Geometry of fractional quantum Hall fluids

Gil Young Cho, Yizhi You, and Eduardo Fradkin
Department of Physics and Institute for Condensed Matter Theory, University of Illinois, 1110 W. Green St., Urbana, Illinois 61801-3080, USA

(Received 14 June 2014; revised manuscript received 9 September 2014; published 22 September 2014)

We use the field theory description of the fractional quantum Hall states to derive the universal response of these topological fluids to shear deformations and curvature of their background geometry, i.e., the Hall viscosity, and the Wen-Zee term. To account for the coupling to the background geometry, we show that the concept of flux attachment needs to be modified and use it to derive the geometric responses from Chern-Simons theories. We show that the resulting composite particles minimally couple to the spin connection of the geometry. We derive a consistent theory of geometric responses from the Chern-Simons effective field theories and from parton constructions, and apply it to both Abelian and non-Abelian states.

DOI: 10.1103/PhysRevB.90.115139 PACS number(s): 73.43.Lp, 11.10.−z, 71.10.Pm

I. INTRODUCTION

The quantized Hall conductivity is the most fundamental transverse response of the incompressible fractional quantum Hall (FQH) states of two-dimensional electron fluids in external magnetic fields [1–9]. The charge current flows perpendicular to the direction of an external in-plane electric field, and the transport is dissipationless. The Hall conductivity does not depend on the microscopic details of the system, but only on the topological properties of the states. The Hall conductivity is one of the key topological properties characterizing the quantum Hall fluids. However, the Hall conductivity does not fully characterize these topological fluids [6,7,10].

A full characterization of Abelian FQH states as topological fluids [6,10] includes the (fractional) charge and statistics of the quasiholes and the ground state degeneracy on closed surfaces, as well as various so-called fusion rules for the quasiholes. These dimensionless universal properties of the FQH state are determined by topological invariants of the topological fluid states. However, a full characterization of the FQH fluid also requires the intrinsic orbital spin s [6], and the associated Hall viscosity η_H. These quantities express the way the fluid couples to geometric properties of the two-dimensional (2D) surface on which it moves [11–24]. They become manifest when a FQH state is put on a curved (and dynamical) surface. The Hall viscosity is the response of the Hall fluid to an external shear deformation of the background surface, under which the Hall fluid develops a momentum density perpendicular to it. As a result, the net energy for the deformation vanishes, resulting in a nondissipative viscosity [23]. It has been argued that the Hall viscosity η_H depends only on the density of electron $\bar{\rho}$ and the orbital spin s through the relation $\eta_H = s\bar{\rho}/2$ [10,12,13,24].

The coupling of the Hall fluid to the curvature of the background surface is the origin of the shift vector associatedwith FQH states on spheres [6,11]. At the level of the effective hydrodynamic theory this coupling is represented in its effective action by the Wen-Zee term (whose coefficient involves the orbital spin s [6,11–13,16]). It represents the universal coupling of the hydrodynamic gauge fields of the fluid to the spin connection of the geometry of the surface. The Wen-Zee term was introduced to account for an additional Berry phase needed to represent the Hall fluid on a sphere, and also predicts that local changes in the curvature of the surface should be accompanied by local accumulation of electric charge. Since the orbital spin s and the geometric response are closely related to each other, a calculation of the geometric response amounts to a derivation of the orbital spin s of the fluid (for a rotationally invariant system [12,13]).

The topological properties of FQH fluids are encoded in the effective hydrodynamic theory which has the form of Chern-Simons gauge theory [6,25]. At a microscopic level, the FQH states are described either from the structure of model wave functions [1,4,26–28], Chern-Simons gauge theories that implement the concept of flux attachment [5,29], or by parton constructions [30]. In the past, the Hall viscosity and Wen-Zee term have been studied in various ways, ranging from the modular properties of FQH wave functions, using AdS/CFT holographic dual methods, to modeling hydrodynamic theories of FQH states (and in topological insulators) [11–24,31–34]. However, so far there has been no consistent derivation of the geometrical properties of FQH fluids from their field theoretic descriptions.

In this paper we derive the Hall viscosity and the Wen-Zee term using the description of Chern-Simons (CS) gauge theories, which embody the concept of flux attachment, and also with the projective parton approach [30,35]. To derive the geometric response from the Chern-Simons gauge theories, we first show that the conventional approach to flux attachment must be modified even for a system of nonrelativistic particles moving in a curved space. We show that the resulting composite particles are minimally coupled to the spin connection of the geometry (even though the microscopic particles are scalars.) The strength of this coupling is identified with the topological spin induced by the flux attached to the electron. We show that the coupling to the spin connection is essential to reproduce the geometric responses of FQH states. We also derive the effective field theory using the parton construction [30,35], including the geometric responses for general FQH states including non-Abelian states. We get a consistent understanding of two-dimensional Abelian topological orders [10] from the effective field theoretic approaches as well as geometric responses of a more general class of the topological states including non-Abelian FQH states.

We then further show that the *straightforward* application of the composite particle theories and projective parton constructions *fail* to reproduce the correct CS action for the spin connection, i.e., the gravitational CS action, whose coefficient should be equal to the central charge of the Virasoro algebra of the edge states of the fluid on a disk geometry. The gravitational CS term [36–39] reflects the gravitational anomaly [40] of the energy-momentum tensor in topological fluids [41–43]. We find that the gravitational CS term resulting from the field theoretic descriptions predicts the central charge of the mean-field theory used in the descriptions, instead of the correct ground state. For example, the correct central charge of the Laughlin state is $c = 1$ because of a single chiral edge state of the state. However, the composite boson theory predicts the central charge of the Laughlin state to be zero because the composite boson theory describes the state as the (approximately) time-reversal symmetric superfluid state. We will show that the composite fermion theory and the projective parton construction suffer the same problem on predicting the correct central charge. Thus, predicting the gravitational CS term of FQH states using the field theoretic approaches remains an open problem.

This paper is organized as follows. In Sec. II we show that on a curved surface (a manifold) a consistent theory of flux attachment necessarily requires us to take into account the topological spin. Here we derive the form of the resulting Chern-Simons gauge theory which now includes a coupling to the spin connection of the manifold. In Sec. III we use this theory for the case of the composite fermion construction of the Laughlin and Jain states and for the multicomponent Abelian FQH states. Here we derive the effective hydrodynamic theories for each case and show that they now predict the correct value of the Hall viscosity and of the Wen-Zee term in each case. However, we also find that in general the composite particle theories do not predict the correct value of the coefficient of the gravitational Chern-Simons term which should be consistent with the value of the central charge of the theory of the chiral edge states. In Sec. IV we present the equivalent description for the theory of composite bosons and in Sec. V we extend this formulation to the parton construction of Abelian and non-Abelian FQH states. Our conclusions are presented in Sec. VI.

II. FLUX ATTACHMENT AND GEOMETRY

In the descriptions of FQH states, the Chern-Simons (CS) term plays an important role: it binds the flux to the charge and induces the statistical transmutation. Another equally important but less appreciated ingredient from the CS term is the topological spin [44–49]. Formally, the spin can be introduced by defining a local frame attached to the worldline of the charge-flux composite particle. The topological spin counts the winding of the charge around the flux. The spin is related to the self-statistical angle of the composite particle. To be precise, we consider a CS gauge theory minimally coupled to the charge current j^μ,

$$\mathcal{L} = \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - j^\mu a_\mu. \tag{2.1}$$

The content of the CS Lagrangian [Eq. (2.1)] is a charge-flux constraint (the Gauss law of this theory) and canonical commutation relations for the gauge fields [38]. The charge is bound with the flux and is turned into a composite particle with the change in the statistical angle $\theta_{\rm stat} = \frac{\pi}{k}$. Then the spin-statistics connection implies that the composite particle will carry the topological spin $S_z = \frac{\theta_{\rm stat}}{2\pi} = \frac{1}{2k}$. Thus the composite particle carries spin polarized along the z direction.

On a surface with a nonflat metric the topological spin of the composite particle couples to the (Abelian) spin connection with a coupling strength dictated by the topological spin. To demonstrate this, we perform a parallel transport of a composite particle along the curve $C : s \to \boldsymbol{r} = (x_1(s),x_2(s),t(s)) \in \Sigma^2 \times \mathbb{R}$ with its arc length s. We are interested in the adiabatic transport of the particle, i.e., $|\frac{d\boldsymbol{x}}{ds}|^2 \ll |\frac{dt}{ds}|^2$ along C. In the pure CS theory, the amplitude for the transport is given by a Wilson line operator [44–46]

$$\Phi[C] = \left\langle e^{i\int_C A_\mu}\right\rangle = e^{i\theta_{\rm stat}W[C]} = e^{i\theta_{\rm stat}L}e^{-i\theta_{\rm stat}T[C]}, \tag{2.2}$$

where we have introduced the writhing number $W[C]$, defined as $W[C] = L - T[C]$, where L is the linking number and $T[C]$ is the torsion [44–47] (or the twist) of the curve C. Because L is always an integer, it is independent of the background metric. The torsion represents how fast the frame of the curve rotates along C,

$$T[C] = \frac{1}{2\pi}\int d\boldsymbol{r}\cdot\left[\boldsymbol{e}_2 \times \frac{\partial \boldsymbol{e}_2}{\partial s}\right]. \tag{2.3}$$

We have chosen the frame along the curve to be $\boldsymbol{e}_1 = \frac{\partial \boldsymbol{r}}{\partial s}$ and $\boldsymbol{e}_2 \perp \boldsymbol{e}_1$. When the curvature is purely spatial, and in the absence of torsion in Σ^2, we can prove that the phase factor Eq. (2.2) reduces to

$$\Phi[C] = \exp(iL\theta_{\rm stat})\exp\left(-iS_z\int d\boldsymbol{r}\cdot\boldsymbol{\omega}\right). \tag{2.4}$$

To prove Eq. (2.4) from Eqs. (2.3) and (2.2), we rewrite the torsion as

$$\begin{aligned} T &= \frac{1}{2\pi}\int d\boldsymbol{r}\cdot\left(\boldsymbol{e}_2 \times \frac{\partial \boldsymbol{e}_2}{\partial s}\right) \\ &= \frac{1}{2\pi}\int ds\frac{\partial \boldsymbol{r}}{\partial s}\cdot\left(\boldsymbol{e}_2 \times \frac{\partial \boldsymbol{e}_2}{\partial s}\right). \end{aligned} \tag{2.5}$$

We write the vectors appearing in Eq. (2.5) explicitly:

$$\begin{aligned} \boldsymbol{r} &= (x_1(s),x_2(s),t(s)), \quad s \in [s_i,s_f], \\ \boldsymbol{e}_1 &= \frac{\partial \boldsymbol{r}}{\partial t} = (v_1,v_2,\alpha), \quad \boldsymbol{e}_2 = \frac{1}{\sqrt{\boldsymbol{v}^2}}(-v_2,v_1,0), \\ \boldsymbol{v} &= (v_1,v_2,0). \end{aligned} \tag{2.6}$$

Here s is the arc length of C, and thus we have taken $\boldsymbol{e}_1 = \frac{\partial \boldsymbol{r}}{\partial s}$. It is clear that $\boldsymbol{e}_2 \cdot \boldsymbol{e}_1 = 0$ from the expression. As we are interested in the adiabatic transport of the particle, we impose the condition

$$\boldsymbol{v}^2 \ll \alpha^2 \tag{2.7}$$

along the curve C. Then this translates as $\alpha = 1 + O(\frac{v^2}{\alpha^2})$. As the space Σ^2 is curved, we introduce a static local frame on the space:

$$\begin{aligned} \boldsymbol{E}_1 &= (u_1(x_1,x_2),u_2(x_1,x_2),0), \\ \boldsymbol{E}_2 &= (-u_2(x_1,x_2),u_1(x_1,x_2),0), \\ \boldsymbol{E}_3 &= (0,0,1). \end{aligned} \tag{2.8}$$

Below we will suppress the dependence of u_i on $(x_1,x_2,0) \in \Sigma^2$. However, it is important to remember that the frame [Eq. (2.8)] depends on the position because of nonzero curvature in Σ^2. When there is no torsion in Σ^2, the frame follows the equation of motion dictated by the spin connection $\omega_{\mu:ab}$,

$$\partial_\mu E_{a;\nu} - \partial_\nu E_{a;\mu} = -\omega_{\mu;ab} E_{b;\nu} + \omega_{\nu;ab} E_{b;\mu}. \tag{2.9}$$

Because the curvature is solely from the space Σ^2, the only nonzero element of the spin connection is $\omega_{\mu:12} = -\omega_{\mu:21} = \omega_\mu$ (we suppress the Lorentz indices ab in the spin connection from here on). The equation of motion Eq. (2.9) implies that we have the following equations when we are translating the frame along $\frac{\partial \boldsymbol{r}}{\partial s}$:

$$\begin{aligned} \left(\frac{\partial \boldsymbol{r}}{\partial s}\cdot\nabla\right)\boldsymbol{E}_1 &= \left(\frac{\partial \boldsymbol{r}}{\partial s}\cdot\boldsymbol{\omega}\right)\boldsymbol{E}_2, \\ \left(\frac{\partial \boldsymbol{r}}{\partial s}\cdot\nabla\right)\boldsymbol{E}_2 &= -\left(\frac{\partial \boldsymbol{r}}{\partial s}\cdot\boldsymbol{\omega}\right)\boldsymbol{E}_1, \end{aligned} \tag{2.10}$$

with ∇ the covariant derivative. Furthermore, we represent $\boldsymbol{e}_2$ in terms of $\boldsymbol{E}_i, i = 1,2$ by introducing an angle $\phi(s)$:

$$\boldsymbol{e}_2 = \cos(\phi(s))\boldsymbol{E}_1 + \sin(\phi(s))\boldsymbol{E}_2. \tag{2.11}$$

The dependence of $\phi(s)$ on the arc length s represents the relative rotation of the frame $\boldsymbol{e}_i$ of the curve to the frame $\boldsymbol{E}_i$ of the space Σ^2. With these in hand, we can proceed to rewrite the twist Eq. (2.5),

$$\begin{aligned} T &= \frac{1}{2\pi}\int ds \frac{\partial \boldsymbol{r}}{\partial s}\cdot\left(\boldsymbol{e}_2 \times \frac{\partial \boldsymbol{e}_2}{\partial s}\right), \\ &= \frac{1}{2\pi}\int ds \frac{\partial \boldsymbol{r}}{\partial s}\cdot\Bigg\{(\boldsymbol{E}_1\cos\phi + \boldsymbol{E}_2\sin\phi) \\ &\quad\times\Bigg[\frac{\partial\phi}{\partial s}(-\boldsymbol{E}_1\sin\phi + \boldsymbol{E}_2\cos\phi) \\ &\quad+\left(\frac{\partial \boldsymbol{E}_1}{\partial s}\cos\phi + \frac{\partial \boldsymbol{E}_2}{\partial s}\sin\phi\right)\Bigg]\Bigg\}, \\ &= \frac{1}{2\pi}\int ds \frac{\partial \boldsymbol{r}}{\partial s}\cdot[(\boldsymbol{E}_1\cos\phi + \boldsymbol{E}_2\sin\phi) \\ &\quad\times(-\boldsymbol{E}_1\sin\phi + \boldsymbol{E}_2\cos\phi)]\left(\frac{\partial\phi}{\partial s} + \frac{\partial \boldsymbol{r}}{\partial s}\cdot\boldsymbol{\omega}\right), \\ &= \frac{1}{2\pi}\int ds\left(\frac{\partial \boldsymbol{r}}{\partial s}\cdot\boldsymbol{E}_3\right)\left(\frac{\partial\phi}{\partial s} + \frac{\partial \boldsymbol{r}}{\partial s}\cdot\boldsymbol{\omega}\right), \\ &= \frac{1}{2\pi}\int ds\left(\frac{\partial\phi}{\partial s} + \frac{\partial \boldsymbol{r}}{\partial s}\cdot\boldsymbol{\omega}\right) + O\left(\frac{\boldsymbol{v}^2}{\alpha^2}\right), \\ &= \frac{1}{2\pi}[\phi(s_f) - \phi(s_i)] + \frac{1}{2\pi}\int d\boldsymbol{r}\cdot\boldsymbol{\omega}. \end{aligned} \tag{2.12}$$

We have used elementary chain rules and Eq. (2.10) in the second and third lines. Between the fourth line and the fifth line we have used Eqs. (2.6) and (2.8). Then the last line is just rewriting the integral in a way that it is apparently parametrization independent within the approximation Eq. (2.7) (this approximation becomes exact if the transport is performed *infinitely slowly* $\boldsymbol{v}^2 \to 0$). The first term in the last line is nonuniversal and depends only on the boundary condition. The term may be dropped out by imposing a periodic boundary condition at s_i and s_f, i.e., we may impose that the configuration of the frame at $s = s_i$ is the same as that of the frame at $s = s_f$. So we drop it in Eq. (2.4) and from here on.

Thus the covariant derivative of the composite particle should also include the spin connection with coupling strength S_z,

$$D_\mu = \partial_\mu + i a_\mu + i S_z \omega_\mu. \tag{2.13}$$

This is one of the key results in this paper. Notice that this spin connection is Abelian (in contrast to the conventional spin connection of relativistic fermions which is non-Abelian.) The composite fermion (CF) and composite boson (CB) CS theories in literature are restricted to flat space, and hence there is no need to introduce the spin connection explicitly. However, the geometric response involves the deformation of the metric, and it is necessary to keep the spin connection explicitly. We will show that inclusion of the spin connection in the covariant derivative leads to the correct Hall viscosity and Wen-Zee term for Abelian FQH fluids.

III. GEOMETRY IN THE COMPOSITE FERMION THEORY

A. Laughlin and Jain states

We first consider the CF theory [4,5] of a FQH state at the filling $\nu = \frac{1}{2p+1}$ in a curved space. We begin with the action of the nonrelativistic Fermi field Ψ_e describing the dynamics of electrons in two dimensions under an uniform magnetic field,

$$\begin{aligned} S = \int d^3x\sqrt{g}\Bigg\{&\frac{i}{2}[(D_0\Psi_e(x))^\dagger\Psi_e - \Psi_e^\dagger(x)(D_0\Psi_e(x))] \\ &-\frac{1}{2}(D_i\Psi_e(x))^\dagger g^{ij}(D_j\Psi_e(x))\Bigg\} + S_{\text{int}}, \end{aligned} \tag{3.1}$$

in which $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative of the electron, and we set the effective mass m_e and the charge of electron to be unity. The electron is a scalar field, and thus does not couple minimally with the spin connection. S_{int} encodes the short-ranged repulsive density-density interaction between electrons. The interaction term will not affect the Hall viscosity [50], and so it can be ignored from here on. The electromagnetic gauge field A_μ can be written as $A_\mu = \bar{A}_\mu + \delta A_\mu$, where $\bar{A}_\mu$ is the uniform magnetic field and δA_μ is a probe field that measures the electromagnetic response of the FQH state.

The fermion Chern-Simons field theory of the FQH states [5,9] consists of attaching an even number of flux quanta to each electron by formally coupling the theory of Eq. (3.1) to an Abelian (statistical) gauge field a_μ whose Lagrangian has the CS form. The resulting action in terms of the composite fermion Ψ is

$$S = \int d^3x\sqrt{g}\left\{\frac{i}{2}[(D_0\Psi(x))^\dagger\Psi - \Psi^\dagger(x)(D_0\Psi(x))] - \frac{1}{2}(D_i\Psi(x))^\dagger g^{ij}(D_j\Psi(x)) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi p}a_\mu\partial_\nu a_\lambda\right\}, \quad (3.2)$$

where $D_\mu = \partial_\mu + iA_\mu + ia_\mu + ip\omega_\mu$ is a covariant derivative which, in addition to the minimal coupling to the statistical gauge field a_μ, includes the minimal coupling to the spin connection ω_μ. The CS term binds the $2p$ flux quanta to the electron Ψ_e and turns the electron into the CF Ψ [4]. Notice that the spin connection enters explicitly in the covariant derivative with a topological spin $p \in \mathbb{Z}$ reflecting the statistical angle $\theta_{\text{stat}} = 2\pi p$.

The FQH state is described in this CF picture [4,5] by noting that if we attached $2p$ flux quanta to each fermion, on average the external vector potential $\bar{A}_j$ is partially screened to $\bar{A}_j + \bar{a}_j = \frac{1}{2p+1}\bar{A}_j$. Thus the CF Ψ is subject to the magnetic field which is $\frac{1}{2p+1}$ of the magnetic field experienced by the bare electron. Then for a system with filling $\nu = \frac{1}{2p+1}$ the CF fill up the lowest effective Landau level [4]. The FQH effect is then obtained by integrating out the CF fluctuations at the one-loop level.

Next we note that integrating out the CF fluctuations at $\nu = \frac{1}{2p+1}$ is formally equivalent to integrating out the electron fluctuations in the integer quantum Hall fluid at $\nu = 1$, i.e., the theory Eq. (3.1) at the filling $\nu = 1$, which is done in Refs. [19,50]. For the integer quantum Hall state, we obtain the effective theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{topo}}$ of the electromagnetic probe δA_μ, and of the metric δg_{ij},

$$\mathcal{L}_0 = \delta A_0\bar{\rho} + \bar{\rho}s_0\omega_0, \quad (3.3)$$

$$\mathcal{L}_{\text{topo}} = \varepsilon^{\mu\nu\lambda}\left[\frac{1}{4\pi}\delta A_\mu\partial_\nu\delta A_\lambda + \frac{s_0}{2\pi}\delta A_\mu\partial_\nu\omega_\lambda + \frac{1}{24\pi}\omega_\mu\partial_\nu\omega_\lambda\right], \quad (3.4)$$

in which $s_0 = \frac{1}{2}$, the average orbital spin of the integer quantum Hall state. Here we have $\omega_i = -\frac{1}{2}\varepsilon^{jk}\partial_j\delta g_{ik}$ and $\omega_t = \frac{1}{2}\varepsilon^{jk}\delta g_{ij}\partial_t\delta g_{ik}$ [19,20,22]. The second term in $\mathcal{L}_0$ is the Berry phase term of the effective action which accounts for the Hall viscosity, and the second term in $\mathcal{L}_{\text{topo}}$ is the Wen-Zee term. The last term in $\mathcal{L}_{\text{topo}}$ is the gravitational CS term [19] (see below).

Having the effective theory Eq. (3.4) in hand, we can easily obtain the effective theory of the fluctuating component δa_μ of the statistical gauge field, of the electromagnetic probe δA_μ, and of the metric δg_{ij} in the FQH state from Eq. (3.4). We replace δA_μ in Eq. (3.4) with $\delta A_\mu + \delta a_\mu + p\omega_\mu$ to obtain that of the FQH state because the CF field minimally couples to $\delta A_\mu + \delta a_\mu + p\omega_\mu$. Then the resulting effective Lagrangian to lowest orders in a gradient expansion again has the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{topo}}$, where [19,50]

$$\mathcal{L}_0 = (\delta A_0 + p\omega_0)\bar{\rho} + \bar{\rho}s_0\omega_0,$$

$$\mathcal{L}_{\text{topo}} = \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}(\delta A_\mu + \delta a_\mu + p\omega_\mu)\partial_\nu(\delta A_\lambda + \delta a_\lambda + p\omega_\lambda) + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}(\delta A_\mu + \delta a_\mu + p\omega_\mu)\partial_\nu\omega_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{24\pi}\omega_\mu\partial_\nu\omega_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi p}\delta a_\mu\partial_\nu\delta a_\lambda, \quad (3.5)$$

where $s_0 = 1/2$ is the orbital spin of a system of fermions at $\nu = 1$, and $\bar{\rho}$ is the electron density. The last term in $\mathcal{L}_{\text{topo}}$, the CS term of δa_μ, comes from the CS term responsible for flux attachment.

In the above discussion we did not include explicitly the short-ranged repulsive density-density interaction. Although the interactions are obviously necessary to stabilize the FQH state, in the effective action of the excitations their contribution interaction enters only in the Maxwell term of the statistical gauge field, which is nontopological and subleading to the Chern-Simons terms. Thus the universal geometric response does not depend explicitly in the form of the interactions.

1. Hydrodynamic theory: Hall viscosity and Wen-Zee term

We can further transform the effective theory of Eq. (3.5) into the hydrodynamic theory of the FQH state [6,7]. To this end we introduce the hydrodynamic field b_μ, rewrite the last term in Eq. (3.5) as

$$\frac{\varepsilon^{\mu\nu\lambda}}{8\pi p}\delta a_\mu\partial_\nu\delta a_\lambda \to -\frac{2p}{4\pi}\varepsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}b_\mu\partial_\nu\delta a_\lambda, \quad (3.6)$$

and integrate out the fluctuation δa_μ from Eq. (3.5) to obtain the hydrodynamic theory for the FQH state [6,7], which now also includes the coupling to a curved space

$$\mathcal{L} = +\bar{\rho}\delta A_0 + \frac{2p+1}{2}\bar{\rho}\omega_0 - \frac{2p+1}{4\pi}\varepsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda - \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}b_\mu\partial_\nu\delta A_\lambda - \frac{2p+1}{2}\frac{\varepsilon^{\mu\nu\lambda}}{2\pi}b_\mu\partial_\nu\omega_\lambda - \frac{\varepsilon^{\mu\nu\lambda}}{48\pi}\omega_\mu\partial_\nu\omega_\lambda. \quad (3.7)$$

This effective Lagrangian has the same form as the hydrodynamic theory [6]. However, Eq. (3.7) also includes the Berry phase term of the Hall viscosity (second term of the first line), and the Wen-Zee term [11] (second term of the second line). The last term has the form of gravitational CS terms [36] and will be discussed below.

The Hall viscosity of the FQH state is obtained by varying $\mathcal{L}$ in Eq. (3.7) with respect to the metric δg_{ij}. We find

$$\eta_H = s\frac{\bar{\rho}}{2} = \frac{2p+1}{2}\frac{\bar{\rho}}{2}, \quad (3.8)$$

which agrees with previous results obtained by other arguments [12,13,15,24]. Here s is the intrinsic orbital spin $s = \frac{2p+1}{2}$.

2. The gravitational Chern-Simons term

We can further identify the central charge of the edge states of this FQH fluid by reading off the coefficient of the gravitational CS term $-\frac{c}{48\pi}\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu\omega_\lambda$ in Eq. (3.7) and find $c = 1$. The gravitational CS term embodies the gravitational anomaly of the energy-momentum tensor in topological fluids [41–43]. The response to the effective action to a change of the metric (and hence of the spin connection) to the expectation value of the energy-momentum tensor [51]. Given the (holographic) correspondence between the bulk and the edge states of quantum Hall fluids [6] (reflecting the holographic nature of Chern-Simons gauge theory [38,52]), the central charge derived from the gravitational CS term should be the same as the central charge of the theory of the edge states, determined by the level of the CS term for the hydrodynamic gauge fields.

At this stage one might conclude that the CF theory can be used to predict the correct central charge of the FQH state. However, this is the *artifact* of the mean-field state that we choose to study. To see this we consider another legitimate CF construction for the Laughlin state at $\nu = \frac{1}{2p+1}$ in which we attach $(2p+2)$ flux quanta to electron Ψ_e in Eq. (3.1). Then the resulting CF sees *on average* one flux quanta, which is directed opposite from the direction of the external magnetic flux, per particle. Thus we choose the mean-field state in which the CF is in the $\nu = -1$ state, in which the CF fills up the lowest Landau level, i.e., $\bar{A}_j + \bar{a}_j = -\frac{1}{2p+1}\bar{A}_j, j = x,y$. The state has the opposite chirality from the integer quantum Hall state at $\nu = 1$.

We can now follow the same steps that we used to derive the effective hydrodynamic theory. The resulting effective action, i.e., the analog of Eq. (3.7), now is

$$\mathcal{L} = +\bar{\rho}\delta A_0 + \frac{2p+1}{2}\bar{\rho}\omega_0 - \frac{2p+1}{4\pi}\varepsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda - \frac{\varepsilon^{\mu\nu\lambda}}{2\pi} b_\mu \partial_\nu \delta A_\lambda - \frac{2p+1}{2}\frac{\varepsilon^{\mu\nu\lambda}}{2\pi} b_\mu \partial_\nu \omega_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{48\pi}\omega_\mu \partial_\nu \omega_\lambda. \tag{3.9}$$

Compared to Eq. (3.7), we see that only the last term has the *wrong* coefficient $c = -1$, which is the central charge of the mean-field state! This implies that the correct central charge in Eq. (3.7) is an artifact of the mean-field state which accidentally has the same central charge as the Laughlin state. Except the coefficient of the gravitational CS term, i.e., central charge, the Hall viscosity and the Wen-Zee term are, here too, correctly reproduced and are independent of the mean-field state of the CF theory. We will see that the CB theory and the projective parton constructions suffer from the same problem and they predict the central charge to be that of the mean-field states, which in general is not the correct central charge of the system.

B. Multicomponent Abelian FQH states

Having the descriptions Eqs. (3.7) and (3.5) of the Laughlin states in hand, we can proceed to study the bilayer quantum Hall state from the composite fermion theory (where repeated indices are summed over):

$$\mathcal{L} = \sqrt{g}\Big\{\frac{i}{2}\big[\big(D_0^a\Psi_a(x)\big)^\dagger\Psi_a - \Psi_a^\dagger(x)\big(D_0^a\Psi_a(x)\big)\big] - \frac{1}{2}\big(D_i^a\Psi_a(x)\big)^\dagger g^{ij}\big(D_j^a\Psi_a(x)\big)\Big\} + \mathcal{L}_{\rm CS}, \tag{3.10}$$

in which $D_\mu^a = \partial_\mu + iA_\mu^a + i\alpha_\mu^a + ip_a\omega_\mu$ is the covariant derivative of the composite fermions in the layer $a = 1,2$. As in the single layer case, the electrons are attached with fluxes of the statistical gauge fields α_μ^a and turned into the composite fermions. Here too, to simplify the notation, we do not include the interaction terms explicitly since they do not affect the topological structure. Naturally the interactions are crucial to stabilize the FQH state and these particles are not free but strongly interacting.

The CS term $\mathcal{L}_{\rm CS}$ in (3.10) is

$$\mathcal{L}_{\rm CS} = \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\alpha_\mu^a[K^{-1}]^{ab}\partial_\nu\alpha_\lambda^b, \quad K = \begin{pmatrix} 2p_1 & n \\ n & 2p_2 \end{pmatrix}, \tag{3.11}$$

and where $a,b = 1,2$, and p_1, p_2, and n are arbitrary integers. Notice that the spins of the composite fermions depend only on the change in the self-statistical angle $\theta_a = 2\pi p_a$ and thus the fermions couple to the spin connection with the strength of p_a, i.e., the diagonal elements of K matrix in $\mathcal{L}_{\rm CS}$.

We smear out the fluxes of the statistical gauge fields into space and assume that the composite fermions are at $\nu^a = 1, a = 1,2$ (the generalization to the other values of $\nu^a \in \mathbb{Z}$ is straightforward). We integrate out the composite fermions and expand the resulting effective Lagrangian $\mathcal{L}$ in terms of the perturbations $\{\delta\alpha_\mu^a, \delta A_\mu^a, \delta g^{ij}\}$ around their mean values

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\rm topo} + \cdots, \\ \mathcal{L}_0 &= (\delta A^a + p_a\omega_t)\bar{\rho}^a + \frac{1}{2}\bar{\rho}^a\omega_t, \\ \mathcal{L}_{\rm topo} &= \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\delta\kappa_\mu^a\partial_\nu\delta\kappa_\mu^a + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\delta\kappa_\mu^a\partial_\nu\omega_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{24\pi}\omega_\mu\partial_\nu\omega_\lambda \\ &\quad + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\delta\alpha_\mu^a[K^{-1}]^{ab}\partial_\nu\delta\alpha_\lambda^b, \end{aligned} \tag{3.12}$$

in which $\delta\kappa_\mu^a = \delta A_\mu^a + \delta\alpha_\mu^a + p_a\omega_\mu$. We can transform this effective theory Eq. (3.12) into the hydrodynamic description by rewriting the last term of $\mathcal{L}_{\rm topo}$ in Eq. (3.12) in terms of the hydrodynamics fields β_μ^a:

$$\begin{aligned} \mathcal{L}_{\rm CS} &= \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\delta\alpha_\mu^a[K^{-1}]^{ab}\partial_\nu\delta\alpha_\lambda^b \\ &\to -\frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\beta_\mu^a K^{ab}\partial_\nu\beta_\lambda^b + \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}\beta_\mu^a\partial_\nu\delta\alpha_\lambda^a. \end{aligned} \tag{3.13}$$

With this result in hand, we integrate out $\delta\alpha_\mu^a$ from (3.12) to find

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\rm topo} + \cdots, \\ \mathcal{L}_0 &= \bar{\rho}^a\delta A_t^a + \left(p_a + \frac{1}{2}\right)\bar{\rho}^a\omega_t, \\ \mathcal{L}_{\rm topo} &= -\frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\beta_\mu^a\widetilde{K}^{ab}\partial_\nu\beta_\lambda^b + -\frac{\varepsilon^{\mu\nu\lambda}}{2\pi}\beta_\mu^a\partial_\nu\delta A_\lambda^a \\ &\quad - \frac{p_a + \frac{1}{2}}{2\pi}\varepsilon^{\mu\nu\lambda}\beta_\mu^a\partial_\nu\omega_\lambda - \frac{\varepsilon^{\mu\nu\lambda}}{48\pi}\omega_\mu\partial_\nu\omega_\lambda, \end{aligned} \tag{3.14}$$

where

$$\widetilde{K} = \begin{pmatrix} 2p_1+1 & n \\ n & 2p_2+1 \end{pmatrix}. \tag{3.15}$$

The Hall viscosity of this bilayer system is

$$\eta_H = \sum_{a=1,2}\left(p_a + \frac{1}{2}\right)\frac{\bar{\rho}^a}{2}, \tag{3.16}$$

in agreement with Refs. [20,24]. Furthermore, these results yield the correct value of the Wen-Zee term in Eq. (3.14) with the correct spin for the bilayer system [6,11]. Thus, as in the CF description of the Laughlin states, the Hall viscosity and the Wen-Zee term are however correctly reproduced, independent of the choice of the mean-field states. Finally, it is straightforward to generalize the theory present here to any Abelian multicomponent FQH states.

Finally, here too, from the coefficient of the gravitational CS term $-\frac{c}{48\pi}\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu\omega_\lambda$ in (3.14), we infer that the chiral central charge of the theory is $c=2$. However, just as in the case of the Laughlin and Jain states, this result is again an artifact of the mean-field states, which accidentally have the same central charge as the physical states that we are studying.

IV. GEOMETRY IN THE COMPOSITE BOSON THEORY

A. Laughlin states

We can do the same analysis in the CB theory [2,29], again for a FQH state with filling fraction $\nu = 1/(2p+1)$. The main difference with the CF theory is that the theory of fermions in a magnetic field is now mapped onto a system of with a Bose field Φ coupled to the CS theory:

$$S = \int d^3x\sqrt{g}\left\{\frac{i}{2}[(D_0\Phi(x))^\dagger\Phi - \Phi^\dagger(x)(D_0\Phi(x))] - \frac{1}{2}(D_i\Phi(x))^\dagger g^{ij}(D_j\Phi(x)) + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi(2p+1)}a_\mu\partial_\nu a_\lambda\right\}, \tag{4.1}$$

in which $D_\mu = \partial_\mu + iA_\mu + ia_\mu + i\frac{2p+1}{2}\omega_\mu$ is the covariant derivative of the CB on a curved manifold [22]. We can perform the standard dual transformation [2] of the CB theory on the action Eq. (4.1). We start by rewriting Eq. (4.1) as

$$\mathcal{L} = \sqrt{g}\left\{\frac{i}{2}[(D_0\Phi(x))^\dagger\Phi - \Phi^\dagger(x)(D_0\Phi(x))] - \frac{1}{2}(D_i\Phi(x))^\dagger g^{ij}(D_j\Phi(x)) + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi(2p+1)}a_\mu\nabla_\nu a_\lambda\right\}. \tag{4.2}$$

Here the CS term is written in a way that it is explicitly invariant under general coordinate transformation by using the covariant derivative ∇_ν,

$$\varepsilon^{\mu\nu\lambda}\nabla_\nu a_\lambda = \varepsilon^{\mu\nu\lambda}\left(\partial_\nu a_\lambda + \Gamma^\sigma_{\nu\lambda}a_\sigma\right) = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda, \tag{4.3}$$

because of the property of the Christoffel symbol $\Gamma^\sigma_{\nu\lambda} = \Gamma^\sigma_{\lambda\nu}$. The Levi-Civita tensor is normalized as $\varepsilon^{txy} = \frac{1}{\sqrt{g}}$. In the composite boson theory, the FQH state of the electron corresponds to the superfluid state of the boson Φ. In the superfluid phase, the average $\bar{A}_\mu$ of the electromagnetic gauge field $A_\mu = \bar{A}_\mu + \delta A_\mu$ is completely canceled by the average $\bar{a}_\mu$ of the statistical gauge field $a_\mu = \bar{a}_\mu + \delta a_\mu$, i.e., $\bar{A}_i + \bar{a}_i = 0, i = x,y$. On the other hand, the average density of the boson is locked with the average magnetic field due to the quantum Hall effect:

$$\langle\Phi^\dagger\Phi\rangle = \bar{\rho} = \frac{1}{2\pi k}\varepsilon^{tij}\nabla_i\bar{A}_j = -\frac{1}{2\pi k}\varepsilon^{tij}\nabla_i\bar{a}_j. \tag{4.4}$$

We can write down the low-energy Lagrangian for the superfluid by expanding $\Phi = \sqrt{\bar{\rho}+\delta\rho}\,e^{i\theta}$ in terms of $\delta\rho$ and θ,

$$\mathcal{L} = \sqrt{g}\left[(\partial_t\theta + \delta\alpha_t)\bar{\rho} + (\partial_t\theta + \delta\alpha_t + \delta a_t)\delta\rho - \frac{\bar{\rho}g^{ij}}{2}(\partial_i\theta + \delta\alpha_i + \delta a_i)(\partial_j\theta + \delta\alpha_j + \delta a_j) + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi(2p+1)}\delta a_\mu\nabla_\nu\delta a_\lambda\right], \tag{4.5}$$

with $\delta\alpha_\mu = \delta A_\mu + \frac{2p+1}{2}\omega_\mu$. Here the first term $\sim\sqrt{g}\bar{\rho}\partial_t\theta$ on the right-hand side can be gauged away. We introduce a Hubbard-Stratonovich field J^i to rewrite the kinetic term of the composite boson,

$$\sqrt{g}\frac{\bar{\rho}g^{ij}}{2}(\partial_i\theta + \delta\alpha_i + \delta a_i)(\partial_j\theta + \delta\alpha_j + \delta a_j) \to \sqrt{g}\left[(\partial_i\theta + \delta\alpha_i + \delta a_i)g^{ij}J_j - \frac{1}{2\bar{\rho}}J_i g^{ij}J_j\right]. \tag{4.6}$$

With these in hand we have

$$\mathcal{L} = \sqrt{g}\left[\bar{\rho}\delta\alpha_t + (\partial_\mu\theta + \delta\alpha_\mu + \delta a_\mu)J^\mu + \frac{1}{2\bar{\rho}}J_i g^{ij}J_j\right] + \sqrt{g}\frac{\varepsilon^{\mu\nu\lambda}}{4\pi(2p+1)}\delta a_\mu\nabla_\nu\delta a_\lambda, \tag{4.7}$$

where $J^\mu = (\delta\rho, -J^i)$ represents the conserved boson current. In the absence of the vortex excitation, we can integrate out the phase variable $\theta\in\mathbb{R}$ to obtain (but inclusion of the vortex can be done easily)

$$\partial_\mu(\sqrt{g}J^\mu) = \sqrt{g}\nabla_\mu J^\mu = 0 \to J^\mu = \varepsilon^{\mu\nu\lambda}\frac{1}{2\pi}\nabla_\nu b_\lambda, \tag{4.8}$$

in which a hydrodynamic (gauge) field b_μ is introduced to solve the equation of motion. By plugging this back to the Lagrangian Eq. (4.7), we find

$$\mathcal{L} = \sqrt{g}\left[\bar{\rho}\delta\alpha_t + \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}(\delta\alpha_\mu + \delta a_\mu)\nabla_\nu b_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi(2p+1)}\delta a_\mu\nabla_\nu\delta a_\lambda - \frac{1}{2\bar{\rho}}e_i g^{ij}e_j\right]. \tag{4.9}$$

Here $e_i = \frac{1}{2\pi}\varepsilon^{i\sigma\lambda}\nabla_\sigma b_\lambda, i = x,y$ is the electric field of b_μ. We integrate out δa_μ and obtain the effective action for the FQH state in the curved space

$$\mathcal{L} = \sqrt{g}\left[\bar{\rho}\delta\alpha_t + \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\delta\alpha_\mu\nabla_\nu b_\lambda - \frac{2p+1}{4\pi}\varepsilon^{\mu\nu\lambda}b_\mu\nabla_\nu b_\lambda - \frac{1}{2\bar{\rho}}e_i g^{ij}e_j\right]. \tag{4.10}$$

Expanding this effective theory to the leading order of δg_{ij} and gauge fields we find

$$\mathcal{L} = \delta A_0\bar{\rho} + \frac{2p+1}{2}\omega_0\bar{\rho} - \frac{2p+1}{4\pi}\varepsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\delta A_\mu\partial_\nu b_\lambda + \frac{2p+1}{4\pi}\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu b_\lambda + \cdots. \tag{4.11}$$

The Hall viscosity and the Wen-Zee term (with the correct orbital spin) of the FQH state are correctly reproduced here. Finally, in our analysis of the composite boson theory we did not include explicitly the short-ranged repulsive density-density interaction just as in the composite fermion picture and for the same reasons. Here too, the interaction, which is crucial for the existence of the FQH states, does not affect the value of the Hall viscosity and of the Wen-Zee term.

However, the expected gravitational CS term [19,20] is apparently absent in the boson theory. Naively this happens because the mean-field state of the CB theory is the time-reversal-invariant superfluid phase. Time-reversal symmetry is unbroken at the mean-field level since the external magnetic field is exactly canceled by the flux of the statistical gauge field. In this picture the breaking of time-reversal invariance enters only through the Chern-Simons term in the effective action.

B. Multicomponent FQH states

As in the CF theory case, we can proceed to describe the bilayer FQH state by the composite boson theory. We have the two species of the Bose fields $\Phi_a, a = 1,2$ (again with repeated indices being summed over)

$$\mathcal{L} = \sqrt{g}\frac{i}{2}\left[\left(D_0^a\Phi_a(x)\right)^\dagger\Phi_a - \Phi_a^\dagger(x)\left(D_0^a\Phi_a(x)\right)\right] - \frac{1}{2}\left(D_i^a\Phi_a(x)\right)^\dagger g^{ij}\left(D_j^a\Phi_a(x)\right) + \mathcal{L}_{\rm CS}, \tag{4.12}$$

in which $D_\mu^a = \partial_\mu + iA_\mu^a + i\alpha_\mu^a + i(p_a + \frac{1}{2})\omega_\mu$ is the covariant derivative of the composite bosons Φ_a in the layer $a = 1,2$. The CS term $\mathcal{L}_{\rm CS}$ in (4.12) is

$$\mathcal{L}_{\rm CS} = \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\alpha_\mu^a[K^{-1}]^{ab}\partial_\nu\alpha_\lambda^b, \quad K = \begin{pmatrix} 2p_1+1 & n \\ n & 2p_2+1\end{pmatrix}, \tag{4.13}$$

where $a,b = 1,2$ and p_1, p_2, and n are arbitrary integers. Notice that the spins of the composite bosons depend only on the change in the self-statistical angle $\theta_a = 2\pi(p_a + \frac{1}{2})$ and thus the bosons couple to the spin connection with the strength of $p_a + \frac{1}{2}$, i.e., the diagonal elements of K matrix in $\mathcal{L}_{\rm CS}$. Then, the FQH state corresponds to the superfluid state of the boson $\Phi_a, a = 1,2$. By performing the dual transformation we find

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\rm topo} + \cdots,\\ \mathcal{L}_0 &= \bar\rho^a\delta A_t^a + \left(p_a + \frac{1}{2}\right)\bar\rho^a\omega_t,\\ \mathcal{L}_{\rm topo} &= -\frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\beta_\mu^a K^{ab}\partial_\nu\beta_\lambda^b + \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}\beta_\mu^a\partial_\nu\delta A_\lambda^a\\ &\quad + \frac{p_a + \frac{1}{2}}{2\pi}\varepsilon^{\mu\nu\lambda}\beta_\mu^a\partial_\nu\omega_\lambda,\end{aligned} \tag{4.14}$$

with the same K matrix in the first term of $\mathcal{L}_{\rm topo}$ appearing in flux attachment Eq. (4.13). The Hall viscosity and Wen-Zee term are reproduced correctly here [20,24]. It is straightforward to generalize to the other Abelian multicomponent FQH states.

V. PROJECTIVE PARTON CONSTRUCTIONS

We will now discuss the Hall viscosity and geometric responses of Abelian and non-Abelian FQH states using the projective parton construction of Refs. [30,35]. In this picture the electron is formally split into several "partons," each with a certain preassigned charge and all coupled to the same uniform magnetic field. This formal enlargement of the Hilbert space leads to a new local gauge symmetry. The action of the associated gauge fields projects the Hilbert space into the physical subspace of the original fermions. This procedure yields a correct effective theory in all cases [30,35] but, as we will see below, has some open issues in the non-Abelian case.

A. Abelian states

We begin with the projective parton description of the Laughlin state $\nu = \frac{1}{2p+1}, p \in \mathbb{Z}$. In this construction the electron operator factorizes into $2p+1$ fermionic partons [30,53]

$$\Psi_e(z) = \psi_1(z)\psi_2(z)\cdots\psi_{2p+1}(z), \tag{5.1}$$

which is a singlet under a local SU$(2p+1)$ gauge symmetry. These "emergent" gauge symmetries are characteristic of parton constructions. Each parton ψ_i carries the fractional electric charge $e/(2p+1)$ and fills up a lowest Landau level. Notice that the electron and the partons are all scalars and thus do not couple with the spin connection minimally. We also need to introduce $2p$ internal $U(1)$ gauge fields [or a SU$(2p+1)$ gauge field] to project out the nonphysical states in the Hilbert space spanned by the partons of Eq. (5.1) [30]. We assume that the partons see the same background metric as the electron. As each parton is in the $\nu = 1$ state and is gapped, we integrate out the partons to express the result in terms of a hydrodynamic theory of the Laughlin state. The resulting theory is identical to those of the composite particle theories, e.g., Eq. (3.7), except the gravitational CS term. Hence we find that the correct Hall viscosity and Wen-Zee term are reproduced in the projective parton approach, but the central charge is overestimated as $c = 2p + 1$.

As a concrete example of this, we study the bosonic Laughlin state at $\nu = \frac{1}{2}$. For this state, we fractionalize a bosonic field b into the two fermionic partons $\psi_i, i = 1,2$ carrying $\frac{1}{2}$ electric charge:

$$b(z) = \psi_1(z)\psi_2(z). \tag{5.2}$$

The Hilbert space of the partons ψ_i has unphysical states, and we need to project out those unphysical states by requiring that $\rho_b = \langle b^\dagger b\rangle$ and $\rho_j^\psi = \langle\psi_j^\dagger\psi_j\rangle, j = 1,2$ are the same, i.e., $\rho_b = \rho_j^\psi, j = 1,2$. This projection can be implemented by introducing an internal $U(1)$ gauge field a_μ. Under the $U(1)$ gauge field [30,53] ψ_1 and ψ_2 are oppositely charged because the fundamental boson b should be invariant under the $U(1)$ gauge transformation. To describe the Laughlin state, we choose the mean field state where the fermionic partons ψ_i are in $\nu = 1$ state. Furthermore, the partons are scalars and thus do not minimally couple with the spin connection:

$$\mathcal{L} = \sum_{j=1}^{2} \sqrt{g}\left\{\frac{i}{2}\left[\left(D_0^j\psi_j(x)\right)^\dagger\Psi_j - \Psi_j^\dagger(x)\left(D_0^j\Psi_j(x)\right)\right] - \frac{1}{2}\left(D_a^j\Psi_j(x)\right)^\dagger g^{ab}\left(D_b^j\Psi_j(x)\right)\right\}, \tag{5.3}$$

in which $D_\mu^j = \partial_\mu + i\frac{1}{2}A_\mu \pm ia_\mu$ are the covariant derivatives of the fermionic partons $\psi_j, j = 1,2$ ($+ia_\mu$ for ψ_1 and $-ia_\mu$ for ψ_2). We integrate out the partons and obtain the effective theory

$$\mathcal{L} = \rho_b\omega_t + \rho_b A_t + \frac{2}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{1}{2}\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda} \times (A_\mu + \bar{s}\omega_\mu)\partial_\nu(A_\lambda + \bar{s}\omega_\lambda) + \cdots, \tag{5.4}$$

with the average orbital spin $\bar{s} = 1$. The effective theory is obtained by replacing δA_μ in Eq. (3.4) (the effective theory of the integer quantum Hall fluid) by $\frac{1}{2}A_\mu \pm a_\mu$ because each parton is at the filling $\nu = 1$ and minimally couples to $\frac{1}{2}A_\mu \pm a_\mu$. The average orbital spin can be deduced from the coefficients of the mutual CS term between A_μ and ω_μ. We also find the correct Hall viscosity from the effective action.

$$\eta_H = \frac{\rho_b}{2}. \tag{5.5}$$

This is consistent with the average orbital spin $s = 1$.

In fact, it is better to recast Eq. (5.4) into the following form which is more amenable to be turned into the hydrodynamic description:

$$\mathcal{L} = \rho_b\omega_t + \rho_b A_t + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\alpha_\mu\partial_\nu\alpha_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\beta_\mu\partial_\nu\beta_\lambda - \frac{2}{48\pi}\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu\omega_\lambda + \cdots, \tag{5.6}$$

in which $\alpha_\mu = \frac{1}{2}A_\mu + \frac{1}{2}\omega_\mu + a_\mu$ and $\beta_\mu = \frac{1}{2}A_\mu + \frac{1}{2}\omega_\mu - a_\mu$. Then we introduce the *two* hydrodynamic fields $b_{1,\mu}$ and $b_{2,\mu}$ to rewrite the CS terms of α_μ and β_μ appearing in Eq. (5.6):

$$\mathcal{L} = \rho_b\omega_t + \rho_b A_t - \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}b_{1,\mu}\partial_\nu b_{1,\lambda} - \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}b_{2,\mu}\partial_\nu b_{2,\lambda} + \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}b_{1,\mu}\partial_\nu\alpha_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}b_{2,\mu}\partial_\nu\beta_\lambda - \frac{2}{48\pi}\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu\omega_\lambda. \tag{5.7}$$

We integrate out a_μ and obtain the equation of motion $b_{1,\mu} = b_{2,\mu}$. We set $b_\mu = b_{1,\mu} = b_{2,\mu}$ and then (5.7) becomes

$$\mathcal{L} = \rho_b\omega_t + \rho_b A_t - \frac{2}{4\pi}\varepsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{2\pi}b_\mu\partial_\nu A_\lambda + \frac{2}{4\pi}\varepsilon^{\mu\nu\lambda}b_\mu\partial_\nu\omega_\lambda - \frac{2}{48\pi}\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu\omega_\lambda, \tag{5.8}$$

where we notice that the Hall viscosity and Wen-Zee term are correctly captured in this projective parton approach. However, it yields the wrong central charge 2, which is doubly larger than the correct value.

Now, for the general Laughlin state at $\nu = \frac{1}{k}, k \in \mathbb{Z}$, we start with the definition for the fundamental particle with k partons carrying the electromagnetic charge $\frac{1}{k}$:

$$\Psi_e(z) = \psi_1(z)\psi_2(z)\cdots\psi_k(z). \tag{5.9}$$

To describe the Laughlin state, each parton should be at the filling $\nu = 1$. This mean-field ansatz and the fundamental particle are invariant under the $(k-1)$ internal $U(1)$ gauge fields $(a_{1,\mu}, a_{2,\mu}, \ldots, a_{k-1,\mu})$. We choose the coupling between the gauge fields and the partons in the way that the jth parton ψ_j $(1 < j < k)$ couples minimally to $\alpha_{j,\mu} = a_{j-1,\mu} - a_{j,\mu}$, and the first parton ψ_1 (the last parton ψ_k) couples only to $\alpha_{1,\mu} = -a_{1,\mu}$ $(\alpha_{k-1,\mu} = a_{k-1,\mu})$. It is convenient to introduce another set of the gauge fields $(\beta_{1,\mu}, \beta_{2,\mu}, \ldots, \beta_{k,\mu})$ such that

$$\beta_{j,\mu} = \alpha_{j,\mu} + \frac{1}{k}A_\mu + \frac{1}{2}\omega_\mu. \tag{5.10}$$

We integrate out the partons to obtain the effective response theory,

$$\mathcal{L} = \sum_{i=1}^{k}\left(\rho_i\frac{1}{k}A_t + \frac{1}{2}\rho_i\omega_t\right) + \sum_{i=1}^{k}\frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\beta_{i,\mu}\partial_\nu\beta_{i,\lambda}. \tag{5.11}$$

Then we introduce the hydrodynamic fields $b_{i,\mu}$ to rewrite the CS terms:

$$\mathcal{L} = \sum_{i=1}^{k}\left(\rho_i\frac{1}{k}A_t + \frac{1}{2}\rho_i\omega_t\right) - \sum_{i=1}^{k}\frac{\varepsilon^{\mu\nu\lambda}}{4\pi}b_{i,\mu}\partial_\nu b_{i,\lambda} + \sum_{i=1}^{k}\frac{\varepsilon^{\mu\nu\lambda}}{2\pi}b_{i,\mu}\partial_\nu\beta_{i,\lambda}. \tag{5.12}$$

Now we integrate out $a_{j,\mu}, j = 1,2,\ldots,k-1$ and this generates the equation of motion $b_\mu = b_{i,\mu}, i = 1,2,\ldots,k$. Then we find the same effective hydrodynamic response theory as that of the composite fermion Eq. (3.7) in the main text except the overestimate of the chiral central charge by k times, i.e., we will find a wrong central charge $c = k$ for the Laughlin state instead of the correct value $c = 1$. Notice that k is the central charge of the mean-field state. Hence the projective parton construction predicts a wrong central charge, which is of the mean-field state.

B. Non-Abelian states

Wen [30] (and Barkeshli and Wen [35]) generalized the parton construction for the non-Abelian $\mathbb{Z}_k$ Read-Rezayi parafermion states [28] (including the $k = 2$ fermionic and bosonic pfaffian states) at filling $\nu = \frac{k}{Mk+2}$. The fundamental particle Ψ_e now is

$$\Psi_e(z) = \psi_1\psi_2\cdots\psi_M\sum_{a=1}^{k} f_{2a-1}f_{2a}. \tag{5.13}$$

Here $\psi_i, i = 1,2,\ldots,M$ carries electric charge $q_\psi = \frac{k}{Mk+2}$, and $f_a, a = 1,\ldots,2k$ carries electric charge $q_f = \frac{1}{Mk+2}$. Thus we introduce the electromagnetic charge matrix

$$Q = \begin{pmatrix} q_\psi I_{M\times M} & 0 \\ 0 & q_f I_{2k\times 2k}\end{pmatrix}. \tag{5.14}$$

All the partons ψ_i and f_j are fermions in a $\nu = 1$ state. The state has $U(M)\times Sp(2k)$ gauge symmetry, under which the electron operator of Eq. (5.13) is invariant. This construction of the electron operator satisfies $SU(2)_k$ current algebra [35] and generates the same $\mathbb{Z}_k$ parafermion state wave function as the $SU(2)_k$ Wess-Zumino-Witten conformal field theory. By integrating out the partons, we obtain the effective field theory

$$\mathcal{L} = \text{Tr}[(Q\delta A_0 + a_0)\rho] + \frac{1}{2}\text{Tr}(\rho)\omega_0 + \frac{1}{4\pi}\text{Tr}(Q^2)\varepsilon^{\mu\nu\lambda}\delta A_\mu \partial_\nu \delta A_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\delta A_\mu \text{Tr}(Q\mathcal{F}_{\nu\lambda}) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi}\text{Tr}(a_\mu \mathcal{F}_{\nu\lambda}) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi}\omega_\mu \text{Tr}(QF_{\nu\lambda} + \mathcal{F}_{\nu\lambda}), \quad (5.15)$$

where

$$\rho = \begin{pmatrix} \rho_\psi I_{M\times M} & 0 \\ 0 & \rho_f I_{2k\times 2k} \end{pmatrix}, \quad (5.16)$$

$F_{\nu\lambda} = \partial_\nu \delta A_\lambda - \partial_\lambda \delta A_\nu$, $\mathcal{F}_{\nu\lambda}$ is the field strength of $a_\mu \in U(M) \times Sp(2k)$, and $\rho^\psi = \rho_e$ and $\rho^f = \rho_e/k$. Using $\text{Tr}(\mathcal{F}_{\nu\lambda}) = 0$, we find the effective response of the FQH state to the external electromagnetic gauge field δA_μ and a distortion of the geometry

$$\mathcal{L}_{\text{eff}} = \rho_e \delta A_0 + \frac{M+2}{2}\rho_e \omega_0 + \frac{k}{Mk+2}\frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}(\delta A_\mu + \bar{s}\omega_\mu)\partial_\nu(\delta A_\lambda + \bar{s}\omega_\lambda) + \cdots, \quad (5.17)$$

where $\bar{s} = (M+2)/2$. This effective Lagrangian yields the average orbital spin $\bar{s}$ for the non-Abelian FQH state, and the Berry phase term $\frac{M+2}{2}\rho\omega_0$ in Eq. (5.17), yields the correct Hall viscosity [13]. However, the gravitational CS term of Eq. (5.17) is an integer although it should be fractional for non-Abelian FQH fluids.

The parton approach can be generalized to general FQH states which have the parton description. The preceding parton approach can be generalized to compute the effective response of general FQH states [30]. Let us consider a set of fermionic partons $\{f_1, f_2, \ldots, f_K\}$ with a definition for the electron operator, e.g., that of Eq. (5.13). Each parton f_i carries the electric charge q_i such that the $K \times K$ electromagnetic charge matrix is given by $Q = q_i\delta_{ij}$. If not, further structure is assumed, the partons are all scalars and thus do not couple minimally to the spin connection. The partons may have different integer filling $m_i \in \mathbb{Z}$, and hence we also define a $K \times K$ filling matrix $M = m_i\delta_{ij}$. This sets the spin *matrix* for partons as $S = \frac{m_i}{2}\delta_{ij}$. The density of partons is the matrix $\rho = \rho_i\delta_{ij}$. Furthermore, we assume that the partons see the same background metric as the electrons. From these assumptions, we deduce that there is a gauge group G which leaves the electron operator and this mean-field state invariant. Thus the internal or statistical gauge field a_μ lives in the algebra of G. As the partons are in integer quantum Hall states, they are gapped and can be integrated out to find an effective theory of the same form as Eq. (5.15) except that the partons couple with a spin matrix S:

$$\mathcal{L} = \text{Tr}[\rho(QA_t + a_t)] + \text{Tr}(\rho S)\omega_t + \frac{1}{4\pi}\text{Tr}(Q^2 M)\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}A_\mu \text{Tr}(MQ\mathcal{F}_{\nu\lambda}) + \frac{\varepsilon^{\mu\nu\lambda}}{8\pi}\text{Tr}(Ma_\mu\mathcal{F}_{\nu\lambda}) + \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\omega_\mu \text{Tr}[MS(QF_{\nu\lambda} + \mathcal{F}_{\nu\lambda})], \quad (5.18)$$

in which $F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$ and $\mathcal{F}_{\nu\lambda}$ is the field strength of a_μ. To obtain this effective action, we first assumed that the gauge field a_μ is taken from the maximally Abelian subgroup of the gauge group and integrate out the fermionic partons. Because the mean-field state does not break the gauge symmetry, we can restore the full gauge invariant action [30]. Here too, the projective parton approach does not yield the consistent value of the gravitational CS term [19,20].

VI. CONCLUSIONS

We derived a theory of the Hall viscosity and Wen-Zee term for FQH states from the composite particle theories and the projective parton approach. The composite particles carry the spin because of the spin-statistics connection, and couple with the background geometry through the spin connection. We derived the correct Hall viscosity and Wen-Zee term for CF and CB theories. In the projective parton construction, we obtained the electromagnetic and geometric response of general FQH states, both Abelian and non-Abelian. We found that the composite particle theories and the projective parton approach do not yield the correct gravitational CS term, while the universal global ground state properties, such as the Hall conductivity and the ground state degeneracy, are correctly reproduced in all cases.

ACKNOWLEDGMENTS

We thank T. Hughes, D. T. Son, and J. Teo for helpful discussions. This work is supported in part by the by the NSF Grants No. DMR-1064319 and No. DMR-1408713 at UIUC and by ICMT.

[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2] S. C. Zhang, Int. J. Mod. Phys. B **06**, 25 (1992).
[3] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[4] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[5] A. López and E. Fradkin, Phys. Rev. B **44**, 5246 (1991).
[6] X. G. Wen, Adv. Phys. **44**, 405 (1995).
[7] X.-G. Wen, Int. J. Mod. Phys. B **06**, 1711 (1992).
[8] X.-G. Wen, *Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2004).
[9] E. Fradkin, *Field Theories of Condensed Matter Systems*, 2nd ed. (Cambridge University Press, Cambridge, UK, 2013).
[10] X.-G. Wen, arXiv:1212.5121.
[11] X. G. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).

[12] N. Read, Phys. Rev. B **79**, 045308 (2009).
[13] N. Read and E. H. Rezayi, Phys. Rev. B **84**, 085316 (2011).
[14] C. Hoyos and D. T. Son, Phys. Rev. Lett. **108**, 066805 (2012).
[15] B. Bradlyn, M. Goldstein, and N. Read, Phys. Rev. B **86**, 245309 (2012).
[16] F. D. M. Haldane, arXiv:0906.1854.
[17] T. L. Hughes, R. G. Leigh, and E. Fradkin, Phys. Rev. Lett. **107**, 075502 (2011).
[18] T. L. Hughes, R. G. Leigh, and O. Parrikar, Phys. Rev. D **88**, 025040 (2013).
[19] A. G. Abanov and A. Gromov, Phys. Rev. B **90**, 014435 (2014).
[20] A. Gromov and A. G. Abanov, arXiv:1403.5809
[21] A. Nicolis and D. T. Son, arXiv:1103.2137.
[22] D. T. Son, arXiv:1306.0638.
[23] J. E. Avron, R. Seiler, and P. G. Zograf, Phys. Rev. Lett. **75**, 697 (1995).
[24] M. Fremling, T. H. Hansson, and J. Suorsa, Phys. Rev. B **89**, 125303 (2014).
[25] X. G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).
[26] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[27] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[28] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[29] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).
[30] X.-G. Wen, Phys. Rev. B **60**, 8827 (1999).
[31] O. Saremi and D. T. Son, J. High Energy Phys. 04 (2012) 091.
[32] D. T. Son and C. Wu, arXiv:1311.4882.
[33] J.-W. Chen, S.-H. Dai, N.-E. Lee, and D. Maity, J. High Energy Phys. 09 (2012) 096.
[34] J.-W. Chen, N.-E. Lee, D. Maity, and W.-Y. Wen, Phys. Lett. B **713**, 47 (2012).
[35] M. Barkeshli and X.-G. Wen, Phys. Rev. B **81**, 155302 (2010).
[36] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982).
[37] E. Witten, Commun. Math. Phys. **117**, 353 (1988).
[38] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[39] E. Witten, arXiv:0706.3359.
[40] L. Alvarez-Gaumé, Nucl. Phys. B **234**, 269 (1984).
[41] S. Ryu, J. E. Moore, and A. W. W. Ludwig, Phys. Rev. B **85**, 045104 (2012).
[42] M. Stone, Phys. Rev. B **85**, 184503 (2012).
[43] J. M. Luttinger, Phys. Rev. **135**, A1505 (1964).
[44] T. H. Hansson, M. Roček, I. Zahed, and S. C. Zhang, Phys. Lett. B **214**, 475 (1988).
[45] J. Grundberg, T. H. Hansson, A. Karlhede, and U. Lindström, Phys. Lett. B **218**, 321 (1989).
[46] A. M. Polyakov, Mod. Phys. Lett. A **03**, 325 (1988).
[47] C.-H. Tze, Int. J. Mod. Phys. A **03**, 1959 (1988).
[48] G. W. Semenoff and P. Sodano, Nucl. Phys. B **328**, 753 (1989).
[49] G. V. Dunne, R. Jackiw, and C. A. Trugenberger, Ann. Phys. **194**, 197 (1989).
[50] Y. You, G. Y. Cho, and E. Fradkin (unpublished).
[51] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, UK, 2005).
[52] S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, Nucl. Phys. B **326**, 108 (1989).
[53] M. Barkeshli and J. McGreevy, Phys. Rev. B **89**, 235116 (2014).