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# Anyon and loop braiding statistics in field theories with a topological $\Theta$ term

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We demonstrate that the anyon statistics and three-loop statistics of various $2d$ and $3d$ topological phases can be derived using semiclassical nonlinear sigma model field theories with a topological $\Theta$ term. In our formalism, the braiding statistics has a natural geometric meaning: The braiding process of anyons or loops leads to a nontrivial field configuration in the space-time, which will contribute a braiding phase factor due to the $\Theta$ term.

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*Introduction*. One of the key properties of topological states is that the gapped topological excitations above the ground state can have nontrivial braiding statistics. In both $2d$ and $3d$, all discrete lattice gauge theories have a deconfined topological phase [1]. $2d$ discrete gauge theories have point particle topological excitations, while $3d$ discrete gauge theories have both particle excitations and loop excitations which correspond to gauge charge and gauge flux loop, respectively. The simplest lattice discrete gauge theory (which we call "plain gauge theory") already has nontrivial braiding statistics [2]. More exotic gauge theories can be constructed by coupling the plain gauge theory to matter fields, and drive the matter fields into certain nontrivial short range entangled (SRE) state or symmetry protected topological (SPT) phase [3,4]. For example, once we couple a $2d\, p+ip$ topological superconductor to a $\mathbb{Z}_2$ gauge field, then the vison of the gauge field would acquire a Majorana fermion zero mode, which will grant the vison a non-Abelian statistics [5,6]. Also, if we couple a $2d$ bosonic SPT phase with $\mathbb{Z}_2$ symmetry to a $\mathbb{Z}_2$ lattice gauge theory, the lattice gauge theory will have both semion and antisemion excitations [7], which is different from a plain lattice gauge theory.

Recently these results have been generalized to $3d$ systems. It was demonstrated that once a $3d$ lattice discrete gauge theory is coupled to a $3d$ SPT state, the loop excitations (fluctuating gauge flux loops) would acquire nontrivial multiloop braiding statistics [8–11], in addition to the standard particle-loop statistics of the plain gauge theory. For example when loop B and loop C are both linked to loop A, namely none of the loops is contractible, the system wave function could acquire a universal phase angle after braiding loop C through loop B as shown in Fig. 1(a). These braiding statistics can be used as a diagnostics for SPT phases [8].

Besides the standard group cohomology description of SPT phases introduced in Refs. [3,4], it was pointed out in Refs. [12–15] that the bosonic SPT phases can also be described by semiclassical nonlinear sigma model (NLSM) field theories with a topological $\Theta$-term. In this theory all the field variables are fluctuating Landau order parameters that transform nontrivially under global symmetry. The goal of this work is to demonstrate that the nontrivial statistics between topological excitations after coupling the SPT phases to a discrete gauge theory can also be described and calculated using this NLSM field theory. Basically the braiding phase factor comes from the $\Theta$ term in the field theory, as long as we carefully analyze the field configuration in the space-time which corresponds to the braiding process. The NLSM field theory with a topological term can be viewed as the continuum limit field theory description for these braiding statistics.

*$2d$ anyon statistics*. We will first look at $2d$ systems, and as an example let us start with the $2d$ SPT state with $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$ symmetry, which can be described by the following $(2+1)d$ $O(4)$ NLSM with a $\Theta$ term at $\Theta = 2\pi$ [15]:

$$S = \int d^2x d\tau \ \frac{1}{g}(\partial_\mu \boldsymbol{n})^2 + \frac{i\Theta}{\Omega_3}\epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d, \quad (1)$$

where $\boldsymbol{n}$ is a four-component vector with unit length, and $\Omega_3 = 2\pi^2$ is the volume of a three-dimensional sphere with unit radius. Under the $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$ symmetry, the vector $\boldsymbol{n}$ transforms as

$$\begin{aligned} &\mathbb{Z}_2^A : n^1,n^2 \to -n^1, -n^2, \quad n^3,n^4 \to n^3,n^4; \\ &\mathbb{Z}_2^B : n^1,n^2 \to n^1,n^2, \quad n^3,n^4 \to -n^3, -n^4. \end{aligned} \quad (2)$$

Now let us couple the vector $\boldsymbol{n}$ to a $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$ gauge field. The excitations that will have nontrivial braiding statistics are the vison excitations ($\pi$-gauge flux) of gauge fields $\mathbb{Z}_2^A$ and $\mathbb{Z}_2^B$. Let us consider the following braiding process: one pair of $\mathbb{Z}_2^A$ visons and one pair of $\mathbb{Z}_2^B$ visons are created in space at one instance in time, then they are annihilated at another later instance after braiding one $\mathbb{Z}_2^A$ vison with one $\mathbb{Z}_2^B$ vison. In the $(2+1)d$ space-time, this process corresponds to one linking between $\mathbb{Z}_2^A$ and $\mathbb{Z}_2^B$ vison loops, as shown in Fig. 1(b). Because the $\mathbb{Z}_2$ gauge fields are coupled to the four-component vector $\boldsymbol{n}$, the $\mathbb{Z}_2^A$ vison is bound with a $\pm 1/2$ vortex of $(n^1,n^2)$, while $\mathbb{Z}_2^B$ vison is bound with a $\pm 1/2$ vortex of $(n^3,n^4)$. Then the braiding process in the space-time can be viewed as a linking configuration between the $(n^1,n^2)$ half-vortex loop and the $(n^3,n^4)$ half-vortex loop. Due to the $\Theta$ term in Eq. (1), this configuration will contribute a phase factor $\exp(\pm i\pi/2) = \pm i$ to the action, which implies the mutual braiding statistics between the $\mathbb{Z}_2^A$ vison and $\mathbb{Z}_2^B$ vison.

To calculate this phase factor explicitly, let us first consider a finite segment of the $\mathbb{Z}_2^A$ vison loop along the $\hat{\tau}$ direction. A vison is always bound with either the $1/2$ vortex or $-1/2$ vortex of $(n_1,n_2)$. Around this segment, the $O(4)$ vector $\boldsymbol{n}$ has the following configuration with cylindrical coordinate $(r,\phi,\tau)$ [$x = r\cos\phi$, $y = r\sin\phi$; see Fig. 1(b) inset]:

$$\begin{aligned} n^1 &= \sin\alpha(r)\cos f(\phi), \\ n^2 &= \sin\alpha(r)\sin f(\phi), \\ n^3 &= \cos\alpha(r) N^1(\tau), \\ n^4 &= \cos\alpha(r) N^2(\tau), \end{aligned} \quad (3)$$

 

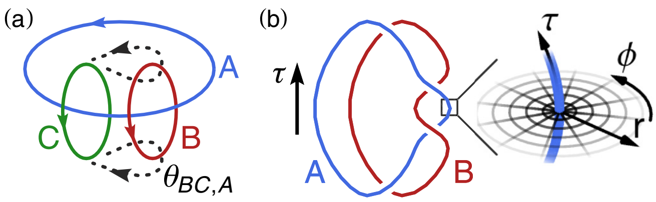


FIG. 1. (Color online) (a) Three-loop braiding process. The loops A, B, and C are colored blue, red, and green, respectively. The braiding path of loop C is indicated by the dotted arrow curve. (b) Two-loop linking in the $(2+1)d$ space-time, which corresponds to creating a pair of $\mathbb{Z}_2^A$ and $\mathbb{Z}_2^B$ visons, and annihilating them after braiding one $\mathbb{Z}_2^A$ and one $\mathbb{Z}_2^B$ vison. The time $\tau$ is along the vertical direction. The inset shows the local cylindrical coordinate system around a segment of the $\mathbb{Z}_2^A$ vison loop.

where $\boldsymbol{N} = (N_1, N_2)$ is an $O(2)$ unit vector $|\boldsymbol{N}|^2 = 1$. $\boldsymbol{N}$ is a function of $\tau$ only. $\alpha(r)$ is a nonnegative continuous function that satisfies $\alpha(0) = 0$, $\alpha(\infty) = \pi/2$. Along the $\hat{\tau}$ axis, i.e., $r = 0$, we have $(n^3, n^4) = \boldsymbol{N}$. Using this configuration, we can compute the $\Theta$ term:

$$\int d^2x d\tau \, \frac{2\pi i}{\Omega_3} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d = \int_0^{2\pi} d\phi \, \partial_\phi f \int d\tau \, \frac{i}{2\pi} \epsilon_{ab} N^a \partial_\tau N^b. \quad (4)$$

If $n^1$ and $n^2$ form a full vortex line along the $\hat{\tau}$ axis, namely $f(\phi) \sim \phi$, the $O(4)$ $\Theta$ term reduces to a $1d$ $O(2)$ NLSM with $\Theta = 2\pi$. If there is a $\mathbb{Z}_2^A$ vison line along the $\hat{\tau}$ axis, i.e., $n^1$ and $n^2$ form a $\pm 1/2$-vortex line along $\hat{\tau}$ axis, namely $f(\phi) \sim \pm\phi/2$, then the $(2+1)d$ $O(4)$ NLSM reduces to a $1d$ $O(2)$ NLSM of vector $\boldsymbol{N}$ with $\Theta = \pm\pi$. Now let us consider two linked vison loops, and in Eq. (4) $\tau$ becomes the parameter along the $\mathbb{Z}_2^A$ vison loop. Since the two loops are linked, vector $\boldsymbol{N}$ will have a $\pm 1/2$-vortex winding along the $\mathbb{Z}_2^A$ vison loop:

$$\oint d\tau \, \epsilon_{ab} N^a \partial_\tau N^b = \pm\pi. \quad (5)$$

Combining Eq. (4) and Eq. (5) together, we conclude that this linking configuration (which corresponds to a braiding process in the space-time) would contribute factor $\pm i$ to the action. In other words, the linking configuration in Fig. 1(b) corresponds to a $\pm 1/4$ instanton of the four-component vector $\boldsymbol{n}$ in the $(2+1)d$ space-time.

Now let us consider a $2d$ SPT state with $\mathbb{Z}_2$ global symmetry only, and couple it to a $\mathbb{Z}_2$ gauge field. This SPT state can be described by the same field theory Eq. (1), and under the $\mathbb{Z}_2$ symmetry $\boldsymbol{n} \to -\boldsymbol{n}$. A vison of this $\mathbb{Z}_2$ gauge field can be viewed as a bound state between the $\mathbb{Z}_2^A$ vison and $\mathbb{Z}_2^B$ vison discussed previously. Then the linking configuration in Fig. 1(b) can be interpreted as creating a pair of visons, self-twisting one vison by $2\pi$, then annihilating them. The phase $\pm i$ corresponds to topological spin-$\pm 1/4$ of the vison, which is consistent with the semion and antisemion statistics of the vison proved in Ref. [7].

All the analysis above can be straightforwardly generalized to $\mathbb{Z}_N$ gauge theory coupled to a $2d$ $\mathbb{Z}_N$ SPT state. The $2d$ $\mathbb{Z}_N$ SPT state is described by the same field theory Eq. (1) [15], where $\Theta = 2\pi k$, $k = 0, 1, \ldots, N-1$. The same analysis above leads to the result that the topological spin of the $2\pi/N$ flux excitations can be $k/N^2$; namely self-twisting such excitation will grant its wave function a phase $\exp(2\pi i k/N^2)$.

*3d loop statistics.* Now we consider $3d$ bosonic SPT states with $\mathbb{Z}_2^A \times \mathbb{Z}_2^B \times \mathbb{Z}_2^C$ symmetry. In terms of field theory, one of these SPT states is described by the following $(3+1)d$ $O(5)$ NLSM:

$$S = \int d^3x d\tau \, \frac{1}{g} (\partial_\mu \boldsymbol{n})^2 + \frac{i\Theta}{\Omega_4} \epsilon_{abcde} n^a \partial_x n^b \partial_y n^c \partial_z n^d \partial_\tau n^e, \quad (6)$$

where $\Omega_4 = 8\pi^2/3$ is the volume of a four-dimensional sphere with unit radius. Under the $\mathbb{Z}_2^A \times \mathbb{Z}_2^B \times \mathbb{Z}_2^C$ symmetry, the five-component vector $\boldsymbol{n}$ transforms as

$$\begin{aligned} &\mathbb{Z}_2^A : n^1, n^2 \to -n^1, -n^2, \quad n^{3,4,5} \to n^{3,4,5}; \\ &\mathbb{Z}_2^B : n^2, n^3 \to -n^2, -n^3, \quad n^{1,4,5} \to n^{1,4,5}; \\ &\mathbb{Z}_2^C : n^4, n^5 \to -n^4, -n^5, \quad n^{1,2,3} \to n^{1,2,3}. \end{aligned} \quad (7)$$

Now let us couple this SPT state to the $\mathbb{Z}_2^A \times \mathbb{Z}_2^B \times \mathbb{Z}_2^C$ gauge field, and consider the statistics between the three loops in Fig. 1(a), in which the base loop is a vison loop of the $\mathbb{Z}_2^A$ gauge field, and it is linked with vison loops of both $\mathbb{Z}_2^B$ and $\mathbb{Z}_2^C$ gauge fields.

A vison loop can be bound with either a $+1/2$ vortex or $-1/2$ vortex; both cases exist in the system, and they correspond to different excitations. As an example let us study the braiding statistics of vison loops bound with the $+1/2$ vortex. The choice of $+1/2$ vortex gives each vison loop an orientation, as marked out in Fig. 1(a). Let us first look at the $\mathbb{Z}_2^B$ vison loop. Following the same calculation as Eq. (4), because the $\mathbb{Z}_2^B$ vison loop is bound with a half-vortex loop of $(n_2, n_3)$, the $O(5)$ NLSM with $\Theta = 2\pi$ is reduced to an $O(3)$ NLSM with $\Theta = \pi$ in the $(1+1)d$ world sheet of the $\mathbb{Z}_2^B$ vison loop, and the three-component vector on this world sheet is $\boldsymbol{N} \sim (n_1, n_4, n_5)$:

$$S_{1d,B} = \int dx d\tau \, \frac{1}{g} (\partial_\mu \boldsymbol{N})^2 + \frac{i\pi}{4\pi} \epsilon_{abc} N^a \partial_x N^b \partial_\tau N^c. \quad (8)$$

On the $(1+1)d$ world sheet of the $\mathbb{Z}_2^B$ vison loop, the braiding between the $\mathbb{Z}_2^B$ and $\mathbb{Z}_2^C$ vison loops corresponds to the space-time configuration $\boldsymbol{N}(x,\tau)$ in Fig. 2, and this configuration carries a $1/2$ $O(3)$ instanton number; thus it will contribute a factor $i$ to the action. This implies that the three-loop braiding statistics angle is $\theta_{BC,A} = \pi/2$. The statistics angle $\theta_{AC,B}$ can be calculated in the same way after interchanging $n_1$ and $n_3$ in the $O(5)$ vector, which will leads to factor $-1$ due to the antisymmetrization in the $\Theta$ term in Eq. (6). Thus $\theta_{AC,B} = -\pi/2$.

The loop braiding statistics can also be understood in a different way. Reference [16] pointed out that the three-loop braiding in Fig. 1(a) can also be viewed as a link of the $\mathbb{Z}_2^B$ and $\mathbb{Z}_2^C$ vison loops braiding with the $\mathbb{Z}_2^A$ vison loop, as illustrated in Fig. 3(a). This link-loop braiding statistics can be described by the NLSM as well. As the vison link braid through the vison loop, the space-time configuration of the $O(5)$ vector $\boldsymbol{n}$ around

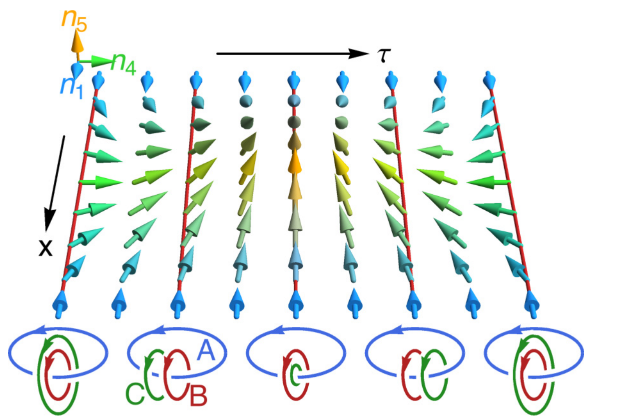


FIG. 2. (Color online) The space-time configuration of $\boldsymbol{N} \sim (n_1, n_4, n_5)$ on the world sheet of the $\mathbb{Z}_2^B$ vison loop (in red) as the $\mathbb{Z}_2^C$ vison loop (in green) braiding around it. Each red line is a time slice, at which moment the corresponding three-loop configuration is shown below.

the vison link can be described as follows:

$$
\begin{aligned}
n^1 &= \cos\alpha(\tau),\\
n^2 &= \sin\alpha(\tau) N^1(x,y,z),\\
n^3 &= \sin\alpha(\tau) N^2(x,y,z),\\
n^4 &= \sin\alpha(\tau) N^3(x,y,z),\\
n^5 &= \sin\alpha(\tau) N^4(x,y,z),
\end{aligned} \tag{9}
$$

where $\boldsymbol{N} = (N^1, N^2, N^3, N^4)$ is an $O(4)$ unit vector $|\boldsymbol{N}|^2 = 1$ that describes the configuration of the (linked) half-vortex loops bound to the vison loops of $\mathbb{Z}_2^B$ and $\mathbb{Z}_2^C$. The time $\tau$ (running from 0 to 1) parametrizes a full braiding of the $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ vison link with the $\mathbb{Z}_2^A$ vison loop. Suppose the $n^1$ component is energetically more favored, then the $\mathbb{Z}_2^A$ branch cut disk bordered by the $\mathbb{Z}_2^A$ vison loop will be bound with a $n^1$ domain wall. Let the braiding of the $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ vison link initiate from one side of the domain wall, and end up at the other side of the domain wall, then $\alpha(\tau)$ will be a continuous function satisfying $\alpha(0) = \pi$, $\alpha(1) = 0$. Plugging the configuration Eq. (9) into the NLSM Eq. (6), the $O(5)$ $\Theta$ term of $\boldsymbol{n}$ is reduced to an $O(4)$ $\Theta$ term of $\boldsymbol{N}$ at $\Theta = 2\pi$:

$$
\begin{aligned}
&-\int_0^1 d\tau\, \partial_\tau \alpha \sin^3\alpha \int d^3x\, \frac{2\pi i}{\Omega_4} \epsilon_{abcd} N^a \partial_x N^b \partial_y N^c \partial_z N^d\\
&\quad = \int d^3x\, \frac{2\pi i}{\Omega_3} \epsilon_{abcd} N^a \partial_x N^b \partial_y N^c \partial_z N^d.
\end{aligned} \tag{10}
$$

According to our previous calculation, the linking configuration between the $(N_1, N_2)$ half-vortex loop and $(N_3, N_4)$ half-vortex loop corresponds to the $1/4$ $O(4)$ soliton in the $3d$ space, so the above $O(4)\Theta$ term in Eq. (10) will result in a $\pi/2$ phase angle accumulated in the link-loop braiding, which equals the three-loop braiding angle $\theta_{BC,A}$ calculated already in this Rapid Communication.

The nontrivial link-loop braiding statistics implies that the $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ vison link must carry the charge of the $\mathbb{Z}_2^A$ gauge field. Let us denote the $\mathbb{Z}_2^A$ charge carried by the $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ vison link as $q_{BC}^A$. It is related to the braiding angle by $\theta_{BC,A} = -\pi q_{BC}^A$. The minus sign is due to the reversed link-loop braiding direction as shown in Fig. 3(a) (which corresponds to the positive three-loop braiding direction). As shown in Fig. 3(b), the torus traced out by the $\mathbb{Z}_2^A$ vison loop through braiding with the $\mathbb{Z}_2^C$ vison loop (in the linking with the $\mathbb{Z}_2^B$ vison loop) actually forms a Gaussian surface enclosing the $\mathbb{Z}_2^C$ vison loop. So the three-loop braiding statistics angle $\theta_{AC,B}$ measures the $\mathbb{Z}_2^A$ charge carried by the $\mathbb{Z}_2^C$ vison loop in the $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ link, denoted $q_C^A$, and $\theta_{AC,B} = \pi q_C^A$. Similarly from Fig. 3(c), the three-loop braiding statistics angle $\theta_{AB,C}$ measures the $\mathbb{Z}_2^A$ charge carried by the $\mathbb{Z}_2^B$ vison loop in the same $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ link, denoted $q_B^A$, and $\theta_{AB,C} = \pi q_B^A$. Obviously, $q_{BC}^A = q_B^A + q_C^A$; thus

$$
\theta_{AB,C} + \theta_{BC,A} + \theta_{AC,B} = 0, \tag{11}
$$

which is precisely the cyclic relation [8,16], and it implies that $\theta_{AB,C} = 0$ (given $\theta_{BC,A} = \pi/2$ and $\theta_{AC,B} = -\pi/2$ as previously calculated).

$\theta_{AB,C}$ can also be computed as follows: $\theta_{AB,C}$ corresponds to braiding $\mathbb{Z}_2^A$ and $\mathbb{Z}_2^B$ vison loops, both of which are linked to a $\mathbb{Z}_2^C$ vison loop. This process can be divided into two steps: first moving the $\mathbb{Z}_2^B$ vison loop through the $\mathbb{Z}_2^A$ vison loop, then moving the $\mathbb{Z}_2^A$ vison loop through the $\mathbb{Z}_2^B$ vison loop. The first step (see Fig. 4) is equivalent to creating a pair of $\mathbb{Z}_2^B$ vison-antivison (vison and antivison have semion and antisemion statistics) at the $2d$ $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ SPT phase, then braiding the $\mathbb{Z}_2^B$ vison (or antivison) around the $\mathbb{Z}_2^C$ vison, and annihilating the vison-antivison pair. This step will contribute a phase factor $i$ to the action. The second step is equivalent

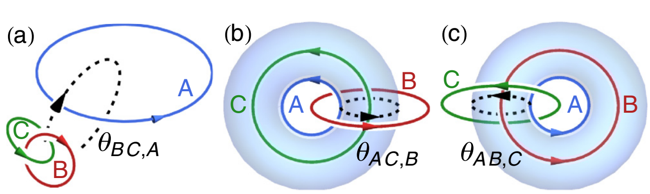


FIG. 3. (Color online) (a) Braiding a link of the $\mathbb{Z}_2^B$ and $\mathbb{Z}_2^C$ vison loops with the $\mathbb{Z}_2^A$ vison loop also accumulates the phase $\theta_{BC,A}$. (b), (c) The three-loop braiding process that corresponds to the statistic angle $\theta_{AC,B}$ ($\theta_{AB,C}$). The light blue torus indicates the surface traced out by the $\mathbb{Z}_2^A$ vison loop through the braiding processes, which can be considered as the Gaussian surface that measures the $\mathbb{Z}_2^A$ charge enclosed. Small arrows on the loops mark out the loop orientation.

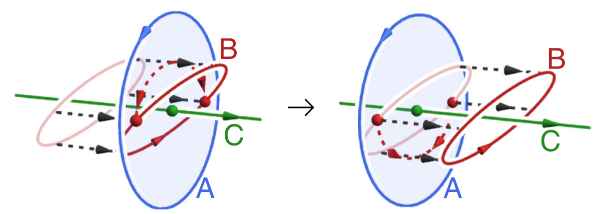


FIG. 4. (Color online) Illustration of moving $\mathbb{Z}_2^B$ vison loop through the $\mathbb{Z}_2^A$ vison loop. The $\mathbb{Z}_2^A$ vison loop borders a branch cut disk, which can be viewed as a $2d\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ SPT. When the $\mathbb{Z}_2^B$ vison loop pokes through this disk, a pair of $\mathbb{Z}_2^B$ semion-antisemion are created, braided with the $\mathbb{Z}_2^C$ vison, and annihilated.

to creating and annihilating a pair of $\mathbb{Z}_2^A$ visons at the $2d$ $\mathbb{Z}_2^A \times \mathbb{Z}_2^C$ SPT phase, and braiding around the $\mathbb{Z}_2^C$ vison in between, which will contribute factor $-i$. The two processes together will lead to a trivial phase factor, namely $\theta_{AB,C} = 0$.

More "conventionally," $\theta_{BC,A}$ and $\theta_{AC,B}$ can be interpreted in the "decorated domain wall" picture [17]. In our NLSM Eq. (6), the $\mathbb{Z}_2^A$ vison loop is the boundary of a $2d$ disk of branch cut of coupling between $n_1$ components. According to Ref. [18], after integrating out $n_1$, the effective field theory on this $2d$ disk is the same as Eq. (1) with $\Theta = 2\pi$, except now the $O(4)$ vector is $(n_2, n_3, n_4, n_5)$; i.e., this $2d$ disk can be viewed as a $2d$ SPT state with $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ symmetry, which is precisely the decorated domain wall picture. Then after gauging the $\mathbb{Z}_2^B$ and $\mathbb{Z}_2^C$ symmetry, the vison loop statistics reduces to the anyon statistics of the $2d$ $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ topological order, which is what we have already computed using Eq. (1).

We can also consider the $3d$ SPT state with $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$ symmetry. There are in total three different nontrivial $3d$ bosonic SPT states with this symmetry [3]. The first state can be constructed using the previously discussed $\mathbb{Z}_2^A \times \mathbb{Z}_2^B \times \mathbb{Z}_2^C$ SPT state, and break its subgroup $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$ down to one diagonal $\mathbb{Z}_2$ symmetry; namely now the $O(5)$ vector $\boldsymbol{n}$ transforms as

$$\begin{aligned} &\mathbb{Z}_2^A : n_1, n_2 \to -n_1, -n_2, \quad n_{3,4,5} \to n_{3,4,5}; \\ &\mathbb{Z}_2^B : n_1 \to n_1, \quad n_{2,3,4,5} \to -n_{2,3,4,5}. \end{aligned} \tag{12}$$

Now a $\mathbb{Z}_2^B$ vison loop corresponds to a bound state between the $\mathbb{Z}_2^C$ and $\mathbb{Z}_2^B$ vison loops in the previous case. Thus [19]

$$\begin{aligned} &\theta_{BB,A} = 2\theta_{BC,A} = \pi, \\ &\theta_{AB,B} = \theta_{AC,B} + \theta_{AB,C} = \pm\pi/2. \end{aligned} \tag{13}$$

All the other braiding angles are zero. The second type of $3d$ SPT state corresponds to interchanging $\mathbb{Z}_2^A$ and $\mathbb{Z}_2^B$ symmetries; thus after gauging the symmetries, $\theta_{AB,A} = \pm\pi/2$, $\theta_{AA,B} = \pi$. The third type of SPT state is equivalent to the two SPT states discussed above weakly coupled together; thus

$$\theta_{AB,A} = \theta_{AB,B} = \pm\pi/2, \quad \theta_{AA,B} = \theta_{BB,A} = \pi. \tag{14}$$

In summary, we have computed the anyon braiding statistics, and three-loop statistics of $2d$ and $3d$ topological phases constructed by coupling plain gauge theories to bosonic SPT states. Our calculation is based on semiclassical field theories, and all the braiding phases naturally come from the topological $\Theta$ term in the field theory.

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[1] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishers, 1987).
[2] A. Y. Kitaev, Ann. Phys. **303**, 2 (2003).
[3] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[4] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[5] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[6] A. Y. Kitaev, Ann. Phys. **321**, 2 (2006).
[7] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).
[8] C. Wang and M. Levin, arXiv:1403.7437.
[9] H. Moradi and X.-G. Wen, arXiv:1404.4618.
[10] J. Wang and X.-G. Wen, arXiv:1404.7854.
[11] S. Jiang, A. Mesaros, and Y. Ran, arXiv:1404.1062.
[12] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[13] C. Xu, Phys. Rev. B **87**, 144421 (2013).
[14] C. Xu and T. Senthil, Phys. Rev. B **87**, 174412 (2013).
[15] Z. Bi, A. Rasmussen, and C. Xu, arXiv:1309.0515.
[16] C.-M. Jian and X.-L. Qi, arXiv:1405.6688.
[17] X. Chen, Y.-M. Lu, and A. Vishwanath, Nat. Commun. **5**, 3507 (2014).
[18] Z. Bi, A. Rasmussen, and C. Xu, Phys. Rev. B **89**, 184424 (2014).
[19] Here $\theta_{BB,A}$ stands for the full braiding statistics angle between two $\mathbb{Z}_2^B$ vison loops while they are both linked with a $\mathbb{Z}_2^A$ vison loop.