be

$$W_{\text{eff}}(g)|\{g_i\}_{\text{edge}},g_0\rangle = \prod_i w^*_{i,i+1}|\{gg_i\}_{\text{edge}},g_0\rangle, \tag{92}$$

which is not an onsite symmetry, with $w_{i,i+1}(g_i,g_{i+1})$ given by

$$w_{i,i+1}(0,0) = 1, \quad w_{i,i+1}(0,1) = -1,$$
$$w_{i,i+1}(1,0) = 1, \quad w_{i,i+1}(1,1) = 1. \tag{93}$$

Such a non-onsite symmetry can protect the gapless edge excitations: the edge excitations must be gapless without breaking the symmetry [45].

If we identify $|0\rangle$ as $|\uparrow\rangle$ and $|1\rangle$ as $|\downarrow\rangle$, we may rewrite the above in an operator form

$$w_{i,i+1} = e^{i\pi[\frac14(\sigma_i^z+1)(\sigma_{i+1}^z-1)]},$$
$$W_{\text{eff}}(1) = \left[\prod_i \sigma_i^x\right]\left[\prod_i e^{i\pi[-\frac14-\frac14\sigma_i^z+\frac14\sigma_{i+1}^z+\frac14\sigma_i^z\sigma_{i+1}^z]}\right] = \left[\prod_i \sigma_i^x\right]\left[\prod_i e^{i\pi[-\frac14+\frac14\sigma_i^z\sigma_{i+1}^z]}\right]. \tag{94}$$

We find

$$[W_{\text{eff}}(1)]^2 = \prod_{i=1}^L e^{i\pi[-\frac12+\frac12\sigma_i^z\sigma_{i+1}^z]} = \prod_{i=1}^L \sigma_i^z\sigma_{i+1}^z = 1. \tag{95}$$

So, $W_{\text{eff}}(0) = 1$ and $W_{\text{eff}}(1)$ indeed form a Z_2 representation. From

$$W_{\text{eff}}^\dagger(1)\sigma_i^x W_{\text{eff}}(1) = -\sigma_{i-1}^z\sigma_i^x\sigma_i^z \tag{96}$$

we find that the following edge Hamiltonian

$$H_{\text{eff}} = \sum_i \left[J_z\sigma_i^z\sigma_{i+1}^z + h_x\left(\sigma_i^x - \sigma_{i-1}^z\sigma_i^x\sigma_{i+1}^z\right)\right] \tag{97}$$

respects the non-onsite Z_2 symmetry on the edge. Such a system either spontaneously breaks the Z_2 symmetry on the edge or has gapless edge excitations [45,61].

For the nontrivial fermionic SPT phase (87), we find the symmetry action on the edge states to be

$$W_{\text{eff}}(g)|\{g_i\}_{\text{edge}},g_0\rangle = (-)^{n_2(g^{-1}g_0,g_0,g_1)\sum_{i=1}^L[n_2(g_0,g_{i+1},g_i)+1]} \times \prod_i w^*_{i,i+1}|\{gg_i\}_{\text{edge}},g_0\rangle, \tag{98}$$

which is also not an onsite symmetry, with $w_{i,i+1}(g_i,g_{i+1})$ given by

$$w_{i,i+1}(0,0) = 1, \quad w_{i,i+1}(0,1) = 1,$$
$$w_{i,i+1}(1,0) = -i, \quad w_{i,i+1}(1,1) = -1. \tag{99}$$

Again, we may rewrite the above in an operator form

$$w_{i,i+1} = e^{i\pi[-\frac38+\frac18\sigma_{i+1}^z+\frac38\sigma_i^z-\frac18\sigma_i^z\sigma_{i+1}^z]}. \tag{100}$$

When $g = 1$ and $g_0 = 0$, we find that $(-)^{n_2(g^{-1}g_0,g_0,g_1)} = \sigma_1^z$. Also, since

$$n_2(g_0,0,0) = 0, \quad n_2(g_0,0,1) = 0,$$
$$n_2(g_0,1,0) = 1, \quad n_2(g_0,1,1) = 0, \tag{101}$$

we find that

$$\sum_{i=1}^L n_2(g_0,g_i,g_{i+1}) = \frac14\sum_{i=1}^L\left(1-\sigma_i^z\sigma_{i+1}^z\right). \tag{102}$$

Therefore,

$$W_{\text{eff}}(1) = \left[\prod_i \sigma_i^x\right]\left[\prod_i e^{i\pi[\frac38-\frac18\sigma_{i+1}^z-\frac38\sigma_i^z+\frac18\sigma_i^z\sigma_{i+1}^z]}\right] \times \left[\sigma_1^z\right]^{\sum_{i=1}^L[\frac14(1-\sigma_i^z\sigma_{i+1}^z)+1]}$$
$$= \left[\prod_i \sigma_i^y\right]\left[\prod_i e^{i\pi[\frac18\sigma_i^z\sigma_{i+1}^z-\frac58]}\right]\left[\sigma_1^z\right]^{\sum_{i=1}^L\frac14(5-\sigma_i^z\sigma_{i+1}^z)}. \tag{103}$$

We note that

$$[W_{\text{eff}}(1)]^2 = e^{i\frac{\pi}{4}\sum_{i=1}^L[5-\sigma_i^z\sigma_{i+1}^z]}\prod_i e^{i\pi[\frac14\sigma_i^z\sigma_{i+1}^z-\frac54]} = 1. \tag{104}$$

We see that $W_{\text{eff}}(g)$ forms a Z_2 representation.

From the expression (G19) of the low energy edge state, we see that the total number of fermions in the low energy edge states $|\{g_i\}_{\text{edge}},g_0\rangle$ is given by

$$N_F = \sum_i n_2(g_a,g_{i+1},g_i) \tag{105}$$

(which is independent of g_a). If we choose $g_a = 0$, we see that only the link $(g_i, g_{i+1}) = (0,1)$ has one fermion, while other links have no fermion. If we choose $g_a = 1$ instead, then only the link $(g_i, g_{i+1}) = (1,0)$ has one fermion, while other links have no fermion. Let us call a link $(i, i+1)$ with $\sigma_i^z \sigma_{i+1}^z = -1$ a domain wall. We see that either only the step-up domain wall has a fermion or the step-down domain wall has a fermion. Since the numbers of step-up domain walls and the step-down domain walls are equal on a ring, the expression $N_F = \sum_i n_2(g_a, g_{i+1}, g_i)$ does not depend on g_a. The number of fermions is equal to half of the number of the domain walls. So, the fermion parity operator $P_f = (-)^{N_F}$ is given by

$$P_f = (-)^{\frac{1}{4}\sum_{i=1}^{L}(1-\sigma_i^z\sigma_{i+1}^z)} = \prod_i e^{i\frac{\pi}{4}(1-\sigma_i^z\sigma_{i+1}^z)}$$
$$= \prod_i e^{i\frac{\pi}{4}}\left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\sigma_i^z\sigma_{i+1}^z\right]. \quad (106)$$

The effective edge Hamiltonian must be invariant under both P_f and $W_{\rm eff}(1)$ transformations.

The edge effective Hamiltonian must be invariant under both the fermion-number parity P_f and the Z_2 symmetry $W_{\rm eff}(1)$ transformations. One example is given by the following (for infinite long edge):

$$H_{\rm edge} = \sum_i \big[- J\sigma_i^z\sigma_{i+1}^z + h_0\sigma_i^x\big(\sigma_{i-1}^z - \sigma_{i+1}^z\big)$$
$$+ h_1\sigma_i^y\big(1 - \sigma_{i-1}^z\sigma_{i+1}^z\big) + ih_2(\sigma_i^+\sigma_{i+2}^+ - \sigma_i^-\sigma_{i+2}^-)$$
$$\times \big(1 + \sigma_{i-1}^z\sigma_{i+1}^z\big)\big(1 + \sigma_{i+1}^z\sigma_{i+3}^z\big)\big], \quad (107)$$

where

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (108)$$

To understand the behavior of such a Hamiltonian, we note that the J, h_0, and h_1 terms in the above $H_{\rm eff}$ cannot change the number of domain walls. The h_0 and h_1 terms can only induce domain wall hopping. So, if $h_2 = 0$, the model will have an effective $U(1)$ symmetry. It has two phases: a gapped phase for large J where $\sigma^z = \pm 1$ and there are no domain walls, and a gapless phase for large h_0, h_1 where the domain walls form a "superfluid." The gapped phase breaks the Z_2 symmetry while the gapless phase is described by a central-charge $c = 1$ nonchiral Luttinger liquid theory.

The h_2 term can only change the domain wall number by ± 4 since the number of fermions on the edge is given by half of the domain wall number. So, the fermion-number-parity conservation only allows the domain wall number to change by a multiple of 4. The h_2 term can be irrelevant. So, the edge excitations can still be gapless.

Now, let us explain why the effective edge Hamiltonian (107) respects the non-onsite Z_2 symmetry (103). We note that $\frac{1}{2}\sum_{i=1}^{L}(1 - \sigma_i^z\sigma_{i+1}^z)$ counts the number of domain walls. Therefore, J, h_0, and h_1 terms in Eq. (107) commute with $\exp(\frac{i\pi}{4}\frac{1}{2}\sum_{i=1}^{L}(1 - \sigma_i^z\sigma_{i+1}^z))$ since those terms do not change the domain wall number, while the h_2 term in Eq. (107) anticommutes with $\exp(\frac{i\pi}{4}\frac{1}{2}\sum_{i=1}^{L}(1 - \sigma_i^z\sigma_{i+1}^z))$ since that term changes the domain wall number by ± 4. We can also show that the J, h_0, and h_1 terms commute with $\prod_{i=1}^{L}\sigma_i^y$, while the h_2 terms anticommute with $\prod_{i=1}^{L}\sigma_i^y$.

For the terms in Eq. (107) that do not act on the site 1, we can treat the σ_1^z in $W_{\rm eff}(1)$ [see Eq. (103)] as a c number. In this case $[\sigma_1^z]^{\sum_{i=1}^{L}\frac{1}{4}(5-\sigma_i^z\sigma_{i+1}^z)}$ are either ± 1 or $\pm P_f$. So, $[\sigma_1^z]^{\sum_{i=1}^{L}\frac{1}{4}(5-\sigma_i^z\sigma_{i+1}^z)}$ always commutes with the terms in Eq. (107) (as long as they do not act on the site 1). After dropping the term $[\sigma_1^z]^{\sum_{i=1}^{L}\frac{1}{4}(5-\sigma_i^z\sigma_{i+1}^z)}$ we find that $W_{\rm eff}(1)$ and $\exp(\frac{i\pi}{4}\frac{1}{2}\sum_{i=1}^{L}(1 - \sigma_i^z\sigma_{i+1}^z))\prod_{i=1}^{L}\sigma_i^y$ only differ by a phase. This way, we show that the edge Hamiltonian is invariant under P_f and $W_{\rm eff}(1)$.

3. Stability and instability of edge theory

Here, we make use of the method developed in [61] to study the stability and instability of the edge theory for the nontrivial fermionic SPT phase. As we know, the $c = 1$ nonchiral Luttinger liquid can be described as

$$\mathcal{L} = \frac{1}{4\pi}(\partial_x\theta\partial_t\phi + \partial_x\phi\partial_t\theta) - \frac{v}{8\pi}\left[K(\partial_x\theta)^2 + \frac{4}{K}(\partial_x\phi)^2\right] \quad (109)$$

with Luttinger parameter K and velocity v.

The key step for understanding the stability and instability of edge theory is to figure out how the low energy fields θ and ϕ transform under the non-onsite Z_2 symmetry. Let us first introduce the domain wall representation

$$\tau_i^z = \sigma_i^z\sigma_{i+1}^z. \quad (110)$$

In principle, we can reexpress everything in terms of the τ's. However, the above duality transformation does not quite work for a system with periodic boundary conditions since the τ_i^z variables obey the global constraint $\prod_{i=1}^{L}\tau_i^z = 1$, and therefore only describe $L - 1$ independent degrees of freedom.

In order to incorporate the missing degree of freedom and make the dual description complete, we introduce an additional Z_2 gauge field $\mu_{i-1,i}^z$ that lives on the links connecting neighboring boundary sites $i - 1, i$. We then define the duality transformation between σ and τ, μ by the relation

$$\mu_{i-1,i}^x = \sigma_i^z \quad (111)$$

together with the gauge invariance constraint

$$\mu_{i-1,i}^x\mu_{i,i+1}^x\tau_i^z = 1. \quad (112)$$

It is easy to check that this duality transformation is complete: there is a one-to-one correspondence between configurations of $\sigma_i^z = \pm 1$ and configurations of $\mu_i^x = \pm 1, \tau_i^z = \pm 1$ obeying the constraint (112). Similarly, there is a one-to-one correspondence between physical operators written in terms of the σ's and gauge invariant combinations of μ, τ [i.e., operators that commute with the left-hand side of (112)]. In particular, the operators $\sigma^x, \sigma^y, \sigma^z$ are given by

$$\sigma_i^x = \tau_{i-1}^x\tau_i^x\mu_{i-1,i}^y,$$
$$\sigma_i^y = \tau_{i-1}^x\tau_i^x\mu_{i-1,i}^z, \quad (113)$$
$$\sigma_i^z = \mu_{i-1,i}^x,$$

while the symmetry transformation S is given by

$$W_{\text{eff}}(1) \sim \prod_{i=1}^{L} \sigma_i^y \exp\left(\frac{i\pi}{4}\frac{1}{2}\sum_{i=1}^{L}\left(1-\sigma_i^z\sigma_{i+1}^z\right)\right) = \prod_{i=1}^{L}\mu_{i-1,i}^z \exp\left(\frac{i\pi}{8}\sum_{i=1}^{L}\left(1-\tau_i^z\right)\right). \quad (114)$$

In the long wavelength limit, the domain wall density is given by

$$\frac{\tau_i^z}{2} \sim \frac{1}{\pi}\partial_x\phi. \quad (115)$$

We also note that (115) implies that

$$\exp\left(-\frac{\pi i}{8}\sum_{i=1}^{L}\tau_i^z\right) = \exp\left(-\frac{i}{4}\int \partial_x\phi\, dx\right). \quad (116)$$

Similarly, we have

$$\prod_{i=1}^{L}\mu_{i-1,i}^z = \exp\left(\frac{i}{2}\int \partial_x\theta\, dx\right). \quad (117)$$

This equality follows from the observation that the periodic/antiperiodic sectors $\prod_{i=1}^{L}\mu_{i-1,i}^z = \pm 1$ correspond to the two boundary conditions $\theta(L)-\theta(0) = 4m\pi$, $(4m+2)\pi$, respectively.

Combining these two results, we see that our expression (114) for S becomes

$$W_{\text{eff}}(1) \sim \exp\left(\frac{i}{2}\int \partial_x\theta\, dx - \frac{i}{4}\int\partial_x\phi\, dx\right). \quad (118)$$

Using the commutation relation $[\theta(x),\partial_x\phi(y)] = 2\pi i\delta(x-y)$, we deduce that

$$W_{\text{eff}}(1)^{-1}\theta W_{\text{eff}}(1) = \theta + \frac{\pi}{2}, \quad W_{\text{eff}}(1)^{-1}\phi W_{\text{eff}}(1) = \phi + \pi. \quad (119)$$

For the fermion parity symmetry P_f, a similar calculation gives out

$$P_f = \exp\left(\frac{i\pi}{4}\sum_{i=1}^{L}\left(1-\tau_i^z\right)\right) = \exp\left(-\frac{i}{2}\int\partial_x\phi\, dx\right), \quad (120)$$

$$P_f^{-1}\theta P_f = \theta + \pi; \quad P_f^{-1}\phi P_f = \phi. \quad (121)$$

The above transformation laws (119) and (121) together with the action (109) give a complete description of the low energy edge physics.

It is easy to see terms like $\cos[4\theta - \alpha(x)]$ or $\cos[2\phi - \alpha(x)]$ are allowed by both the non-onsite Z_2 symmetry and fermion parity P_f. Obviously, the Z_2 symmetry will be broken if a mass gap is generated by these terms. By performing a simple scaling dimension calculation, we find that both terms are irrelevant and the edge theory remains gapless when $2 < K < 8$. To this end, we see that in contrast to the bosonic Z_2 SPT phase [61], the fermionic Z_2 SPT phases have a stable gapless edge.

X. AN EXAMPLE OF 3D FERMIONIC SPT STATES WITH $T^2 = 1$ TIME-REVERSAL SYMMETRY

In Appendix F, we have also calculated the group supercohomology classes $\mathscr{H}^4[Z_2^T \times Z_2^f, U(1)] = \mathbb{Z}_2$. So, interacting fermion systems with a $T^2 = 1$ time-reversal symmetry can have (at least) one nontrivial SPT phase. This is described by the $Z_2^T \times Z_2^f$ row and $d_{sp} = 3$ column of Table II.

Let us list the fermionic 4-cocycles $[\nu_4(g_0,g_1,g_2,g_3,g_4), n_3(g_0,g_1,g_2,g_3)]$ that describe the intrinsic fermionic SPT phase that can neither be realized by free fermion models nor by interacting boson models:

$$n_3(0,1,0,1) = n_3(1,0,1,0) = 1, \quad \text{other } n_3 = 0,$$
$$\nu_4(0,1,0,1,0) = -\nu_4(1,0,1,0,1) = \pm i, \quad \text{other } \nu_4 = 1. \quad (122)$$

Using the above cocycles, we can write the corresponding wave functions and exactly solvable Hamiltonians. However, the explicit Hamiltonian is very complicated. We wonder if there exists a better basis, in which the Hamiltonian will have a simpler form. We will address this problem in future publications.

Moreover, we note that the above two fermionic 4-cocycles only differ by the following bosonic 4-cocycle:

$$n_3(g_0,g_1,g_2,g_3) = 0,$$
$$\nu_4(0,1,0,1,0) = \nu_4(1,0,1,0,1) = \pm 1, \quad \text{other } \nu_4 = 1, \quad (123)$$

which describes a $T^2 = 1$ (nontrivial) bosonic SPT phase. [We note that in the limit with tightly bounded fermion pairs, e.g., $n_3(g_0,g_1,g_2,g_3) = 0$, a fermionic system can always be viewed as a bosonic system.] Surprisingly, in Appendix F, we found such a solution can be generated by a fermionic coboundary. Physically, such a statement indicates that the $T^2 = 1$ (nontrivial) bosonic SPT phase can be connected to a trivial product state or an atomic insulator state without breaking the corresponding $T^2 = 1$ time-reversal symmetry through interacting fermion systems. In addition, such a result also implies the two fermionic 4-cocycles in Eq. (122) actually describe the same fermionic SPT phase.

XI. SUMMARY

It was shown in [46] that generalized topological nonlinear σ models with symmetry can be constructed from group cohomology theory of the symmetry group. This leads to a systematic construction of bosonic SPT phases in any dimensions and for any symmetry groups. This result allows us to construct new topological insulators [with symmetry group $U(1) \times Z_2^T$] and new topological superconductors (with symmetry group Z_2^T) for interacting boson systems (or qubit systems). It also leads to a complete classification of all gapped phases in 1D interacting boson/fermion systems.

In this paper, we introduce a special group supercohomology theory which is a generalization of the standard group cohomology theory. Using the special group supercohomology theory, we can construct new discrete fermionic topological nonlinear σ models with symmetry. This leads to a systematic construction of fermionic SPT phases in any dimensions and for certain symmetry groups G_f where the fermions form a 1D representation. The discrete fermionic topological nonlinear σ model, when defined on a space-time with boundary, can be viewed as a "nonlocal" boundary effective Lagrangian, which is a fermionic and discrete generalization of the bosonic continuous Wess-Zumino-Witten term. Thus, the boundary excitations of a nontrivial SPT phase are described by a "nonlocal" boundary effective Lagrangian, which, we believe, implies that the boundary excitations are gapless or topologically ordered if the symmetry is not broken.

As a simple application of our group supercohomology theory, we constructed a nontrivial SPT phase in 3D for interacting fermionic superconductors with coplanar spin order. Such a topological superconductor has time-reversal Z_2^T and fermion-number-parity Z_2^f symmetries described by the full symmetry group $G_f = Z_2^T \times Z_2^f$ [48]. The nontrivial SPT phase should have gapless or fractionalized excitations on the 2D surface if the time-reversal symmetry is not broken. It is known that such a nontrivial 3D gapped topological superconductor does not exist if the fermions are noninteracting. In addition, such a nontrivial SPT phase can not be realized in any interacting bosonic models either. So, the constructed nontrivial fermionic SPT phase is totally new.

We also constructed three nontrivial SPT phases in 2D for interacting fermionic systems with the full symmetry group $G_f = Z_2 \times Z_2^f$. We show that the three nontrivial SPT phases indeed have gapless excitations on the 1D edge which are described by central-charge $c = 1$ conformal field theory, if the Z_2 symmetry is not broken. In several recent works [60,61,74,75], it has been further shown that by "gauging" the Z_2 global symmetry, each of the three nontrivial SPT phases can be uniquely identified by the braiding statistics of the corresponding gauge flux.

Clearly, more work is needed to generalize the special group supercohomology theory to the yet-to-be-defined full group supercohomology theory, so that we can handle the cases when the fermions do not form a 1D representation in the fixed-point wave functions. This will allow us to construct more general interacting fermionic SPT phases.

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APPENDIX A: TOPOLOGICAL INVARIANCE OF THE PARTITION AMPLITUDES

In this Appendix, we will prove the topological invariance of the partition amplitudes (38) in $(1+1)$D and $(2+1)$D under their corresponding fermionic group cocycle condition. The $(3+1)$D case is much more complicated but can still be checked by computer.

1. $(1+1)$D

Similarly as we prove the topological invariance of the partition amplitudes that describe bosonic SPT phases [46], we can check all the admissible branching $2 \leftrightarrow 2$ and $1 \leftrightarrow 3$ moves for the Grassmann graded 2-cocycle function:

$$\mathcal{V}_2^{\pm}(g_i,g_j,g_k) = \nu_2^{\pm}(g_i,g_j,g_k)\theta_{(i,j)}^{n_1(g_i,g_j)}\theta_{(j,k)}^{n_1(g_j,g_k)}\bar{\theta}_{(i,k)}^{n_1(g_i,g_k)}. \quad \text{(A1)}$$

For admissible branching $2 \leftrightarrow 2$ moves, we can have three different equations (up to the orientation conjugate $+ \to -; - \to +$):

$$\int d\theta_{(12)}^{n_1(g_1,g_2)} d\bar{\theta}_{(12)}^{n_1(g_1,g_2)} \mathcal{V}_2^{+}(g_1,g_2,g_3)\mathcal{V}_2^{-}(g_0,g_1,g_2) = \int d\theta_{(03)}^{n_1(g_0,g_3)} d\bar{\theta}_{(03)}^{n_1(g_0,g_3)} \mathcal{V}_2^{+}(g_0,g_2,g_3)\mathcal{V}_2^{-}(g_0,g_1,g_3), \quad \text{(A2)}$$

$$\int d\theta_{(13)}^{n_1(g_1,g_3)} d\bar{\theta}_{(13)}^{n_1(g_1,g_3)} \mathcal{V}_2^{+}(g_1,g_2,g_3)\mathcal{V}_2^{+}(g_0,g_1,g_3) = \int d\theta_{(02)}^{n_1(g_0,g_2)} d\bar{\theta}_{(02)}^{n_1(g_0,g_2)} \mathcal{V}_2^{+}(g_0,g_2,g_3)\mathcal{V}_2^{+}(g_0,g_1,g_2), \quad \text{(A3)}$$

and

$$\int d\theta_{(23)}^{n_1(g_2,g_3)} d\bar{\theta}_{(23)}^{n_1(g_2,g_3)} \mathcal{V}_2^{-}(g_1,g_2,g_3)\mathcal{V}_2^{+}(g_0,g_2,g_3) = \int d\theta_{(01)}^{n_1(g_0,g_1)} d\bar{\theta}_{(01)}^{n_1(g_0,g_1)} \mathcal{V}_2^{+}(g_0,g_1,g_3)\mathcal{V}_2^{-}(g_0,g_1,g_2). \quad \text{(A4)}$$

For admissible branching $1 \leftrightarrow 3$ moves, we can have four different equations (up to the orientation conjugate $+ \to -; - \to +$):

$$\mathcal{V}_2^{+}(g_0,g_1,g_3) = \int d\theta_{(02)}^{n_1(g_0,g_2)} d\bar{\theta}_{(02)}^{n_1(g_0,g_2)} d\theta_{(12)}^{n_1(g_1,g_2)} d\bar{\theta}_{(12)}^{n_1(g_1,g_2)} \times d\theta_{(23)}^{n_1(g_2,g_3)} d\bar{\theta}_{(23)}^{n_1(g_2,g_3)} (-)^{m_0(g_2)} \mathcal{V}_2^{-}(g_1,g_2,g_3)\mathcal{V}_2^{+}(g_0,g_2,g_3)\mathcal{V}_2^{+}(g_0,g_1,g_2), \quad \text{(A5)}$$

$$\mathcal{V}_2^{+}(g_0,g_2,g_3) = \int d\theta_{(01)}^{n_1(g_0,g_1)} d\bar{\theta}_{(01)}^{n_1(g_0,g_1)} d\theta_{(12)}^{n_1(g_1,g_2)} d\bar{\theta}_{(12)}^{n_1(g_1,g_2)} \times d\theta_{(13)}^{n_1(g_1,g_3)} d\bar{\theta}_{(13)}^{n_1(g_1,g_3)} (-)^{m_0(g_1)} \mathcal{V}_2^{+}(g_1,g_2,g_3)\mathcal{V}_2^{+}(g_0,g_1,g_3)\mathcal{V}_2^{-}(g_0,g_1,g_2), \quad \text{(A6)}$$

and

$$\mathcal{V}_2^+(g_1,g_2,g_3) = \int d\theta_{(01)}^{n_1(g_0,g_1)} d\bar{\theta}_{(01)}^{n_1(g_0,g_1)} d\theta_{(02)}^{n_1(g_0,g_2)} d\bar{\theta}_{(02)}^{n_1(g_0,g_2)} \times d\theta_{(03)}^{n_1(g_0,g_3)} d\bar{\theta}_{(03)}^{n_1(g_0,g_3)} (-)^{m_0(g_0)} \mathcal{V}_2^+(g_0,g_2,g_3)\mathcal{V}_2^-(g_0,g_1,g_3)\mathcal{V}_2^+(g_0,g_1,g_2), \quad \text{(A7)}$$

$$\mathcal{V}_2^+(g_0,g_1,g_2) = \int d\theta_{(03)}^{n_1(g_0,g_3)} d\bar{\theta}_{(03)}^{n_1(g_0,g_3)} d\theta_{(13)}^{n_1(g_1,g_3)} d\bar{\theta}_{(13)}^{n_1(g_1,g_3)} \times d\theta_{(23)}^{n_1(g_2,g_3)} d\bar{\theta}_{(23)}^{n_1(g_2,g_3)} (-)^{m_0(g_3)} \mathcal{V}_2^+(g_1,g_2,g_3)\mathcal{V}_2^-(g_0,g_2,g_3)\mathcal{V}_2^+(g_0,g_1,g_3). \quad \text{(A8)}$$

Note that the first two equations of $2 \leftrightarrow 2$ and $1 \leftrightarrow 3$ moves can be induced by a global time ordering [see in Figs. 16(a) and 17(a)] while the rest can not [see in Figs. 16(b) and 17(b)]. Here, g^0 is defined on the vertex with no incoming edge, g^1 with one incoming edge, etc.

Let us use the definition of $\mathcal{V}_2^\pm$ and integral out the Grassmann variables. For the branching $2 \leftrightarrow 2$ moves, we have

$$\nu_2^+(g_1,g_2,g_3)\nu_2^-(g_0,g_1,g_2) = \nu_2^+(g_0,g_2,g_3)\nu_2^-(g_0,g_1,g_3), \quad \text{(A9)}$$

$$\nu_2^+(g_1,g_2,g_3)\nu_2^+(g_0,g_1,g_3) = \nu_2^+(g_0,g_2,g_3)\nu_2^+(g_0,g_1,g_2), \quad \text{(A10)}$$

and

$$\nu_2^-(g_1,g_2,g_3)\nu_2^+(g_0,g_2,g_3) = (-)^{n_1(g_1,g_2)}\nu_2^+(g_0,g_1,g_3)\nu_2^-(g_0,g_1,g_2). \quad \text{(A11)}$$

Similarly, for the branching $1 \leftrightarrow 3$ moves, we have

$$\nu_2^+(g_0,g_1,g_3) = (-)^{m_0(g_2)}\nu_2^-(g_1,g_2,g_3)\nu_2^+(g_0,g_2,g_3)\nu_2^+(g_0,g_1,g_2), \quad \text{(A12)}$$

$$\nu_2^+(g_0,g_2,g_3) = (-)^{m_0(g_1)}\nu_2^+(g_1,g_2,g_3)\nu_2^+(g_0,g_1,g_3)\nu_2^-(g_0,g_1,g_2) \quad \text{(A13)}$$

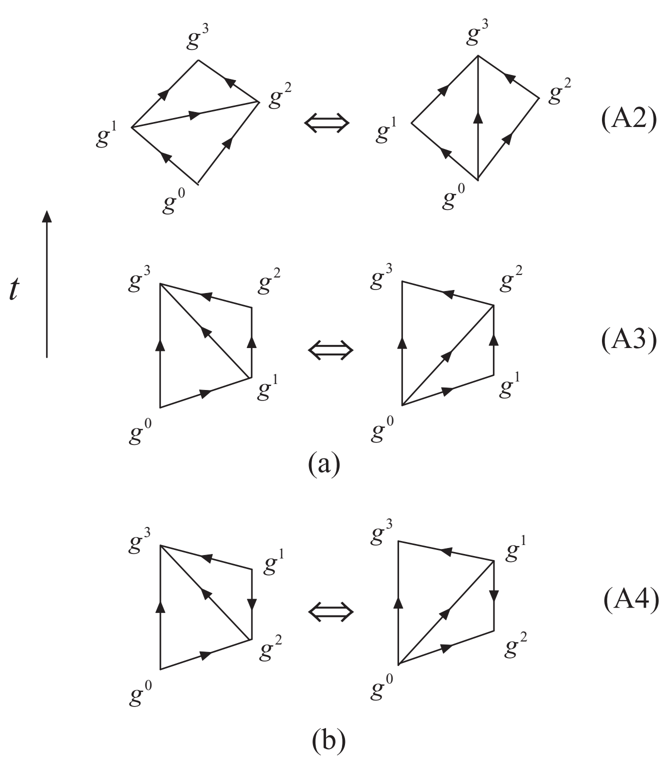

FIG. 16. The admissible branching $2 \leftrightarrow 2$ moves. (a) Branching moves that can be induced by a global time ordering. (b) Branching moves that can not be induced by a global time ordering.

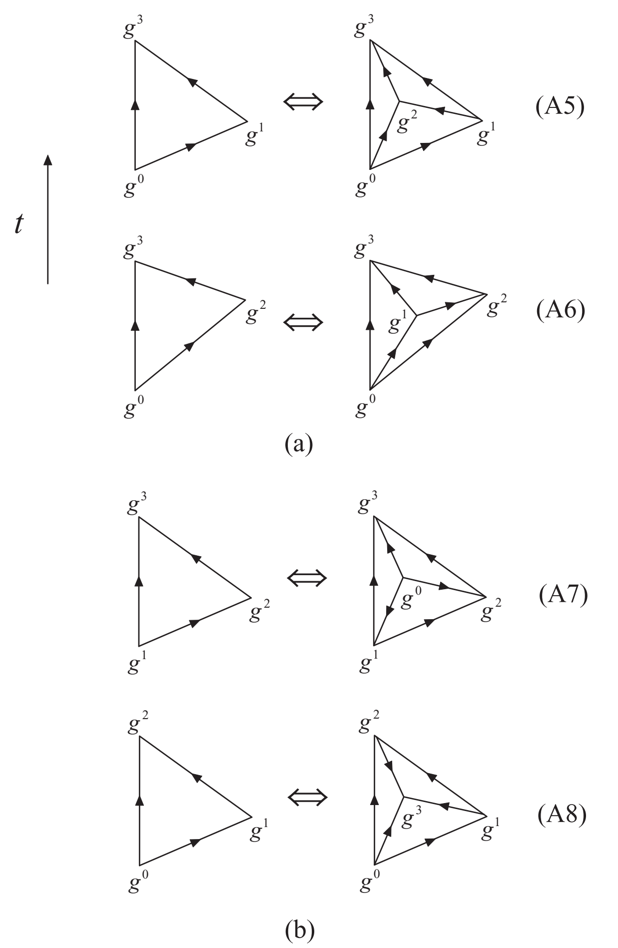

FIG. 17. The admissible branching $1 \leftrightarrow 3$ moves. (a) Branching moves that can be induced by a global time ordering. (b) Branching moves that can not be induced by a global time ordering.

and

$$\nu_2^+(g_1,g_2,g_3) = (-)^{m_0(g_1)}\nu_2^+(g_0,g_2,g_3)\nu_2^-(g_0,g_1,g_3)\nu_2^+(g_0,g_1,g_2), \tag{A14}$$

$$\nu_2^+(g_0,g_1,g_2) = (-)^{m_0(g_2)}\nu_2^+(g_1,g_2,g_3)\nu_2^-(g_0,g_2,g_3)\nu_2^+(g_0,g_1,g_3). \tag{A15}$$

If we use the definition of $\nu_2^\pm$,

$$\nu_2^+(g_0,g_1,g_2) = \nu_2(g_0,g_1,g_2), \quad \nu_2^-(g_0,g_1,g_2) = (-)^{m_0(g_1)}/\nu_2(g_0,g_1,g_2), \tag{A16}$$

All the above admissible branching moves will be equivalent to a single fermionic 2-cocycle equation of ν_2:

$$\nu_2(g_0,g_1,g_3)\nu_2(g_1,g_2,g_3) = \nu_2(g_0,g_1,g_2)\nu_2(g_0,g_2,g_3). \tag{A17}$$

We note that this equation is the same as the 2-cocycle equation in bosonic systems.

2. (2 + 1)D

In (2 + 1)D, there are in total 10 admissible branching 2 ↔ 3 moves and 5 admissible branching 1 ↔ 4 moves. Let us show all theses moves will lead to the same fermionic 3-cocycle condition.

For example, the admissible 2 ↔ 3 in Fig. 10 represents the following equation for $\mathcal{V}_3^\pm$:

$$\begin{aligned}&\int d\theta_{(123)}^{n(g_1,g_2,g_3)} d\bar\theta_{(123)}^{n(g_1,g_2,g_3)} \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_1,g_2,g_3,g_4)\\ &= \int d\theta_{(014)}^{n(g_0,g_1,g_4)} d\bar\theta_{(014)}^{n(g_0,g_1,g_4)} d\theta_{(024)}^{n(g_0,g_2,g_4)} d\bar\theta_{(024)}^{n(g_0,g_2,g_4)} d\theta_{(034)}^{n(g_0,g_3,g_4)} d\bar\theta_{(034)}^{n(g_0,g_3,g_4)} (-)^{m_1(g_0,g_4)}\\ &\quad\times \mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_0,g_1,g_3,g_4).\end{aligned} \tag{A18}$$

Note that in the above expression, we put the sign factor $(-)^{m_1(g_0,g_4)}$ on the interior link and integral out the Grassmann variables on the interior faces. To simplify the representation, we can formally rewrite the above equation as [see Eq. (31)]

$$\int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_1,g_2,g_3,g_4) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_0,g_1,g_3,g_4). \tag{A19}$$

In such a way, we can formally write all admissible branching 2 ↔ 3 moves in terms of $\mathcal{V}_3^\pm$ (up to the orientation conjugate $+ \to -; - \to +$):

$$\int \mathcal{V}_3^-(g_0,g_1,g_2,g_4)\mathcal{V}_3^-(g_0,g_2,g_3,g_4) = \int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_3,g_4)\mathcal{V}_3^-(g_1,g_2,g_3,g_4), \tag{A20}$$

$$\int \mathcal{V}_3^-(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_1,g_2,g_3,g_4) = \int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4), \tag{A21}$$

$$\int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_2,g_3,g_4) = \int \mathcal{V}_3^-(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_1,g_2,g_3,g_4), \tag{A22}$$

$$\int \mathcal{V}_3^+(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_1,g_2,g_3,g_4) = \int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4), \tag{A23}$$

$$\int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_1,g_3,g_4) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_1,g_2,g_3,g_4), \tag{A24}$$

$$\int \mathcal{V}_3^-(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_1,g_3,g_4) = \int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_1,g_2,g_3,g_4), \tag{A25}$$

$$\int \mathcal{V}_3^-(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_1,g_2,g_3,g_4), \tag{A26}$$

$$\int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_1,g_2,g_3,g_4) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_0,g_1,g_3,g_4), \tag{A27}$$

and

$$\int \mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_1,g_2,g_3,g_4) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_1,g_3,g_4), \tag{A28}$$

$$\int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_1,g_2,g_4) = \int \mathcal{V}_3^-(g_0,g_2,g_3,g_4)\mathcal{V}_3^+(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_1,g_2,g_3,g_4). \tag{A29}$$

Similarly, we can formally write all the admissible branching $1 \leftrightarrow 4$ moves in terms of $\mathcal{V}_3^{\pm}$ (up to the orientation conjugate $+ \to -; - \to +$):

$$\mathcal{V}_3^+(g_0,g_1,g_2,g_4) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_1,g_3,g_4)\mathcal{V}_3^-(g_0,g_2,g_3,g_4)\mathcal{V}_3^+(g_1,g_2,g_3,g_4), \tag{A30}$$

$$\mathcal{V}_3^+(g_0,g_2,g_3,g_4) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_1,g_2,g_3,g_4), \tag{A31}$$

$$\mathcal{V}_3^+(g_0,g_1,g_3,g_4) = \int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_1,g_2,g_3,g_4) \tag{A32}$$

and

$$\mathcal{V}_3^+(g_1,g_2,g_3,g_4) = \int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^-(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4), \tag{A33}$$

$$\mathcal{V}_3^+(g_0,g_1,g_2,g_3) = \int \mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^-(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\mathcal{V}_3^-(g_1,g_2,g_3,g_4). \tag{A34}$$

Here, the symbol $\int$ means that we put the sign factors $(-)^{m_1(g_i,g_j)}$ on all four interior links and integrate over the Grassmann variables on all six interior faces. Note that the first eight $2 \leftrightarrow 3$ moves (Fig. 18) and first three $1 \leftrightarrow 4$ can be induced by a global time ordering while the rest can not. Again, here g^0 is defined on the vertex with no incoming edge, g^1 with one incoming edge, etc.

Expressing $\mathcal{V}_3^{\pm}$ in terms of $\nu_3^{\pm}$ and integrating out all the Grassmann variables, the admissible branching $2 \leftrightarrow 3$ (Fig. 18) moves lead to the following equations:

$$\nu_3^-(g_0,g_1,g_2,g_4)\nu_3^-(g_0,g_2,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+m_1(g_1,g_3)}\nu_3^-(g_0,g_1,g_2,g_3)\nu_3^-(g_0,g_1,g_3,g_4)\nu_3^-(g_1,g_2,g_3,g_4), \tag{A35}$$

$$\nu_3^-(g_0,g_1,g_2,g_4)\nu_3^+(g_1,g_2,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+m_1(g_0,g_3)}\nu_3^-(g_0,g_1,g_2,g_3)\nu_3^-(g_0,g_1,g_3,g_4)\nu_3^+(g_0,g_2,g_3,g_4), \tag{A36}$$

$$\nu_3^-(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_2,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+m_1(g_1,g_4)}\nu_3^-(g_0,g_1,g_2,g_4)\nu_3^+(g_0,g_1,g_3,g_4)\nu_3^+(g_1,g_2,g_3,g_4), \tag{A37}$$

$$\nu_3^+(g_0,g_1,g_3,g_4)\nu_3^+(g_1,g_2,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+m_1(g_0,g_2)}\nu_3^-(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_1,g_2,g_4)\nu_3^+(g_0,g_2,g_3,g_4), \tag{A38}$$

$$\nu_3^+(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_1,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+m_1(g_2,g_4)}\nu_3^+(g_0,g_1,g_2,g_4)\nu_3^+(g_0,g_2,g_3,g_4)\nu_3^-(g_1,g_2,g_3,g_4), \tag{A39}$$

$$\begin{aligned}\nu_3^-(g_0,g_1,g_2,g_4)\nu_3^+(g_0,g_1,g_3,g_4) = {} & (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_1,g_2,g_3)+n_2(g_1,g_2,g_4)+m_1(g_2,g_3)} \\ & \times \nu_3^-(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_2,g_3,g_4)\nu_3^-(g_1,g_2,g_3,g_4),\end{aligned} \tag{A40}$$

$$\begin{aligned}\nu_3^-(g_0,g_1,g_3,g_4)\nu_3^+(g_0,g_2,g_3,g_4) = {} & (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_0,g_1,g_2)+n_2(g_0,g_1,g_3)+m_1(g_1,g_2)} \\ & \times \nu_3^+(g_0,g_1,g_2,g_3)\nu_3^-(g_0,g_1,g_2,g_4)\nu_3^+(g_1,g_2,g_3,g_4),\end{aligned} \tag{A41}$$

$$\begin{aligned}\nu_3^+(g_0,g_1,g_2,g_3)\nu_3^+(g_1,g_2,g_3,g_4) = {} & (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_0,g_3,g_4)+n_2(g_1,g_3,g_4)+m_1(g_0,g_4)} \\ & \times \nu_3^+(g_0,g_1,g_2,g_4)\nu_3^+(g_0,g_2,g_3,g_4)\nu_3^-(g_0,g_1,g_3,g_4)\end{aligned} \tag{A42}$$

and (Fig. 19)

$$\begin{aligned}\nu_3^+(g_0,g_2,g_3,g_4)\nu_3^-(g_1,g_2,g_3,g_4) = {} & (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_0,g_1,g_2)+n_2(g_1,g_2,g_4)+m_1(g_0,g_1)} \\ & \times \nu_3^+(g_0,g_1,g_2,g_3)\nu_3^-(g_0,g_1,g_2,g_4))\nu_3^+(g_0,g_1,g_3,g_4),\end{aligned} \tag{A43}$$

$$\begin{aligned}\nu_3^-(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_1,g_2,g_4) = {} & (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_0,g_2,g_4)+n_2(g_0,g_3,g_4)+m_1(g_3,g_4)} \\ & \times \nu_3^-(g_0,g_2,g_3,g_4)\nu_3^+(g_0,g_1,g_3,g_4)\nu_3^+(g_1,g_2,g_3,g_4).\end{aligned} \tag{A44}$$

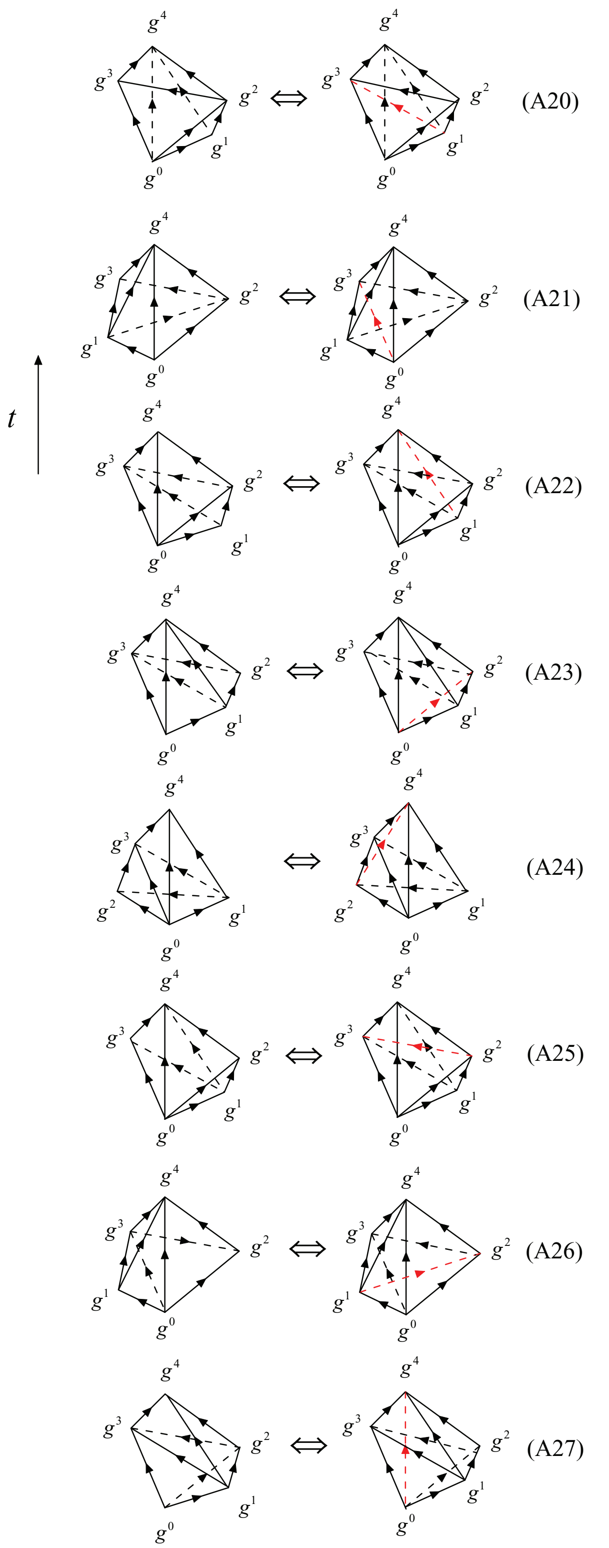

FIG. 18. (Color online) The admissible branching 2 ↔ 3 moves that can be induced by a global time ordering.

We note that the sign factor $(-)^{m_1(g_i,g_j)}$ comes from the definition of the fermionic path integral. Later, we will see such a factor is very important to make all the branching moves to be self-consistent.

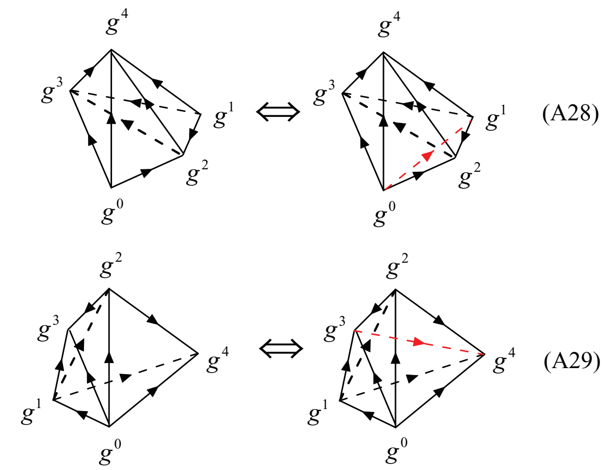

FIG. 19. (Color online) The admissible branching 2 ↔ 3 moves that can not be induced by a global time ordering.

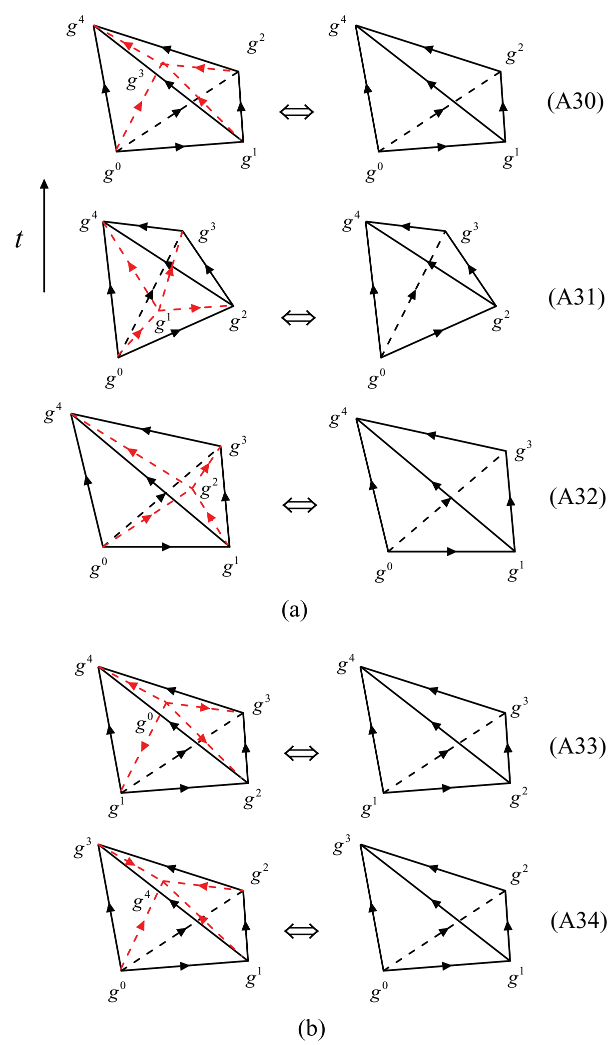

FIG. 20. (Color online) The admissible branching 1 ↔ 4 moves that (a) can be induced by a global time ordering and (b) can not be induced by a global time ordering.

Similarly, all the admissible branching $1 \leftrightarrow 4$ (Fig. 20) moves lead to the following equations for $\nu_3^{\pm}$:

$$\nu_3^+(g_0,g_1,g_2,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_2,g_3,g_4)}(-)^{m_1(g_0,g_3)+m_1(g_1,g_3)+m_1(g_2,g_3)+m_1(g_3,g_4)} \times \nu_3^+(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_1,g_3,g_4)\nu_3^-(g_0,g_2,g_3,g_4)\nu_3^+(g_1,g_2,g_3,g_4), \tag{A45}$$

$$\nu_3^+(g_0,g_2,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_0,g_1,g_2)}(-)^{m_1(g_0,g_1)+m_1(g_1,g_2)+m_1(g_1,g_3)+m_1(g_1,g_4)} \times \nu_3^+(g_0,g_1,g_2,g_3)\nu_3^-(g_0,g_1,g_2,g_4)\nu_3^+(g_0,g_1,g_3,g_4)\nu_3^+(g_1,g_2,g_3,g_4), \tag{A46}$$

$$\nu_3^+(g_0,g_1,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_1,g_2,g_3)}(-)^{m_1(g_0,g_2)+m_1(g_1,g_2)+m_1(g_2,g_3)+m_1(g_2,g_4)} \times \nu_3^-(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_1,g_2,g_4)\nu_3^+(g_0,g_2,g_3,g_4)\nu_3^-(g_1,g_2,g_3,g_4) \tag{A47}$$

and

$$\nu_3^+(g_1,g_2,g_3,g_4) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_0,g_1,g_4)}(-)^{m_1(g_0,g_1)+m_1(g_0,g_2)+m_1(g_0,g_3)+m_1(g_0,g_4)} \times \nu_3^-(g_0,g_1,g_2,g_3)\nu_3^+(g_0,g_1,g_2,g_4)\nu_3^-(g_0,g_1,g_3,g_4)\nu_3^+(g_0,g_2,g_3,g_4), \tag{A48}$$

$$\nu_3^+(g_0,g_1,g_2,g_3) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)+n_2(g_0,g_3,g_4)}(-)^{m_1(g_0,g_4)+m_1(g_1,g_4)+m_1(g_2,g_4)+m_1(g_3,g_4)} \times \nu_3^+(g_0,g_1,g_2,g_4)\nu_3^-(g_0,g_1,g_3,g_4)\nu_3^+(g_0,g_2,g_3,g_4)\nu_3^-(g_1,g_2,g_3,g_4). \tag{A49}$$

Amazingly, if we define

$$\nu_3^+(g_0,g_1,g_2,g_3) = (-)^{m_1(g_0,g_2)}\nu_3(g_0,g_1,g_2,g_3),$$
$$\nu_3^-(g_0,g_1,g_2,g_3) = (-)^{m_1(g_1,g_3)}/\nu_3(g_0,g_1,g_2,g_3), \tag{A50}$$

we find all the above equations are equivalent to the following single equation, which is the fermionic 3-cocycle condition of ν_3:

$$\nu_3(g_1,g_2,g_3,g_4)\nu_3(g_0,g_1,g_3,g_4)\nu_3(g_0,g_1,g_2,g_3) = (-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)}\nu_3(g_0,g_2,g_3,g_4)\nu_3(g_0,g_1,g_2,g_4). \tag{A51}$$

3. (3 + 1)D

The admissible branching moves for (3 + 1)D case is much more complicated. There are in total 10 $3 \leftrightarrow 3$ moves, 15 $2 \leftrightarrow 4$ moves, and 6 $1 \leftrightarrow 5$ moves.

For example, Fig. 12 represents one admissible branching $2 \leftrightarrow 4$ move, which leads to the following equation for $\mathcal{V}_4^{\pm}$:

$$\int d\theta_{(1234)}^{n(g_1,g_2,g_3,g_4)} d\bar\theta_{(1234)}^{n(g_1,g_2,g_3,g_4)} \mathcal{V}_4^+(g_1,g_2,g_3,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_2,g_3,g_4)$$
$$= \int d\theta_{(0125)}^{n(g_0,g_1,g_2,g_5)} d\bar\theta_{(0125)}^{n(g_0,g_1,g_2,g_5)} d\theta_{(0135)}^{n(g_0,g_1,g_3,g_5)} d\bar\theta_{(0135)}^{n(g_0,g_1,g_3,g_5)} d\theta_{(0145)}^{n(g_0,g_1,g_4,g_5)} d\bar\theta_{(0145)}^{n(g_0,g_1,g_4,g_5)} d\theta_{(0235)}^{n(g_0,g_2,g_3,g_5)} d\bar\theta_{(0235)}^{n(g_0,g_2,g_3,g_5)}$$
$$\times d\theta_{(0245)}^{n(g_0,g_2,g_4,g_5)} d\bar\theta_{(0245)}^{n(g_0,g_2,g_4,g_5)} d\theta_{(0345)}^{n(g_0,g_3,g_4,g_5)} d\bar\theta_{(0345)}^{n(g_0,g_3,g_4,g_5)} (-)^{m_2(g_0,g_1,g_5)+m_2(g_0,g_2,g_5)+m_2(g_0,g_3,g_5)+m_2(g_0,g_4,g_5)}$$
$$\times \mathcal{V}_4^+(g_0,g_2,g_3,g_4,g_5)\mathcal{V}_4^+(g_0,g_1,g_2,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_3,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_2,g_3,g_5). \tag{A52}$$

Here, we integrate out the Grassmann variables on the interior tetrahedra and put the sign factor on the interior surfaces. We can formally denote the above equation as [see Eq. (31)]

$$\int \mathcal{V}_4^+(g_1,g_2,g_3,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_2,g_3,g_4) = \int \mathcal{V}_4^+(g_0,g_2,g_3,g_4,g_5)\mathcal{V}_4^+(g_0,g_1,g_2,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_3,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_2,g_3,g_5). \tag{A53}$$

Similarly, Fig. 13 represents one admissible branching $1 \leftrightarrow 5$ move and can be formally written down in terms of $\mathcal{V}_4^{\pm}$ as

$$\mathcal{V}_4^-(g_0,g_1,g_2,g_3,g_4) = \int \mathcal{V}_4^+(g_0,g_2,g_3,g_4,g_5)\mathcal{V}_4^+(g_0,g_1,g_2,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_3,g_4,g_5)\mathcal{V}_4^-(g_0,g_1,g_2,g_3,g_5)\mathcal{V}_4^-(g_1,g_2,g_3,g_4,g_5). \tag{A54}$$

We note that here the symbol $\int$ means integrating over all Grassmann variables on 10 interior tetrahedra and put the sign factor $(-)^{m_2(g_i,g_j,g_k)}$ on 10 interior surfaces.

In addition, there are $3 \leftrightarrow 3$ moves. For example, one of such moves gives rise the following formal equation for $\mathcal{V}_4^{\pm}$:

$$\int \mathcal{V}_4^+(g_1,g_2,g_3,g_4,g_5)\mathcal{V}_4^+(g_0,g_1,g_3,g_4,g_5)\mathcal{V}_4^+(g_0,g_1,g_2,g_3,g_5) = \int \mathcal{V}_4^+(g_0,g_1,g_2,g_3,g_4)\mathcal{V}_4^+(g_0,g_2,g_3,g_4,g_5)\mathcal{V}_4^+(g_0,g_1,g_2,g_4,g_5). \tag{A55}$$

In this case, the symbol $\int$ on both sides means integrating over all Grassmann variables on three interior tetrahedra and put the sign factor $(-)^{m_2(g_i,g_j,g_k)}$ on one interior surface.

After integrating over all the Grassmann variables, we can express the above three equations in terms of $\nu_4^{\pm}$:

$$\begin{aligned}
&\nu_4^+(g_1,g_2,g_3,g_4,g_5)\nu_4^-(g_0,g_1,g_2,g_3,g_4)\\
&\quad= (-)^{n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5)+n_3(g_0,g_1,g_4,g_5)}\\
&\qquad\times (-)^{m_2(g_0,g_1,g_5)+m_2(g_0,g_2,g_5)+m_2(g_0,g_3,g_5)+m_2(g_0,g_4,g_5)}\\
&\qquad\times \nu_4^+(g_0,g_2,g_3,g_4,g_5)\nu_4^+(g_0,g_1,g_2,g_4,g_5)\nu_4^-(g_0,g_1,g_3,g_4,g_5)\nu_4^-(g_0,g_1,g_2,g_3,g_5),
\end{aligned} \tag{A56}$$

$$\begin{aligned}
&\nu_4^-(g_0,g_1,g_2,g_3,g_4)\\
&\quad= (-)^{n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5)+n_3(g_0,g_1,g_4,g_5)+n_3(g_1,g_2,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)}\\
&\qquad\times (-)^{m_2(g_0,g_1,g_5)+m_2(g_0,g_2,g_5)+m_2(g_0,g_3,g_5)+m_2(g_0,g_4,g_5)+m_2(g_1,g_2,g_5)+m_2(g_1,g_3,g_5)+m_2(g_1,g_4,g_5)+m_2(g_2,g_3,g_5)+m_2(g_2,g_4,g_5)+m_2(g_3,g_4,g_5)}\\
&\qquad\times \nu_4^+(g_0,g_2,g_3,g_4,g_5)\nu_4^+(g_0,g_1,g_2,g_4,g_5)\nu_4^-(g_0,g_1,g_3,g_4,g_5)\nu_4^-(g_0,g_1,g_2,g_3,g_5)\nu_4^-(g_1,g_2,g_3,g_4,g_5),
\end{aligned} \tag{A57}$$

$$\begin{aligned}
&\nu_4^+(g_1,g_2,g_3,g_4,g_5)\nu_4^+(g_0,g_1,g_3,g_4,g_5)\nu_4^+(g_0,g_1,g_2,g_3,g_5)(-)^{m_2(g_1,g_3,g_5)}\\
&\quad= (-)^{n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5)+m_2(g_0,g_2,g_4)}\\
&\qquad\times \nu_4^+(g_0,g_1,g_2,g_3,g_4)\nu_4^+(g_0,g_2,g_3,g_4,g_5)\nu_4^+(g_0,g_1,g_2,g_4,g_5).
\end{aligned} \tag{A58}$$

Surprisingly, if we define $\nu_4^{\pm}$ as

$$\begin{aligned}
\nu_4^+(g_0,g_1,g_2,g_3,g_4) &= (-)^{m_2(g_0,g_1,g_3)+m_2(g_1,g_3,g_4)+m_2(g_1,g_2,g_3)}\nu_4(g_0,g_1,g_2,g_3,g_4),\\
\nu_4^-(g_0,g_1,g_2,g_3,g_4) &= (-)^{m_2(g_0,g_2,g_4)}/\nu_4(g_0,g_1,g_2,g_3,g_4),
\end{aligned} \tag{A59}$$

we find all the above three equations will be equivalent to the following single equation of ν_4, which is the fermionic 4-cocycle condition

$$\begin{aligned}
&\nu_4(g_1,g_2,g_3,g_4,g_5)\nu_4(g_0,g_1,g_3,g_4,g_5)\nu_4(g_0,g_1,g_2,g_3,g_5)\\
&\quad= (-)^{n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5)}\\
&\qquad\times \nu_4(g_0,g_2,g_3,g_4,g_5)\nu_4(g_0,g_1,g_2,g_4,g_5)\nu_4(g_0,g_1,g_2,g_3,g_4).
\end{aligned} \tag{A60}$$

Indeed, it can be verified by computer that all other admissible $3 \leftrightarrow 3$, $2 \leftrightarrow 4$, and $1 \leftrightarrow 5$ branching moves will give rise to the same fermionic 4-cocycle condition! Again, the phase factor $(-)^{m_2(g_i,g_j,g_k)}$ in the fermionic path integral is crucial for the self-consistency of all admissible branching moves.

APPENDIX B: FIXED-POINT ACTION ON A CLOSED COMPLEX AND THE UNITARY CONDITION

In the following, we will show that $\int \mathcal{V}_d^+\mathcal{V}_d^-$ is always equal to 1. In $(1+1)$D, we have

$$\begin{aligned}
\int \mathcal{V}_2^+(g_0,g_1,g_2)\mathcal{V}_2^-(g_0,g_1,g_2) &= \int d\theta_{(01)}^{n_1(g_0,g_1)}d\bar\theta_{(01)}^{n_1(g_0,g_1)}d\theta_{(02)}^{n_1(g_0,g_2)}d\bar\theta_{(02)}^{n_1(g_0,g_2)}d\theta_{(12)}^{n_1(g_1,g_2)}d\bar\theta_{(12)}^{n_1(g_1,g_2)}(-)^{m_0(g_0)+m_0(g_1)+m_0(g_2)}\\
&\quad\times \nu_2^+(g_0,g_1,g_2)\theta_{(12)}^{n_1(g_1,g_2)}\theta_{(01)}^{n_1(g_0,g_1)}\bar\theta_{(02)}^{n_1(g_0,g_2)}\nu_2^-(g_0,g_1,g_2)\theta_{(02)}^{n_1(g_0,g_2)}\bar\theta_{(01)}^{n_1(g_0,g_1)}\bar\theta_{(12)}^{n_1(g_1,g_2)}\\
&= \int d\theta_{(01)}^{n_1(g_0,g_1)}d\bar\theta_{(01)}^{n_1(g_0,g_1)}d\theta_{(02)}^{n_1(g_0,g_2)}d\bar\theta_{(02)}^{n_1(g_0,g_2)}d\theta_{(12)}^{n_1(g_1,g_2)}d\bar\theta_{(12)}^{n_1(g_1,g_2)}(-)^{n_1(g_0,g_2)}\\
&\quad\times \theta_{(12)}^{n_1(g_1,g_2)}\theta_{(01)}^{n_1(g_0,g_1)}\bar\theta_{(02)}^{n_1(g_0,g_2)}\theta_{(02)}^{n_1(g_0,g_2)}\bar\theta_{(01)}^{n_1(g_0,g_1)}\bar\theta_{(12)}^{n_1(g_1,g_2)}\\
&= 1.
\end{aligned} \tag{B1}$$

In the above calculation, we can move the pair $d\theta_{(01)}^{n_1(g_0,g_1)} d\bar\theta_{(01)}^{n_1(g_0,g_1)}$ to the front of $\bar\theta_{(02)}^{n_1(g_0,g_2)}\theta_{(02)}^{n_1(g_0,g_2)}$ without generating any signs. We then can evaluate $\int d\theta_{(02)}^{n_1(g_0,g_1)} d\bar\theta_{(02)}^{n_1(g_0,g_1)}\bar\theta_{(02)}^{n_1(g_0,g_2)}\theta_{(02)}^{n_1(g_0,g_2)} = 1$. We next move $d\theta_{(01)}^{n_1(g_0,g_1)} d\bar\theta_{(01)}^{n_1(g_0,g_1)}$ to the front of $\theta_{(01)}^{n_1(g_0,g_1)}\bar\theta_{(01)}^{n_1(g_0,g_1)}$ without generating any signs. We then evaluate $\int d\theta_{(01)}^{n_1(g_0,g_1)} d\bar\theta_{(01)}^{n_1(g_0,g_1)}\theta_{(01)}^{n_1(g_0,g_1)}\bar\theta_{(01)}^{n_1(g_0,g_1)} = (-)^{n_1(g_0,g_1)}$, which has a sign factor. The Grassmann integral $\int d\theta_{(12)}^{n_1(g_1,g_2)} d\bar\theta_{(12)}^{n_1(g_1,g_2)}\theta_{(12)}^{n_1(g_1,g_2)}\bar\theta_{(12)}^{n_1(g_1,g_2)} = (-)^{n_1(g_1,g_2)}$ also generates a sign factor. Such two sign factors $(-)^{n_1(g_0,g_1)}(-)^{n_1(g_1,g_2)}$ cancel the sign factor $(-)^{n_1(g_0,g_2)}$ due to the condition (23) on n_d.

Similarly, in $(2+1)$D and in $(3+1)$D, we find that the sign factors in the integration measure (31) and in the $\nu_d^\pm$ expressions (48), when combined with those from exchanging the Grassmann numbers, just cancel each other:

$$\begin{aligned}
&\int \mathcal{V}_3^+(g_0,g_1,g_2,g_3)\mathcal{V}_3^-(g_0,g_1,g_2,g_3)\\
&= \int d\theta_{(012)}^{n_2(g_0,g_1,g_2)} d\bar\theta_{(012)}^{n_2(g_0,g_1,g_2)} d\theta_{(013)}^{n_2(g_0,g_1,g_3)} d\bar\theta_{(013)}^{n_2(g_0,g_1,g_3)} d\theta_{(023)}^{n_2(g_0,g_2,g_3)} d\bar\theta_{(023)}^{n_2(g_0,g_2,g_3)} d\theta_{(123)}^{n_2(g_1,g_2,g_3)} d\bar\theta_{(123)}^{n_2(g_1,g_2,g_3)}\\
&\quad \times (-)^{m_1(g_0,g_1)+m_1(g_0,g_2)+m_1(g_0,g_3)+m_1(g_1,g_2)+m_1(g_1,g_3)+m_1(g_2,g_3)}\nu_3^+(g_0,g_1,g_2,g_3)\theta_{(123)}^{n_2(g_1,g_2,g_3)}\theta_{(013)}^{n_2(g_0,g_1,g_3)}\bar\theta_{(023)}^{n_2(g_0,g_2,g_3)}\bar\theta_{(012)}^{n_2(g_0,g_1,g_2)}\\
&\quad \times \nu_3^-(g_0,g_1,g_2,g_3)\theta_{(012)}^{n_2(g_0,g_1,g_2)}\theta_{(023)}^{n_2(g_0,g_2,g_3)}\bar\theta_{(013)}^{n_2(g_0,g_1,g_3)}\bar\theta_{(123)}^{n_2(g_1,g_2,g_3)}\\
&= \int d\theta_{(012)}^{n_2(g_0,g_1,g_2)} d\bar\theta_{(012)}^{n_2(g_0,g_1,g_2)} d\theta_{(013)}^{n_2(g_0,g_1,g_3)} d\bar\theta_{(013)}^{n_2(g_0,g_1,g_3)} d\theta_{(023)}^{n_2(g_0,g_2,g_3)} d\bar\theta_{(023)}^{n_2(g_0,g_2,g_3)} d\theta_{(123)}^{n_2(g_1,g_2,g_3)} d\bar\theta_{(123)}^{n_2(g_1,g_2,g_3)}\\
&\quad \times (-)^{n_2(g_0,g_1,g_3)+n_2(g_1,g_2,g_3)}\theta_{(123)}^{n_2(g_1,g_2,g_3)}\theta_{(013)}^{n_2(g_0,g_1,g_3)}\bar\theta_{(023)}^{n_2(g_0,g_2,g_3)}\bar\theta_{(012)}^{n_2(g_0,g_1,g_2)}\theta_{(012)}^{n_2(g_0,g_1,g_2)}\theta_{(023)}^{n_2(g_0,g_2,g_3)}\bar\theta_{(013)}^{n_2(g_0,g_1,g_3)}\bar\theta_{(123)}^{n_2(g_1,g_2,g_3)}\\
&= 1; \qquad\qquad (\text{B2})
\end{aligned}$$

$$\begin{aligned}
&\int \mathcal{V}_4^+(g_0,g_1,g_2,g_3,g_4)\mathcal{V}_4^-(g_0,g_1,g_2,g_3,g_4)\\
&= \int d\theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)} d\bar\theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)} d\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)} d\bar\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)} d\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)} d\bar\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)}\\
&\quad \times d\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)} d\bar\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)} d\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)} d\bar\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)}(-)^{m_2(g_0,g_1,g_2)+m_2(g_0,g_1,g_3)}\\
&\quad \times (-)^{m_2(g_0,g_1,g_4)+m_2(g_0,g_2,g_3)+m_2(g_0,g_2,g_4)+m_2(g_0,g_3,g_4)+m_2(g_1,g_2,g_3)+m_2(g_1,g_2,g_4)+m_2(g_1,g_3,g_4)+m_2(g_2,g_3,g_4)}\\
&\quad \times \nu_4^+(g_0,g_1,g_2,g_3,g_4)\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)}\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)}\theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)}\bar\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)}\bar\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)}\\
&\quad \times \nu_4^-(g_0,g_1,g_2,g_3,g_4)\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)}\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)}\bar\theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)}\bar\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)}\bar\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)}\\
&= \int d\theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)} d\bar\theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)} d\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)} d\bar\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)} d\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)} d\bar\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)}\\
&\quad \times d\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)} d\bar\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)} d\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)} d\bar\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)}(-)^{n_3(g_0,g_1,g_2,g_4)+n_3(g_0,g_2,g_3,g_4)}\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)}\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)}\\
&\quad \times \theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)}\bar\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)}\bar\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)}\theta_{(0124)}^{n_3(g_0,g_1,g_2,g_4)}\theta_{(0234)}^{n_3(g_0,g_2,g_3,g_4)}\bar\theta_{(0123)}^{n_3(g_0,g_1,g_2,g_3)}\bar\theta_{(0134)}^{n_3(g_0,g_1,g_3,g_4)}\bar\theta_{(1234)}^{n_3(g_1,g_2,g_3,g_4)}\\
&= 1. \qquad\qquad (\text{B3})
\end{aligned}$$

We also note that the bosonic topological nonlinear σ models are characterized by action amplitudes that are pure $U(1)$ phases $|\nu_d(g_0,\ldots,g_d)| = 1$ [where the action amplitude on "+" oriented simplexes is given by $\nu_d(g_0,\ldots,g_d)$, and the action amplitude on "−" oriented simplexes is given by $\nu_d^{-1}(g_0,\ldots,g_d) = \nu_d^*(g_0,\ldots,g_d)$]. The action amplitude being a pure $U(1)$ phase ensures that the model defined by the path integral to be unitary theory.

But what is the analog of the pure $U(1)$ phase condition on the Grassmann amplitude $\mathcal{V}_d^\pm$ {or on the pair of functions $[\nu_d(g_0,\ldots,g_d),m_{d-2}(g_0,\ldots,g_{d-2})]$}? Clearly, the Grassmann amplitude $\mathcal{V}_d^\pm$ is not even a complex number. It is hard to say when $\mathcal{V}_d^\pm$ behaves like $U(1)$ phases.

To address this issue, let us introduce the complex conjugate of a quantity that contains Grassmann numbers:

$$(\nu\theta_1\bar\theta_2\theta_3\ldots)^* \equiv \nu^*\ldots\bar\theta_3\theta_2\bar\theta_1. \qquad (\text{B4})$$

We see that under the complex conjugate (a) the complex coefficients are complex conjugated, (b) the order of the Grassmann number is reversed, and (c) θ and $\bar\theta$ are exchanged. With this definition, one is tempting to require $\mathcal{V}_d^+(\mathcal{V}_d^+)^* = 1$, in order for $\mathcal{V}_d^+$ to be a $U(1)$ phase. But, $\mathcal{V}_d^+(\mathcal{V}_d^+)^*$ still contains Grassmann numbers and we cannot require it to be 1. So, we try to require $\int \mathcal{V}_d^+(\mathcal{V}_d^+)^* = 1$ where the Grassmann integral is defined in Eq. (31). Now, $\int \mathcal{V}_d^+(\mathcal{V}_d^+)^*$ is a complex number. But $\int \mathcal{V}_d^+(\mathcal{V}_d^+)^*$ is not non-negative. So, we cannot treat $\int \mathcal{V}_d^+(\mathcal{V}_d^+)^*$ as a norm-square of $\mathcal{V}_d^+$. After some considerations, we find that we need to define a different complex conjugate:

$$\begin{aligned} [\mathcal{V}_1^{\pm}(g_0,g_1)]^\dagger &= [\mathcal{V}_1^{\pm}(g_0,g_1)]^*,\\ [\mathcal{V}_2^{\pm}(g_0,g_1,g_2)]^\dagger &= (-)^{m_0(g_1)}[\mathcal{V}_2^{\pm}(g_0,g_1,g_2)]^*,\\ [\mathcal{V}_3^{\pm}(g_0,\ldots,g_3)]^\dagger &= (-)^{m_1(g_0,g_2)+m_1(g_1,g_3)}[\mathcal{V}_3^{\pm}(g_0,\ldots,g_3)]^*,\\ [\mathcal{V}_4^{\pm}(g_0,\ldots,g_4)]^\dagger &= [\mathcal{V}_4^{\pm}(g_0,\ldots,g_4)]^*(-)^{m_1(g_0,g_1,g_3)+m_1(g_1,g_3,g_4)+m_1(g_1,g_2,g_3)+m_1(g_0,g_2,g_4)}. \end{aligned} \tag{B5}$$

Using the new complex conjugate and the evaluation of $\int \mathcal{V}_d^+\mathcal{V}_d^-$ described above, we find that

$$\int \mathcal{V}_d^+[\mathcal{V}_d^+]^\dagger = \int \mathcal{V}_d^-[\mathcal{V}_d^-]^\dagger = \nu_d\nu_d^*. \tag{B6}$$

Thus, we would like to require the Grassmann amplitude $\mathcal{V}_d^{\pm}$ to satisfy

$$\int \mathcal{V}_d^{\pm}[\mathcal{V}_d^{\pm}]^\dagger = 1, \tag{B7}$$

which is to require $|\nu_d(g_0,\ldots,g_d)| = 1$. Equation (B7) is the analog of the pure $U(1)$ phase condition on the Grassmann amplitude $\mathcal{V}_d^{\pm}$. For the Grassmann amplitude $\mathcal{V}_d^{\pm}$ that satisfies the pure $U(1)$ phase condition (B7), we have

$$[\mathcal{V}_d^{\pm}]^\dagger = \mathcal{V}_d^{\mp}. \tag{B8}$$

APPENDIX C: GROUP SUPERCOHOMOLOGY

In Sec. V, we studied the condition on $\nu_d(g_0,g_1,\ldots)$ so that the fermionic path integral constructed from $\nu_d(g_0,g_1,\ldots)$ corresponds to a fermionic topological nonlinear σ model which is a fixed-point theory under the RG flow. Those conditions on $\nu_d(g_0,g_1,\ldots)$ actually define a generalization of group cohomology [46,47]. We will call such a generalization group supercohomology. In this Appendix, we will study the group supercohomology in detail.

A d-cohomology class of group supercohomology is a set that depends on a full symmetry group G_f and a fermionic G-module M. We will denote the fermionic d-cohomology class as $\mathscr{H}^d(G_f,M)$.

We note that a fermion system always has a Z_2 symmetry which corresponds to the conservation of fermion-number parity. Such a symmetry group is denoted as Z_2^f which is generated by $P_f = (-)^{N_f}$ where N_f is the fermion number. The full symmetry group G_f always contains Z_2^f as a subgroup. We will define $G_b = G_f/Z_2^f$.

1. Graded structure of a group G_b

First, let us introduce the graded structure of a group G_b. A dD-graded structure of a group G_b is an integer function $n_{d-1}(g_1,\ldots,g_d)$ of d variables, whose values are 0,1. Here, we choose g_i to be an element of G_b rather than an element of the full symmetry group G_f. This is because g_i corresponds to the *bosonic* fields in the path integral, and the bosonic field is invariant under Z_2^f.

The function $n_{d-1}(g_1,\ldots,g_d)$ satisfies

$$\begin{aligned} &n_{d-1}(gg_1,\ldots,gg_d) = n_{d-1}(g_1,\ldots,g_d), \quad \forall\, g\in G_b,\\ &\sum_{i=0}^{d} n_{d-1}(g_0,\ldots,g_{i-1},g_{i+1},\ldots g_d) = \text{ even}, \quad \forall\, g_0,\ldots,g_d \in G. \end{aligned} \tag{C1}$$

We see that a dD-graded structure of a group G_b is a $(d-1)$-cocycle $n_{d-1}\in \mathcal{Z}^{d-1}(G_b,\mathbb{Z}_2)$. If a function n_{d-1} only satisfies the first condition in Eq. (C1), then it will be called a $(d-1)$-cochain. The space of all $(d-1)$-cochains is denoted as $\mathcal{C}^{d-1}(G_b,\mathbb{Z}_2)$.

The coboundary is given by

$$\mathcal{B}^{d-1}(G_b,\mathbb{Z}_2) = \left\{ n'_{d-1}(g_1,\ldots,g_d) = \sum_i m'_{d-2}(g_1,\ldots,g_{i-1},g_{i+1},\ldots,g_d) \bmod 2 \,|\, m'_{d-2}(gg_1,\ldots,gg_{d-1}) = m'_{d-2}(g_1,\ldots,g_{d-1}),\ \forall\, g,g_1,\ldots,g_{d-1}\in G_b \right\}.$$

We say that two graded structures differing by an above coboundary are equivalent. Thus, different classes of dD-graded structures are given by $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2) = \mathcal{Z}^{d-1}(G_b,\mathbb{Z}_2)/\mathcal{B}^{d-1}(G_b,\mathbb{Z}_2)$.

2. G-module

For a group G_f, let M be a G_f-module, which is an Abelian group (with multiplication operation) on which G_f acts compatibly with the multiplication operation (i.e., the Abelian group structure):

$$g\cdot(ab) = (g\cdot a)(g\cdot b), \quad g\in G_f, \quad a,b\in M. \tag{C2}$$

For the most cases studied in this paper, M is simply the $U(1)$ group and a a $U(1)$ phase. The multiplication operation ab is the usual multiplication of the $U(1)$ phases. The group action is trivial: $g \cdot a = a$, $g \in G_f$, $a \in U(1)$. We will denote such a trivial G_f-module as $M = U(1)$.

For a group G_f that contains time-reversal operation, we can define a nontrivial G_f-module which is denoted as $U_T(1)$. $U_T(1)$ is also a $U(1)$ group whose elements are the $U(1)$ phases. The multiplication operation ab, $a,b \in U_T(1)$, is still the usual multiplication of the $U(1)$ phases. However, the group action is nontrivial now: $g \cdot a = a^{s(g)}$, $g \in G_f$, $a \in U_T(1)$. Here, $s(g) = 1$ if the number of time-reversal operations in the group operation g is even and $s(g) = -1$ if the number of time-reversal operations in g is odd.

3. Fermionic d-cochain

A fermionic d-cochain is described by a set of three functions $\nu_d(g_0,\ldots,g_d)$, $n_{d-1}(g_1,\ldots,g_d)$, and $u^g_{d-1}(g_1,\ldots,g_d)$. $n_{d-1}(g_1,\ldots,g_d)$ is $(d-1)$-cochain in $\mathcal{C}^{d-1}(G_b,\mathbb{Z}_2)$ which has been discussed above. $\nu_d(g_0,\ldots,g_d)$ is a function of $1+d$ variables whose value is in a G-module M, $\nu_d : G_b^{1+d} \to M$. Again, note that g_i is an element of G_b rather than the full symmetry group G_f. For cases studied in this paper, M is always a $U(1)$ group [i.e., $\nu_d(g_0,\ldots,g_d)$ is a complex phase].

Since the action amplitude is invariant under the symmetry transformation in G_f, $\nu_d(g_0,\ldots,g_d)$ must satisfy certain conditions. Note that a transformation g in G_f will generate a transformation in G_b, $gG_b \to G_b$, since $G_b = G_f/Z_2^f$. So, $\nu_d(g_0,\ldots,g_d)$ transforms as $\nu_d(g_0,\ldots,g_d) \to \nu_d^{s(g)}(gg_0,\ldots,gg_d)$.

The invariance under the symmetry transformation in G_f is discussed in Secs. V E and VII A. This motivates us to require that the fermionic d-cochain $\nu_d(g_0,\ldots,g_d)$ satisfy the following symmetry condition:

$$\begin{aligned}
&\nu_d^{s(g)}(gg_0,\ldots,gg_d)\\
&\quad= \nu_d(g_0,\ldots,g_d)\prod_i \left[u^g_{d-1}(g_0,\ldots,\hat{g}_i,\ldots,g_d)\right]^{(-)^i},\\
&n_{d-1}(gg_0,\ldots,gg_{d-1})\\
&\quad= n_{d-1}(g_0,\ldots,g_{d-1}), \quad g \in G_f
\end{aligned} \tag{C3}$$

where the sequence $g_0,\ldots,\hat{g}_i,\ldots,g_d$ is the sequence $g_0,\ldots,g_d$ with g_i removed, and $u^g_{d-1}(g_i,g_j,\ldots,g_k)$ is a 1D representation of G_f:

$$\begin{aligned}
&u^{g_a}_{d-1}(g_i,\ldots,g_k)\left[u^{g_b}_{d-1}(g_i,\ldots,g_k)\right]^{s(g_a)}\\
&\quad= u^{g_a g_b}_{d-1}(g_i,\ldots,g_k),\\
&u^{g_a}_{d-1}(gg_i,gg_j,\ldots,gg_k)\\
&\quad= \left[u^{g_a}_{d-1}(g_i,g_j,\ldots,g_k)\right]^{s(g)},
\end{aligned} \tag{C4}$$

such that $u^{P_f}_{d-1}(g_i,g_j,\ldots,g_k) = (-)^{n_{d-1}(g_i,g_j,\ldots,g_k)}$. The triple functions $(\nu_d,n_{d-1},u^g_{d-1})$ that satisfy Eqs. (C3) and (C4) are called fermionic cochains. We will denote the collection of all those fermionic cochains $(\nu_d,n_{d-1},u^g_{d-1})$ as $\mathscr{C}^d[G_f,U_T(1)]$.

4. Fermionic d-cocycle

With the above setup, now we are ready to define fermionic d-cocycle. The fermionic d-cocycles are fermionic d-cochains $[\nu_d,n_{d-1},u^g_{d-1}]$ in $\mathscr{C}^d[G_f,U_T(1)]$ that satisfy some additional conditions.

First, we require $n_{d-1}(g_1,\ldots,g_d)$ to be $(d-1)$-cocycles in $\mathcal{Z}^{d-1}(G_f,\mathbb{Z}_2)$. Note that when $d-1$, we have $n_0(g_1) = 0$ or $n_0(g_1) = 1$ for any G_f. We also require $\nu_d(g_0,\ldots,g_d)$ to satisfy

$$\nu_1(g_1,g_2)\nu_1^{-1}(g_0,g_2)\nu_1(g_0,g_1) = 1, \tag{C5}$$

$$\nu_2(g_1,g_2,g_3)\nu_2^{-1}(g_0,g_2,g_3)\nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_1,g_2) = 1, \tag{C6}$$

$$(-)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)}\nu_3(g_1,g_2,g_3,g_4)\nu_3^{-1}(g_0,g_2,g_3,g_4)\nu_3(g_0,g_1,g_3,g_4)\nu_3^{-1}(g_0,g_1,g_2,g_4)\nu_3(g_0,g_1,g_2,g_3) = 1, \tag{C7}$$

$$\begin{aligned}
&(-)^{n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5)}\nu_4(g_1,g_2,g_3,g_4,g_5)\\
&\quad\times \nu_4^{-1}(g_0,g_2,g_3,g_4,g_5)\nu_4(g_0,g_1,g_3,g_4,g_5)\nu_4^{-1}(g_0,g_1,g_2,g_4,g_5)\nu_4(g_0,g_1,g_2,g_3,g_5)\nu_4^{-1}(g_0,g_1,g_2,g_3,g_4)\\
&\quad= 1.
\end{aligned} \tag{C8}$$

We will denote the collections of all d-cocycles $[\nu_d(g_0,\ldots,g_d),u^g_{d-1}(g_1,\ldots,g_d)]$ that satisfy the above conditions as $\mathscr{Z}^d[G_f,U_T(1)]$.

5. A "linearized" representation of group supercohomology

Since the coefficient of the group supercohomology is always $U(1)$, we can map the fermionic cochains $[\nu_d,n_{d-1},u^g_{d-1}]$ into two real vectors and an integer vector $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}^g_{d-1})$, where the components of $\boldsymbol{v}_d$ and $\boldsymbol{h}^g_{d-1}$ are in $[0,1)$ and the components of $\boldsymbol{n}_{d-1}$ are 0,1. Then, we can rewrite the cochain and the cocycle conditions on $[\nu_d(g_0,\ldots,g_d),n_{d-1}(g_1,\ldots,g_d),u^g_{d-1}(g_1,\ldots,g_d)]$ as "linear equations" on $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}^g_{d-1})$. Those linear equations can be solved numerically more easily.

To map $\nu_d(g_0,\ldots,g_d)$ into a real vector $\boldsymbol{v}_d$, we can introduce the log of $\nu_d(g_0,\ldots,g_d)$:

$$v_d(g_0,g_1,\ldots,g_d) = \frac{1}{2\pi i}\ln \nu_d(g_0,g_1,\ldots,g_d) \bmod 1. \tag{C9}$$

So, we define $\boldsymbol{v}_d$ as a $|G_b|^{d+1}$-dimensional vector, whose components are given by

$$(\boldsymbol{v}_d)_{(g_0,\ldots,g_d)} = v_d(g_0,\ldots,g_d). \tag{C10}$$

Similarly, $\boldsymbol{n}_{d-1}$ is a $|G_b|^d$-dimensional integer vector

$$(\boldsymbol{n}_{d-1})_{(g_0,\ldots,g_{d-1})} = n_d(g_0,\ldots,g_{d-1}), \tag{C11}$$

and $\boldsymbol{h}^g_{d-1}$ is a $|G_b|^d$-dimensional vector

$$(\boldsymbol{h}^g_{d-1})_{(g_0,\dots,g_{d-1})} = h^g_{d-1}(g_0,\dots,g_{d-1}),$$
$$h^g_{d-1}(g_0,\dots,g_{d-1}) = \frac{1}{2\pi i}\ln u^g_{d-1}(g_0,\dots,g_{d-1}) \bmod 1. \quad \text{(C12)}$$

Now, the cochain conditions (i.e., the symmetry condition) (C3), (C1), and (C4) can be written in a "linearized" form

$$\begin{aligned} s(g)S^g_d \boldsymbol{v}_d &= \boldsymbol{v}_d + D_{d-1}\boldsymbol{h}^g_{d-1} \bmod 1,\\ S^g_{d-1}\boldsymbol{n}_{d-1} &= \boldsymbol{n}_{d-1},\\ s(g)S^g_{d-1}\boldsymbol{h}^{g_1}_{d-1} &= \boldsymbol{h}^{g_1}_{d-1},\\ \boldsymbol{h}^{g_1g_2}_{d-1} &= \boldsymbol{h}^{g_1}_{d-1} + s(g_1)\boldsymbol{h}^{g_2}_{d-1} \bmod 1,\\ \boldsymbol{h}^{P_f}_{d-1} &= \tfrac{1}{2}\boldsymbol{n}_{d-1} \bmod 1,\\ g,g_1,g_2 &\in G_f, \end{aligned} \quad \text{(C13)}$$

where S^g_d is a $(|G_b|^{d+1}\times|G_b|^{d+1})$-dimensional matrix, whose elements are

$$(S^g_d)_{(g_0,\dots,g_d),(g'_0,\dots,g'_d)} = \delta_{(g'_0,\dots,g'_d),(gg_0,\dots,gg_{d+1})}. \quad \text{(C14)}$$

Also, D_d is a $(|G_b|^{d+2}\times|G_b|^{d+1})$-dimensional integer matrix, whose elements are all 0 or ±1:

$$(D_d)_{(g_0,\dots,g_{d+1}),(g'_0,\dots,g'_d)} = (-)^i\delta_{(g'_0,\dots,g'_d),(g_0,\dots,\hat{g}_i,\dots,g_{d+1})}, \quad \text{(C15)}$$

where $(g_0,\dots,\hat{g}_i,\dots,g_{d+1})$ is the sequence obtained from the sequence $(g_0,\dots,g_{d+1})$ with the element g_i removed. The matrices D_d satisfy

$$D_{d+1}D_d = 0. \quad \text{(C16)}$$

A triple $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}^g_{d-1})$ that satisfies Eq. (C13) corresponds to a fermionic d-cochain in $\mathscr{C}^d[G_f,U_T(1)]$.

We can also rewrite the cocycle conditions (59) as a "linear" equation:

$$\begin{aligned} D_{d-1}\boldsymbol{n}_{d-1} &= 0 \bmod 2,\\ D_d\boldsymbol{v}_d + \tfrac{1}{2}\boldsymbol{f}_{d+1} &= 0 \bmod 1. \end{aligned} \quad \text{(C17)}$$

Here, $\boldsymbol{f}_d$ is a $|G_b|^{d+1}$-dimensional integer vector whose components are given by

$$(\boldsymbol{f}_d)_{(g_0,\dots,g_d)} = f_d(g_0,\dots,g_d) \bmod 2, \quad \text{(C18)}$$

where $f_d(g_0,\dots,g_d)$ is obtained from $n_{d-2}(g_0,\dots,g_{d-2})$ according to the relation (57). A triple $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}^g_{d-1})$ that satisfies Eqs. (C13) and (C17) corresponds to a fermionic d-cocycle in $\mathscr{Z}^d[G_f,U_T(1)]$.

6. Fermionic *d*-cohomology class

As discussed in Sec. V, each $(d+1)$-cocycle will define a fermionic topological nonlinear σ model in d spatial dimensions which is a fixed point of the RG flow. Those $(d+1)$-cocycles give rise to new fermionic SPT phases. However, different $(d+1)$-cocycles may give rise to the same fermionic SPT phase. Those $(d+1)$-cocycles are said to be equivalent. To find (or guess) the equivalence relation between cocycles, in this Appendix, we will discuss some natural choices of possible equivalence relations. A more physical discussion of this issue is given in Appendix G 1.

First, we would like to show that (for $d \leqslant 4$)

$$D_d\boldsymbol{f}_d = 0 \bmod 2, \quad \text{(C19)}$$

where $\boldsymbol{f}_d$ is determined from $n_{d-1}(g_1,\dots,g_d)$ in $\mathcal{Z}^{d-1}(G_f,\mathbb{Z}_2)$ [see Eq. (57)]. For $1\leqslant d\leqslant 3$, the above is trivially satisfied since the corresponding $f_d = 0$. To show the above for $d=4$, we note that $n_{d-2}(g_0,\dots,g_{d-2})$ (which give rise to $\boldsymbol{f}_d$) is a $(d-2)$-cocycle in $\mathcal{H}^{d-2}(G_b,\mathbb{Z}_2)$. Let us assume that a 2-cocycle $n_2(g_0,g_1,g_2)$ gives rise to $\boldsymbol{f}_4$. The expression $f_4(g_0,g_1,g_2,g_3,g_4) = n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)$ implies that the 4-cochain $f_4(g_0,g_1,g_2,g_3,g_4)$ is the cup product of two 2-cocycles $n_2(g_0,g_1,g_2)$ and $n_2(g_0,g_1,g_2)$ in $\mathcal{Z}^2(G_f,\mathbb{Z}_2)$. So, $f_4(g_0,g_1,g_2,g_3,g_4)$ is also a cocycle in $\mathcal{Z}^4(G_f,\mathbb{Z}_2)$. Therefore, $D_4\boldsymbol{f}_4 = 0 \mod 2$. We can also show that $D_5\boldsymbol{f}_5 = 0 \mod 2$ by an explicit calculation using *Mathematica*. Therefore, for $d\leqslant 5$,

$$D_d\big[D_{d-1}\boldsymbol{v}_{d-1} + \tfrac{1}{2}\boldsymbol{f}_d\big] = 0 \bmod 1 \quad \text{(C20)}$$

for any fermionic $(d-1)$-cochains $(\boldsymbol{v}_{d-1},\boldsymbol{n}_{d-1},\boldsymbol{h}^g_{d-1})$.

There is another important relation. Let us assume that a d-cocycle $n_d(g_0,g_1,\dots,g_d)$ gives rise to $\boldsymbol{f}_{d+2}$, and another d-cocycle $\tilde{n}_2(g_0,g_1,\dots,g_d)$ gives rise to $\tilde{\boldsymbol{f}}_{d+2}$ (assuming $d\leqslant 3$). If $n_d(g_0,g_1,\dots,g_d)$ and $\tilde{n}_d(g_0,g_1,\dots,g_d)$ are related by a coboundary in $\mathcal{B}^d(G_b,\mathbb{Z}_2)$,

$$\begin{aligned} &n_d(g_0,g_1,\dots,g_d) + \sum_i m'_{d-1}(g_0,\dots,\hat{g}_i,\dots,g_d)\\ &= n_d(g_0,g_1,\dots,g_d) + (\delta m'_{d-1})(g_0,\dots,g_d)\\ &= \tilde{n}_d(g_0,g_1,\dots,g_d) \bmod 2, \end{aligned} \quad \text{(C21)}$$

then what is the relation between $\boldsymbol{f}_{d+2}$ and $\tilde{\boldsymbol{f}}_{d+2}$? In Appendix I, we will show that $\boldsymbol{f}_{d+2}$ and $\tilde{\boldsymbol{f}}_{d+2}$ also differ by a coboundary $D_{d+1}\boldsymbol{g}_{d+1}$ in $\mathcal{B}^{d+2}(G_b,\mathbb{Z}_2)$:

$$\tilde{\boldsymbol{f}}_{d+2} = \boldsymbol{f}_{d+2} + D_{d+1}\boldsymbol{g}_{d+1} \bmod 2, \quad \text{(C22)}$$

where $\boldsymbol{g}_{d+1}$ is determined from $\boldsymbol{n}_d$ and $\boldsymbol{m}_{d-1}$ (see Appendix I).

Using the above two relations, we can define the first equivalence relation between two fermionic d-cocycles. Starting from a fermionic d-cocycle $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}^g_{d-1})$, we can use a fermionic $(d-1)$-cochain $(\boldsymbol{v}_{d-1},\boldsymbol{n}_{d-2},\boldsymbol{h}^g_{d-2})$ to transform the above d-cocycle to an equivalent d-cocycle $(\tilde{\boldsymbol{v}}_d,\tilde{\boldsymbol{n}}_{d-1},\tilde{\boldsymbol{h}}^g_{d-1})$:

$$\begin{aligned} \tilde{\boldsymbol{v}}_d &= \boldsymbol{v}_d - \tfrac{1}{2}\boldsymbol{g}_d + D_{d-1}\boldsymbol{v}_{d-1},\\ \tilde{\boldsymbol{n}}_{d-1} &= \boldsymbol{n}_{d-1} + D_{d-2}\boldsymbol{n}_{d-2}\\ \tilde{\boldsymbol{h}}^g_{d-1} &= \boldsymbol{h}^g_{d-1} + D_{d-2}\boldsymbol{h}^g_{d-2}, \end{aligned} \quad \text{(C23)}$$

where $\boldsymbol{g}_d$ is determined from $\boldsymbol{n}_{d-1}$ and $\boldsymbol{n}_{d-2}$ [see Appendix I, Eq. (I6)].

First, we would like to show that if $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}^g_{d-1})$ satisfies the cochain condition (C13), then $(\tilde{\boldsymbol{v}}_d,\tilde{\boldsymbol{n}}_{d-1},\tilde{\boldsymbol{h}}^g_{d-1})$ obtained above also satisfies the same cochain condition. Let us first consider the first equation in Eq. (C13):

$$\begin{aligned}
&s(g)S_d^g\tilde{\boldsymbol{v}}_d\\
&\quad= s(g)S_d^g\boldsymbol{v}_d - \tfrac{1}{2}\boldsymbol{g}_d + s(g)S_d^g D_{d-1}\boldsymbol{v}_{d-1}\\
&\quad= \boldsymbol{v}_d + D_{d-1}\boldsymbol{h}_{d-1}^g - \tfrac{1}{2}\boldsymbol{g}_d + D_{d-1}s(g)S_{d-1}^g\boldsymbol{v}_{d-1} \bmod 1\\
&\quad= \tilde{\boldsymbol{v}}_d + D_{d-1}\big(\boldsymbol{h}_{d-1}^g + D_{d-2}\boldsymbol{h}_{d-2}^g\big) \bmod 1\\
&\quad= \tilde{\boldsymbol{v}}_d + D_{d-1}\tilde{\boldsymbol{h}}_{d-1}^g \bmod 1,
\end{aligned} \tag{C24}$$

where we have used

$$S_d^g\boldsymbol{g}_d = \boldsymbol{g}_d;\quad S_d^g\boldsymbol{f}_d = \boldsymbol{f}_d, \tag{C25}$$

and the relation

$$S_d^g D_{d-1} = D_{d-1}S_{d-1}^g. \tag{C26}$$

It is easy to show that $(\tilde{\boldsymbol{v}}_d,\tilde{\boldsymbol{n}}_{d-1},\tilde{\boldsymbol{h}}_{d-1}^g)$ also satisfies other equations in Eq. (C13).

Now we like to show that if $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}_{d-1}^g)$ satisfies the cocycle condition (C17), then $(\tilde{\boldsymbol{v}}_d,\tilde{\boldsymbol{n}}_{d-1},\tilde{\boldsymbol{h}}_{d-1}^g)$ obtained above also satisfies the same cocycle condition:

$$\begin{aligned}
D_d\tilde{\boldsymbol{v}}_d + \tfrac{1}{2}\tilde{\boldsymbol{f}}_{d+1} &= D_d\boldsymbol{v}_d - \tfrac{1}{2}D_d\boldsymbol{g}_d + \tfrac{1}{2}(\boldsymbol{f}_{d+1} + D_d\boldsymbol{g}_d)\\
&= 0 \bmod 1.
\end{aligned} \tag{C27}$$

We can also define the second equivalence relation between two fermionic d-cocycles. Let $\bar{h}_{d-1}^g(g_0,\dots,g_{d-1})$ be representations of G_b that satisfy

$$\begin{aligned}
\bar{h}_{d-1}^{g_a}(g_0,g_1,\dots)+s(g_a)\bar{h}_{d-1}^{g_b}(g_0,g_1,\dots) &= \bar{h}_{d-1}^{g_ag_b}(g_0,g_1,\dots),\\
s(g)\bar{h}_{d-1}^{g_a}(gg_0,\dots,gg_{d-1}) &= \bar{h}_{d-1}^{g_a}(g_0,\dots,g_{d-1}),\\
g,g_a,g_b &\in G_b.
\end{aligned} \tag{C28}$$

Then, starting from a fermionic d-cocycle $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1},\boldsymbol{h}_{d-1}^g)$, we can use such a $\bar{h}_{d-1}^g(g_0,\dots,g_{d-1})$ to transform the above d-cocycle to an equivalent d-cocycle $(\tilde{\boldsymbol{v}}_d,\tilde{\boldsymbol{n}}_{d-1},\tilde{\boldsymbol{h}}_{d-1}^g)$:

$$\begin{aligned}
\tilde{v}_d(g_0,g_1,\dots,g_d) &= v_d(g_0,g_1,\dots,g_d) + s(g_1)\bar{h}_{d-1}^{g_1}(g_1,\dots,g_d) + s(g_0)\sum_{i=1}^{d}(-)^i\bar{h}_{d-1}^{g_0}(g_0,\dots,\hat{g}_i,\dots,g_d),\\
\tilde{n}_{d-1}(g_0,g_1,\dots,g_{d-1}) &= n_{d-1}(g_0,g_1,\dots,g_{d-1}),\\
\tilde{h}_{d-1}^g(g_0,\dots,g_{d-1}) &= h_{d-1}^g(g_0,\dots,g_{d-1}) + s(gg_0)\bar{h}_{d-1}^g(g_0,\dots,g_{d-1}).
\end{aligned} \tag{C29}$$

The above can also be written in a more compact form

$$\tilde{\boldsymbol{v}}_d = \boldsymbol{v}_d + D_{d-1}\bar{\boldsymbol{h}}_{d-1},\quad \tilde{\boldsymbol{n}}_{d-1} = \boldsymbol{n}_{d-1},\quad \tilde{\boldsymbol{h}}_{d-1}^g = \boldsymbol{h}_{d-1}^g + \bar{\boldsymbol{h}}_{d-1}^g, \tag{C30}$$

where $\bar{\boldsymbol{h}}_{d-1}$ and $\bar{\boldsymbol{h}}_{d-1}^g$ are the $|G_b|^d$-dimensional vectors

$$(\bar{\boldsymbol{h}}_{d-1})_{(g_0,\dots,g_{d-1})} = s(g_0)h_{d-1}^{g_0}(g_0,\dots,g_{d-1}),\quad \big(\bar{\boldsymbol{h}}_{d-1}^g\big)_{(g_0,\dots,g_{d-1})} = s(gg_0)h_{d-1}^g(g_0,\dots,g_{d-1}). \tag{C31}$$

Let us first show that $(\tilde{\boldsymbol{v}}_d,\tilde{\boldsymbol{n}}_{d-1},\tilde{\boldsymbol{h}}_{d-1}^g)$ is a d-cochain:

$$\begin{aligned}
&s(g)\tilde{v}_d(gg_0,gg_1,\dots,gg_d)\\
&= s(g)v_d(gg_0,gg_1,\dots,gg_d) + s(g)s(gg_1)\bar{h}_{d-1}^{gg_1}(gg_1,\dots,gg_d) + s(g)s(gg_0)\sum_{i=1}^{d}(-)^i\bar{h}_{d-1}^{gg_0}(gg_0,\dots,g\hat{g}_i,\dots,gg_d)\\
&= v_d(g_0,g_1,\dots,g_d) + \sum_{i=0}^{d}(-)^i h_{d-1}^g(g_0,\dots,\hat{g}_i,\dots,g_d) + s(gg_1)\bar{h}_{d-1}^{gg_1}(g_1,\dots,g_d) + s(gg_0)\sum_{i=1}^{d}(-)^i\bar{h}_{d-1}^{gg_0}(g_0,\dots,\hat{g}_i,\dots,g_d)\\
&= v_d(g_0,g_1,\dots,g_d) + \sum_{i=0}^{d}(-)^i h_{d-1}^g(g_0,\dots,\hat{g}_i,\dots,g_d) + s(gg_1)\bar{h}_{d-1}^{g}(g_1,\dots,g_d) + s(gg_0)\sum_{i=1}^{d}(-)^i\bar{h}_{d-1}^{g}(g_0,\dots,\hat{g}_i,\dots,g_d)\\
&\quad + s(g_1)\bar{h}_{d-1}^{g_1}(g_1,\dots,g_d) + s(g_0)\sum_{i=1}^{d}(-)^i\bar{h}_{d-1}^{g_0}(g_0,\dots,\hat{g}_i,\dots,g_d)\\
&= \tilde{v}_d(g_0,g_1,\dots,g_d) + \sum_{i=0}^{d}(-)^i\tilde{h}_{d-1}^g(g_0,\dots,\hat{g}_i,\dots,g_d).
\end{aligned} \tag{C32}$$

Second, since $\tilde{\boldsymbol{v}}_d = \boldsymbol{v}_d + D_{d-1}\bar{\boldsymbol{h}}_{d-1}$, it also clear that $(\tilde{\boldsymbol{v}}_d,\tilde{\boldsymbol{n}}_{d-1},\tilde{\boldsymbol{h}}_{d-1}^g)$ is a d-cocycle. The equivalence classes of fermionic d-cocycles obtained from the above two kinds of equivalence relations (C24) and (C29) form the group d-supercohomology $\mathcal{H}^d[G_f,U_T(1)]$.

We note that, when $\boldsymbol{n}_{d-1}=0$, the representations of G_f, $\boldsymbol{h}^g_{d-1}$ are also representations of G_b. So, the second equivalence relation implies that a d-cocycle $(\boldsymbol{v}_d,\boldsymbol{n}_{d-1}=0,\boldsymbol{h}^g_{d-1})$ is always equivalent to a simpler d-cocycle $(\tilde{\boldsymbol{v}}_d,\boldsymbol{n}_{d-1}=0,\boldsymbol{h}^g_{d-1}=0)$. Such kinds of d-cocycles are described by group cohomology and form group cohomology class $\mathcal{H}^d[G_b,U_T(1)]$.

We also note that, due to the requirement $\boldsymbol{h}^{P_f}_{d-1}=\frac{1}{2}\boldsymbol{n}_{d-1}$ mod 1, for each fixed $\boldsymbol{n}_{d-1}$, all allowed choices of $\boldsymbol{h}^g_{d-1}$ belong to one class under the second equivalence relation. So, we just need to choose one allowed $\boldsymbol{h}^g_{d-1}$ for each $\boldsymbol{n}_{d-1}$.

APPENDIX D: ABELIAN GROUP STRUCTURE OF THE GROUP SUPERCOHOMOLOGY CLASSES

Similar to the standard group cohomology, the elements of supercohomology classes $\mathscr{H}^d[G_f,U_T(1)]$ should form an Abelian group. This Abelian group structure has a very physical meaning. Let us consider two SPT phases labeled by a,b. If we stack the phase a and the phase b on top of each other, we still have a gapped SPT phase which should be labeled by c. Such a stacking operation $a+b\to c$ generates the Abelian group structure of the group supercohomology classes. So, if our construction of the fermionic SPT phases is in some sense complete, stacking the phase a and the phase b in $\mathscr{H}^d[G_f,U_T(1)]$ will result in a SPT phase c still in $\mathscr{H}^d[G_f,U_T(1)]$. In other words, $\mathscr{H}^d[G_f,U_T(1)]$ should be an Abelian group.

The key step to showing $\mathscr{H}^d[G_f,U_T(1)]$ to be an Abelian group is to show that all the valid graded structures labeled by the elements in $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ form an Abelian group. Since $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ is an Abelian group, we only need to show that for any two elements $n_{d-1},n'_{d-1}\in B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, their summation $n''_{d-1}=n_{d-1}+n'_{d-1}\in B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$. In Appendix J, we prove that $f''_{d+1}-f_{d+1}-f'_{d+1}$ is a coboundary in $\mathcal{B}^{d+1}(G_b,\mathbb{Z}_2)$ (given by a_d below), where f_d is obtained from n_d, f'_d is obtained from n'_d, and f''_d is obtained from n''_d, respectively. Therefore, if f_{d+1} and f'_{d+1} are $(d+1)$-coboundaries in $\mathcal{B}^{d+1}[G_b,U_T(1)]$, f''_{d+1} is also a $(d+1)$-coboundary in $\mathcal{B}^{d+1}[G_b,U_T(1)]$. On the other hand, $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ is a subset of $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$, hence, $B\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$ forms an (Abelian) subgroup of $\mathcal{H}^{d-1}(G_b,\mathbb{Z}_2)$.

Using the above result, we can show that if $(\nu_d,n_{d-1},u^g_{d-1})$ and $(\nu'_d,n'_{d-1},u'^g_{d-1})$ are elements in $\mathscr{H}^d[G_f,U_T(1)]$, then using an additive operation we can produce a $(\nu''_d,n''_{d-1},u''^g_{d-1})$:

$$\begin{aligned} n''_{d-1}&=n_{d-1}+n'_{d-1},\\ u''^g_{d-1}&=u^g_{d-1}u'^g_{d-1},\\ \nu''_d&=\nu_d\nu'_d(-)^{a_d}\end{aligned}\tag{D1}$$

such that it is also an element in $\mathscr{H}^d[G_f,U_T(1)]$. Here, the phase factor $(-)^{a_d}$ is given by

$$\begin{aligned} &a_0=a_1=a_2=0,\\ &a_3=n_2(g_0,g_1,g_2)n'_2(g_0,g_2,g_3)+n_2(g_1,g_2,g_3)n'_2(g_0,g_1,g_3),\\ &a_4=n_3(g_0,g_1,g_2,g_3)n'_3(g_0,g_1,g_3,g_4)\\ &\quad+n'_3(g_0,g_1,g_2,g_4)n_3(g_0,g_2,g_3,g_4)\\ &\quad+n_3(g_0,g_1,g_3,g_4)n'_3(g_1,g_2,g_3,g_4)\\ &\quad+n_3(g_0,g_1,g_2,g_3)n'_3(g_1,g_2,g_3,g_4).\end{aligned}\tag{D2}$$

APPENDIX E: CALCULATE $\mathcal{H}^d(G_b,\mathbb{Z}_2)$, THE GRADED STRUCTURE

In this Appendix, we will discuss several methods that allow us to calculate the graded structure $\mathcal{H}^d(G_b,\mathbb{Z}_2)$ in general.

1. Calculate $\mathcal{H}^d(Z_n,\mathbb{Z}_2)$

The cohomology group $\mathcal{H}^d(Z_n,M)$ has a very simple form. To describe the simple form in a more general setting, let us define Tate cohomology groups $\hat{\mathcal{H}}^d(G,M)$. For d to be 0 or -1, we have

$$\begin{aligned}\hat{\mathcal{H}}^0(G,M)&=M^G/\text{Img}(N_G,M),\\ \hat{\mathcal{H}}^{-1}(G,M)&=\text{Ker}(N_G,M)/I_GM.\end{aligned}\tag{E1}$$

Here, M^G, $\text{Img}(N_G,M)$, $\text{Ker}(N_G,M)$, and I_GM are submodules of M. M^G is the maximal submodule that is invariant under the group action. Let us define a map $N_G: M\to M$ as

$$a\to\sum_{g\in G}g\cdot a,\quad a\in M.\tag{E2}$$

$\text{Img}(N_G,M)$ is the image of the map and $\text{Ker}(N_G,M)$ is the kernel of the map. The submodule I_GM is given by

$$I_GM=\left\{\sum_{g\in G}n_g g\cdot a\,\middle|\,\sum_{g\in G}n_g=0,\,a\in M\right\}.\tag{E3}$$

In other words, I_GM is generated by $g\cdot a-a$, $\forall\ g\in G$, $a\in M$.

For d other than 0 and -1, the Tate cohomology group $\hat{\mathcal{H}}^d(G,M)$ is given by

$$\begin{aligned}\hat{\mathcal{H}}^d(G,M)&=\mathcal{H}^d(G,M)\ \text{ for } d>0,\\ \hat{\mathcal{H}}^d[G,M]&=\mathcal{H}_{-d-1}[G,M]\ \text{ for } d<-1.\end{aligned}\tag{E4}$$

For cyclic group Z_n, its (Tate) group cohomology over a generic Z_n module M is given by [47,76]

$$\hat{\mathcal{H}}^d(Z_n,M)=\begin{cases}\hat{\mathcal{H}}^0(Z_n,M) & \text{if } d=0 \bmod 2,\\ \hat{\mathcal{H}}^{-1}(Z_n,M) & \text{if } d=1 \bmod 2,\end{cases}\tag{E5}$$

where

$$\begin{aligned}\hat{\mathcal{H}}^0(Z_n,M)&=M^{Z_n}/\text{Img}(N_{Z_n},M),\\ \hat{\mathcal{H}}^{-1}(Z_n,M)&=\text{Ker}(N_{Z_n},M)/I_{Z_n}M.\end{aligned}\tag{E6}$$

For example, when the group action is trivial, we have $M^{Z_n}=M$ and $I_{Z_n}M=\mathbb{Z}_1$. The map N_{Z_n} becomes $N_{Z_n}: a\to na$. For $M=\mathbb{Z}_2$, we have $\text{Img}(N_{Z_n},\mathbb{Z}_2)=n\mathbb{Z}_2$ and $\text{Ker}(N_{Z_n},\mathbb{Z}_2)=\mathbb{Z}_{(n,2)}$ [where (n,m) is the greatest common divisor of n and m]. So, we have

$$\mathcal{H}^d(Z_n,\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{if } d=0,\\ \mathbb{Z}_{(n,2)} & \text{if } d>0.\end{cases}\tag{E7}$$

What is the nontrivial cocycle in $\mathcal{H}^d[Z_2,\mathbb{Z}_2]$? In [46], it is shown that a d-cocycle can be chosen to satisfy

$$\begin{aligned}\nu_d(\mathbf{g},\mathbf{g},g_2,g_3,\ldots,g_{d-2},g_{d-1},g_d) &= 1,\\ \nu_d(g_0,\mathbf{g},\mathbf{g},g_3,\ldots,g_{d-2},g_{d-1},g_d) &= 1\\ &\cdots\\ \nu_d(g_0,g_1,g_2,g_3,\ldots,g_{d-2},\mathbf{g},\mathbf{g}) &= 1 \end{aligned} \tag{E8}$$

by adding a proper coboundary.

Let us denote the two elements in Z_2 as $\{E,\sigma\}$ with $\sigma^2=E$. If we choose the cocycle $\nu_d(g_0,g_1,g_2,\ldots)$ in $\mathcal{H}^d(Z_2,\mathbb{Z}_2)$ to satisfy the above condition, then only $\nu_d(E,\sigma,E,\ldots)=\nu_d(\sigma,E,\sigma,\ldots)$ can be nonzero. Since $\mathcal{H}^d(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$, so $\nu_d(E,\sigma,E,\ldots)=\nu_d(\sigma,E,\sigma,\ldots)=0$ must correspond to the trivial class and $\nu_d(E,\sigma,E,\ldots)=\nu_d(\sigma,E,\sigma,\ldots)=1$ must correspond to the nontrivial class in $\mathcal{H}^d(Z_2,\mathbb{Z}_2)$.

2. Calculate $\mathcal{H}^d(G,\mathbb{Z}_2)$ from $\mathcal{H}^d(G,\mathbb{Z})$ using Künneth formula for group cohomology

We can also calculate $\mathcal{H}^d(G,\mathbb{Z}_2)$ from $\mathcal{H}^d(G,\mathbb{Z})$ using the Künneth formula for group cohomology. Let M (resp. M') be an arbitrary G-module (resp. G'-module) over a principal ideal domain R. We also assume that either M or M' is R free. Then, we have a Künneth formula for group cohomology [77,78]

$$\begin{aligned}&\mathbb{Z}_1 \to \prod_{p=0}^{d}\mathcal{H}^p(G,M)\otimes_R \mathcal{H}^{d-p}(G',M')\\ &\to \mathcal{H}^d(G\times G',M\otimes_R M')\\ &\to \prod_{p=0}^{d+1}\text{Tor}_1^R[\mathcal{H}^p(G,M),\mathcal{H}^{d-p+1}(G',M')]\to \mathbb{Z}_1. \end{aligned}\tag{E9}$$

If both M and M' are R free, then the sequence splits and we have

$$\begin{aligned}&\mathcal{H}^d(G\times G',M\otimes_R M')\\ &=\left[\prod_{p=0}^{d}\mathcal{H}^p(G,M)\otimes_R\mathcal{H}^{d-p}(G',M')\right]\\ &\quad\times\left[\prod_{p=0}^{d+1}\text{Tor}_1^R[\mathcal{H}^p(G,M),\mathcal{H}^{d-p+1}(G',M')]\right]. \end{aligned}\tag{E10}$$

If R is a field K, we have

$$\begin{aligned}&\mathcal{H}^d(G\times G',M\otimes_K M')\\ &=\prod_{p=0}^{d}[\mathcal{H}^p(G,M)\otimes_K\mathcal{H}^{d-p}(G',M')]. \end{aligned}\tag{E11}$$

Let us choose $R=\mathbb{Z}$, $M=\mathbb{Z}$, and $M'=\mathbb{Z}_2$. Note that $M=\mathbb{Z}$ is a free module over $R=\mathbb{Z}$ but $M'=\mathbb{Z}_2$ is not a free module over $R=\mathbb{Z}$. We also note that $M\otimes_\mathbb{Z} M'=\mathbb{Z}\otimes_\mathbb{Z}\mathbb{Z}_2=\mathbb{Z}_2$. Let us choose $G'=Z_1$ as the trivial group with only identity. We have

$$\mathcal{H}^d(Z_1,\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{if } d=0,\\ \mathbb{Z}_1 & \text{if } d>0.\end{cases}\tag{E12}$$

Thus, we obtain

$$\begin{aligned}&\mathbb{Z}_1\to\mathcal{H}^d(G,\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Z}_2\to\mathcal{H}^d(G,\mathbb{Z}_2)\\ &\to\text{Tor}_1^\mathbb{Z}[\mathcal{H}^{d+1}(G,\mathbb{Z}),\mathbb{Z}_2]\to\mathbb{Z}_1.\end{aligned}\tag{E13}$$

The above can be calculated using the following simple properties of the tensor product $\otimes_\mathbb{Z}$ and $\text{Tor}_1^\mathbb{Z}$ functor:

$$\begin{aligned}\mathbb{Z}\otimes_\mathbb{Z} M &= M\otimes_\mathbb{Z}\mathbb{Z}=M,\\ \mathbb{Z}_n\otimes_\mathbb{Z} M &= M\otimes_\mathbb{Z}\mathbb{Z}_n = M/nM,\\ \mathbb{Z}_m\otimes_\mathbb{Z}\mathbb{Z}_n &= \mathbb{Z}_{(m,n)},\\ (A\times B)\otimes_R M &= (A\otimes_R M)\times(B\otimes_R M),\\ M\otimes_R(A\times B) &= (M\otimes_R A)\times(M\otimes_R B);\end{aligned}\tag{E14}$$

$$\begin{aligned}\text{Tor}_1^\mathbb{Z}(\mathbb{Z},M) &= \text{Tor}_1^\mathbb{Z}(M,\mathbb{Z})=\mathbb{Z}_1,\\ \text{Tor}_1^\mathbb{Z}(\mathbb{Z}_n,M) &= \text{Tor}_1^\mathbb{Z}(M,\mathbb{Z}_n)=M/nM,\\ \text{Tor}_1^\mathbb{Z}(\mathbb{Z}_m,\mathbb{Z}_n) &= \mathbb{Z}_{(m,n)},\\ \text{Tor}_1^R(A\times B,M) &= \text{Tor}_1^R(A,M)\times\text{Tor}_1^R(B,M),\\ \text{Tor}_1^R(M,A\times B) &= \text{Tor}_1^R(M,A)\times\text{Tor}_1^R(M,B).\end{aligned}\tag{E15}$$

Here, (m,n) is the greatest common divisor of m and n.

Using

$$\mathcal{H}^d(Z_n,\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{if } d=0,\\ \mathbb{Z}_n & \text{if } d=0 \bmod 2,\\ \mathbb{Z}_1 & \text{if } d=1 \bmod 2,\end{cases}\tag{E16}$$

we find that for $d=$ even,

$$\begin{aligned}&\mathbb{Z}_1\to\mathbb{Z}_n\otimes_\mathbb{Z}\mathbb{Z}_2\to\mathcal{H}^d(\mathbb{Z}_n,\mathbb{Z}_2)\to\text{Tor}_1^\mathbb{Z}(\mathbb{Z}_1,\mathbb{Z}_2)\to\mathbb{Z}_1\\ \text{or }\ &\mathbb{Z}_1\to\mathbb{Z}_{(n,2)}\to\mathcal{H}^d(\mathbb{Z}_n,\mathbb{Z}_2)\to\mathbb{Z}_1\to\mathbb{Z}_1;\end{aligned}\tag{E17}$$

and for $d=$ odd,

$$\begin{aligned}&\mathbb{Z}_1\to\mathbb{Z}_1\otimes_\mathbb{Z}\mathbb{Z}_2\to\mathcal{H}^d(\mathbb{Z}_n,\mathbb{Z}_2)\to\text{Tor}_1^\mathbb{Z}(\mathbb{Z}_n,\mathbb{Z}_2)\to\mathbb{Z}_1\\ \text{or }\ &\mathbb{Z}_1\to\mathbb{Z}_1\to\mathcal{H}^d(\mathbb{Z}_n,\mathbb{Z}_2)\to\mathbb{Z}_{(n,2)}\to\mathbb{Z}_1.\end{aligned}\tag{E18}$$

This allows us to obtain Eq. (E7) using a different approach.

Using

$$\mathcal{H}^d[Z_n\rtimes Z_2,\mathbb{Z}]\leqslant\begin{cases}\mathbb{Z}, & d=0,\\ \mathbb{Z}_2\times\mathbb{Z}_n\times\mathbb{Z}_{(2,n)}^{\frac{d-2}{2}}, & d=0 \bmod 4,\\ \mathbb{Z}_{(2,n)}^{\frac{d-1}{2}}, & d=1 \bmod 2,\\ \mathbb{Z}_2\times\mathbb{Z}_{(2,n)}^{\frac{d}{2}}, & d=2 \bmod 4,\end{cases}\tag{E19}$$

we can show that for $d=0 \bmod 4$, $d>0$,

$$\begin{aligned}&\mathbb{Z}_1\to\left(\mathbb{Z}_2\times\mathbb{Z}_n\times\mathbb{Z}_{(2,n)}^{\frac{d-2}{2}}\right)\otimes_\mathbb{Z}\mathbb{Z}_2\to\mathcal{H}^d[Z_n\rtimes Z_2,\mathbb{Z}_2]\\ &\to\text{Tor}_1^\mathbb{Z}\left(\mathbb{Z}_{(2,n)}^{\frac{d}{2}},\mathbb{Z}_2\right)\to\mathbb{Z}_1\\ \text{or }\ &\mathbb{Z}_1\to\mathbb{Z}_2\times\mathbb{Z}_{(2,n)}^{\frac{d}{2}}\to\mathcal{H}^d[Z_n\rtimes Z_2,\mathbb{Z}_2]\to\mathbb{Z}_{(2,n)}^{\frac{d}{2}}\to\mathbb{Z}_1;\end{aligned}\tag{E20}$$

for $d=1 \bmod 4$, $d>0$,

$$\begin{aligned}&\mathbb{Z}_1\to\mathbb{Z}_{(2,n)}^{\frac{d-1}{2}}\otimes_\mathbb{Z}\mathbb{Z}_2\to\mathcal{H}^d[Z_n\rtimes Z_2,\mathbb{Z}_2]\\ &\to\text{Tor}_1^\mathbb{Z}\left(\mathbb{Z}_2\times\mathbb{Z}_{(2,n)}^{\frac{d+1}{2}},\mathbb{Z}_2\right)\to\mathbb{Z}_1\\ \text{or }\ &\mathbb{Z}_1\to\mathbb{Z}_{(2,n)}^{\frac{d-1}{2}}\to\mathcal{H}^d[Z_n\rtimes Z_2,\mathbb{Z}_2]\to\mathbb{Z}_2\times\mathbb{Z}_{(2,n)}^{\frac{d+1}{2}}\to\mathbb{Z}_1;\end{aligned}\tag{E21}$$

for $d = 2 \bmod 4$, $d > 0$,

$$\mathbb{Z}_1 \to \left(\mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^{\frac{d}{2}}\right) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathcal{H}^d[Z_n \rtimes Z_2,\mathbb{Z}_2] \to \mathrm{Tor}_1^{\mathbb{Z}}\left(\mathbb{Z}_{(2,n)}^{\frac{d}{2}},\mathbb{Z}_2\right) \to \mathbb{Z}_1$$

$$\text{or } \mathbb{Z}_1 \to \mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^{\frac{d}{2}} \to \mathcal{H}^d[Z_n \rtimes Z_2,\mathbb{Z}_2] \to \mathbb{Z}_{(2,n)}^{\frac{d}{2}} \to \mathbb{Z}_1; \tag{E22}$$

for $d = 3 \bmod 4$, $d > 0$,

$$\mathbb{Z}_1 \to \mathbb{Z}_{(2,n)}^{\frac{d-1}{2}} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathcal{H}^d[Z_n \rtimes Z_2,\mathbb{Z}_2] \to \mathrm{Tor}_1^{\mathbb{Z}}\left(\mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_{(2,n)}^{\frac{d-1}{2}},\mathbb{Z}_2\right) \to \mathbb{Z}_1$$

$$\text{or } \mathbb{Z}_1 \to \mathbb{Z}_{(2,n)}^{\frac{d-1}{2}} \to \mathcal{H}^d[Z_n \rtimes Z_2,\mathbb{Z}_2] \to \mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^{\frac{d+1}{2}} \to \mathbb{Z}_1. \tag{E23}$$

Combining the above results, we find that, for $d > 0$,

$$\mathcal{H}^d[Z_n \rtimes Z_2,\mathbb{Z}_2] = \mathbb{Z}_2 \times \mathbb{Z}_{(2,n)}^{d}. \tag{E24}$$

Using

$$\mathcal{H}^d[U(1),\mathbb{Z}] = \begin{cases} \mathbb{Z} & \text{if } d = 0 \bmod 2, \\ \mathbb{Z}_1 & \text{if } d = 1 \bmod 2, \end{cases} \tag{E25}$$

we find that for $d =$ even,

$$\mathbb{Z}_1 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathcal{H}^d[U(1),\mathbb{Z}_2] \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_1,\mathbb{Z}_2) \to \mathbb{Z}_1$$

$$\text{or } \mathbb{Z}_1 \to \mathbb{Z}_2 \to \mathcal{H}^d[U(1),\mathbb{Z}_2] \to \mathbb{Z}_1 \to \mathbb{Z}_1; \tag{E26}$$

and for $d =$ odd,

$$\mathbb{Z}_1 \to \mathbb{Z}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathcal{H}^d[U(1),\mathbb{Z}_2] \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}_2) \to \mathbb{Z}_1$$

$$\text{or } \mathbb{Z}_1 \to \mathbb{Z}_1 \to \mathcal{H}^d(U(1),\mathbb{Z}_2) \to \mathbb{Z}_1 \to \mathbb{Z}_1. \tag{E27}$$

This allows us to obtain

$$\mathcal{H}^d[U(1),\mathbb{Z}_2] = \begin{cases} \mathbb{Z}_2 & \text{if } d = 0 \bmod 2, \\ \mathbb{Z}_1 & \text{if } d = 1 \bmod 2. \end{cases} \tag{E28}$$

Note that in $\mathcal{H}^d[U(1),\mathbb{Z}]$ and $\mathcal{H}^d[U(1),\mathbb{Z}_2]$, the cocycles $\nu_d : [U(1)]^{d+1} \to \mathbb{Z}$ or $\nu_d : [U(1)]^{d+1} \to \mathbb{Z}_2$ are Borel measurable functions.

Let us choose $R = M = M' = \mathbb{Z}_2$. Since R is a field and $\mathbb{Z}_2 \otimes_{\mathbb{Z}_2} \mathbb{Z}_2$, we have

$$\mathcal{H}^d(G \times G',\mathbb{Z}_2) = \prod_{p=0}^{d} [\mathcal{H}^p(G,\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \mathcal{H}^{d-p}(G',\mathbb{Z}_2)]. \tag{E29}$$

Note that

$$\mathbb{Z}_2 \otimes_{\mathbb{Z}_2} \mathbb{Z}_2 = \mathbb{Z}_2, \quad \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} \mathbb{Z}_1 = \mathbb{Z}_1. \tag{E30}$$

We find

$$\mathcal{H}^d[Z_2 \times Z_2,\mathbb{Z}_2] = \mathbb{Z}_2^{d+1}. \tag{E31}$$

APPENDIX F: CALCULATE GROUP SUPERCOHOMOLOGY CLASS

In this Appendix, we will demonstrate how to calculate group supercohomology classes, following the general outline described in Sec. VI. We will do so by performing explicit calculations from some simple groups.

1. Calculate $\mathscr{H}^d[Z_1 \times Z_2^f, U(1)]$

Here, we choose $G_b = Z_1$ (the trivial group) which corresponds to fermion systems with no symmetry. Since $\mathcal{H}^{d-1}(Z_1,\mathbb{Z}_2) = \mathbb{Z}_2$ for $d = 1$ and $\mathcal{H}^{d-1}(Z_1,\mathbb{Z}_2) = \mathbb{Z}_1 = 0$ for $d > 1$. We find that there are two 1D-graded structures and only one trivial dD-graded structure for $d > 1$. For each dD-graded structure, we can choose a $u_{d-1}^{g}(g_0,\dots,g_{d-1})$ that $u_{d-1}^{P_f}(g_0,\dots,g_{d-1}) = (-)^{n_{d-1}(g_0,\dots,g_{d-1})}$.

For $d > 1$ and for the only trivial graded structure, the equivalent classes of $\boldsymbol{v}_d$ are described by $\mathcal{H}^d[Z_1,U(1)] = \mathbb{Z}_1$. Therefore, $\mathscr{H}^d[Z_1 \times Z_2^f, U(1)] = \mathbb{Z}_1$ for $d > 1$.

For $d = 1$, $\mathcal{H}^0(Z_1,\mathbb{Z}_2) = \mathbb{Z}_2$. The trivial graded structure is given by $n^{(0)}(E) = 0$ and the other nontrivial graded structure is given by $n^{(1)}(E) = 1$. The corresponding $f_2^{(a)}(g_0,\dots,g_d)$ are given by $f_2^{(a)} = 0$, $a = 0,1$. Thus, for each graded structure, the equivalent classes of $\boldsymbol{v}_d$ are described by $\mathcal{H}^1[Z_1,U(1)] = \mathbb{Z}_1$. Therefore, $\mathscr{H}^1[Z_1 \times Z_2^f, U(1)] = \mathbb{Z}_2$. To summarize,

$$\mathscr{H}^d\left[Z_1 \times Z_2^f, U(1)\right] = \begin{cases} \mathbb{Z}_2, & d = 1 \\ \mathbb{Z}_1, & d > 1. \end{cases} \tag{F1}$$

$\mathscr{H}^1[Z_1 \times Z_2^f, U(1)] = \mathbb{Z}_2$ implies that there are two possible fermionic SPT phases in $d_{sp} = 0$ spatial dimension if there is no symmetry. One has an even number of fermions and the other has an odd number of fermions.

$\mathscr{H}^2[Z_1 \times Z_2^f, U(1)] = \mathbb{Z}_1$ implies that there is only one trivial gapped fermionic SPT phase in $d_{sp} = 1$ spatial dimension if there is no symmetry. It is well known that there are two gapped fermionic phases in $d_{sp} = 1$ spatial dimension if there is no symmetry: the trivial phase and the phase with boundary Majorana zero mode [59]. One way to see this result is to use the Jordan-Wigner transformation to map the 1D fermion system with no symmetry to a 1D boson system with Z_2^f symmetry. A 1D boson system with Z_2^f symmetry can have only two gapped phases: the symmetry unbroken one (which corresponds to the trivial fermionic phase) and the symmetry breaking one (which corresponds to the Majorana chain) [44]. Both gapped phases can be realized by noninteracting fermions (see Table II). However, the nontrivial gapped phase (the bosonic Z_2^f symmetry breaking phase) has nontrivial intrinsic topological orders. So, the nontrivial gapped phase is not a fermionic SPT phase. This is why we only have one trivial fermionic SPT phase in 1D if there is no symmetry.

$\mathscr{H}^3[Z_1 \times Z_2^f, U(1)] = \mathbb{Z}_1$ implies that there is only one trivial gapped fermionic SPT phase in $d_{sp} = 2$ spatial dimensions if there is no symmetry. It is well known that even noninteracting fermions have infinite gapped phases labeled by $\mathbb{Z}$ in $d_{sp} = 2$ spatial dimensions if there is no symmetry (see Table II). Those phases correspond to integer quantum Hall states and/or $p + ip$, $d + id$ superconductors. Again, all those gapped phases have nontrivial intrinsic topological orders. So, we only have one trivial fermionic SPT phase in $d_{sp} = 2$ spatial dimensions if there is no symmetry.

2. Calculate $\mathscr{H}^d[Z_{2k+1}\times Z_2^f,U(1)]$

Next, we choose $G_f = Z_{2k+1}\times Z_2^f$. Again, since $\mathcal{H}^{d-1}(Z_{2k+1},\mathbb{Z}_2)=\mathbb{Z}_2$ for $d=1$ and $\mathcal{H}^{d-1}(Z_{2k+1},\mathbb{Z}_2)=\mathbb{Z}_1$ for $d>1$, there are two 1D-graded structures and only one trivial dD-graded structure for $d>1$. For each dD-graded structure, we can choose a $u_{d-1}^g(g_0,\dots,g_{d-1})$ that satisfies $u_{d-1}^g(g_0,\dots,g_{d-1})=1$ for $g\in G_b$ and $u_{d-1}^{P_f}(g_0,\dots,g_{d-1})=(-)^{n_{d-1}(g_0,\dots,g_{d-1})}$.

For $d>1$ and for the only trivial graded structure, the equivalent classes of $\boldsymbol{v}_d$ are described by $\mathcal{H}^d[Z_{2k+1},U(1)]=\mathbb{Z}_1$ for $d=$ even and $\mathcal{H}^d[Z_{2k+1},U(1)]=\mathbb{Z}_{2k+1}$ for $d=$ odd. Therefore, $\mathscr{H}^d[Z_1\times Z_2^f,U(1)]=\mathbb{Z}_1$ for $d=$ even and $\mathscr{H}^d[Z_1\times Z_2^f,U(1)]=\mathbb{Z}_{2k+1}$ for $d=$ odd.

For $d=1$, $\mathcal{H}^0(Z_{2k+1},\mathbb{Z}_2)=\mathbb{Z}_2$. The trivial graded structure is given by $n^{(0)}(g_0)=0$ and the other nontrivial graded structure is given by $n^{(1)}(g_0)=1$. The corresponding $f_2^{(a)}(g_0,\dots,g_d)$ is given by $f_2^{(a)}=0$, $a=0,1$. Thus, for the each graded structure, the equivalent classes of $\boldsymbol{v}_d$ are described by $\mathcal{H}^1[Z_{2k+1},U(1)]=\mathbb{Z}_{2k+1}$. Therefore, $\mathscr{H}^1[Z_{2k+1}\times Z_2^f,U(1)]=\mathbb{Z}_2\times\mathbb{Z}_{2k+1}=\mathbb{Z}_{4k+2}$. To summarize,

$$\mathscr{H}^d\big[Z_{2k+1}\times Z_2^f,U(1)\big]=\begin{cases}\mathbb{Z}_{4k+2}, & d=1\\ \mathbb{Z}_1, & d=\text{even}, d>0\\ \mathbb{Z}_{2k+1}, & d=\text{odd}.\end{cases} \tag{F2}$$

$\mathscr{H}^1[Z_{2k+1}\times Z_2^f,U(1)]=\mathbb{Z}_{4k+2}$ implies that there are $4k+2$ possible fermionic SPT phases in $d_{sp}=0$ spatial dimension if there is a Z_{2k+1} symmetry. $2k+1$ of them have an even number of fermions and the other $2k+1$ of them have an odd number of fermions. The $2k+1$ phases are separated by the $2k+1$ possible Z_{2k+1} quantum numbers.

$\mathscr{H}^2[Z_{2k+1}\times Z_2^f,U(1)]=\mathbb{Z}_1$ implies that there is no nontrivial gapped fermionic SPT phases in $d_{sp}=1$ spatial dimension if there is a Z_{2k+1} symmetry. We can use the Jordan-Wigner transformation to map 1D fermion systems with Z_{2k+1} symmetry to 1D boson systems with $Z_{2k+1}\times Z_2^f=Z_{4k+2}$ symmetry. Since $\mathcal{H}^2[Z_{4k+2},U(1)]=\mathbb{Z}_1$, 1D boson systems have only two phases that do not break the Z_{2k+1} symmetry: one does not break the Z_2^f symmetry and one breaks the Z_2^f symmetry. The fermionic SPT phase has no intrinsic topological order and corresponds to the bosonic phase that does not break the Z_2^f symmetry (and the Z_{2k+1} symmetry). Thus, 1D fermion systems with Z_{2k+1} symmetry indeed have only one trivial fermionic SPT phase.

$\mathscr{H}^3[Z_{2k+1}\times Z_2^f,U(1)]=\mathbb{Z}_{2k+1}$ implies that our construction gives rise to $2k+1$ gapped fermionic SPT phases in $d_{sp}=2$ spatial dimensions if there is a Z_{2k+1} symmetry. The constructed $2k+1$ gapped fermionic SPT phases actually correspond to the $2k+1$ gapped bosonic SPT phases with Z_{2k+1} symmetry [since $\mathcal{H}^2(Z_{2k+1},\mathbb{Z}_2)=\mathbb{Z}_1$ and there is no nontrivial graded structure].

3. Calculate $\mathscr{H}^d[Z_2\times Z_2^f,U(1)]$

Now, let us choose $G_f=Z_2\times Z_2^f$. Following the first step in Sec. VI, we find that $\mathcal{H}^{d-1}(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$ and there are two graded structures in all dimensions. For each dD-graded structure, we can choose a $u_{d-1}^g(g_0,\dots,g_{d-1})$ that satisfies $u_{d-1}^g(g_0,\dots,g_{d-1})=1$ for $g\in G_b$ and $u_{d-1}^{P_f}(g_0,\dots,g_{d-1})=(-)^{n_{d-1}(g_0,\dots,g_{d-1})}$.

Following the second step in Sec. VI, we note that the trivial graded structure is given by $n_{d-1}(g_0,\dots,g_{d-1})=0$. The nontrivial graded structure is given by

$$n_{d-1}(e,o,e,\dots)=n_{d-1}(o,e,o,\dots)=1,\quad \text{others}=0, \tag{F3}$$

where e represents the identity element in Z_2 and o represents the other element in Z_2.

For $d=1,2$, $\mathcal{H}^{d-1}(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$. For each graded structure $n_{d-1}(g_0,\dots,g_{d-1})$, the corresponding $f_{d+1}(g_0,\dots,g_{d+1})=0$. Thus, $B\mathcal{H}^{d-1}(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$. For $d=3$, $\mathcal{H}^2(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$. For the nontrivial 3D-graded structure, $n_2(e,o,e)=n_2(o,e,o)=1$ and others $n_2=0$, the corresponding f_4 has a form

$$f_4(e,o,e,o,e)=f_4(o,e,o,e,o)=1,\quad \text{others}=0. \tag{F4}$$

One can show that $(-)^{f_4(g_0,\dots,g_4)}$ is a cocycle in $\mathcal{Z}^4[Z_2,U(1)]$. Since $\mathcal{H}^4[Z_2,U(1)]=\mathbb{Z}_1$, $(-)^{f_4(g_0,\dots,g_4)}$ is also a coboundary in $\mathcal{B}^4[Z_2,U(1)]$. Therefore, $B\mathcal{H}^2(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$.

For $d=4$, $\mathcal{H}^3(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$. For the nontrivial 4D-graded structure, $n_3(e,o,e,o)=n_3(o,e,o,e)=1$ and others $n_3=0$, the corresponding f_5 has a form

$$f_5(e,o,e,o,e,o)=f_5(o,e,o,e,o,e)=1,\quad \text{others}=0. \tag{F5}$$

One can show that $(-)^{f_5(g_0,\dots,g_5)}$ is a cocycle in $\mathcal{Z}^5[Z_2,U(1)]$. Since $\mathcal{H}^5[Z_2,U(1)]=\mathbb{Z}_2$ and $(-)^{f_5(g_0,\dots,g_5)}$ corresponds to a nontrivial cocycle in $\mathcal{H}^5[Z_2,U(1)]=\mathbb{Z}_2$, we find that $B\mathcal{H}^3(Z_2,\mathbb{Z}_2)=\mathbb{Z}_1$.

Following the third step in Sec. VI, we can show that the elements in $\mathscr{H}^d[Z_2\times Z_2^f,U(1)]$ can be labeled by $\mathcal{H}^d[Z_2\times Z_2^f,U(1)]\times B\mathcal{H}^{d-1}(Z_2,\mathbb{Z}_2)$. Following the fourth step in Sec. VI, we would like to show that the above labeling is one to one. For $d=1,2,3$, $f_d(g_0,\dots,g_d)=0$. Thus, the labeling is one to one. For $d=4$, a 3D-graded structure in $B\mathcal{H}^2(Z_2,\mathbb{Z}_2)$ gives rise to a $f_4(g_0,\dots,g_4)$. Since $(-)^{f_4(g_0,\dots,g_4)}$ is a cocycle in $\mathcal{Z}^4[Z_2,U(1)]$ and since $\mathcal{H}^4[Z_2,U(1)]=\mathbb{Z}_1$, we find that $(-)^{f_4(g_0,\dots,g_4)}$ is also a coboundary in $\mathcal{B}^4[Z_2,U(1)]$ and the labeling is one to one.

So, using

$$\mathcal{H}^d[Z_n,U(1)]=\begin{cases}\mathbb{Z}_1, & d=0 \bmod 2,\ d>0\\ \mathbb{Z}_n, & d=1 \bmod 2,\end{cases} \tag{F6}$$

we find that [46]

$$\mathscr{H}^d\big[Z_2\times Z_2^f,U(1)\big]=\begin{cases}\mathbb{Z}_2^2, & d=1\\ \mathbb{Z}_2, & d=2\\ \mathbb{Z}_4, & d=3\\ \mathbb{Z}_1, & d=4.\end{cases} \tag{F7}$$

Here, the $\mathbb{Z}_4$ group structure for $(2+1)$D case comes from the fact that two copies of a fermionic SPT phase give rise to the nontrival bosonic SPT phase [see Eqs. (88) and (89)].

$\mathscr{H}^1[Z_2\times Z_2^f,U(1)]=\mathbb{Z}_2^2$ implies that there are four possible fermionic SPT phases in $d_{sp}=0$ spatial dimension if there is a Z_2 symmetry. Two of them have an even number of fermions and the other two of them have an odd number of fermions. The two phases with an even number of fermions are separated by the two possible Z_2 quantum numbers. The two phases with odd number of fermions are also separated by the two possible Z_2 quantum numbers.

$\mathscr{H}^2[Z_2 \times Z_2^f, U(1)] = \mathbb{Z}_2$ implies that there are two gapped fermionic SPT phases in $d_{sp} = 1$ spatial dimension if there is a Z_2 symmetry. We can use the Jordan-Wigner transformation to map 1D fermion systems with Z_2 symmetry to 1D boson systems with $Z_2 \times Z_2^f$ symmetry. Since $\mathcal{H}^2[Z_2 \times Z_2, U(1)] = \mathbb{Z}_2$, 1D boson systems have two phases that do not break the $Z_2 \times Z_2^f$ symmetry. 1D boson systems also have another phase that does not break the Z_2 symmetry: the phase that breaks the Z_2^f symmetry. The fermionic SPT phases have no intrinsic topological order and correspond to the bosonic phases that do not break the Z_2^f symmetry (and the Z_2 symmetry). Thus, 1D fermion systems with Z_2 symmetry indeed have two fermionic SPT phases.

$\mathscr{H}^3[Z_2 \times Z_2^f, U(1)] = \mathbb{Z}_4$ implies that our construction gives rise to four gapped fermionic SPT phases in $d_{sp} = 2$ spatial dimensions if there is a Z_2 symmetry.

4. Calculate $\mathscr{H}^d[Z_{2k} \times Z_2^f, U(1)]$

Similarly, let us choose $G_f = Z_{2k} \times Z_2^f$. Following the first step in Sec. VI, we find that $\mathcal{H}^{d-1}(Z_{2k}, \mathbb{Z}_2) = \mathbb{Z}_2$ and there are two graded structures in all dimensions. For each dD-graded structure, we can choose a $u_{d-1}^g(g_0, \dots, g_{d-1})$ that satisfies $u_{d-1}^g(g_0, \dots, g_{d-1}) = 1$ for $g \in G_b$ and $u_{d-1}^{P_f}(g_0, \dots, g_{d-1}) = (-)^{n_{d-1}(g_0, \dots, g_{d-1})}$.

Following the second step in Sec. VI, we note that the trivial graded structure is given by $n_{d-1}(g_0, \dots, g_{d-1}) = 0$. The nontrivial graded structure is given by

$$n_{d-1}(e,o,e,\dots) = n_{d-1}(o,e,o,\dots) = 1, \quad \text{others} = 0, \quad \text{(F8)}$$

where e represents the even elements in Z_{2k} and o represents the odd elements in Z_{2k}.

For $d = 1,2$, $\mathcal{H}^{d-1}(Z_{2k}, \mathbb{Z}_2) = \mathbb{Z}_2$. For each graded structure $n_{d-1}(g_0, \dots, g_{d-1})$, the corresponding $f_{d+1}(g_0, \dots, g_{d+1}) = 0$. Thus, $B\mathcal{H}^{d-1}(Z_{2k}, \mathbb{Z}_2) = \mathbb{Z}_2$.

For $d = 3$, $\mathcal{H}^2(Z_{2k}, \mathbb{Z}_2) = \mathbb{Z}_2$. For the nontrivial 3D-graded structure, $n_2(e,o,e) = n_2(o,e,o) = 1$ and others $n_2 = 0$, the corresponding f_4 has a form

$$f_4(e,o,e,o,e) = f_4(o,e,o,e,o) = 1, \quad \text{others} = 0. \quad \text{(F9)}$$

One can show that $(-)^{f_4(g_0, \dots, g_4)}$ is a cocycle in $\mathcal{Z}^4[Z_{2k}, U(1)]$. Since $\mathcal{H}^4[Z_{2k}, U(1)] = \mathbb{Z}_1$, $(-)^{f_4(g_0, \dots, g_4)}$ is also a coboundary in $\mathcal{B}^4[Z_{2k}, U(1)]$. Therefore, $B\mathcal{H}^2(Z_{2k}, \mathbb{Z}_2) = \mathbb{Z}_2$.

For $d = 4$, $\mathcal{H}^3(Z_{2k}, \mathbb{Z}_2) = \mathbb{Z}_2$. For the nontrivial 4D-graded structure, $n_3(e,o,e,o) = n_3(o,e,o,e) = 1$ and others $n_3 = 0$, the corresponding f_5 has a form

$$f_5(e,o,e,o,e,o) = f_5(o,e,o,e,o,e) = 1, \quad \text{others} = 0. \quad \text{(F10)}$$

One can show that $(-)^{f_5(g_0, \dots, g_5)}$ is a cocycle in $\mathcal{Z}^5[Z_{2k}, U(1)]$. Since $\mathcal{H}^5[Z_{2k}, U(1)] = \mathbb{Z}_{2k}$ and $(-)^{f_5(g_0, \dots, g_5)}$ corresponds to a nontrivial cocycle in $\mathcal{H}^5[Z_{2k}, U(1)] = \mathbb{Z}_{2k}$, we find that $B\mathcal{H}^3(Z_{2k}, \mathbb{Z}_2) = \mathbb{Z}_1$.

Following the third step in Sec. VI, we can show that the elements in $\mathscr{H}^d[Z_{2k} \times Z_2^f, U(1)]$ can be labeled by $\mathcal{H}^d[Z_{2k}, U(1)] \times B\mathcal{H}^{d-1}(Z_{2k}, \mathbb{Z}_2)$.

Following the fourth step in Sec. VI, we would like to show that the above labeling is one to one. For $d = 1,2,3$, $f_d(g_0, \dots, g_d) = 0$. Thus, the labeling is one to one. For $d = 4$, a 3D-graded structure in $B\mathcal{H}^2(Z_{2k}, \mathbb{Z}_2)$ gives rise to a $f_4(g_0, \dots, g_4)$. Since $(-)^{f_4(g_0, \dots, g_4)}$ is a cocycle in $\mathcal{Z}^4[Z_{2k}, U(1)]$ and since $\mathcal{H}^4[Z_{2k}, U(1)] = \mathbb{Z}_1$, we find that $(-)^{f_4(g_0, \dots, g_4)}$ is also a coboundary in $\mathcal{B}^4[Z_{2k}, U(1)]$ and the labeling is one to one.

So, using

$$\mathcal{H}^d[Z_n, U(1)] = \begin{cases} \mathbb{Z}_1, & d = 0 \bmod 2,\ d > 0 \\ \mathbb{Z}_n, & d = 1 \bmod 2 \end{cases} \quad \text{(F11)}$$

we find that [46]

$$\mathscr{H}^d\big[Z_{2k} \times Z_2^f, U(1)\big] = \begin{cases} \mathbb{Z}_{2k} \times \mathbb{Z}_2, & d = 1 \\ \mathbb{Z}_2, & d = 2 \\ \mathbb{Z}_{4k}, & d = 3 \\ \mathbb{Z}_1, & d = 4. \end{cases} \quad \text{(F12)}$$

Here, the group structure $\mathbb{Z}_{4k}$ could be derived from a topological field theory approach [74]. We note that it is a central extension of $\mathbb{Z}_{2k}$ (which classifies bosonic SPT phases) over a $\mathbb{Z}_2$ graded structure.

$\mathscr{H}^1[Z_{2k} \times Z_2^f, U(1)] = \mathbb{Z}_{2k} \times \mathbb{Z}_2$ implies that there are $4k$ possible fermionic SPT phases in $d_{sp} = 0$ spatial dimension if there is a Z_{2k} symmetry. $2k$ of them have an even number of fermions and the other $2k$ of them have an odd number of fermions. The $2k$ phases are separated by the $2k$ possible Z_{2k} quantum numbers.

$\mathscr{H}^2[Z_{2k} \times Z_2^f, U(1)] = \mathbb{Z}_2$ implies that there are two gapped fermionic SPT phases in $d_{sp} = 1$ spatial dimension if there is a Z_{2k} symmetry. We can use the Jordan-Wigner transformation to map 1D fermion systems with Z_{2k} symmetry to 1D boson systems with $Z_{2k} \times Z_2^f$ symmetry. The fermionic SPT phases have no intrinsic topological order and correspond to the bosonic phases that do not break the Z_2^f symmetry (and the Z_{2k} symmetry). Since $\mathcal{H}^2[Z_{2k} \times Z_2, U(1)] = \mathbb{Z}_2$, 1D boson systems have two phases that do not break the $Z_{2k} \times Z_2^f$ symmetry. Thus, 1D fermion systems with Z_{2k} symmetry indeed have two fermionic SPT phases.

$\mathscr{H}^3[Z_{2k} \times Z_2^f, U(1)] = \mathbb{Z}_{4k}$ implies that our construction gives rise to $4k$ gapped fermionic SPT phases in $d_{sp} = 2$ spatial dimensions if there is a Z_{2k} symmetry.

5. Calculate $\mathscr{H}^d[Z_2^T \times Z_2^f, U_T(1)]$

Now, let us choose $G_f = Z_2^T \times Z_2^f$. Since $\mathcal{H}^{d-1}(Z_2^T, \mathbb{Z}_2) = \mathbb{Z}_2$, there are two graded structures in all dimensions. The trivial graded structure is given by $n_{d-1}(g_0, \dots, g_{d-1}) = 0$. The nontrivial graded structure is given by

$$n_{d-1}(e,o,e,\dots) = n_{d-1}(o,e,o,\dots) = 1, \quad \text{others} = 0, \quad \text{(F13)}$$

where e is the identity element in Z_2^T and o is the nontrivial element in Z_2^T. For each dD-graded structure, we can choose a $u_{d-1}^g(g_0, \dots, g_{d-1})$ that satisfies $u_{d-1}^g(g_0, \dots, g_{d-1}) = 1$ for $g \in G_b$.

For $d = 1,2$, $\mathcal{H}^{d-1}(Z_2^T, \mathbb{Z}_2) = \mathbb{Z}_2$. For each graded structure $n_{d-1}(g_0, \dots, g_{d-1})$, the corresponding $f_{d+1}(g_0, \dots, g_{d+1}) = 0$, and the equivalent classes of $\boldsymbol{v}_d$ are described by $\mathcal{H}^d[Z_2^T, U_T(1)]$, Therefore, $\mathscr{H}^1[Z_2^T \times Z_2^f, U_T(1)] = \mathbb{Z}_2$ and $\mathscr{H}^2[Z_2^T \times Z_2^f, U_T(1)] = \mathbb{Z}_4$.

For $d=3$, $\mathcal{H}^2(Z_2^T,\mathbb{Z}_2)=\mathbb{Z}_2$. For the nontrivial 3D-graded structure, $n_2(e,o,e)=n_2(o,e,o)=1$ and others $n_2=0$, the corresponding f_4 has a form

$$f_4(e,o,e,o,e)=f_4(o,e,o,e,o)=1,\quad \text{others}=0. \tag{F14}$$

One can show that $(-)^{f_4(g_0,\dots,g_4)}$ is a cocycle in $\mathcal{Z}^4[Z_2^T,U_T(1)]$. Since $\mathcal{H}^4[Z_2^T,U_T(1)]=\mathbb{Z}_2$ and $(-)^{f_4(g_0,\dots,g_4)}$ is a nontrivial cocycle in $\mathcal{H}^4[Z_2^T,U_T(1)]$, so for the nontrivial 3D-graded structure, Eq. (C17) for $\boldsymbol{v}_d$ has no solutions. For the trivial 3D-graded structure, the equivalent classes of $\boldsymbol{v}_d$ are described by $\mathcal{H}^3[Z_2^T,U_T(1)]$. Thus, $\mathscr{H}^3[Z_2^T\times Z_2^f,U_T(1)]=\mathbb{Z}_1$.

For $d=4$, $\mathcal{H}^3(Z_2^T,\mathbb{Z}_2)=\mathbb{Z}_2$. For the nontrivial 4D-graded structure, $n_3(e,o,e,o)=n_3(o,e,o,e)=1$ and others $n_3=0$, the corresponding f_5 has a form

$$f_5(e,o,e,o,e,o)=f_5(o,e,o,e,o,e)=1,\quad \text{others}=0. \tag{F15}$$

One can show that $(-)^{f_5(g_0,\dots,g_5)}$ is a cocycle in $\mathcal{Z}^5[Z_2^T,U_T(1)]$. Since $\mathcal{H}^5[Z_2^T,U_T(1)]=\mathbb{Z}_1$, $(-)^{f_5(g_0,\dots,g_5)}$ is also a coboundary in $\mathcal{B}^5[Z_2^T,U_T(1)]$. So, for the nontrivial 4D-graded structure, the equivalent classes of $\boldsymbol{v}_d$ are labeled by $\mathcal{H}^4[Z_2^T,U_T(1)]=\mathbb{Z}_2$. On the other hand, if $(-)^{f_4(g_0,\dots,g_4)}$ is a nontrivial cocycle in $\mathcal{B}^4[Z_2^T,U_T(1)]$, then the labeling is *not* one to one. In fact, $(-)^{f_4(g_0,\dots,g_4)}$ is a nontrivial cocycle in $\mathcal{B}^4[Z_2^T,U_T(1)]$ which implies that the nontrivial bosonic SPT phase protected by $T^2=1$ time-reversal symmetry can be connected to a trivial direct product state or an atomic insulator state via interacting fermion systems. (We note that such a "collapsing" can happen because local unitary transformations in interacting fermion systems have a much more general meaning [27].) Thus, $\mathscr{H}^4[Z_2^T\times Z_2^f,U_T(1)]=\mathbb{Z}_2$. To summarize,

$$\mathscr{H}^d\left[Z_2^T\times Z_2^f,U_T(1)\right]=\begin{cases}\mathbb{Z}_2, & d=1\\ \mathbb{Z}_4, & d=2\\ \mathbb{Z}_1, & d=3\\ \mathbb{Z}_2, & d=4.\end{cases} \tag{F16}$$

$\mathscr{H}^1[Z_2^T\times Z_2^f,U_T(1)]=\mathbb{Z}_2$ implies that there are two possible fermionic SPT phases in $d_{sp}=0$ spatial dimension if there is a Z_2^T symmetry. One has even an number of fermions and the other one has an odd number of fermions.

$\mathscr{H}^2[Z_2^T\times Z_2^f,U_T(1)]=\mathbb{Z}_4$ implies that there are four gapped fermionic SPT phases in $d_{sp}=1$ spatial dimension if there is a Z_2^T symmetry. We can use the Jordan-Wigner transformation to map 1D fermion systems with Z_2^T symmetry to 1D boson systems with $Z_2^T\times Z_2^f$ symmetry. Since $\mathcal{H}^2[Z_2^T\times Z_2,U_T(1)]=\mathbb{Z}_2\times\mathbb{Z}_2$, 1D boson systems have four phases that do not break the $Z_2^T\times Z_2^f$ symmetry. These four bosonic phases correspond to the four gapped fermionic SPT phases described by $\mathscr{H}^2[Z_2^T\times Z_2^f,U_T(1)]=\mathbb{Z}_4$.

Note that the stacking operation of 1D systems and the Jordan-Wigner transformation of the 1D systems do not commute. The state obtained by stacking two 1D fermion systems then performing the Jordan-Wigner transformation is different from the state obtained by performing the Jordan-Wigner transformation on each 1D fermion systems then stacking the two resulting boson systems. This is why, although $\mathscr{H}^2[Z_2^T\times Z_2^f,U_T(1)]$ and $\mathcal{H}^2[Z_2^T\times Z_2,U_T(1)]$ classify the same four states, their Abelian group structure is different. The Abelian group multiplication operation in $\mathscr{H}^2[Z_2^T\times Z_2^f,U_T(1)]$ corresponds to stacking 1D fermion systems, while the Abelian group multiplication operation in $\mathcal{H}^2[Z_2^T\times Z_2,U_T(1)]$ corresponds to stacking 1D boson systems. Both types of group multiplication operations give rise to Abelian group structures, but they give rise to different Abelian groups.

1D boson systems also have four phases that do break the Z_2^f symmetry: two do not break the time-reversal T symmetry and two do not break the TP_f symmetry. The fermionic SPT phases have no intrinsic topological order and correspond to the bosonic phases that do not break the Z_2^f symmetry (and the Z_2^T symmetry). Thus, 1D fermion systems with Z_2^T symmetry indeed have only four fermionic SPT phases.

$\mathscr{H}^3[Z_2^T\times Z_2^f,U_T(1)]=\mathbb{Z}_1$ implies that our construction only gives rise to one trivial gapped fermionic SPT phase in $d_{sp}=2$ spatial dimensions if there is a Z_2^T symmetry. $\mathscr{H}^4[Z_2^T\times Z_2^f,U_T(1)]=\mathbb{Z}_2$ implies that our construction gives rise to one nontrivial gapped fermionic SPT phases in $d_{sp}=3$ spatial dimensions if there is a Z_2^T symmetry. We note that, using noninteracting fermions, we cannot construct any nontrivial gapped phases with Z_2^T symmetry (the $T^2=1$ time-reversal symmetry).

APPENDIX G: PROPERTIES OF THE CONSTRUCTED SPT STATES

1. Equivalence between ground state wave functions

In Sec. VII, we have demonstrated how to construct an ideal ground state wave function $\Phi(\{g_i\},\{n_{ij\dots k}\})$ in d spatial dimensions from a $(d+1)$-cocycle (ν_{d+1},n_d,u_d^g). However, a natural question is as follows: How do we know these wave functions describe the same fermionic SPT phases or not? Here, we will address this important question based on the new mathematical structure: group supercohomology class invented in Appendix C.

a. A special equivalent relation

Let us choose arbitrary $(\nu_d,n_{d-1},u_{d-1}^g)$ in one lower dimension. Here, $(\nu_d,n_{d-1},u_{d-1}^g)$ is not a d-cocycle. They just satisfy the symmetry condition (55). In Appendix C, we have shown that $(\tilde\nu_{d+1},\tilde n_d,\tilde u_d^g)$ obtained from (ν_{d+1},n_d,u_d^g) through Eq. (C23) is also a $(d+1)$-cocycle. Here, we will assume that $n_{d-1}=0$ and $u_{d-1}^g=1$, and the resulting

$$\tilde\nu_{d+1}=\nu_{d+1}\delta\nu_d,\quad \tilde n_d=n_d,\quad \tilde u_d^g=u_d^g \tag{G1}$$

is a $(d+1)$-cocycle. The new $(d+1)$-cocycle $(\tilde\nu_{d+1},\tilde n_d,\tilde u_d^g)$ gives rise to a new ideal wave function $\Phi'(\{g_i\},\{n_{ij\dots k}\})$. Should we view the two wave functions $\Phi(\{g_i\},\{n_{ij\dots k}\})$ and $\Phi'(\{g_i\},\{n_{ij\dots k}\})$ as wave functions for different SPT phases?

From the definition of $\delta\nu_d$ [Eq. (58)], we can show that the two wave functions $\Phi(\{g_i\},\{n_{ijk}\})$ and $\Phi'(\{g_i\},\{n_{ijk}\})$ are related through

$$\Phi'(\{g_i\},\{n_{ijk}\})=\Phi(\{g_i\},\{n_{ijk}\})\prod_{(ij\dots k)}(-)^{f_{d+1}(g_0,g_i,g_j,\dots,g_k)}\times\nu_d^{s_{ij\dots k}}(g_i,g_j,\dots,g_k), \tag{G2}$$

where $\prod_{(ij\dots k)}$ is the product over all the d-simplexes that form the d-dimensional space [in our (2+1)D example,

$\prod_{(ij...k)}$ is the product over all triangles]. Also, $s_{ij...k} = \pm 1$ depending on the orientation of the d-simplex $(ij\dots k)$. Since $\nu_d^{s_{ij...k}}(g_i,g_j,\dots,g_k)$ is a pure $U(1)$ phase that satisfies $\nu_d^{s_{ij...k}}(gg_i,gg_j,\dots,gg_k) = \nu_d^{s_{ij...k}}(g_i,g_j,\dots,g_k)$, it represents a bosonic *symmetric* local unitary (LU) transformation. Therefore, the new wave function Φ' induced by $(\nu_d \neq 0, n_{d-1} = 0, u_{d-1}^g = 1)$ belongs to the same SPT phase as the original wave function Φ. Also, f_{d+1} for $d < 3$ is always zero. Therefore, below three spatial dimensions, the new wave function Φ' belongs to the same SPT phase as the original wave function Φ.

In three spatial dimensions and above ($d \geqslant 3$), if we choose $(\nu_d = 0, n_{d-1} \neq 0)$, the new induced wave function Φ' will differ from the original wave function Φ by a phase factor $\prod_{(ij...k)}(-)^{f_{d+1}(g_0,g_i,g_j,\dots,g_k)}$. $(-)^{f_{d+1}(g_0,g_i,g_j,\dots,g_k)}$ represents a LU transformation, and the two wave functions Φ' and Φ have the same intrinsic topological order. But, since $(-)^{f_{d+1}(g_0,gg_i,gg_j,\dots,gg_k)} \neq (-)^{f_{d+1}(g_0,g_i,g_j,\dots,g_k)}$, $(-)^{f_{d+1}(g_0,g_i,g_j,\dots,g_k)}$ does not represent a bosonic symmetric LU transformation. So, we do not know whether Φ' and Φ belong to the same fermionic SPT phase or not.

We would like to point out that in order for Φ' and Φ to belong to the same *fermionic* SPT phase, the two wave functions can only differ by a *fermionic* symmetric LU transformation (which is defined in [27]). The *bosonic* symmetric LU transformations are a subset of *fermionic* symmetric LU transformations. So, even though $(-)^{f_{d+1}(g_0,g_i,g_j,\dots,g_k)}$ is not a *bosonic* symmetric LU transformation, we do not know whether it is a *fermionic* symmetric LU transformation or not. The structure of group supercohomology theory developed in Appendix C shows that $f_{d+1}(g_0,g_i,g_j,\dots,g_k)$ is a (d+1)-cocycle with $\mathbb{Z}_2$ coefficient, suggesting the sign factor $(-)^{f_{d+1}(g_0,g_i,g_j,\dots,g_k)}$ can be generated by *fermionic* symmetric LU transformation.

b. Generic equivalent relations

In the above, we discussed a special case of $(\tilde{\nu}_{d+1},\tilde{n}_d,\tilde{u}_d^g)$ where $\tilde{n}_d$ and $\tilde{u}_d^g$ are the same as n_d and u_d^g. However, the equivalent class of group supercohomology has a more complicated structure. In Appendix C, we have shown that $(\tilde{\nu}_{d+1},\tilde{n}_d,\tilde{u}_d^g)$ obtained from

$$\tilde{\nu}_{d+1} = \nu_{d+1}(-)^{g_d}, \quad \tilde{n}_d = n_d + \delta m'_{d-1}, \quad \tilde{u}_d^g = u_d^g \tag{G3}$$

[with $g_d(g_0,\dots,g_d) = (\boldsymbol{g}_d)_{(g_0,\dots,g_d)} \bmod 2$] also belongs to the same equivalent class of group supercohomology. We note that here the graded structure n_d changes into $\tilde{n}_d$. It would be much harder to understand why the corresponding new wave function still describes the same fermionic SPT phase. Let us consider the $(2+1)$D wave function on a sphere as a simple example. Since the wave function derived from the fixed-point amplitude (43) is a fixed-point wave function with zero correlation length, it is enough to only consider a minimal wave function with four sites on a sphere (labeled as 1,2,3,4). Such minimal wave functions will contain four triangles 123, 124, 134, and 234. From the definition of $\tilde{\nu}_d$, we can show that the two wave functions $\Phi(\{g_i\},\{n_{ijk}\})$ and $\tilde{\Phi}(\{g_i\},\{\tilde{n}_{ijk}\})$ are related through $\tilde{\Phi}(\{g_i\},\{n'_{ijk}\}) = (-)^{g_4(g_1,g_2,g_3,g_4)}\Phi(\{g_i\},\{n_{ijk}\})$. To see this explicitly, let us construct the ground state wave function by adding one more point 0 inside the sphere. In this way, the wave function can be formally expressed as

$$\begin{aligned}\Psi(g_1,g_2,g_3,g_4,\theta_{234},\theta_{124},\bar{\theta}_{134},\bar{\theta}_{123}) &= (-)^{m_1(g_1,g_3)}\int \mathcal{V}_3^-(g_0,g_1,g_2,g_3)\mathcal{V}_3^+(g_0,g_1,g_2,g_4)\mathcal{V}_3^-(g_0,g_1,g_3,g_4)\mathcal{V}_3^+(g_0,g_2,g_3,g_4)\\ &= (-)^{m_1(g_1,g_3)}\mathcal{V}_3^+(g_1,g_2,g_3,g_4) = \nu_3(g_1,g_2,g_3,g_4)\theta_{234}^{n_2(g_2,g_3,g_4)}\theta_{124}^{n_2(g_1,g_2,g_4)}\bar{\theta}_{234}^{n_2(g_2,g_3,g_4)}\bar{\theta}_{123}^{n_2(g_1,g_2,g_3)}.\end{aligned} \tag{G4}$$

Again, the symbol $\int$ is defined as integrating over the internal Grassmann variables with respect to the weights m_1 on the internal links [see Eq. (31)]. Similarly, the wave function corresponding to a different solution $(\tilde{\nu}_{d+1},\tilde{n}_d,\tilde{u}_d^g)$ takes a form

$$\tilde{\Psi}(g_1,g_2,g_3,g_4,\theta_{234},\theta_{124},\bar{\theta}_{134},\bar{\theta}_{123}) = \nu_3(g_1,g_2,g_3,g_4)(-)^{g_4(g_1,g_2,g_3,g_4)}\theta_{234}^{n'_2(g_2,g_3,g_4)}\theta_{124}^{n'_2(g_1,g_2,g_4)}\bar{\theta}_{234}^{n'_2(g_2,g_3,g_4)}\bar{\theta}_{123}^{n'_2(g_1,g_2,g_3)}. \tag{G5}$$

We note that $g_4(gg_1,gg_2,gg_3,gg_4) = g_4(g_1,g_2,g_3,g_4)$ implies the phase factor $(-)^{g_4(g_1,g_2,g_3,g_4)}$ is a *symmetric* LU transformation. However, we still do not know whether the two wave functions with different patterns of fermion number $\{\tilde{n}_{ijk}\}$ and $\{n_{ijk}\}$ describe the same fermionic SPT phase or not.

Indeed, we find that up to some *symmetric* phase factors, the state $|\Phi\rangle$ and $|\tilde{\Phi}\rangle$ can be related through a *symmetric* LU transformation $|\tilde{\Phi}\rangle \sim \hat{U}'|\Phi\rangle$, where

$$\begin{aligned}\hat{U}' = \prod_{(0ij)} c_{(0ij)}^{m'_1(g_i,g_j)}\bar{c}_{(0ij)}^{m'_1(g_i,g_j)}\prod_{\triangle} c_{(0ij)}^{\dagger m'_1(g_i,g_j)}c_{(0jk)}^{\dagger m'_1(g_j,g_k)}\bar{c}_{(0ik)}^{\dagger m'_1(g_i,g_k)}(c_{(ijk)}^{\dagger})^{\tilde{n}_2(g_i,g_j,g_k)-n_2(g_i,g_j,g_k)}\\ \times\prod_{\triangledown}(c_{(ijk)}^{\dagger})^{\tilde{n}_2(g_i,g_j,g_k)-n_2(g_i,g_j,g_k)}c_{(0ik)}^{\dagger m'_1(g_i,g_k)}\bar{c}_{(0jk)}^{\dagger m'_1(g_j,g_k)}\bar{c}_{(0ij)}^{\dagger m'_1(g_i,g_j)},\end{aligned} \tag{G6}$$

with $\tilde{n}_2(g_i,g_j,g_k) = n_2(g_i,g_j,g_k) + m'_1(g_i,g_j) + m'_1(g_j,g_k) + m'_1(g_k,g_i)$. We note that both n_2 and m'_1 are invariant under symmetry transformation. Here, $(c_{(ijk)}^{\dagger})^{-1}$ is defined as $(c_{(ijk)}^{\dagger})^{-1} = c_{(ijk)}$. $\sim$ means equivalent up to some symmetric sign factors which will arise when we reorder the fermion creation/annihilation operators in $\hat{U}'$. We also note that $\hat{U}'$ is a generalized LU transformation which only has nonzero action on the subspace with fixed fermion occupation pattern n_2. The above discussions can be generalized into any dimension. Thus, we have shown the wave function constructed from the above $(\tilde{\nu}_{d+1},\tilde{n}_d,\tilde{u}_2^g)$ describes the same fermionic SPT phase.

Finally, we can consider the more general case where u_d^g also changes into $\tilde{u}_d^g$. It is very easy to see that such changes in the wave function can be realized by symmetric LU transformation since those u_d^g evolve "g_0 are all canceled in the wave function.

In conclusion, we have shown that (ν_{d+1},n_d,u_d^g) and $(\tilde{\nu}_{d+1},\tilde{n}_d,\tilde{u}_d^g)$ belonging to the same group supercohomology class will give rise to fixed-point wave functions describing the same fermionic SPT phase.

2. Symmetry transformations and generic SPT states

In Sec. VII B, we have constructed an ideal fermion SPT state $|\Psi\rangle = \hat{U}|\Phi_0\rangle$ using the the fermionic LU transformation $\hat{U}$ constructed from a cocycle (ν_3,n_2). In this Appendix, we are going to discuss how to construct a more generic SPT state $|\Psi'\rangle$ that is in the same phase as the ideal state $|\Psi\rangle$. Clearly, $|\Psi'\rangle$ and $|\Psi\rangle$ have the same symmetry. So, to construct $|\Psi'\rangle$, we need to first discuss how symmetry transformation changes under the fermionic LU transformation $\hat{U}$.

The symmetry G_b that acts on $|\Psi\rangle$ is represented by the following (anti)unitary operators:

$$\hat{W}(g) = \sum_{\{g_i\}} \prod_{\bigtriangledown} [u_2^g(g_i,g_j,g_k)]^{-1} \prod_{\bigtriangleup} u_2^g(g_i,g_j,g_k) \times \prod_i |g_i\rangle\langle gg_i|\, K^{\frac{1-s(g)}{2}}, \tag{G7}$$

where $\hat{W}_i$ is the symmetry transformation acting on a single site i, and K is the antiunitary operator

$$KiK^{-1} = -i, \quad KcK^{-1} = c, \quad Kc^\dagger K^{-1} = c^\dagger. \tag{G8}$$

We find that

$$\begin{aligned}
\hat{W}^0(g) = \hat{U}^\dagger \hat{W}(g)\hat{U} = & \sum_{\{g_i\}} \prod_i |g_i\rangle\langle gg_i| \prod_{\bigtriangledown} [u_2^g(g_i,g_j,g_k)]^{-1} \prod_{\bigtriangleup} u_2^g(g_i,g_j,g_k) \\
& \times \prod_{\bigtriangledown} c_{(ijk)}^{n_2(g_i,g_j,g_k)} c_{(0ik)}^{n_2(g_0,g_i,g_k)} \bar{c}_{(0jk)}^{n_2(g_0,g_j,g_k)} \bar{c}_{(0ij)}^{n_2(g_0,g_i,g_j)} \prod_{\bigtriangleup} c_{(0ij)}^{n_2(g_0,g_i,g_j)} c_{(0jk)}^{n_2(g_0,g_j,g_k)} \bar{c}_{(0ik)}^{n_2(g_0,g_i,g_k)} \bar{c}_{(ijk)}^{n_2(g_i,g_j,g_k)} \\
& \times \prod_{\bigtriangleup} \nu_3(g_0,g_i,g_j,g_k) \prod_{\bigtriangledown} \nu_3^{-1}(g_0,g_i,g_j,g_k) \prod_{(0ij)} c_{(0ij)}^{\dagger n_2(g_0,g_i,g_j)} \bar{c}_{(0ij)}^{\dagger n_2(g_0,g_i,g_j)} \\
& \times \prod_{(0ij)} c_{(0ij)}^{n_2(g_0,gg_i,gg_j)} \bar{c}_{(0ij)}^{n_2(g_0,gg_i,gg_j)} \prod_{\bigtriangleup} [\nu_3^{-1}(g_0,gg_i,gg_j,gg_k)]^{s(g)} \prod_{\bigtriangledown} [\nu_3(g_0,gg_i,gg_j,gg_k)]^{s(g)} K^{\frac{1-s(g)}{2}} \\
& \times \prod_{\bigtriangleup} c_{(0ij)}^{\dagger n_2(g_0,gg_i,gg_j)} c_{(0jk)}^{\dagger n_2(g_0,gg_j,gg_k)} \bar{c}_{(0ik)}^{\dagger n_2(g_0,gg_i,gg_k)} \bar{c}_{(ijk)}^{\dagger n_2(gg_i,gg_j,gg_k)} \prod_{\bigtriangledown} c_{(ijk)}^{\dagger n_2(gg_i,gg_j,gg_k)} c_{(0ik)}^{\dagger n_2(g_0,gg_i,gg_k)} \bar{c}_{(0jk)}^{\dagger n_2(g_0,gg_j,gg_k)} \bar{c}_{(0ij)}^{\dagger n_2(g_0,gg_i,gg_j)}.
\end{aligned} \tag{G9}$$

Using the fact that $\hat{U}$ is independent of g_0, we can change the g_0 in the second half of the right-hand side of the above expression to gg_0. Then, we can use the symmetry condition (55) on (ν_3,n_2,u_2^g) to show that the above is reduced to

$$\hat{W}^0(g) = \hat{U}^\dagger \hat{W}(g)\hat{U} = \sum_{\{g_i\}} \prod_i |g_i\rangle\langle gg_i|K^{\frac{1-s(g)}{2}} \tag{G10}$$

in the subspace with no fermions. Note that $\hat{W}^0(g)$ acts on $|\Phi_0\rangle$. If we choose a new no-fermion state $|\Psi_0'\rangle$ that is symmetric under the symmetry G_b: $\hat{W}^0(g)|\Psi_0'\rangle = |\Psi_0'\rangle$, then the resulting $|\Psi'\rangle = \hat{U}|\Psi_0'\rangle$ will be symmetric under $\hat{W}(g)$: $\hat{W}(g)|\Psi'\rangle = |\Psi'\rangle$. Since $\hat{W}^0(g) = \prod_i |g_i\rangle\langle gg_i|K^{\frac{1-s(g)}{2}}$ has a simple form, it is easy to construct the deformed $|\Phi_0\rangle$ that has the same symmetry under $\hat{W}^0(g) = \prod_i |g_i\rangle\langle gg_i|$. Then, after the fermion LU transformation $\hat{U}$, we can obtain generic SPT states that are in the same phase as $|\Psi\rangle$.

Here, we would like to stress that, only on a system without boundary, the total symmetry transformation $\hat{W}$ is mapped into a simple onsite symmetry $\hat{W}^0$ the LU transformation $\hat{U}$. For a system with boundary, under the LU transformation $\hat{U}$, the total symmetry transformation $\hat{W}$ will be mapped into a complicated symmetry transformation which does not have an onsite form on the boundary [45].

3. Entanglement density matrix

The nontrivialness of the SPT states is in their symmetry-protected gapless boundary excitations. To study the gapless boundary excitations, in this Appendix, we are going to study entanglement density matrix ρ_E and its entanglement Hamiltonian H_E: $\rho_E = e^{-H_E}$, from the above constructed ground state wave functions for SPT states. The entanglement Hamiltonian H_E can be viewed as the effective Hamiltonian for the gapless boundary excitations.

To calculate the entanglement density matrix, we first cut the system into two halves along a horizontal line. To do cutting, we first split each site on the cutting line into two sites: $|g_i\rangle \to |g_i\rangle \otimes |g_{i'}\rangle$ [see Fig. 21(a)]. The ground state lives in the subspace where $g_i = g_{i'}$. We then cut between the splitted sites.

Because the entanglement spectrum does not change if we perform LU transformations within each half of the system,

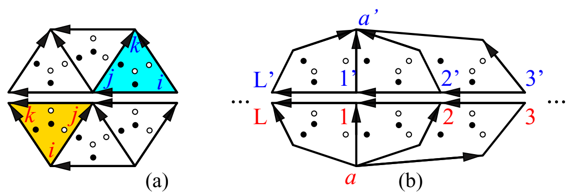

FIG. 21. (Color online) (a) We divide the system into two halves along a line by doubling the sites on the line $|g_i\rangle \to |g_i\rangle \otimes |g_i\rangle$ and then cut between the pairs. (b) The interior of each region is deformed into a simpler lattice without affecting the spectrum of the entanglement density matrix.

we can use the LU transformations to deform the lattice into a simpler one in Fig. 21(b), where the sites are labeled by 1, 2, . . ., L, $1'$, $2'$, . . ., L', along the edge which forms a ring, and the only two interior sites are labeled by a and a'.

On the deformed lattice, the ground state wave function can be written as $\hat{U}|\Phi_0\rangle$, where

$$|\Phi_0\rangle = |\phi_a\rangle\langle\phi_a| \otimes |\phi_{a'}\rangle\langle\phi_{a'}| \otimes_{i=1}^{L} |\phi_i\rangle\langle\phi_i| \otimes_{i'=1'}^{L'} |\phi_{i'}\rangle\langle\phi_{i'}| \tag{G11}$$

and

$$\begin{aligned}\hat{U} = &\prod_{i'} \nu_3^{-1}(g_0,g_{i'+1},g_{i'},g_{a'}) \prod_i \nu_3(g_0,g_a,g_{i+1},g_i) \prod_i \bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)} c_{(0,i'+1,i')}^{n_2(g_0,g_{i'+1},g_{i'})} \prod_i \bar{c}_{(0ai)}^{n_2(g_0,g_a,g_i)} c_{(0ai)}^{n_2(g_0,g_a,g_i)} \\ &\times \prod_{i'} \bar{c}_{(0i'a')}^{n_2(g_0,g_{i'},g_{a'})} c_{(0i'a')}^{n_2(g_0,g_{i'},g_{a'})} \prod_i c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)} c_{(0ai)}^{\dagger n_2(g_0,g_a,g_i)} \bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)} \bar{c}_{(0,a,i+1)}^{\dagger n_2(g_0,g_a,g_{i+1})} \\ &\times \prod_{i'} c_{(0,i'+1,i')}^{\dagger n_2(g_0,g_{i'+1},g_{i'})} c_{(0i'a')}^{\dagger n_2(g_0,g_{i'},g_{a'})} \bar{c}_{(0,i'+1,a')}^{\dagger n_2(g_0,g_{i'+1},g_{a'})} \bar{c}_{(i'+1,i',a')}^{\dagger n_2(g_{i'+1},g_{i'},g_{a'})}.\end{aligned} \tag{G12}$$

Now, let us rewrite $\hat{U}$ as $\hat{U} = \hat{U}_{\text{up}}\hat{U}_{\text{down}}$, where $\hat{U}_{\text{up}}$ acts on the upper half and $\hat{U}_{\text{down}}$ acts on the lower half of the system:

$$\begin{aligned}\hat{U} = &\Bigg[(-)^{f(g_{1'},\dots,g_{L'})} \prod_{i'} \nu_3^{-1}(g_0,g_{i'+1},g_{i'},g_{a'}) \prod_{i'} c_{(0,i'+1',i')}^{n_2(g_0,g_{i'+1'},g_{i'})} \prod_{i'} \bar{c}_{(0i'a')}^{n_2(g_0,g_{i'},g_{a'})} c_{(0i'a')}^{n_2(g_0,g_{i'},g_{a'})} \\ &\times \prod_{i'} c_{(0,i'+1,i')}^{\dagger n_2(g_0,g_{i'+1},g_{i'})} c_{(0i'a')}^{\dagger n_2(g_0,g_{i'},g_{a'})} \bar{c}_{(0,i'+1,a')}^{\dagger n_2(g_0,g_{i'+1},g_{a'})} \bar{c}_{(i'+1,i',a')}^{\dagger n_2(g_{i'+1},g_{i'},g_{a'})}\Bigg] \\ &\times \Bigg[\prod_i \nu_3(g_0,g_a,g_{i+1},g_i) \prod_i \bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)} \prod_i \bar{c}_{(0ai)}^{n_2(g_0,g_a,g_i)} c_{(0ai)}^{n_2(g_0,g_a,g_i)} \prod_i c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)} c_{(0ai)}^{\dagger n_2(g_0,g_a,g_i)} \bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)} \bar{c}_{(0,a,i+1,)}^{\dagger n_2(g_0,g_a,g_{i+1})}\Bigg],\end{aligned}$$

where the pure sign factor $(-)^{f(g_{1'},\dots,g_{L'})}$ arises from rewriting $\prod_i \bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)} c_{(0,i'+1,i')}^{n_2(g_0,g_{i'+1},g_{i'})}$ as $\prod_{i'} c_{(0,i'+1',i')}^{n_2(g_0,g_{i'+1'},g_{i'})} \prod_i c_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)}$, and we have used the fact that $g_i = g_{i'}$ in the ground state subspace. Thus, Eq. (G13) is valid only when $\hat{U}$ acts within the ground state subspace.

Now, the entanglement density matrix ρ_E can be written as

$$\rho_E = \text{Tr}_{\text{upper}}\hat{U}_{\text{up}}\hat{U}_{\text{down}}|\Phi_0\rangle\langle\Phi_0|\hat{U}_{\text{down}}^\dagger\hat{U}_{\text{up}}^\dagger = \text{Tr}_{\text{upper}}\hat{U}_{\text{down}}|\Phi_0\rangle\langle\Phi_0|\hat{U}_{\text{down}}^\dagger, \tag{G13}$$

where Tr_{upper} is the trace over the degrees of freedom on the upper half of the system. Since $g_i = g_{i'}$ along the edge in the ground state, ρ_E (as an operator) does not change g_i along the edge (i.e., ρ_E is diagonal in the $|g_i\rangle$ basis). So it is sufficient to discuss ρ_E in the subspace of fixed $|g_i\rangle$'s:

$$\begin{aligned}\rho_E(g_1,\dots) \propto &\sum_{g_a,g_a'} \prod_i \nu_3(g_0,g_a,g_{i+1},g_i)\nu_3^*(g_0,g_a',g_{i+1},g_i) \prod_{i=1}^{L} \bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)} \prod_i \bar{c}_{(0ai)}^{n_2(g_0,g_a,g_i)} c_{(0ai)}^{n_2(g_0,g_a,g_i)} \\ &\times \prod_i c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)} c_{(0aj)}^{\dagger n_2(g_0,g_a,g_j)} \bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)} \bar{c}_{(0,a,i+1)}^{\dagger n_2(g_0,g_a,g_{i+1})} |g_a\rangle\langle g_a'| \\ &\times \prod_i \bar{c}_{(0,a,i+1)}^{n_2(g_0,g_a',g_{i+1})} \bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)} c_{(0ai)}^{n_2(g_0,g_a',g_i)} c_{(a,i+1,i)}^{n_2(g_a',g_{i+1},g_i)} \prod_i c_{(0ai)}^{\dagger n_2(g_0,g_a',g_i)} \bar{c}_{(0ai)}^{\dagger n_2(g_0,g_a',g_i)} \prod_{i=L}^{1} \bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)}.\end{aligned} \tag{G14}$$

We see that ρ_E has a form

$$\rho_E = \sum_{\{g_i\}} \rho_E(g_1,\dots,g_L) \otimes (\otimes_i |g_i\rangle\langle g_i|) = \sum_{\{g_i\}} |\{g_i\}_{\text{edge}},g_0\rangle\langle\{g_i\}_{\text{edge}},g_0|, \tag{G15}$$

$$|\{g_i\}_{\text{edge}},g_0\rangle = |G_b|^{-1/2}\sum_{g_a}\prod_{\bigtriangledown}\nu_3(g_0,g_a,g_i,g_j)\prod_{i=1}^{L}\bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)}\prod_i \bar{c}_{(0ai)}^{n_2(g_0,g_a,g_i)}c_{(0ai)}^{n_2(g_0,g_a,g_i)}$$
$$\times\prod_i c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)}c_{(0ai)}^{\dagger n_2(g_0,g_a,g_i)}\bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)}\bar{c}_{(0,a,i+1)}^{\dagger n_2(g_0,g_a,g_{i+1})}|g_a\rangle\otimes(\otimes_i|g_i\rangle). \quad (G16)$$

This is a key result of this paper that allows us to understand gapless edge excitations. In fact, $|\{g_i\}_{\text{edge}}\rangle$ is a basis of the low energy subspace of the edge excitations. We see that the low energy edge excitations are described by g_i on the edge.

We can simplify the above expression (G16). First, we note that

$$\prod_{i=1}^{L}\bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)}\prod_i \bar{c}_{(0ai)}^{n_2(g_0,g_a,g_i)}c_{(0ai)}^{n_2(g_0,g_a,g_i)}$$
$$= \prod_{i=1}^{L}\bar{c}_{(0ai)}^{n_2(g_0,g_a,g_i)}c_{(0ai)}^{n_2(g_0,g_a,g_i)}\bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)} = (-)^{n_2(g_0,g_a,g_1)+n_2(g_0,g_a,g_1)\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)}\prod_{i=1}^{L}c_{(0ai)}^{n_2(g_0,g_a,g_i)}\bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)}\bar{c}_{(0,a,i+1)}^{n_2(g_0,g_a,g_{i+1})}.$$
$$(G17)$$

We also note that

$$c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)}c_{(0ai)}^{\dagger n_2(g_0,g_a,g_i)}\bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)}\bar{c}_{(0,a,i+1)}^{\dagger n_2(g_0,g_a,g_{i+1})}$$
$$= \bar{c}_{(0,a,i+1)}^{\dagger n_2(g_0,g_a,g_{i+1})}\bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)}c_{(0ai)}^{\dagger n_2(g_0,g_a,g_i)}c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)}(-)^{[n_2(g_a,g_{i+1},g_i)+n_2(g_0,g_a,g_i)+n_2(g_0,g_{i+1},g_i)+n_2(g_0,g_a,g_{i+1})]/2}. \quad (G18)$$

Therefore,

$$\prod_{i=1}^{L}\bar{c}_{(0,i+1,i)}^{n_2(g_0,g_{i+1},g_i)}\prod_i \bar{c}_{(0ai)}^{n_2(g_0,g_a,g_i)}c_{(0ai)}^{n_2(g_0,g_a,g_i)}\prod_i c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)}c_{(0ai)}^{\dagger n_2(g_0,g_a,g_i)}\bar{c}_{(0,i+1,i)}^{\dagger n_2(g_0,g_{i+1},g_i)}\bar{c}_{(0,a,i+1)}^{\dagger n_2(g_0,g_a,g_{i+1})}$$
$$= (-)^{n_2(g_0,g_a,g_1)+n_2(g_0,g_a,g_1)\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)}\prod_{i=1}^{L}(-)^{\frac{n_2(g_0,g_{i+1},g_i)+n_2(g_a,g_{i+1},g_i)}{2}+n_2(g_0,g_a,g_i)}c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)}.$$

We find

$$|\{g_i\}_{\text{edge}},g_0\rangle = |G_b|^{-1/2}\sum_{g_a}\prod_i\nu_3(g_0,g_a,g_{i+1},g_i)(-)^{n_2(g_0,g_a,g_1)[1+\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)]}$$
$$\times\prod_{i=1}^{L}(-)^{\frac{n_2(g_0,g_{i+1},g_i)+n_2(g_a,g_{i+1},g_i)}{2}+n_2(g_0,g_a,g_i)}c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)}|g_a\rangle\otimes(\otimes_i|g_i\rangle). \quad (G19)$$

We see that the total number of fermions in $|\{g_i\}_{\text{edge}},g_0\rangle$ is given by $N_F = \sum_i n_2(g_a,g_{i+1},g_i)$. So, the fermion number is not fixed for fixed g_i's (due to the g_a dependence). Since $N_F = \sum_i n_2(g_0,g_{i+1},g_i)$ mod 2, the fermion-number parity is fixed for fixed g_i's.

To see how $|\{g_i\}_{\text{edge}},g_0\rangle$ depends on g_0, let us use Eq. (G19) to calculate $\langle\{g_i\}_{\text{edge}},g_0'|\{g_i\}_{\text{edge}},g_0\rangle$:

$$\langle\{g_i\}_{\text{edge}},g_0'|\{g_i\}_{\text{edge}},g_0\rangle = |G_b|^{-1}\sum_{g_a}(-)^{\sum_{i=1}^{L}\frac{n_2(g_0',g_{i+1},g_i)+n_2(g_0,g_{i+1},g_i)}{2}+n_2(g_a,g_{i+1},g_i)+n_2(g_0',g_a,g_i)+n_2(g_0,g_a,g_i)}$$
$$\times(-)^{[n_2(g_0',g_a,g_1)+n_2(g_0,g_a,g_1)][1+\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)]}\prod_i\nu_3^{-1}(g_0',g_a,g_{i+1},g_i)\nu_3(g_0,g_a,g_{i+1},g_i)$$
$$= |G_b|^{-1}(-)^{\sum_{i=1}^{L}\frac{n_2(g_0',g_{i+1},g_i)+n_2(g_0,g_{i+1},g_i)}{2}}\sum_{g_a}(-)^{\sum_{i=1}^{L}n_2(g_a,g_{i+1},g_i)+n_2(g_0,g_0',g_i)+n_2(g_0,g_0',g_a)}$$
$$\times(-)^{[n_2(g_0,g_0',g_1)+n_2(g_0,g_0',g_a)][1+\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)]}\prod_i\nu_3^{-1}(g_0',g_a,g_{i+1},g_i)\nu_3(g_0,g_a,g_{i+1},g_i). \quad (G20)$$

Using $(-)^{\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)} = (-)^{\sum_{i=1}^{L}n_2(g_a,g_{i+1},g_i)}$ and the cocycle condition

$$\nu_3^{-1}(g_0',g_a,g_{i+1},g_i)\nu_3(g_0,g_a,g_{i+1},g_i) = \nu_3(g_0,g_0',g_{i+1},g_i)\nu_3^{-1}(g_0,g_0',g_a,g_i)\nu_3(g_0,g_0',g_a,g_{i+1})(-)^{n_2(g_0,g_0',g_a)n_2(g_a,g_{i+1},g_i)}, \quad (G21)$$

we can rewrite the above as

$$\langle\{g_i\}_{\text{edge}},g_0'|\{g_i\}_{\text{edge}},g_0\rangle = (-)^{\sum_{i=1}^L \frac{n_2(g_0',g_{i+1},g_i)+n_2(g_0,g_{i+1},g_i)}{2}}(-)^{\sum_{i=1}^L n_2(g_0,g_{i+1},g_i)+n_2(g_0,g_0',g_i)} \times (-)^{n_2(g_0,g_0',g_1)[1+\sum_{i=1}^L n_2(g_0,g_{i+1},g_i)]}\prod_i \nu_3(g_0,g_0',g_{i+1},g_i)|G_b|^{-1}\sum_{g_a}(-)^{(L+1)n_2(g_0,g_0',g_a)}. \tag{G22}$$

We see that when $L=$ odd, $|\{g_i\}_{\text{edge}},g_0\rangle$ and $|\{g_i\}_{\text{edge}},g_0'\rangle$ only differ by a phase

$$|\{g_i\}_{\text{edge}},g_0\rangle = (-)^{\sum_{i=1}^L \frac{n_2(g_0',g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)}{2}+n_2(g_0,g_0',g_i)} \times (-)^{n_2(g_0,g_0',g_1)[1+\sum_{i=1}^L n_2(g_0,g_{i+1},g_i)]}\prod_i \nu_3(g_0,g_0',g_{i+1},g_i)\ |\{g_i\}_{\text{edge}},g_0'\rangle. \tag{G23}$$

From the above results, and using the relation between the entanglement density matrix and gapless edge excitations, we learn two things. (1) The low energy edge degrees of freedom are labeled by $\{g_1,\dots,g_L\}$ on the boundary since $\rho_E(g_1,\dots,g_L)$ has one and only one nonzero eigenvalue for each $\{g_1,\dots,g_L\}$. (2) The low energy edge degrees of freedom are entangled with the bulk degrees of freedom since the states on the site a (in the bulk) are different for different $\{g_1,\dots,g_L\}$.

Using the expression (G19), we calculate the low energy effective Hamiltonian H_{eff} on the edge from a physical Hamiltonian H_{edge} on the edge:

$$(H_{\text{eff}})_{\{g_i'\},\{g_i\}} = \langle\{g_i'\}_{\text{edge}},g_0|H_{\text{edge}}|\{g_i\}_{\text{edge}},g_0\rangle. \tag{G24}$$

H_{eff} has a short-range interaction and satisfies certain symmetry conditions if H_{edge} are symmetric. In the following, we are going to study how $|\{g_i\}_{\text{edge}},g_0\rangle$ transforms under the symmetry transformation.

4. Symmetry transformation on edge states

Let us apply the symmetry operation $\hat W(g)$ [(G7)] to the edge state $|\{gg_i\}_{\text{edge}},g_0\rangle$:

$$\begin{aligned}
&\hat W(g)|\{gg_i\}_{\text{edge}},g_0\rangle\\
&= |G_b|^{-1/2}\sum_{g_a}\prod_i\left[u_2^g(g_a,g_{i+1},g_i)\right]^{-1}\nu_3^{s(g)}(g_0,gg_a,gg_{i+1},gg_i)\prod_{i=1}^L \bar c_{(0,i+1,i)}^{n_2(g_0,gg_{i+1},gg_i)}\\
&\quad\times\prod_i \bar c_{(0ai)}^{n_2(g_0,gg_a,gg_i)}c_{(0ai)}^{n_2(g_0,gg_a,gg_i)}\prod_i c_{(a,i+1,i)}^{\dagger n_2(gg_a,gg_{i+1},gg_i)}c_{(0ai)}^{\dagger n_2(g_0,gg_a,gg_i)}\bar c_{(0,i+1,i)}^{\dagger n_2(g_0,gg_{i+1},gg_i)}\bar c_{(0,a,i+1)}^{\dagger n_2(g_0,gg_a,gg_{i+1})}|g_a\rangle\otimes(\otimes_i|g_i\rangle)\\
&= |G_b|^{-1/2}\sum_{g_a}\prod_i\left[u_2^g(g^{-1}g_0,g_{i+1},g_i)\right]^{-1}\nu_3(g^{-1}g_0,g_a,g_{i+1},g_i)\prod_{i=1}^L \bar c_{(0,i+1,i)}^{n_2(g^{-1}g_0,g_{i+1},g_i)}\prod_i \bar c_{(0ai)}^{n_2(g^{-1}g_0,g_a,g_i)}c_{(0ai)}^{n_2(g^{-1}g_0,g_a,g_i)}\\
&\quad\times\prod_i c_{(a,i+1,i)}^{\dagger n_2(g_a,g_{i+1},g_i)}c_{(0ai)}^{\dagger n_2(g^{-1}g_0,g_a,g_i)}\bar c_{(0,i+1,i)}^{\dagger n_2(g^{-1}g_0,g_{i+1},g_i)}\bar c_{(0,a,i+1)}^{\dagger n_2(g^{-1}g_0,g_a,g_{i+1})}|g_a\rangle\otimes(\otimes_i|g_i\rangle)\propto|\{g_i\}_{\text{edge}},g^{-1}g_0\rangle.
\end{aligned}\tag{G25}$$

When $L=$ odd, $|\{g_i\}_{\text{edge}},g^{-1}g_0\rangle\propto|\{g_i\}_{\text{edge}},g_0\rangle$ and we find

$$\begin{aligned}
\hat W(g)|\{gg_i\}_{\text{edge}},g_0\rangle &= w(\{g_i\},g)|\{g_i\}_{\text{edge}},g_0\rangle,\\
w(\{g_i\},g) &= \frac{\langle\{g_i\}_{\text{edge}},g_0|\{g_i\}_{\text{edge}},g^{-1}g_0\rangle\prod_{i=1}^L\left[u_2^g(g^{-1}g_0,g_{i+1},g_i)\right]^{-1}}{\langle\{g_i\}_{\text{edge}},g_0|\{g_i\}_{\text{edge}},g_0\rangle}\\
&= (-)^{\sum_{i=1}^L\frac{n_2(g^{-1}g_0,g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)}{2}+n_2(g^{-1}g_0,g_0,g_i)}(-)^{n_2(g^{-1}g_0,g_0,g_1)[L+\sum_{i=1}^L n_2(g^{-1}g_0,g_{i+1},g_i)]}\\
&\quad\times\prod_i\nu_3(g^{-1}g_0,g_0,g_{i+1},g_i)\prod_{i=1}^L\left[u_2^g(g^{-1}g_0,g_{i+1},g_i)\right]^{-1}.
\end{aligned}\tag{G26}$$

Note that we have rewritten $(-)^{n_2(g^{-1}g_0,g_0,g_1)[1+\sum_{i=1}^L n_2(g^{-1}g_0,g_{i+1},g_i)]}$ as $(-)^{n_2(g^{-1}g_0,g_0,g_1)[L+\sum_{i=1}^L n_2(g^{-1}g_0,g_{i+1},g_i)]}$ since L is odd.

Although the above expression for the action of the edge effective symmetry is obtained for $L=$ odd, we can show that the expression actually forms a representation of G_f for both $L=$ odd and $L=$ even. Let us consider

$$\begin{aligned}
\hat W(g')|\{g'g_i\}_{\text{edge}},g_0\rangle &= \hat W(g^{-1}g')\hat W(g)|\{gg^{-1}g'g_i\}_{\text{edge}},g_0\rangle = \hat W(g^{-1}g')w(\{g^{-1}g'g_i\},g)|\{g^{-1}g'g_i\}_{\text{edge}},g_0\rangle\\
&= w(\{g^{-1}g'g_i\},g)w(\{g_i\},g^{-1}g')|\{g_i\}_{\text{edge}},g_0\rangle = w(\{g_i\},g')|\{g_i\}_{\text{edge}},g_0\rangle.
\end{aligned}\tag{G27}$$

We see that in order for $\hat{W}(g)$ to form a representation, we require that $w(\{g^{-1}g'g_i\},g)w(\{g_i\},g^{-1}g') = w(\{g_i\},g')$. So, let us examine

$$
\begin{aligned}
w(\{g^{-1}g'g_i\},g)w(\{g_i\},g^{-1}g') = {}& (-)^{\sum_{i=1}^{L}\frac{n_2(g^{-1}g_0,g^{-1}g'g_{i+1},g^{-1}g'g_i)-n_2(g_0,g^{-1}g'g_{i+1},g^{-1}g'g_i)}{2}+n_2(g^{-1}g_0,g_0,g^{-1}g'g_i)} \\
&\times (-)^{n_2(g^{-1}g_0,g_0,g^{-1}g'g_1)[L+\sum_{i=1}^{L}n_2(g^{-1}g_0,g^{-1}g'g_{i+1},g^{-1}g'g_i)]}\prod_i \nu_3(g^{-1}g_0,g_0,g^{-1}g'g_{i+1},g^{-1}g'g_i) \\
&\times \left[\prod_{i=1}^{L}\left[u_2^{g}(g^{-1}g_0,g^{-1}g'g_{i+1},g^{-1}g'g_i)\right]^{-1}\right](-)^{\sum_{i=1}^{L}\frac{n_2(g'^{-1}gg_0,g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)}{2}+n_2(g'^{-1}gg_0,g_0,g_i)} \\
&\times (-)^{n_2(g'^{-1}gg_0,g_0,g_1)[L+\sum_{i=1}^{L}n_2(g'^{-1}gg_0,g_{i+1},g_i)]}\prod_i \nu_3(g'^{-1}gg_0,g_0,g_{i+1},g_i) \\
&\times \prod_{i=1}^{L}\left[u_2^{g^{-1}g'}(g'^{-1}gg_0,g_{i+1},g_i)\right]^{-1}.
\end{aligned}
\tag{G28}
$$

Using

$$
\begin{aligned}
\nu_3(g^{-1}g_0,g_0,g^{-1}g'g_{i+1},g^{-1}g'g_i) = {}& \nu_3(g'^{-1}g_0,g'^{-1}gg_0,g_{i+1},g_i)u_2^{g^{-1}g'}(g'^{-1}gg_0,g_{i+1},g_i)\left[u_2^{g^{-1}g'}(g'^{-1}g_0,g_{i+1},g_i)\right]^{-1} \\
&\times u_2^{g^{-1}g'}(g'^{-1}g_0,g'^{-1}gg_0,g_i)\left[u_2^{g^{-1}g'}\left(g'^{-1}g_0,g'^{-1}gg_0,g_{i+1}\right]^{-1}\right),
\end{aligned}
\tag{G29}
$$

we can simplify the above as

$$
\begin{aligned}
&w(\{g^{-1}g'g_i\},g)w(\{g_i\},g^{-1}g') \\
&\quad = (-)^{[n_2(g_0,g'g_0,g'g_1)+n_2(g_0,gg_0,g'g_0)][L+\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)]}(-)^{\sum_{i=1}^{L}\frac{n_2(g'^{-1}g_0,g_{i+1},g_i)-n_2(g'^{-1}gg_0,g_{i+1},g_i)}{2}+n_2(g'^{-1}g_0,g'^{-1}gg_0,g_i)} \\
&\qquad \times (-)^{\sum_{i=1}^{L}\frac{n_2(g'^{-1}gg_0,g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)}{2}+n_2(g'^{-1}gg_0,g_0,g_i)}\prod_i \nu_3(g'^{-1}g_0,g'^{-1}gg_0,g_{i+1},g_i)\nu_3(g'^{-1}gg_0,g_0,g_{i+1},g_i) \\
&\qquad \times \prod_{i=1}^{L}\left[u_2^{g'}(g'^{-1}g_0,g_{i+1},g_i)\right]^{-1} \\
&\quad = (-)^{[n_2(g_0,g'g_0,g'g_1)+n_2(g_0,gg_0,g'g_0)][L+\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)]}(-)^{\sum_{i=1}^{L}\frac{n_2(g'^{-1}g_0,g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)}{2}+n_2(g'^{-1}g_0,g_0,g_i)+n_2(g'^{-1}g_0,g'^{-1}gg_0,g_0)} \\
&\qquad \times \prod_i \nu_3(g'^{-1}g_0,g'^{-1}gg_0,g_{i+1},g_i)\nu_3(g'^{-1}gg_0,g_0,g_{i+1},g_i)\prod_{i=1}^{L}\left[u_2^{g'}(g'^{-1}g_0,g_{i+1},g_i)\right]^{-1}.
\end{aligned}
\tag{G30}
$$

Using the cocycle condition

$$
\begin{aligned}
&\nu_3(g'^{-1}gg_0,g_0,g_{i+1},g_i)\nu_3(g'^{-1}g_0,g'^{-1}gg_0,g_{i+1},g_i) \\
&\quad = \nu_3(g'^{-1}g_0,g_0,g_{i+1},g_i)\nu_3(g'^{-1}g_0,g'^{-1}gg_0,g_0,g_i)\nu_3^{-1}(g'^{-1}g_0,g'^{-1}gg_0,g_0,g_{i+1})(-)^{n_2(g'^{-1}g_0,g'^{-1}gg_0,g_0)n_2(g_0,g_{i+1},g_i)},
\end{aligned}
\tag{G31}
$$

we can rewrite the above as

$$
\begin{aligned}
w(\{g^{-1}g'g_i\},g)w(\{g_i\},g^{-1}g') = {}& (-)^{n_2(g'^{-1}g_0,g_0,g_1)[L+\sum_{i=1}^{L}n_2(g_0,g_{i+1},g_i)]}(-)^{\sum_{i=1}^{L}\frac{n_2(g'^{-1}g_0,g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)}{2}+n_2(g'^{-1}g_0,g_0,g_i)} \\
&\times \prod_i \nu_3(g'^{-1}g_0,g_0,g_{i+1},g_i)\prod_{i=1}^{L}\left[u_2^{g'}(g'^{-1}g_0,g_{i+1},g_i)\right]^{-1}.
\end{aligned}
\tag{G32}
$$

We see that we indeed have $w(\{g^{-1}g'g_i\},g)w(\{g_i\},g^{-1}g') = w(\{g_i\},g')$. The strange factor $(-)^{n_2(g^{-1}g_0,g_0,g_1)[L+\sum_{i=1}^{L}n_2(g^{-1}g_0,g_{i+1},g_i)]}$ is important to make $w(\{g_i\},g)$ to be consistent with $\hat{W}(g)$ being a representation of G_f.

So, if $\hat{W}^\dagger(g)H_{\text{edge}}\hat{W}(g) = H_{\text{edge}}$, then $(H_{\text{eff}})_{\{g_i'\},\{g_i\}}$ satisfies

$$
\begin{aligned}
(H_{\text{eff}})_{\{gg_i'\},\{gg_i\}} &= \langle\{gg_i'\}_{\text{edge}},g_0|H_{\text{edge}}|\{gg_i\}_{\text{edge}},g_0\rangle = \langle\{gg_i'\}_{\text{edge}},g_0|\hat{W}^\dagger(g)H_{\text{edge}}\hat{W}(g)|\{gg_i\}_{\text{edge}},g_0\rangle \\
&= w^*(\{g_i'\},g)w(\{g_i\},g)\langle\{g_i'\}_{\text{edge}},g_0|H_{\text{edge}}|\{g_i\}_{\text{edge}},g_0\rangle = w^*(\{g_i'\},g)(H_{\text{eff}})_{\{g_i'\},\{g_i\}}w(\{g_i\},g).
\end{aligned}
\tag{G33}
$$

In the operator form, the above can be rewritten as

$$
\begin{aligned}
&W_{\text{eff}}^\dagger(g)H_{\text{eff}}W_{\text{eff}}(g) = H_{\text{eff}}, \\
&W_{\text{eff}}(g)|\{g_i\}_{\text{edge}},g_0\rangle = w^*(\{g_i\},g)|\{gg_i\}_{\text{edge}},g_0\rangle.
\end{aligned}
\tag{G34}
$$

We see that the symmetry transformation of H_{eff} contains additional phase factor $w(\{g_i\},g)$. In particular, the phase factor cannot be written as an onsite form $w(\{g_i\},g) = \prod_i f(g_i,g)$. So, the symmetry transformation on the effective edge degrees of freedom is not an onsite symmetry transformation. The non-onsite symmetry of the edge state ensures the gaplessness of the edge excitation if the symmetry is not broken [45]. Note that for bosonic case $n_2 = 0$ and $u_2^g = 1$. We have a simple result

$$w(\{g_i\},g) = \textstyle\prod_i \nu_3(g^{-1}g_0,g_0,g_{i+1},g_i). \tag{G35}$$

Let us write $w(\{g_i\},g)$ as

$$w(\{g_i\},g) = (-)^{n_2(g^{-1}g_0,g_0,g_1)[L+\sum_{i=1}^{L} n_2(g^{-1}g_0,g_{i+1},g_i)]} \prod_i w_{i,i+1}, \tag{G36}$$

where

$$w_{i,i+1} = i^{n_2(g^{-1}g_0,g_{i+1},g_i)-n_2(g_0,g_{i+1},g_i)+2n_2(g^{-1}g_0,g_0,g_i)} \times \nu_3(g^{-1}g_0,g_0,g_{i+1},g_i)\big[u_2^g(g^{-1}g_0,g_i,g_{i+1})\big]^{-1}. \tag{G37}$$

We see that for fermion cases, we cannot write the phase factor $w(\{g_i\},g)$ as a product of local non-onsite phase factors $w_{i,i+1}$. The nonlocal phase factor $(-)^{n_2(g^{-1}g_0,g_0,g_1)[L+\sum_{i=1}^{L} n_2(g^{-1}g_0,g_{i+1},g_i)]}$ must appear.

Each term in the edge effective Hamiltonian $H_{\text{eff}} = \sum_i H_{\text{eff}}(i)$ must satisfy

$$W_{\text{eff}}^\dagger(g)H_{\text{eff}}(i)W_{\text{eff}}(g) = H_{\text{eff}}(i), \tag{G38}$$

where $H_{\text{eff}}(i)$ only acts on sites near site i. $H_{\text{eff}}(i)$ must also preserve the fermion-number parity $(-)^{N_F} = (-)^{\sum_{i=1}^{L} n_2(g_0,g_{i+1},g_i)}$. In other words, $H_{\text{eff}}(i)$ commutes with $(-)^{\sum_{i=1}^{L} n_2(g_0,g_{i+1},g_i)}$. So, if $H_{\text{eff}}(i)$ is far away from the site 1, then $H_{\text{eff}}(i)$ commutes with the nonlocal phase factor $(-)^{n_2(g^{-1}g_0,g_0,g_1)[L+\sum_{i=1}^{L} n_2(g^{-1}g_0,g_{i+1},g_i)]}$. Thus, $H_{\text{eff}}(i)$ is invariant under

$$\tilde W_{\text{eff}}^\dagger(g)H_{\text{eff}}(i)\tilde W_{\text{eff}}(g) = H_{\text{eff}}, \tag{G39}$$

where

$$\tilde W_{\text{eff}}(g)|\{g_i\}_{\text{edge}},g_0\rangle = \tilde w^*(\{g_i\},g)|\{gg_i\}_{\text{edge}},g_0\rangle \tag{G40}$$

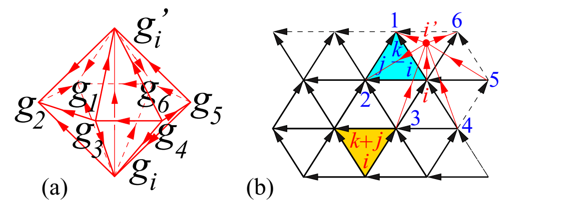

FIG. 22. (Color online) (a) The matrix elements of the Hamiltonian term H_i can be obtained from the fermionic path integral of the fixed-point action amplitude $\mathcal{V}_3$ on the complex $\langle g_1g_2g_3g_4g_5g_6g_ig_i'\rangle$. The branching structure on the complex is chosen to match that of the triangular lattice in Fig. 14. (b) The action of H_i can be obtained by attaching the complex in (a) to the triangular lattice in Fig. 14. The Grassmann numbers on the six overlapping triangles are integrated out.

with

$$\tilde w(\{g_i\},g) = \prod_i w_{i,i+1}.$$

APPENDIX H: IDEAL HAMILTONIAN FROM PATH INTEGRAL

After constructing the fermionic SPT state $|\Psi\rangle$ in Sec. VII A, here we would like to construct a local Hamiltonian

$$H = -\sum_i H_i \tag{H1}$$

that has the G_f symmetry, such that $|\Psi\rangle$ is the unique ground state of the Hamiltonian.

We note that the path integral of the fixed-point action amplitude not only gives rise to an ideal ground state wave function (as discussed above), it also gives rise to an ideal Hamiltonian. From the structure of the fixed-point path integral, we find that H_i has a structure that it acts on a seven-spin cluster labeled by i, 1–6 in blue in Fig. 14, and on the six-fermion cluster on the six triangles inside the hexagon $\langle 123456\rangle$ in Fig. 14.

The matrix elements of H_i can be obtained by evaluating the fixed-point action amplitude $\mathcal{V}_3^\pm$ on the complex in Fig. 22(a) since the action of H_i can be realized by attaching the complex Fig. 22(a) to the triangular lattice and then performing the fermionic path integral on the new complex [see Fig. 22(b)]. The evaluation of $\mathcal{V}_3^\pm$ on the complex gives us

$$\begin{aligned}
&(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)}(-)^{m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)} \int \prod_\Sigma \mathcal{V}_3 \\
&= (-)^{m_1(g_4,g_i')+m_1(g_i',g_1)+m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)} \int \prod_{j=1,2,6} d\theta_{(ii'j)}^{n_2(g_i,g_i',g_j)} d\bar\theta_{(ii'j)}^{n_2(g_i,g_i',g_j)} \prod_{j=3,4,5} d\theta_{(jii')}^{n_2(g_j,g_i,g_i')} d\bar\theta_{(jii')}^{n_2(g_j,g_i,g_i')} \\
&\quad\times (-)^{m_1(g_4,g_i)+m_1(g_5,g_i')+m_1(g_i,g_6)+m_1(g_i',g_1)+m_1(g_i,g_2)+m_1(g_3,g_i')}(-)^{m_1(g_i,g_i')} \\
&\quad\times \mathcal{V}_3^+(g_5,g_i,g_i',g_6)\mathcal{V}_3^+(g_4,g_5,g_i,g_i')\mathcal{V}_3^-(g_4,g_3,g_i,g_i')\mathcal{V}_3^-(g_3,g_i,g_i',g_2)\mathcal{V}_3^-(g_i,g_i',g_2,g_1)\mathcal{V}_3^+(g_i,g_i',g_6,g_1) \\
&= (-)^{m_1(g_4,g_i')+m_1(g_i',g_1)+m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)} \int \prod_{j=1,2,6} d\theta_{(ii'j)}^{n_2(g_i,g_i',g_j)} d\bar\theta_{(ii'j)}^{n_2(g_i,g_i',g_j)} \prod_{j=3,4,5} d\theta_{(jii')}^{n_2(g_j,g_i,g_i')} d\bar\theta_{(jii')}^{n_2(g_j,g_i,g_i')} \\
&\quad\times (-)^{m_1(g_4,g_i)+m_1(g_5,g_i')+m_1(g_i,g_6)+m_1(g_i',g_1)+m_1(g_i,g_2)+m_1(g_3,g_i')}(-)^{m_1(g_i,g_i')}
\end{aligned}$$

$$\times \nu_3(g_5,g_i,g_i',g_6)\theta_{(ii'6)}^{n_2(g_i,g_i',g_6)}\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}\bar\theta_{(5i'6)}^{n_2(g_5,g_i',g_6)}\bar\theta_{(5ii')}^{n_2(g_5,g_i,g_i')}\nu_3(g_4,g_5,g_i,g_i')\theta_{(5ii')}^{n_2(g_5,g_i,g_i')}\theta_{(45i')}^{n_2(g_4,g_5,g_i')}\bar\theta_{(4ii')}^{n_2(g_4,g_i,g_i')}\bar\theta_{(45i)}^{n_2(g_4,g_5,g_i)}$$
$$\times \nu_3^{-1}(g_4,g_3,g_i,g_i')\theta_{(43i)}^{n_2(g_4,g_3,g_i)}\theta_{(4ii')}^{n_2(g_4,g_i,g_i')}\bar\theta_{(43i')}^{n_2(g_4,g_3,g_i')}\bar\theta_{(3ii')}^{n_2(g_3,g_i,g_i')}\nu_3^{-1}(g_3,g_i,g_i',g_2)\theta_{(3ii')}^{n_2(g_3,g_i,g_i')}\theta_{(3i'2)}^{n_2(g_3,g_i',g_2)}\bar\theta_{(3i2)}^{n_2(g_3,g_i,g_2)}\bar\theta_{(ii'2)}^{n_2(g_i,g_i',g_2)}$$
$$\times \nu_3^{-1}(g_i,g_i',g_2,g_1)\theta_{(ii'2)}^{n_2(g_i,g_i',g_2)}\theta_{(i21)}^{n_2(g_i,g_2,g_1)}\bar\theta_{(ii'1)}^{n_2(g_i,g_i',g_1)}\bar\theta_{(i'21)}^{n_2(g_i',g_2,g_1)}\nu_3(g_i,g_i',g_6,g_1)\theta_{(i'61)}^{n_2(g_i',g_6,g_1)}\theta_{(ii'1)}^{n_2(g_i,g_i',g_1)}\bar\theta_{(i61)}^{n_2(g_i,g_6,g_1)}\bar\theta_{(ii'6)}^{n_2(g_i,g_i',g_6)}. \tag{H2}$$

Just like the wave function, we have included an extra factor $(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)}(-)^{m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)}$ in the path integral. With this extra factor, multiplying Eq. (H2) to the wave function can then be viewed as attaching the complex Fig. 22(a) to the triangular lattice [see Fig. 22(b)]. Note that the factor $(-)^{m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)}$ plus the factor that we included in the wave function $(-)^{m_1(g_4,g_i)+m_1(g_i,g_1)}$ give us a factor $(-)^{m_1(g_4,g_i)+m_1(g_i,g_1)+m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)}$ on all the six interior edges $\{i,1\}$, $\{i,2\}$,..., $\{i,6\}$. This factor is needed in the path integral on the complex in Fig. 22(b) obtained by attaching the complex in Fig. 22(a) to the triangular lattice. Also, note that the factor $(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)}$ is what we need in the new wave function after the action of attaching the complex Fig. 22(a) to the triangular lattice [see Fig. 22(b)]. By combining all the m_1 factors on the right-hand side of Eq. (H2), we find that all those m_1 factors become $(-)^{n_2(g_3,g_i,g_i')+n_2(g_4,g_i,g_i')+n_2(g_5,g_i,g_i')}$. So, we find

$$(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)}(-)^{m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)}\int\prod_\Sigma \mathcal{V}_3$$
$$= (-)^{n_2(g_3,g_i,g_i')+n_2(g_4,g_i,g_i')+n_2(g_5,g_i,g_i')}\int\prod_{j=1,2,6} d\theta_{(ii'j)}^{n_2(g_i,g_i',g_j)}d\bar\theta_{(ii'j)}^{n_2(g_i,g_i',g_j)}\prod_{j=3,4,5} d\theta_{(jii')}^{n_2(g_j,g_i,g_i')}d\bar\theta_{(jii')}^{n_2(g_j,g_i,g_i')}$$
$$\times(-)^{n_2(g_4,g_5,g_i)n_2(g_4,g_i,g_i')+n_2(g_4,g_i,g_i')n_2(g_4,g_3,g_i)+n_2(g_i',g_2,g_1)n_2(g_i,g_i',g_1)+n_2(g_i,g_i',g_1)n_2(g_i',g_6,g_1)}$$
$$\times \nu_3(g_5,g_i,g_i',g_6)\theta_{(ii'6)}^{n_2(g_i,g_i',g_6)}\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}\bar\theta_{(5i'6)}^{n_2(g_5,g_i',g_6)}\bar\theta_{(5ii')}^{n_2(g_5,g_i,g_i')}\nu_3(g_4,g_5,g_i,g_i')\theta_{(5ii')}^{n_2(g_5,g_i,g_i')}\theta_{(45i')}^{n_2(g_4,g_5,g_i')}\bar\theta_{(45i)}^{n_2(g_4,g_5,g_i)}\bar\theta_{(4ii')}^{n_2(g_4,g_i,g_i')}$$
$$\times \nu_3^{-1}(g_4,g_3,g_i,g_i')\theta_{(4ii')}^{n_2(g_4,g_i,g_i')}\theta_{(43i)}^{n_2(g_4,g_3,g_i)}\bar\theta_{(43i')}^{n_2(g_4,g_3,g_i')}\bar\theta_{(3ii')}^{n_2(g_3,g_i,g_i')}\nu_3^{-1}(g_3,g_i,g_i',g_2)\theta_{(3ii')}^{n_2(g_3,g_i,g_i')}\theta_{(3i'2)}^{n_2(g_3,g_i',g_2)}\bar\theta_{(3i2)}^{n_2(g_3,g_i,g_2)}\bar\theta_{(ii'2)}^{n_2(g_i,g_i',g_2)}$$
$$\times \nu_3^{-1}(g_i,g_i',g_2,g_1)\theta_{(ii'2)}^{n_2(g_i,g_i',g_2)}\theta_{(i21)}^{n_2(g_i,g_2,g_1)}\bar\theta_{(i'21)}^{n_2(g_i',g_2,g_1)}\bar\theta_{(ii'1)}^{n_2(g_i,g_i',g_1)}\nu_3(g_i,g_i',g_6,g_1)\theta_{(ii'1)}^{n_2(g_i,g_i',g_1)}\theta_{(i'61)}^{n_2(g_i',g_6,g_1)}\bar\theta_{(i61)}^{n_2(g_i,g_6,g_1)}\bar\theta_{(ii'6)}^{n_2(g_i,g_i',g_6)}. \tag{H3}$$

In the above, we have also rearranged the order of the Grassmann numbers within some simplexes to bring, say, $\bar\theta_{(4ii')}^{n_2(g_4,g_i,g_i')}$ next to $\theta_{(4ii')}^{n_2(g_4,g_i,g_i')}$. Now, we can integrate out $d\theta\, d\bar\theta$'s and obtain

$$(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)}(-)^{m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)}\int\prod_\Sigma \mathcal{V}_3 = (-)^{n_2(g_i,g_i',g_6)}$$
$$\times(-)^{n_2(g_3,g_i,g_i')+n_2(g_4,g_i,g_i')+n_2(g_5,g_i,g_i')+n_2(g_4,g_5,g_i)n_2(g_4,g_i,g_i')+n_2(g_4,g_i,g_i')n_2(g_4,g_3,g_i)+n_2(g_i',g_2,g_1)n_2(g_i,g_i',g_1)+n_2(g_i,g_i',g_1)n_2(g_i',g_6,g_1)}$$
$$\times \nu_3(g_5,g_i,g_i',g_6)\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}\bar\theta_{(5i'6)}^{n_2(g_5,g_i',g_6)}\nu_3(g_4,g_5,g_i,g_i')\theta_{(45i')}^{n_2(g_4,g_5,g_i')}\bar\theta_{(45i)}^{n_2(g_4,g_5,g_i)}\nu_3^{-1}(g_4,g_3,g_i,g_i')\theta_{(43i)}^{n_2(g_4,g_3,g_i)}\bar\theta_{(43i')}^{n_2(g_4,g_3,g_i')}$$
$$\times \nu_3^{-1}(g_3,g_i,g_i',g_2)\theta_{(3i'2)}^{n_2(g_3,g_i',g_2)}\bar\theta_{(3i2)}^{n_2(g_3,g_i,g_2)}\nu_3^{-1}(g_i,g_i',g_2,g_1)\theta_{(i21)}^{n_2(g_i,g_2,g_1)}\bar\theta_{(i'21)}^{n_2(g_i',g_2,g_1)}\nu_3(g_i,g_i',g_6,g_1)\theta_{(i'61)}^{n_2(g_i',g_6,g_1)}\bar\theta_{(i61)}^{n_2(g_i,g_6,g_1)}, \tag{H4}$$

where the factor $(-)^{n_2(g_i,g_i',g_6)}$ comes from bringing $\bar\theta_{(ii'6)}^{n_2(g_i,g_i',g_6)}$ in Eq. (H3) all the way from the back to the front. Let us rearrange the order of the Grassmann numbers to bring, say, $\bar\theta_{(5i'6)}^{n_2(g_5,g_i',g_6)}$ in front of $\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}$:

$$(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)}(-)^{m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)}\int\prod_\Sigma \mathcal{V}_3$$
$$= (-)^{n_2(g_i,g_i',g_6)+n_2(g_3,g_i,g_i')+n_2(g_4,g_i,g_i')+n_2(g_5,g_i,g_i')}$$
$$\times(-)^{n_2(g_4,g_5,g_i)n_2(g_4,g_i,g_i')+n_2(g_4,g_i,g_i')n_2(g_4,g_3,g_i)+n_2(g_i',g_2,g_1)n_2(g_i,g_i',g_1)+n_2(g_i,g_i',g_1)n_2(g_i',g_6,g_1)}$$
$$\times\frac{\nu_3(g_4,g_5,g_i,g_i')\nu_3(g_5,g_i,g_i',g_6)\nu_3(g_i,g_i',g_6,g_1)}{\nu_3(g_i,g_i',g_2,g_1)\nu_3(g_3,g_i,g_i',g_2)\nu_3(g_4,g_3,g_i,g_i')}(-)^{n_2(g_5,g_i',g_6)n_2(g_5,g_i,g_6)+n_2(g_4,g_3,g_i')n_2(g_4,g_3,g_i)+n_2(g_i',g_2,g_1)n_2(g_i,g_2,g_1)}$$
$$\times\bar\theta_{(5i'6)}^{n_2(g_5,g_i',g_6)}\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}\theta_{(45i')}^{n_2(g_4,g_5,g_i')}\bar\theta_{(45i)}^{n_2(g_4,g_5,g_i)}\bar\theta_{(43i')}^{n_2(g_4,g_3,g_i')}\theta_{(43i)}^{n_2(g_4,g_3,g_i)}$$
$$\times\theta_{(3i'2)}^{n_2(g_3,g_i',g_2)}\bar\theta_{(3i2)}^{n_2(g_3,g_i,g_2)}\bar\theta_{(i'21)}^{n_2(g_i',g_2,g_1)}\theta_{(i21)}^{n_2(g_i,g_2,g_1)}\theta_{(i'61)}^{n_2(g_i',g_6,g_1)}\bar\theta_{(i61)}^{n_2(g_i,g_6,g_1)}. \tag{H5}$$

Indeed, the above expression can be regarded as the fermion coherent state representation of the ideal Hamiltonian. We can replace, for example, $\bar\theta_{(5i'6)}^{n_2(g_5,g_i',g_6)}\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}$ as

$$\hat C = (c_{(5i6)}^\dagger)^{n_2(g_5,g_i',g_6)}c_{(5i6)}^{n_2(g_5,g_i,g_6)} - [1-n_2(g_5,g_i',g_6)][1-n_2(g_5,g_i,g_6)]c_{(5i6)}^\dagger c_{(5i6)}. \tag{H6}$$

We have

$$\begin{aligned}&\hat C = c_{(5i6)}c_{(5i6)}^\dagger \text{ for } [n_2(g_5,g_i',g_6),n_2(g_5,g_i,g_6)]=[0,0], \quad \hat C = c_{(5i6)} \text{ for } [n_2(g_5,g_i',g_6),n_2(g_5,g_i,g_6)]=[0,1],\\ &\hat C = c_{(5i6)}^\dagger \text{ for } [n_2(g_5,g_i',g_6),n_2(g_5,g_i,g_6)]=[1,0], \quad \hat C = c_{(5i6)}^\dagger c_{(5i6)} \text{ for } [n_2(g_5,g_i',g_6),n_2(g_5,g_i,g_6)]=[1,1].\end{aligned} \tag{H7}$$

We note that the fermion coherent state is defined as $|\theta_{(5i6)}\rangle = |0\rangle - \theta_{(5i6)}c_{(5i6)}^\dagger|0\rangle$. It is easy to check

$$\langle\bar\theta_{(5i'6)}|(c_{(5i6)}^\dagger)^{n_2(g_5,g_i',g_6)}c_{(5i6)}^{n_2(g_5,g_i,g_6)} - [1-n_2(g_5,g_i',g_6)][1-n_2(g_5,g_i,g_6)]c_{(5i6)}^\dagger c_{(5i6)}|\theta_{(5i6)}\rangle = \bar\theta_{(5i'6)}^{n_2(g_5,g_i',g_6)}\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}. \tag{H8}$$

However, a sign factor $(-)^{n_2(g_5,g_i,g_6)}$ is required when the Hamiltonian acts on the coherent state since $d\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}d\bar\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}$ should be reordered as $d\bar\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}d\theta_{(5i6)}^{n_2(g_5,g_i,g_6)}$. Similarly, for the face $(45i)$, the fermion coherent state is defined as $|\bar\theta_{(45i)}\rangle = |0\rangle - \bar\theta_{(45i)}c_{(45i)}^\dagger|0\rangle$. In this case, $d\theta_{(45i)}^{n_2(g_4,g_5,g_i)}d\bar\theta_{(45i)}^{n_2(g_4,g_5,g_i)}$ is the correct ordering and we do not need to introduce the extra sign factor.

By applying the same discussions to other triangles, we find that $43i$ and $i12$ also contribute sign factors $(-)^{n_2(g_4,g_3,g_i)}$ and $(-)^{n_2(g_i,g_2,g_1)}$. We also note that

$$(-)^{n_2(g_i,g_i',g_6)+n_2(g_3,g_i,g_i')+n_2(g_4,g_i,g_i')+n_2(g_5,g_i,g_i')}(-)^{n_2(g_5,g_i,g_6)+n_2(g_4,g_3,g_i)+n_2(g_i,g_2,g_1)} = (-)^{n_2(g_4,g_3,g_i')+n_2(g_5,g_i',g_6)+n_2(g_i,g_2,g_1)} \tag{H9}$$

and finally obtain

$$H_i = \sum_{g_i,g_i'}|g_i',g_1g_2g_3g_4g_5g_6\rangle\langle g_i,g_1g_2g_3g_4g_5g_6|\,\frac{\nu_3(g_4,g_5,g_i,g_i')\nu_3(g_5,g_i,g_i',g_6)\nu_3(g_i,g_i',g_6,g_1)}{\nu_3(g_i,g_i',g_2,g_1)\nu_3(g_3,g_i,g_i',g_2)\nu_3(g_4,g_3,g_i,g_i')} \tag{H10}$$

$$\begin{aligned}&\times(-)^{n_2(g_4,g_5,g_i)n_2(g_4,g_i,g_i')+n_2(g_4,g_i,g_i')n_2(g_4,g_3,g_i)+n_2(g_i',g_2,g_1)n_2(g_i,g_i',g_1)+n_2(g_i,g_i',g_1)n_2(g_i',g_6,g_1)}\\
&\times(-)^{n_2(g_5,g_i',g_6)n_2(g_5,g_i,g_6)+n_2(g_4,g_3,g_i')n_2(g_4,g_3,g_i)+n_2(g_i',g_2,g_1)n_2(g_i,g_2,g_1)}(-)^{n_2(g_4,g_3,g_i')+n_2(g_5,g_i',g_6)+n_2(g_i,g_2,g_1)}\\
&\times\big[(c_{(5i6)}^\dagger)^{n_2(g_5,g_i',g_6)}c_{(5i6)}^{n_2(g_5,g_i,g_6)} - (1-n_2(g_5,g_i',g_6))(1-n_2(g_5,g_i,g_6))c_{(5i6)}^\dagger c_{(5i6)}\big]\\
&\times\big[(c_{(45i)}^\dagger)^{n_2(g_4,g_5,g_i')}c_{(45i)}^{n_2(g_4,g_5,g_i)} - (1-n_2(g_4,g_5,g_i'))(1-n_2(g_4,g_5,g_i))c_{(45i)}^\dagger c_{(45i)}\big]\\
&\times\big[(c_{(43i)}^\dagger)^{n_2(g_4,g_3,g_i')}c_{(43i)}^{n_2(g_4,g_3,g_i)} - (1-n_2(g_4,g_3,g_i'))(1-n_2(g_4,g_3,g_i))c_{(43i)}^\dagger c_{(43i)}\big]\\
&\times\big[(c_{(3i2)}^\dagger)^{n_2(g_3,g_i',g_2)}c_{(3i2)}^{n_2(g_3,g_i,g_2)} - (1-n_2(g_3,g_i',g_2))(1-n_2(g_3,g_i,g_2))c_{(3i2)}^\dagger c_{(3i2)}\big]\\
&\times\big[(c_{(i21)}^\dagger)^{n_2(g_i',g_2,g_1)}c_{(i21)}^{n_2(g_i,g_2,g_1)} - (1-n_2(g_i',g_2,g_1))(1-n_2(g_i,g_2,g_1))c_{(i21)}^\dagger c_{(i21)}\big]\\
&\times\big[(c_{(i61)}^\dagger)^{n_2(g_i',g_6,g_1)}c_{(i61)}^{n_2(g_i,g_6,g_1)} - (1-n_2(g_i',g_6,g_1))(1-n_2(g_i,g_6,g_1))c_{(i61)}^\dagger c_{(i61)}\big].\end{aligned} \tag{H11}$$

We can rewrite H_i as

$$H_i = \textstyle\sum_{g_i,g_i'}|g_i',g_1g_2g_3g_4g_5g_6\rangle\langle g_i,g_1g_2g_3g_4g_5g_6|O_{56;g_5g_6}^{g_{i'}g_i}O_{45;g_4g_5}^{g_{i'}g_i}O_{43;g_4g_3}^{g_{i'}g_i}O_{32;g_3g_2}^{g_{i'}g_i}O_{21;g_2g_1}^{g_{i'}g_i}O_{61;g_6g_1}^{g_{i'}g_i}, \tag{H12}$$

where $O_{mn;g_mg_n}^{g_{i'}g_i}$ are given by

$$\begin{aligned}O_{21;g_2g_1}^{g_{i'}g_i} = {}&\nu_3^{-1}(g_i,g_i',g_2,g_1)(-)^{n_2(g_i',g_2,g_1)n_2(g_i,g_i',g_1)+n_2(g_i',g_2,g_1)n_2(g_i,g_2,g_1)+n_2(g_i,g_2,g_1)}\\ &\times\big[(c_{(i21)}^\dagger)^{n_2(g_i',g_2,g_1)}c_{(i21)}^{n_2(g_i,g_2,g_1)} - (1-n_2(g_i',g_2,g_1))(1-n_2(g_i,g_2,g_1))c_{(i21)}^\dagger c_{(i21)}\big],\end{aligned} \tag{H13}$$

$$O_{32;g_3g_2}^{g_{i'}g_i} = \nu_3^{-1}(g_3,g_i,g_i',g_2)\big[(c_{(3i2)}^\dagger)^{n_2(g_3,g_i',g_2)}c_{(3i2)}^{n_2(g_3,g_i,g_2)} - (1-n_2(g_3,g_i',g_2))(1-n_2(g_3,g_i,g_2))c_{(3i2)}^\dagger c_{(3i2)}\big], \tag{H14}$$

$$\begin{aligned}O_{43;g_4g_3}^{g_{i'}g_i} = {}&\nu_3^{-1}(g_4,g_3,g_i,g_i')(-)^{+n_2(g_4,g_i,g_i')n_2(g_4,g_3,g_i)+n_2(g_4,g_3,g_i')n_2(g_4,g_3,g_i)+n_2(g_4,g_3,g_i')}\\ &\times\big[(c_{(43i)}^\dagger)^{n_2(g_4,g_3,g_i')}c_{(43i)}^{n_2(g_4,g_3,g_i)} - (1-n_2(g_4,g_3,g_i'))(1-n_2(g_4,g_3,g_i))c_{(43i)}^\dagger c_{(43i)}\big],\end{aligned} \tag{H15}$$

$$O^{g_{i'}g_i}_{45;g_4g_5} = \nu_3(g_4,g_5,g_i,g_i')(-)^{n_2(g_4,g_5,g_i)n_2(g_4,g_i,g_i')} \times \big[(c^\dagger_{(45i)})^{n_2(g_4,g_5,g_i')} c^{n_2(g_4,g_5,g_i)}_{(45i)} - (1-n_2(g_4,g_5,g_i'))(1-n_2(g_4,g_5,g_i))c^\dagger_{(45i)}c_{(45i)}\big], \tag{H16}$$

$$O^{g_{i'}g_i}_{56;g_5g_6} = \nu_3(g_5,g_i,g_i',g_6)(-)^{n_2(g_5,g_i',g_6)n_2(g_5,g_i,g_6)+n_2(g_5,g_i',g_6)} \times \big[(c^\dagger_{(5i6)})^{n_2(g_5,g_i',g_6)} c^{n_2(g_5,g_i,g_6)}_{(5i6)} - (1-n_2(g_5,g_i',g_6))(1-n_2(g_5,g_i,g_6))c^\dagger_{(5i6)}c_{(5i6)}\big], \tag{H17}$$

$$O^{g_{i'}g_i}_{61;g_6g_1} = \nu_3(g_i,g_i',g_6,g_1)(-)^{+n_2(g_i,g_i',g_1)n_2(g_i',g_6,g_1)} \times \big[(c^\dagger_{(i61)})^{n_2(g_i',g_6,g_1)} c^{n_2(g_i,g_6,g_1)}_{(i61)} - (1-n_2(g_i',g_6,g_1))(1-n_2(g_i,g_6,g_1))c^\dagger_{(i61)}c_{(i61)}\big]. \tag{H18}$$

The above expression for H_i is valid in the subspace where the fermion occupation number n_{ijk} on each triangle (ijk) satisfies $n_{ijk} = n_2(g_i,g_j,g_k)$. The ideal fermionic SPT state $|\Psi\rangle$ is also in this subspace. We can add a term

$$H_{(ijk)} = U|g_ig_jg_k\rangle\langle g_ig_jg_k|[c^\dagger_{(ijk)}c_{(ijk)} - n_2(g_i,g_j,g_k)]^2 \tag{H19}$$

with a large positive U on each triangle to put the ground state in the subspace.

Next, we would like to show that H_i is Hermitian. This property is very important and it makes the whole theory to be unitary. In the above, we express the matrix element $H_{i;g_i,g_i'}(g_1g_2g_3g_4g_5g_6)$ as

$$(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)+m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)} \int \prod_\Sigma \mathcal{V}_3, \tag{H20}$$

hence, its Hermitian conjugate $H^*_{i;g_i',g_i}(g_1g_2g_3g_4g_5g_6)$ reads as

$$\begin{aligned}
&(-)^{m_1(g_4,g_i)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i')+m_1(g_5,g_i')+m_1(g_i',g_6)} \int \prod_{j=1,2,6} d\theta^{n_2(g_i',g_i,g_j)}_{(ii'j)} d\bar\theta^{n_2(g_i',g_i,g_j)}_{(ii'j)} \prod_{j=3,4,5} d\theta^{n_2(g_j,g_i',g_i)}_{(jii')} d\bar\theta^{n_2(g_j,g_i',g_i)}_{(jii')}\\
&\times (-)^{m_1(g_4,g_i')+m_1(g_5,g_i)+m_1(g_i',g_6)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i)}(-)^{m_1(g_i',g_i)}\nu_3^{-1}(g_5,g_i',g_i,g_6)\theta^{n_2(g_5,g_i',g_i)}_{(5ii')}\theta^{n_2(g_5,g_i,g_6)}_{(5i'6)}\bar\theta^{n_2(g_5,g_i',g_6)}_{(5i6)}\\
&\times \bar\theta^{n_2(g_i',g_i,g_6)}_{(ii'6)}\nu_3^{-1}(g_4,g_5,g_i',g_i)\theta^{n_2(g_4,g_5,g_i')}_{(45i)}\theta^{n_2(g_4,g_i',g_i)}_{(4ii')}\bar\theta^{n_2(g_4,g_5,g_i)}_{(45i')}\bar\theta^{n_2(g_5,g_i',g_i)}_{(5ii')}\nu_3(g_4,g_3,g_i',g_i)\theta^{n_2(g_3,g_i',g_i)}_{(3ii')}\theta^{n_2(g_4,g_3,g_i)}_{(43i')}\bar\theta^{n_2(g_4,g_i',g_i)}_{(4ii')}\\
&\times \bar\theta^{n_2(g_4,g_3,g_i')}_{(43i)}\nu_3(g_3,g_i',g_i,g_2)\theta^{n_2(g_i',g_i,g_2)}_{(ii'2)}\theta^{n_2(g_3,g_i',g_2)}_{(3i2)}\bar\theta^{n_2(g_3,g_i,g_2)}_{(3i'2)}\bar\theta^{n_2(g_3,g_i',g_i)}_{(3ii')}\nu_3(g_i',g_i,g_2,g_1)\bar\theta^{n_2(g_i,g_2,g_1)}_{(i'21)}\bar\theta^{n_2(g_i',g_i,g_1)}_{(ii'1)}\\
&\times \theta^{n_2(g_i',g_2,g_1)}_{(i21)}\theta^{n_2(g_i',g_i,g_2)}_{(ii'2)}\nu_3^{-1}(g_i',g_i,g_6,g_1)\theta^{n_2(g_i',g_i,g_6)}_{(ii'6)}\theta^{n_2(g_i',g_6,g_1)}_{(i61)}\bar\theta^{n_2(g_i',g_i,g_1)}_{(ii'1)}\bar\theta^{n_2(g_i,g_6,g_1)}_{(i'61)}.
\end{aligned} \tag{H21}$$

Note $\nu^* = \nu^{-1}$ [ν is a $U(1)$ phase factor] and $(\theta_1\theta_2\bar\theta_3\bar\theta_4)^* = \theta_4\theta_3\bar\theta_2\bar\theta_1$. However, we can not directly compare the above expression with $H_{i;g_i,g_i'}(g_1g_2g_3g_4g_5g_6)$ since it is a function of new pairs of Grassmann variable $\theta^{n_2(g_5,g_i',g_6)}_{(5i'6)}\bar\theta^{n_2(g_5,g_i,g_6)}_{(5i6)}$, $\bar\theta^{n_2(g_4,g_5,g_i')}_{(45i')}\theta^{n_2(g_4,g_5,g_i)}_{(45i)}$, $\theta^{n_2(g_4,g_3,g_i')}_{(43i')}\bar\theta^{n_2(g_4,g_3,g_i)}_{(43i)}$, $\bar\theta^{n_2(g_3,g_i',g_2)}_{(3i'2)}\theta^{n_2(g_3,g_i,g_2)}_{(3i2)}$, $\theta^{n_2(g_i',g_2,g_1)}_{(i'21)}\bar\theta^{n_2(g_i,g_2,g_1)}_{(i21)}$, and $\bar\theta^{n_2(g_i',g_6,g_1)}_{(i'61)}\theta^{n_2(g_i,g_6,g_1)}_{(i61)}$. Such a difference is simply because the action of H_i maps, for example, $\bar\theta_{(5i6)}$ to $\bar\theta_{(5i'6)}$ while the action of $H_i^\dagger$ maps $\bar\theta_{(5i'6)}$ back to $\bar\theta_{(5i6)}$. Thus, to see whether they are the same mapping in the Hilbert space or not, we need to redefine $\bar\theta_{(5i6)}(\theta_{(5i6)})$ as $\bar\theta_{(5i'6)}(\theta_{(5i'6)})$ and $\bar\theta_{(5i'6)}(\theta_{(5i'6)})$ as $\bar\theta_{(5i6)}(\theta_{(5i6)})$. A simple way to do so is just replacing $i(i')$ by $i'(i)$ in the above expression:

$$\begin{aligned}
&(-)^{m_1(g_4,g_i)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i')+m_1(g_5,g_i')+m_1(g_i',g_6)} \int \prod_{j=1,2,6} d\theta^{n_2(g_i',g_i,g_j)}_{(i'ij)} d\bar\theta^{n_2(g_i',g_i,g_j)}_{(i'ij)} \prod_{j=3,4,5} d\theta^{n_2(g_j,g_i',g_i)}_{(ji'i)} d\bar\theta^{n_2(g_j,g_i',g_i)}_{(ji'i)}\\
&\times (-)^{m_1(g_4,g_i')+m_1(g_5,g_i)+m_1(g_i',g_6)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i)}(-)^{m_1(g_i',g_i)}\nu_3^{-1}(g_5,g_i',g_i,g_6)\theta^{n_2(g_5,g_i',g_i)}_{(5i'i)}\theta^{n_2(g_5,g_i,g_6)}_{(5i6)}\\
&\times \bar\theta^{n_2(g_5,g_i',g_6)}_{(5i'6)}\bar\theta^{n_2(g_i',g_i,g_6)}_{(i'i6)}\nu_3^{-1}(g_4,g_5,g_i',g_i)\theta^{n_2(g_4,g_5,g_i')}_{(45i')}\theta^{n_2(g_4,g_i',g_i)}_{(4i'i)}\bar\theta^{n_2(g_4,g_5,g_i)}_{(45i)}\bar\theta^{n_2(g_5,g_i',g_i)}_{(5i'i)}\nu_3(g_4,g_3,g_i',g_i)\theta^{n_2(g_3,g_i',g_i)}_{(3i'i)}\\
&\times \theta^{n_2(g_4,g_3,g_i)}_{(43i)}\bar\theta^{n_2(g_4,g_i',g_i)}_{(4i'i)}\bar\theta^{n_2(g_4,g_3,g_i')}_{(43i')}\nu_3(g_3,g_i',g_i,g_2)\theta^{n_2(g_i',g_i,g_2)}_{(i'i2)}\theta^{n_2(g_3,g_i',g_2)}_{(3i'2)}\bar\theta^{n_2(g_3,g_i,g_2)}_{(3i2)}\bar\theta^{n_2(g_3,g_i',g_i)}_{(3i'i)}\nu_3(g_i',g_i,g_2,g_1)\\
&\times \bar\theta^{n_2(g_i,g_2,g_1)}_{(i21)}\bar\theta^{n_2(g_i',g_i,g_1)}_{(i'i1)}\theta^{n_2(g_i',g_2,g_1)}_{(i'21)}\theta^{n_2(g_i',g_i,g_2)}_{(i'i2)}\nu_3^{-1}(g_i',g_i,g_6,g_1)\theta^{n_2(g_i',g_i,g_6)}_{(i'i6)}\theta^{n_2(g_i',g_6,g_1)}_{(i'61)}\bar\theta^{n_2(g_i',g_i,g_1)}_{(i'i1)}\bar\theta^{n_2(g_i,g_6,g_1)}_{(i61)}\\
&= -^{m_1(g_4,g_i)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i')+m_1(g_5,g_i')+m_1(g_i',g_6)} \int \prod_{j=1,2,6} d\theta^{n_2(g_i',g_i,g_j)}_{(i'ij)} d\bar\theta^{n_2(g_i',g_i,g_j)}_{(i'ij)} \prod_{j=3,4,5} d\theta^{n_2(g_j,g_i',g_i)}_{(ji'i)} d\bar\theta^{n_2(g_j,g_i',g_i)}_{(ji'i)}\\
&\times (-)^{m_1(g_4,g_i')+m_1(g_5,g_i)+m_1(g_i',g_6)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i)}(-)^{m_1(g_i',g_i)}
\end{aligned}$$

$$\times \mathcal{V}_3^-(g_5,g_i',g_i,g_6)\mathcal{V}_3^-(g_4,g_5,g_i',g_i)\mathcal{V}_3^+(g_4,g_3,g_i',g_i)\mathcal{V}_3^+(g_3,g_i',g_i,g_2)\mathcal{V}_3^+(g_i',g_i,g_2,g_1)\mathcal{V}_3^-(g_i',g_i,g_6,g_1)$$
$$= (-)^{m_1(g_4,g_i)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i')+m_1(g_5,g_i')+m_1(g_i',g_6)} \int \prod_\Sigma \mathcal{V}_3. \tag{H22}$$

We note that in the last line we evaluate the complex Fig. 22(a) in a different way (by choosing opposite orientation for the internal link). The topological invariance of the $\mathcal{V}_3$ path integral implies the two different ways must give out the same results. It is also not hard to see that the new sign factor $(-)^{m_1(g_4,g_i)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i')+m_1(g_5,g_i')+m_1(g_i',g_6)}$ is equivalent to $(-)^{m_1(g_4,g_i')+m_1(g_i',g_1)+m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)}$ since

$$\begin{aligned}
&m_1(g_4,g_i)+m_1(g_i,g_1)+m_1(g_i',g_2)+m_1(g_3,g_i')+m_1(g_5,g_i')+m_1(g_i',g_6)\\
&\quad+m_1(g_4,g_i')+m_1(g_i',g_1)+m_1(g_i,g_2)+m_1(g_3,g_i)+m_1(g_5,g_i)+m_1(g_i,g_6)\\
&=n_2(g_5,g_i',g_6)+n_2(g_5,g_i,g_6)+n_2(g_4,g_5,g_i')+n_2(g_4,g_5,g_i)+n_2(g_4,g_3,g_i')+n_2(g_4,g_3,g_i)\\
&\quad+n_2(g_3,g_i',g_2)+n_2(g_3,g_i,g_2)+n_2(g_i',g_2,g_1)+n_2(g_i,g_2,g_1)+n_2(g_i',g_6,g_1)+n_2(g_i,g_6,g_1)=0 \bmod 2.
\end{aligned} \tag{H23}$$

Thus, we have proved that the matrix element $H_{i;g_i,g_i'}(g_1g_2g_3g_4g_5g_6)$ is the same as $H^*_{i;g_i',g_i}(g_1g_2g_3g_4g_5g_6)$, implying H_i is a Hermitian operator.

Finally, we note that the Hamiltonian only depends on ν_3 and n_2. It does not depend on m_1. So, the Hamiltonian is symmetric under the G_f symmetry. By construction, the ideal ground state wave function is an eigenstate of H_i with eigenvalue 1. The topological invariance of $\mathcal{V}_3$ path integral implies that $H_i^2 = H_i$ and $H_iH_j = H_jH_i$. Thus, H_i is a Hermitian projection operator, and the set $\{H_i\}$ is a set of commuting projectors. Therefore, $H = -\sum_i H_i$ is an exactly solvable Hamiltonian which realizes the fermionic SPT state described by $(\nu_3,n_2,u_2^g) \in \mathscr{H}^3[G_f,U_T(1)]$.

APPENDIX I: THE MAPPING $n_d \to f_{d+2}$ INDUCES A MAPPING $\mathcal{H}^d(G_b,\mathbb{Z}_2) \to \mathcal{H}^{d+2}(G_b,\mathbb{Z}_2)$

We note that Eq. (57) defines a mapping from a d-cochain $n_d \in \mathcal{C}^d(G_b,\mathbb{Z}_2)$ to a $(d+2)$-cochain $f_d \in \mathcal{C}^{d+2}(G_b,\mathbb{Z}_2)$. We can show that, if n_d is a cocycle $n_d \in \mathcal{Z}^d(G_b,\mathbb{Z}_2)$, then the corresponding f_{d+2} is also a cocycle $f_d \in \mathcal{Z}^{d+2}(G_b,\mathbb{Z}_2)$. Thus, Eq. (57) defines a mapping from a d-cocycle $n_d \in \mathcal{Z}^d(G_b,\mathbb{Z}_2)$ to a $(d+2)$-cocycle $f_d \in \mathcal{Z}^{d+2}(G_b,\mathbb{Z}_2)$. In this Appendix, we are going to show that Eq. (57) actually defines a mapping from d-cohomology classes to $(d+2)$-cohomology classes: $\mathcal{H}^d(G_b,\mathbb{Z}_2) \to \mathcal{H}^{d+2}(G_b,\mathbb{Z}_2)$. This is because if n_d and $\tilde n_d$ differ by a coboundary, then the corresponding f_{d+2} and $\tilde f_{d+2}$ also differ by a coboundary.

To show this, let us first assume $d=2$. Since the 4-cochain $f_4(g_0,g_1,g_2,g_3,g_4)$ in $\mathcal{C}^4(G_b,\mathbb{Z}_2)$ is the cup product to two 2-cocycles $n_2(g_0,g_1,g_2)$ and $n_2(g_0,g_1,g_2)$ in $\mathcal{Z}^2(G_b,\mathbb{Z}_2)$, so if we change the two cocycles by a coboundary, the 4-cochain f_4 will also change by a coboundary. Thus, there exists a vector $\boldsymbol{g}_3$ such that

$$\tilde{\boldsymbol{f}}_4 = \boldsymbol{f}_4 + D_3\boldsymbol{g}_3 \bmod 2 \tag{I1}$$

and Eq. (C22) is valid for $d=2$.

To show Eq. (C22) to be valid for $d=3$, we need to show when $n_3(g_0,g_1,g_2,g_3)$ and $\tilde n_3(g_0,g_1,g_2,g_3)$ are related by a coboundary in $\mathcal{B}^3(G_b,\mathbb{Z}_2)$:

$$\begin{aligned}\tilde n_3(g_0,g_1,g_2,g_3) &= n_3(g_0,g_1,g_2,g_3)+m_2'(g_1,g_2,g_3)\\ &\quad+m_2'(g_0,g_2,g_3)+m_2'(g_0,g_1,g_3)\\ &\quad+m_2'(g_0,g_1,g_2) \bmod 2,\end{aligned} \tag{I2}$$

the corresponding $\boldsymbol{f}_5$ and $\tilde{\boldsymbol{f}}_5$ are also related by a coboundary in $\mathcal{B}^5(G_b,\mathbb{Z}_2)$:

$$\boldsymbol{f}_5 = \tilde{\boldsymbol{f}}_5 + D_4\boldsymbol{g}_4 \bmod 2. \tag{I3}$$

Let us introduce

$$\begin{aligned}n_3'(g_0,g_1,g_2,g_3) &= m_2'(g_1,g_2,g_3)+m_2'(g_0,g_2,g_3)\\ &\quad+m_2'(g_0,g_1,g_3)+m_2'(g_0,g_1,g_2).\end{aligned} \tag{I4}$$

Then, we can express $\tilde f_5$ as $\tilde f_5 = f_5 + f_5^I + f_5^{II} + f_5^{III}$, with

$$\begin{aligned}
f_5^I(g_0,g_1,g_2,g_3,g_4,g_5) &= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5),\\
f_5^{II}(g_0,g_1,g_2,g_3,g_4,g_5) &= n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5)+n_3'(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5),\\
f_5^{III}(g_0,g_1,g_2,g_3,g_4,g_5) &= n_3'(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)+n_3'(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5).
\end{aligned} \tag{I5}$$

In the following, we would like to show g_4 can be constructed as $g_4 = g_4^I + g_4^{II} + g_4^{III}$, with

$$\begin{aligned}
g_4^I(g_0,g_1,g_2,g_3,g_4) &= n_3(g_0,g_1,g_2,g_3)m_2'(g_0,g_3,g_4)+n_3(g_1,g_2,g_3,g_4)m_2'(g_0,g_1,g_4),\\
g_4^{II}(g_0,g_1,g_2,g_3,g_4) &= n_3(g_0,g_2,g_3,g_4)m_2'(g_0,g_1,g_2)+n_3(g_0,g_1,g_2,g_4)m_2'(g_2,g_3,g_4)+n_3(g_0,g_1,g_3,g_4)m_2'(g_1,g_2,g_3),\\
g_4^{III}(g_0,g_1,g_2,g_3,g_4) &= n_3'(g_0,g_1,g_2,g_3)m_2'(g_0,g_3,g_4)+n_3'(g_1,g_2,g_3,g_4)m_2'(g_0,g_1,g_4)+m_2'(g_0,g_1,g_2)m_2'(g_2,g_3,g_4).
\end{aligned} \tag{I6}$$

Let us prove the above statement in three steps:

First step. After plug-in, we find that $\boldsymbol{f}_5^I - D_4\boldsymbol{g}_4^I$ is given by

$$\begin{aligned}
&f_5^I(g_0,g_1,g_2,g_3,g_4,g_5) - g_4^I(g_0,g_1,g_2,g_3,g_4) - g_4^I(g_0,g_1,g_2,g_3,g_5) - g_4^I(g_0,g_1,g_2,g_4,g_5)\\
&\quad - g_4^I(g_0,g_1,g_3,g_4,g_5) - g_4^I(g_0,g_2,g_3,g_4,g_5) - g_4^I(g_1,g_2,g_3,g_4,g_5)\\
&= n_3(g_0,g_1,g_2,g_3)[m_2'(g_0,g_3,g_4) + m_2'(g_0,g_3,g_5) + m_2'(g_0,g_4,g_5) + m_2'(g_3,g_4,g_5)]\\
&\quad + n_3(g_1,g_2,g_3,g_4)[m_2'(g_0,g_1,g_4) + m_2'(g_0,g_1,g_5) + m_2'(g_0,g_4,g_5) + m_2'(g_1,g_4,g_5)]\\
&\quad + n_3(g_2,g_3,g_4,g_5)[m_2'(g_0,g_1,g_2) + m_2'(g_0,g_1,g_5) + m_2'(g_0,g_2,g_5) + m_2'(g_1,g_2,g_5)]\\
&\quad - n_3(g_0,g_1,g_2,g_3)m_2'(g_0,g_3,g_4) - n_3(g_1,g_2,g_3,g_4)m_2'(g_0,g_1,g_4) - n_3(g_0,g_1,g_2,g_3)m_2'(g_0,g_3,g_5)\\
&\quad - n_3(g_1,g_2,g_3,g_5)m_2'(g_0,g_1,g_5) - n_3(g_0,g_1,g_2,g_4)m_2'(g_0,g_4,g_5) - n_3(g_1,g_2,g_4,g_5)m_2'(g_0,g_1,g_5)\\
&\quad - n_3(g_0,g_1,g_3,g_4)m_2'(g_0,g_4,g_5) - n_3(g_1,g_3,g_4,g_5)m_2'(g_0,g_1,g_5) - n_3(g_0,g_2,g_3,g_4)m_2'(g_0,g_4,g_5)\\
&\quad - n_3(g_2,g_3,g_4,g_5)m_2'(g_0,g_2,g_5) - n_3(g_1,g_2,g_3,g_4)m_2'(g_1,g_4,g_5) - n_3(g_2,g_3,g_4,g_5)m_2'(g_1,g_2,g_5).
\end{aligned}$$

We note that there are five terms containing $m_2'(g_0,g_4,g_5)$ and five terms containing $m_2'(g_0,g_1,g_5)$. Using the condition that $n_3(g_0,g_1,g_2,g_3)$ is a 3-cocycle

$$n_3(g_0,g_1,g_2,g_3) + n_3(g_0,g_1,g_2,g_3) + n_3(g_0,g_1,g_2,g_3) + n_3(g_0,g_1,g_2,g_3) + n_3(g_0,g_1,g_2,g_3) = 0 \text{ mode } 2, \tag{I7}$$

we find that those terms cancel out. Also, there are two terms containing $m_2'(g_0,g_3,g_4)$, two terms containing $m_2'(g_0,g_4,g_5)$, ..., etc. Those terms also cancel out under the mod 2 calculation. But, the $m_2'(g_0,g_1,g_2)$ term and the $m_2'(g_3,g_4,g_5)$ term appear only once. Thus, we simplify the above to

$$\begin{aligned}
&f_5^I(g_0,g_1,g_2,g_3,g_4,g_5) - g_4^I(g_0,g_1,g_2,g_3,g_4) - g_4^I(g_0,g_1,g_2,g_3,g_5) - g_4^I(g_0,g_1,g_2,g_4,g_5)\\
&\quad - g_4^I(g_0,g_1,g_3,g_4,g_5) - g_4^I(g_0,g_2,g_3,g_4,g_5) - g_4^I(g_1,g_2,g_3,g_4,g_5)\\
&= n_3(g_0,g_1,g_2,g_3)m_2'(g_3,g_4,g_5) + n_3(g_2,g_3,g_4,g_5)m_2'(g_0,g_1,g_2) \text{ mod } 2.
\end{aligned} \tag{I8}$$

Second step. Similarly, we have

$$\begin{aligned}
&f_5^{II}(g_0,g_1,g_2,g_3,g_4,g_5) - g_4^{II}(g_0,g_1,g_2,g_3,g_4) - g_4^{II}(g_0,g_1,g_2,g_3,g_5) - g_4^{II}(g_0,g_1,g_2,g_4,g_5)\\
&\quad - g_4^{II}(g_0,g_1,g_3,g_4,g_5) - g_4^{II}(g_0,g_2,g_3,g_4,g_5) - g_4^{II}(g_1,g_2,g_3,g_4,g_5)\\
&= -n_3(g_0,g_1,g_2,g_3)m_2'(g_3,g_4,g_5) - n_3(g_2,g_3,g_4,g_5)m_2'(g_0,g_1,g_2) \text{ mod } 2.
\end{aligned} \tag{I9}$$

Third step. Finally, by using the results derived in the first step, we have

$$\begin{aligned}
&f_5^{III}(g_0,g_1,g_2,g_3,g_4,g_5) - g_4^{III}(g_0,g_1,g_2,g_3,g_4) - g_4^{III}(g_0,g_1,g_2,g_3,g_5) - g_4^{III}(g_0,g_1,g_2,g_4,g_5) - g_4^{III}(g_0,g_1,g_3,g_4,g_5)\\
&\quad - g_4^{III}(g_0,g_2,g_3,g_4,g_5) - g_4^{III}(g_1,g_2,g_3,g_4,g_5)\\
&= n_3'(g_0,g_1,g_2,g_3)m_2'(g_3,g_4,g_5) + n_3'(g_2,g_3,g_4,g_5)m_2'(g_0,g_1,g_2) + m_2'(g_0,g_1,g_2)m_2'(g_2,g_3,g_4)\\
&\quad + m_2'(g_0,g_1,g_2)m_2'(g_2,g_3,g_5) + m_2'(g_0,g_1,g_2)m_2'(g_2,g_4,g_5) + m_2'(g_0,g_1,g_3)m_2'(g_3,g_4,g_5)\\
&\quad + m_2'(g_0,g_2,g_3)m_2'(g_3,g_4,g_5) + m_2'(g_1,g_2,g_3)m_2'(g_3,g_4,g_5) \text{ mod } 2 = 0 \text{ mod } 2.
\end{aligned} \tag{I10}$$

Combining the results from the above three steps, we can show

$$D_4\boldsymbol{g}_4 = \boldsymbol{f}_5^I + \boldsymbol{f}_5^{II} + \boldsymbol{f}_5^{III} \text{ mod } 2. \tag{I11}$$

Thus, we prove Eq. (C22) for $d = 3$.

We would like to mention that the induced mapping $\mathcal{H}^d(G_b,\mathbb{Z}_2) \to \mathcal{H}^{d+2}(G_b,\mathbb{Z}_2)$ by the $n_d \to f_{d+2}$ mapping (57) appears to be the Steenrod square Sq^2 [72,73]. This realization will allow us to generalize Eq. (57) to higher dimensions.

APPENDIX J: THE MAPPING $n_d \to f_{d+2}$ PRESERVES ADDITIVITY

In this Appendix, we will show that the operation (D1) defines an Abelian group structure in $\mathscr{H}^d[G_f,U_T(1)]$. The key step is to show that if n_d maps to f_{d+2}, n_d' maps to f_{d+2}', and $n_d'' = n_d + n_d'$ maps to f_{d+2}'', then $f_{d+2}'' - f_{d+2} - f_{d+2}'$ is a coboundary in $\mathcal{B}^{d+1}(G_b,\mathbb{Z}_2)$. {It is also a coboundary in $\mathcal{B}^{d+1}[G_b,U_T(1)]$.}

Let us first consider the case $d = 2$ and we have

$$\begin{aligned}
&f_4(g_0,g_1,g_2,g_3,g_4) = n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4), \quad f_4'(g_0,g_1,g_2,g_3,g_4) = n_2'(g_0,g_1,g_2)n_2'(g_2,g_3,g_4),\\
&f_4''(g_0,g_1,g_2,g_3,g_4) = [n_2(g_0,g_1,g_2) + n_2'(g_0,g_1,g_2)][n_2(g_2,g_3,g_4) + n_2'(g_2,g_3,g_4)].
\end{aligned} \tag{J1}$$

So,

$$f_4'' - f_4 - f_4' = n_2(g_0,g_1,g_2)n_2'(g_2,g_3,g_4) + n_2'(g_0,g_1,g_2)n_2(g_2,g_3,g_4). \tag{J2}$$

The above is indeed a coboundary:

$$\begin{aligned} n_2(g_0,g_1,g_2)n_2'(g_2,g_3,g_4) + n_2'(g_0,g_1,g_2)n_2(g_2,g_3,g_4) &= (da_3)(g_0,\ldots,g_4),\\ a_3(g_0,\ldots,g_3) &= n_2(g_0,g_1,g_2)n_2'(g_0,g_2,g_3) + n_2(g_1,g_2,g_3)n_2'(g_0,g_1,g_3), \end{aligned} \tag{J3}$$

where d is a mapping from $(d+1)$-variable functions $f_d(g_0,\ldots,g_d)$ to $(d+2)$-variable functions $(df_d)(g_0,\ldots,g_{d+1})$:

$$(df_d)(g_0,\ldots,g_{d+1}) \equiv \sum_{i=0}^{d+1}(-)^i f_d(g_0,\ldots,\hat{g}_i,\ldots,g_{d+1}).$$

It is easy to check that

$$\begin{aligned}
&n_2(g_0,g_1,g_2)n_2'(g_0,g_2,g_3) + n_2(g_1,g_2,g_3)n_2'(g_0,g_1,g_3) + n_2(g_0,g_1,g_2)n_2'(g_0,g_2,g_4) + n_2(g_1,g_2,g_4)n_2'(g_0,g_1,g_4)\\
&\quad + n_2(g_0,g_1,g_3)n_2'(g_0,g_3,g_4) + n_2(g_1,g_3,g_4)n_2'(g_0,g_1,g_4) + n_2(g_0,g_2,g_3)n_2'(g_0,g_3,g_4) + n_2(g_2,g_3,g_4)n_2'(g_0,g_2,g_4)\\
&\quad + n_2(g_1,g_2,g_3)n_2'(g_1,g_3,g_4) + n_2(g_2,g_3,g_4)n_2'(g_1,g_2,g_4)\\
&= n_2(g_0,g_1,g_2)[n_2'(g_2,g_3,g_4) + n_2'(g_0,g_3,g_4)] + n_2(g_1,g_2,g_3)n_2'(g_0,g_1,g_3) + n_2(g_1,g_2,g_4)n_2'(g_0,g_1,g_4)\\
&\quad + n_2(g_0,g_1,g_3)n_2'(g_0,g_3,g_4) + n_2(g_1,g_3,g_4)n_2'(g_0,g_1,g_4) + n_2(g_0,g_2,g_3)n_2'(g_0,g_3,g_4) + n_2(g_2,g_3,g_4)n_2'(g_0,g_2,g_4)\\
&\quad + n_2(g_1,g_2,g_3)n_2'(g_1,g_3,g_4) + n_2(g_2,g_3,g_4)n_2'(g_1,g_2,g_4) \text{ mod } 2\\
&= n_2(g_0,g_1,g_2)n_2'(g_2,g_3,g_4) + n_2(g_1,g_2,g_3)n_2'(g_0,g_3,g_4) + n_2(g_1,g_2,g_3)n_2'(g_0,g_1,g_3) + n_2(g_1,g_2,g_4)n_2'(g_0,g_1,g_4)\\
&\quad + n_2(g_1,g_3,g_4)n_2'(g_0,g_1,g_4) + n_2(g_2,g_3,g_4)n_2'(g_0,g_2,g_4) + n_2(g_1,g_2,g_3)n_2'(g_1,g_3,g_4) + n_2(g_2,g_3,g_4)n_2'(g_1,g_2,g_4) \text{ mod } 2\\
&= n_2(g_0,g_1,g_2)n_2'(g_2,g_3,g_4) + n_2(g_1,g_2,g_3)n_2'(g_0,g_1,g_4) + n_2(g_1,g_3,g_4)n_2'(g_0,g_1,g_4) + n_2(g_1,g_2,g_4)n_2'(g_0,g_1,g_4)\\
&\quad + n_2(g_2,g_3,g_4)n_2'(g_0,g_2,g_4) + n_2(g_2,g_3,g_4)n_2'(g_1,g_2,g_4) \text{ mod } 2\\
&= n_2(g_0,g_1,g_2)n_2'(g_2,g_3,g_4) + n_2(g_2,g_3,g_4)n_2'(g_0,g_1,g_4) + n_2(g_2,g_3,g_4)n_2'(g_0,g_2,g_4) + n_2(g_2,g_3,g_4)n_2'(g_1,g_2,g_4) \text{ mod } 2\\
&= n_2(g_0,g_1,g_2)n_2'(g_2,g_3,g_4) + n_2(g_2,g_3,g_4)n_2'(g_0,g_1,g_2) \text{ mod } 2.
\end{aligned} \tag{J4}$$

Here, we make use of the fact that $n_2(g_0,g_1,g_2)$ and $n_2(g_0,g_1,g_2)'$ satisfy the 2-cocycle condition

$$\begin{aligned} n_2(g_0,g_1,g_2) + n_2(g_0,g_1,g_3) + n_2(g_0,g_2,g_3) + n_2(g_1,g_2,g_3) &= 0 \text{ mod } 2,\\ n_2(g_0,g_1,g_2)' + n_2(g_0,g_1,g_3)' + n_2(g_0,g_2,g_3)' + n_2(g_1,g_2,g_3)' &= 0 \text{ mod } 2. \end{aligned} \tag{J5}$$

Thus, we have shown that $f_4'' - f_4 - f_4'$ is a coboundary.

Next, we consider the case $d = 3$, we note that

$$\begin{aligned}
f_5(g_0,g_1,g_2,g_3,g_4,g_5) &= n_3(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5) + n_3(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5),\\
f_5'(g_0,g_1,g_2,g_3,g_4,g_5) &= n_3'(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5) + n_3'(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5),\\
f_5''(g_0,g_1,g_2,g_3,g_4,g_5) &= [n_3(g_0,g_1,g_2,g_3) + n_3'(g_0,g_1,g_2,g_3)][n_3(g_0,g_3,g_4,g_5) + n_3'(g_0,g_3,g_4,g_5)]\\
&\quad + [n_3(g_1,g_2,g_3,g_4) + n_3'(g_1,g_2,g_3,g_4)][n_3(g_0,g_1,g_4,g_5) + n_3'(g_0,g_1,g_4,g_5)]\\
&\quad + [n_3(g_2,g_3,g_4,g_5) + n_3'(g_2,g_3,g_4,g_5)][n_3(g_0,g_1,g_2,g_5) + n_3'(g_0,g_1,g_2,g_5)].
\end{aligned} \tag{J6}$$

So,

$$\begin{aligned}
f_5'' - f_5 - f_5' &= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5) + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5)\\
&\quad + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3'(g_1,g_2,g_3,g_4)n_3(g_0,g_1,g_4,g_5) + n_3'(g_2,g_3,g_4,g_5)n_3(g_0,g_1,g_2,g_5),
\end{aligned} \tag{J7}$$

which is indeed a coboundary

$$\begin{aligned}
f_5'' - f_5 - f_5' &= da_4,\\
a_4(g_0,\ldots,g_4) &= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_1,g_3,g_4) + n_3'(g_0,g_1,g_2,g_4)n_3(g_0,g_2,g_3,g_4)\\
&\quad + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_2,g_3,g_4).
\end{aligned} \tag{J8}$$

First, it is easy to check that

$$\begin{aligned}
&n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_1,g_3,g_4)+n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_1,g_3,g_5)+n_3(g_0,g_1,g_2,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_0,g_2,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_2,g_4)n_3(g_0,g_2,g_3,g_4)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_3,g_5)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_0,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\\
&=n_3(g_0,g_1,g_2,g_3)[n_3'(g_0,g_3,g_4,g_5)+n_3'(g_0,g_1,g_4,g_5)+n_3'(g_1,g_3,g_4,g_5)]+n_3(g_0,g_1,g_2,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_0,g_2,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_2,g_4)n_3(g_0,g_2,g_3,g_4)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_3,g_5)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_0,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\text{ mod }2\\
&=n_3(g_0,g_1,g_2,g_3)[n_3'(g_0,g_3,g_4,g_5)+n_3'(g_1,g_3,g_4,g_5)]+[n_3(g_0,g_2,g_3,g_4)+n_3(g_1,g_2,g_3,g_4)]n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_0,g_2,g_3,g_4)n_3'(g_0,g_2,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)+n_3'(g_0,g_1,g_2,g_4)n_3(g_0,g_2,g_3,g_4)\\
&\quad+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_3,g_5)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_4,g_5)+n_3'(g_0,g_1,g_3,g_5)n_3(g_0,g_3,g_4,g_5)\\
&\quad+n_3'(g_0,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\text{ mod }2\\
&=n_3(g_0,g_1,g_2,g_3)[n_3'(g_0,g_3,g_4,g_5)+n_3'(g_1,g_3,g_4,g_5)]+n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_0,g_2,g_3,g_4)[n_3'(g_0,g_1,g_2,g_5)+n_3'(g_1,g_2,g_4,g_5)]+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_3,g_5)\\
&\quad+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_4,g_5)+n_3'(g_0,g_1,g_3,g_5)n_3(g_0,g_3,g_4,g_5)\\
&\quad+n_3'(g_0,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\text{ mod }2\\
&=n_3(g_0,g_1,g_2,g_3)[n_3'(g_0,g_3,g_4,g_5)+n_3'(g_1,g_3,g_4,g_5)]+n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)+[n_3(g_0,g_3,g_4,g_5)\\
&\quad+n_3(g_2,g_3,g_4,g_5)]n_3'(g_0,g_1,g_2,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)+n_3'(g_0,g_1,g_3,g_5)n_3(g_0,g_3,g_4,g_5)\\
&\quad+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)+n_3'(g_0,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\text{ mod }2\\
&=n_3(g_0,g_1,g_2,g_3)[n_3'(g_0,g_3,g_4,g_5)+n_3'(g_1,g_3,g_4,g_5)]+n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+[n_3'(g_0,g_1,g_2,g_3)+n_3'(g_1,g_2,g_3,g_5)]n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\text{ mod }2\\
&=n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5)+n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)\text{ mod }2.
\end{aligned}\tag{J9}$$

In the above calculations, we make use of the fact that $n_3(g_0,g_1,g_2,g_3)$ and $n_3(g_0,g_1,g_2,g_3)'$ satisfy the 4-cocycle condition

$$\begin{aligned}
&n_3(g_0,g_1,g_2,g_3)+n_3(g_0,g_1,g_2,g_4)+n_3(g_0,g_1,g_3,g_4)+n_3(g_0,g_2,g_3,g_4)+n_3(g_1,g_2,g_3,g_4)=0\text{ mod }2,\\
&n_3(g_0,g_1,g_2,g_3)'+n_3(g_0,g_1,g_2,g_4)'+n_3(g_0,g_1,g_3,g_4)'+n_3(g_0,g_2,g_3,g_4)'+n_3(g_1,g_2,g_3,g_4)'=0\text{ mod }2.
\end{aligned}\tag{J10}$$

Next, by using the same trick, we can see that

$$\begin{aligned}
&n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_1,g_3,g_4)+n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_1,g_3,g_5)+n_3(g_0,g_1,g_2,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_0,g_2,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_2,g_4)n_3(g_0,g_2,g_3,g_4)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_3,g_5)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_0,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4)+n_3(g_0,g_1,g_3,g_5)n_3'(g_1,g_2,g_3,g_5)+n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_3,g_4,g_5)+n_3(g_0,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5)+n_3(g_1,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5)
\end{aligned}$$

$$
\begin{aligned}
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_3,g_4,g_5) + n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5) + n_3'(g_1,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_3,g_5)n_3'(g_1,g_2,g_3,g_5) + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_3,g_4,g_5) + n_3(g_0,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5) + n_3(g_1,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2\\
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_3,g_4,g_5) + n_3'(g_1,g_2,g_3,g_5)[n_3(g_0,g_1,g_4,g_5) + n_3(g_0,g_1,g_3,g_4)] + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_3,g_4,g_5)\\
&\quad + n_3(g_0,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5) + n_3(g_1,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2\\
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_3,g_4,g_5) + n_3'(g_1,g_2,g_3,g_5)[n_3(g_0,g_1,g_4,g_5) + n_3(g_0,g_1,g_3,g_4)]\\
&\quad + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_3,g_4,g_5) + n_3(g_0,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5) + n_3(g_1,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2\\
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_3,g_4,g_5) + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_5) + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_3,g_4) + [n_3(g_0,g_1,g_2,g_5) + n_3(g_0,g_1,g_2,g_4)]n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2\\
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_2,g_5)n_3'(g_2,g_3,g_4,g_5) + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5)\\
&\quad + [n_3(g_0,g_1,g_2,g_4) + n_3(g_0,g_1,g_2,g_3) + n_3(g_0,g_1,g_3,g_4)]n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_3,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_5) + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_2,g_4)n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2\\
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_2,g_5)n_3'(g_2,g_3,g_4,g_5) + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5)\\
&\quad + [n_3(g_0,g_1,g_2,g_4) + n_3(g_0,g_1,g_2,g_3)]n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_3,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_3,g_4)[n_3'(g_1,g_3,g_4,g_5) + n_3'(g_2,g_3,g_4,g_5)] + n_3(g_0,g_1,g_2,g_4)n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2\\
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_2,g_5)n_3'(g_2,g_3,g_4,g_5) + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5)\\
&\quad + n_3(g_0,g_1,g_2,g_4)n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_1,g_2,g_3)[n_3'(g_1,g_2,g_3,g_4) + n_3'(g_1,g_2,g_3,g_5) + n_3'(g_2,g_3,g_4,g_5)]\\
&\quad + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_3,g_4,g_5) + [n_3(g_0,g_1,g_2,g_3) + n_3(g_0,g_2,g_3,g_4) + n_3(g_1,g_2,g_3,g_4)]n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2\\
&= n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5) + n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5) + n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad + n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_3,g_4) + n_3(g_0,g_1,g_2,g_5)n_3'(g_2,g_3,g_4,g_5) + n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5)\\
&\quad + n_3(g_0,g_1,g_2,g_4)n_3'(g_1,g_2,g_4,g_5) + n_3(g_0,g_1,g_2,g_3)[n_3'(g_1,g_2,g_3,g_4) + n_3'(g_1,g_2,g_3,g_5)]\\
&\quad + n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_3,g_4,g_5) + [n_3(g_0,g_2,g_3,g_4) + n_3(g_1,g_2,g_3,g_4)]n_3'(g_2,g_3,g_4,g_5) \text{ mod } 2. \qquad \text{(J11)}
\end{aligned}
$$

To this end, we can finally show that

$$\begin{aligned}
&n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_1,g_3,g_4)+n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_1,g_3,g_5)+n_3(g_0,g_1,g_2,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_0,g_2,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_2,g_4)n_3(g_0,g_2,g_3,g_4)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_3,g_5)+n_3'(g_0,g_1,g_2,g_5)n_3(g_0,g_2,g_4,g_5)\\
&\quad+n_3'(g_0,g_1,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_0,g_2,g_3,g_5)n_3(g_0,g_3,g_4,g_5)+n_3'(g_1,g_2,g_3,g_5)n_3(g_1,g_3,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_2,g_3,g_4)+n_3(g_0,g_1,g_3,g_5)n_3'(g_1,g_2,g_3,g_5)+n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_3,g_4,g_5)+n_3(g_0,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5)+n_3(g_1,g_2,g_4,g_5)n_3'(g_2,g_3,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_2,g_3,g_4)+n_3(g_0,g_1,g_2,g_3)n_3'(g_1,g_2,g_3,g_5)+n_3(g_0,g_1,g_2,g_4)n_3'(g_1,g_2,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_3,g_4)n_3'(g_1,g_3,g_4,g_5)+n_3(g_0,g_2,g_3,g_4)n_3'(g_2,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_2,g_3,g_4,g_5)\\
&=n_3(g_0,g_1,g_2,g_3)n_3'(g_0,g_3,g_4,g_5)+n_3'(g_0,g_1,g_2,g_3)n_3(g_0,g_3,g_4,g_5)+n_3(g_1,g_2,g_3,g_4)n_3'(g_0,g_1,g_4,g_5)\\
&\quad+n_3(g_0,g_1,g_4,g_5)n_3'(g_1,g_2,g_3,g_4)+n_3(g_0,g_1,g_2,g_5)n_3'(g_2,g_3,g_4,g_5)+n_3(g_2,g_3,g_4,g_5)n_3'(g_0,g_1,g_2,g_5) \text{ mod } 2. \qquad \text{(J12)}
\end{aligned}$$

Thus, $f_5''-f_5-f_5'=da_4$ is a coboundary. For general d, we believe such a statement is still correct, however, since only the cases with $d=2$ and 3 are relevant to physical reality, we are not going to prove it for general d and leave it as an open mathematical problem. Once we have $f_d''-f_d-f_d'=da_d$, it is quite easy to show Eq. (D1).

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