Topological insulating phases of non-Abelian anyonic chains

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Boundary conformal field theory is brought to bear on the study of topological insulating phases of non-Abelian anyonic chains. These phases display protected anyonic end modes. We consider spin-$1/2$ su$(2)_k$ chains at any level k, focusing on the most prominent examples: the case $k = 2$ describes Ising anyons (equivalent to Majorana fermions) and $k = 3$ corresponds to Fibonacci anyons. The method we develop is quite general and rests on a deep connection between boundary conformal field theory and topological symmetry. This method tightly constrains the nature of the topological insulating phases of these chains for general k. Emergent anyons which arise at domain walls are shown to have the same braiding properties as the physical quasiparticles. This suggests a "solid-state" topological quantum computation scheme in which emergent anyons are braided by tuning the couplings of non-Abelian quasiparticles in a fixed network.

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I. INTRODUCTION

The notion of topological order has emerged as a powerful paradigm for the classification and discovery of new phases of matter [1–3]. One manifestation of topological order is the existence of quasiparticles known as anyons. The exchange of two (Abelian) anyons gives $\psi(r_2,r_1) = e^{i\theta}\psi(r_1,r_2)$, where ψ is the many-body wave function describing the system and θ can (in principle) take any value; these particles are intermediate between bosons ($\theta = 0$) and fermions ($\theta = \pi$) [1,2]. Even richer behavior arises in the case of non-Abelian anyons; the exchange of these objects enacts unitary transformations on the space of degenerate ground states [4]. This physics underlies *topological quantum computation* (TQC) which proposes using a topologically ordered system as a robust quantum memory. The braiding of non-Abelian anyons has been suggested as a means of implementing fault-tolerant quantum gates in these systems [1,2,4,5].

In this work, we study linear chains of spin-$1/2$ su$(2)_k$ anyonic chains at any level k, objects of study in non-Abelian Chern-Simons theories [1,6,7]. This family of models includes two of the most prominent examples of non-Abelian anyons: Ising anyons which are equivalent to Majorana fermions (at level $k = 2$) and Fibonacci anyons for $k = 3$. The Fibonacci chain (also dubbed the "golden chain") has attracted a great deal of interest with studies focusing on its behavior at criticality as well as the effects of disorder [6–13]. Fibonacci anyons capture the non-Abelian character of the quasiparticles of the $\mathbb{Z}_3$-parafermion "Read-Rezayi" state, a candidate theory for the $\nu = \frac{12}{5}$ fractional quantum Hall plateau [4]. Fibonacci anyons are of particular interest since, unlike Majorana fermions, they are capable of performing *universal* TQC [1].

Interacting anyons are thought to exhibit a wide spectrum of behavior [6,7,14–18]. In this work, we apply boundary conformal field theory (CFT) to the study of topological insulating phases of anyonic chains [19–24]. That BCFT is useful in this context is natural since at its heart boundary CFT is a manifestation of the bulk-boundary correspondence and holography [25]. In contrast to other applications of boundary CFT such as the multichannel Kondo problem [19–21], here the anyonic chain is most conveniently described with nonlocal degrees of freedom. In these models an end mode represents not a local degree of freedom but the topological state of the chain. This suggests that boundary CFT may serve as a powerful probe of topological phases.

Indeed, in this work we uncover a deep connection between boundary CFT and the notion of topological symmetry [6,8–11]. Our primary result is the characterization of the topological phases of an open chain of spin-$1/2$ su$(2)_k$ anyons [see Fig. (1a)] [24,26]. Previous work on anyonic chains has focused on critical behavior [6,8,10,15]. We focus on the cases $k = 2$ and $k = 3$, although our method is quite general and greatly constrains the nature of topological insulating phases for these chains for all k. We find that for AFM chains, there are two possible phases, one with no nontrivial end modes and those with spin-$1/2$ anyon end modes. Ferromagnetically coupled chains are expected to exhibit richer behavior. In particular, for $k > 3$, we find that any topological insulating phase that exists necessarily exhibits end modes with spins j, $j > 1/2$ (i.e., the end modes differe in type from the anyons composing the chain). The essence of our method is to consider a chain at criticality and apply the renormalization group (RG) to track boundary degrees of freedom as a bulk gap is opened. The possible fixed points are shown to be an indicator of the topological properties of the chain. Some of our theoretical predictions are verified with numerical diagonalizations of the Ising and Fibonacci chains.

This phenomenon suggests a "solid-state" version of TQC in which emergent anyons can be manipulated in a T-junction network of coupled quasiparticles [Fig. 1(c)] [5,27]. Here, protected anyonic modes are manipulated by tuning the couplings between quasiparticles. Building on previous work [5,27], we present an argument demonstrating that the emergent anyons exhibit the same braiding properties as the physical quasiparticles. This scheme has the virtue of not requiring the large-scale motion of the physical quasiparticles composing the network [1,2,4,28].

The outline of this paper is as follows. In Sec. II we provide a brief but pedagogical discussion of coupled anyonic chains and give explicit forms of the Hamiltonians of the Ising and Fibonacci chains. In Sec. III we characterize the spectra of chains at criticality. Section IV presents an analysis of the topological insulating phases exhibited by anyonic chains and contains the central results of this paper. In Sec. V, we

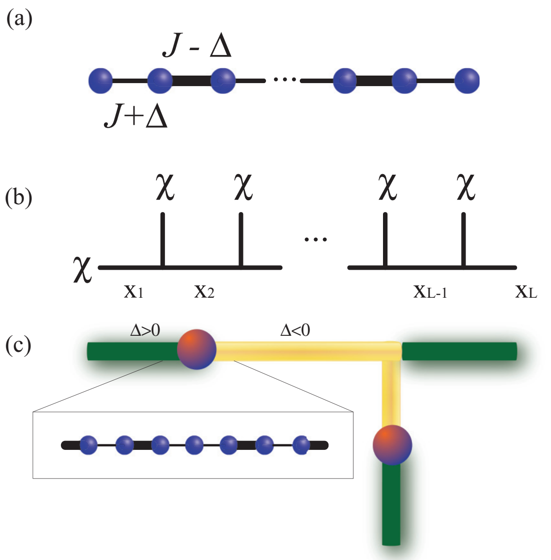

FIG. 1. (Color online) (a) Linear chain of L (even) coupled spin-1/2 ($j = 1/2$) su$(2)_k$ anyons (generically denoted by χ). The Hamiltonian of the chain [cf. Eq. (2)] is conveniently expressed in terms of (b) the fusion tree basis, where $x_n = \chi_j$ reflects the cumulative fusion product of the first n anyons in the chain. (c) Emergent anyons may be braided in a T-junction network [26,27].

illustrate how the emergent anyons which arise at the ends of a topological insulating chain can implement TQC. Finally, we present our conclusions in Sec. VI.

II. CHAINS OF COUPLED ANYONS

We consider a linear array of L non-Abelian anyons (generically denoted χ) [see Fig. 1(b)] with L even. For $L \gg 1$, these anyons encode an $\sim(d_\chi)^L$-dimensional subspace, where d_χ is the *quantum dimension* of χ [4]. In this work, we will specialize to the case of spin-1/2 anyons (which we denote by χ) belonging to su$(2)_k$ for finite k [1,2]. For a given k, these theories contain anyon types χ_j where $j = 0,\frac{1}{2},1,\ldots,\frac{k}{2}$ and we define $\chi \equiv \chi_{\frac{1}{2}}$ and $I \equiv \chi_0$. For k odd, not all these particles are distinct: the anyons $\chi_j \equiv \chi_{\frac{k}{2}-j}$ are identified. The fusion rules of this theory obey an analog of angular momentum addition, i.e., $j_1 \times j_2 = |j_1 - j_2| + (|j_1 - j_2| + 1) + \cdots + \min(j_1 + j_2, k - j_1 - j_2)$.

For anyons separated by a distance D with $D \lesssim \xi$ where ξ is the correlation length of the system, this degeneracy is lifted [1]. The correlation length ξ is related to the system gap $\tilde{\Delta}$ and a characteristic velocity v; i.e., $\xi = \hbar v/\tilde{\Delta}$. Akin to a Heisenberg spin interaction, the interaction energy between two anyons depends on their mutual fusion product [6,8,10]. For instance, two χ anyons fuse to give

$$\chi \times \chi = I + \chi_1. \tag{1}$$

The coupling between anyons lifts the degeneracy between these states. The dominant terms in the Hamiltonian of a chain are likely to arise from nearest-neighbor couplings

$$H_{NN} = \sum_n J_n \Pi_n^{(I)}, \tag{2}$$

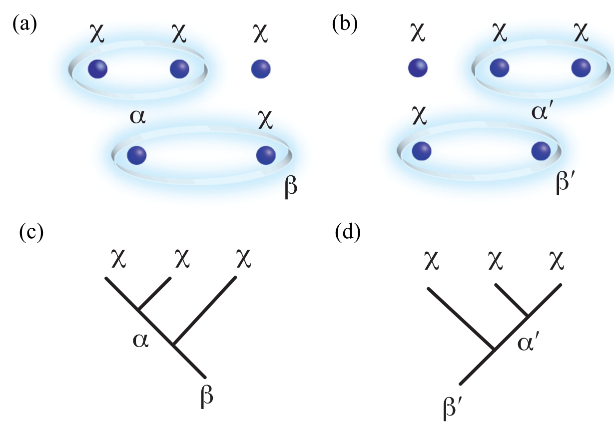

FIG. 2. (Color online) Hilbert space of three anyons may be described in two possible ways. In (a) and its fusion diagram (c), the fusion of the left and middle χ's lead to α (another anyon type) and then α and the rightmost χ fuse to β. In (b), the same system is described by first fusing the two anyons on the right to α'. The corresponding fusion diagram appears in (d). For the same set of three anyons, the total fusion product is independent of the order in which they fuse and thus $\beta = \beta'$. However, α and α' are in general distinct.

where $\Pi_n^{(I)} = |I\rangle\langle I|$ is the projector onto the identity fusion channel for anyons connected by the nth link [see Fig. 1(a)] [6,7,29]. For $J_n < 0$, we refer to the coupling as antiferromagnetic (AFM) and for $J_n > 0$ as ferromagnetic (FM).

In order to describe the collective state of a chain, we employ the fusion tree basis which tracks the fusion of the first n anyons on the left-hand side of the chain [1,6]. The transformations between the local and fusion tree basis can be accomplished by unitary transformations described by so-called F matrices [1,2]. The role of these transformations may be understood by considering two inequivalent ways of specifying the state of three anyons. As illustrated in Fig. 2, a system of three anyons can be described by specifying the fusion product of the leftmost anyons or the rightmost anyons. The symbol $F_\beta^{\chi\chi\chi}$ represents the unitary transformation between these two bases. In general, it is an $n \times n$ matrix, where n is the number of possible fusion products of χ with χ (i.e., the number of possible states α in Fig. 2).

When expressed in the fusion tree basis, the Hamiltonian in Eq. (2) can be mapped to the $p = k + 2$ restricted solid-on-solid (RSOS) model at criticality [6,7]. The quantum RSOS model is defined on a 1D lattice and at every site n there is an integer degree of freedom (a 'height') y_n ($1 \leqslant y_n < p$) which satisfies the condition $|y_n - y_{n+1}| = 1$ [30]. The mapping between these models is given by $y_n = 2j + 1$ for x_n given by χ_j [6,7]. In the context of the anyon model, the restriction that adjacent heights must differ by one enforces the fusion rules. The CFTs describing the critical RSOS model are fully understood in both the FM and AFM cases. This allows for a characterization of the anyonic chains at criticality which will be carefully explained in Sec. III [7].

We now give explicit forms for the Hamiltonian of Ising and Fibonacci chains.

TABLE I. Conformal data for Ising and Fibonacci theories. The fields are listed with their corresponding j values in parentheses. The nontrivial F matrices are shown in the basis $\alpha' = I,\psi$ for the Ising case and in the basis $\alpha' = I,\tau$ for the Fibonacci case. All other F matrices are 1×1 matrices with $F = \pm1$ (or zero if inconsistent with the fusion rules).

	Ising	Fibonacci
Fields	σ $(\frac{1}{2})$, ψ (1)	τ $(\frac{1}{2},1)$
d_χ	$d_\sigma = \sqrt{2}$	$d_\tau = \frac{1+\sqrt{5}}{2}$
Fusion rules	$\sigma\times\sigma = I+\psi$	$\tau\times\tau = I+\tau$
	$\sigma\times\psi = \sigma$	
F matrices	$F^{\sigma\sigma\sigma}_\sigma = \frac{1}{\sqrt{2}}(s^z+s^x)$	$F^{\tau\tau\tau}_\tau = \frac{1}{\varphi}s^z + \frac{1}{\sqrt{\varphi}}s^x$

A. Ising anyons

There are three anyons for $k=2$ denoted by I, $\sigma = \chi$, and $\psi = \chi_1$. The conformal data for these anyons is given in Table I. The fusion rules dictate that $x_n = \sigma$ for n odd and $x_n = I,\psi$ for n even (we take $x_1 = \sigma$) [30]. For the Ising anyons, it is convenient to use the following modified version of Eq. (2):

$$H'_{NN} = \frac{1}{2}\sum_n J_n\big(\Pi_n^{(I)} - \Pi_n^{(\psi)}\big). \tag{3}$$

This Hamiltonian can be written in the fusion tree basis by applying $F^{\sigma\sigma\sigma}_\sigma$ given in Table I. We obtain

$$H_\sigma = \frac{J_1}{2}(|I_2\rangle\langle I_2| - |\psi_2\rangle\langle\psi_2|) + H'_\sigma, \tag{4}$$

where

$$\begin{aligned} H'_\sigma = \frac{1}{2}\sum_{n=1}^{L/2-1} [&J_{2n}(|\sigma I\sigma\rangle\langle\sigma\psi\sigma| + \text{H.c.}) \\ &+ J_{2n+1}(|I\sigma I\rangle\langle I\sigma I| + |\psi\sigma\psi\rangle\langle\psi\sigma\psi|) \\ &- J_{2n+1}(|I\sigma\psi\rangle\langle I\sigma\psi| + |\psi\sigma I\rangle\langle\psi\sigma I|)]. \end{aligned} \tag{5}$$

The first two terms in Eq. (4) arise from the fusion of the first two anyons in the chain [see Fig. 1(a)] and the subscripts indicate that these projectors only act on the fusion product described by x_2. The sums in Eq. (5) are over all possible triples of states along the fusion tree. This Hamiltonian is equivalent to the quantum Ising model. Introducing pseudospins $I = |{\downarrow}\rangle_z$ and $\psi = |{\uparrow}\rangle_z$, Eq. (5) takes the form

$$H_\sigma = \frac{J_1}{2}s_2^z + \frac{1}{2}\sum_{n=1}^{L/2-1}\big(J_{2n}s_{2n}^x + J_{2n+1}s_{2n}^z s_{2n+2}^z\big). \tag{6}$$

The Hamiltonian in Eq. (6) can be mapped to a spinless p-wave superconductor via a Jordan-Wigner transformation [31,32]. We should point out Eqs. (5) and (6) are nonlocal descriptions of the anyonic chain since they describe degrees of freedom in the fusion tree basis. Indeed, it will be seen that local degrees of freedom in H_σ describe topological properties of the chain.

B. Fibonacci anyons

There are only two anyons for the $k=3$ family: the identity (I) and the Fibonacci anyon τ since $\chi_j \equiv \chi_{\frac{3}{2}-j}$ (Table I). The state of the system is characterized by $x_n = I,\tau$ with $x_1 = \tau$. However, two adjacent I's in the fusion tree would violate the fusion rules. Equation (2) when expressed in the fusion tree basis is given by

$$H_\tau = J_1|I_2\rangle\langle I_2| + \sum_{n=2}^{L-1} J_n H_2^n, \tag{7}$$

where

$$\begin{aligned} H_2^n = &-|\tau I\rangle\langle I\tau I| - \phi^{-2}|\tau I\tau\rangle\langle\tau I\tau| - \phi^{-1}|\tau\tau\tau\rangle\langle\tau\tau\tau| \\ &- \phi^{-3/2}(|\tau I\tau\rangle\langle\tau\tau\tau| + \text{H.c.}). \end{aligned} \tag{8}$$

An interesting modification of the golden chain includes three anyon fusion terms and is the Fibonacci analog of the Majumdar-Ghosh chain [10]. The Hamiltonian of this system is given by

$$\begin{aligned} H_{\text{MG}} = &J\cos\Theta\left(|I_2\rangle\langle I_2| + \sum_n H_2^n\right) \\ &+ J\sin\Theta\left(|I_3\rangle\langle I_3| + \sum_n H_3^n\right), \end{aligned} \tag{9}$$

where H_2^n is described by Eq. (4) and H_3^n describes the fusion of three τ anyons [10]. In the fusion tree basis this is given by

$$\begin{aligned} H_3^n = &P_{\tau I\tau I} + P_{I\tau I\tau} + P_{\tau\tau\tau I} + P_{I\tau\tau\tau} + 2\phi^{-2}P_{\tau^4} \\ &+ \phi^{-1}(P_{\tau I\tau\tau} + P_{\tau\tau I\tau}) - (|\tau\tau 1\tau\rangle\langle\tau I\tau\tau| + \text{H.c.}) \\ &+ \phi^{-5/2}(|\tau I\tau\tau\rangle\langle\tau\tau\tau\tau| + |\tau\tau I\tau\rangle\langle\tau\tau\tau\tau| + \text{H.c.}), \end{aligned} \tag{10}$$

where P_a is the projector onto the state $|a\rangle$, e.g., $P_{\tau\tau\tau} = |\tau\tau\tau\rangle\langle\tau\tau\tau|$ ($P_{\tau^4} = |\tau\tau\tau\tau\rangle\langle\tau\tau\tau\tau|$) [7,10].

III. CHAINS AT CRITICALITY

A prominent feature of broad classes of anyonic chains is that, for uniform nearest-neighbor couplings, their spectra are critical [9,11]. This feature can be explained by the notion of *topological symmetry* [9–11]. Below, we will define this symmetry as it plays an important role in our analysis. In this section we review what is known regarding the CFTs describing the chains as well as the relevant boundary CFT results.

We will describe a given chain by its action $\mathcal{S}_0 = \mathcal{S}_{0,\text{bulk}} + \mathcal{S}_{0,R} + \mathcal{S}_{0,L}$. The term $\mathcal{S}_{0,\text{bulk}}$ represents the bulk action of the system, while $\mathcal{S}_{0,L/R}$ describes the left(L)/right(R) ends of the fusion tree. It should be emphasized that this is a continuum description of the fusion tree basis; $\mathcal{S}_{0,R}$ represents the total fusion product of the chain and not a physical end mode. In an experimental setting in which a chain is formed far from the boundary of the system (i.e., the edge states in a FQHE realization), the total fusion product of the chain will likely be fixed. In what follows, we will relax this condition. However, our definition of topologically trivial and nontrivial phases is not affected by whether the fusion product of a chain is fixed; we define a topologically nontrivial phase as one which exhibits protected anyonic end modes (see Sec. IV). For a broad class of chains the bulk action $\mathcal{S}_{0,\text{bulk}}$ has been identified. For the case of spin-1/2 su(2)$_k$ chains of interest here, this knowledge arises from the mapping to the RSOS model described above [7,9]. We briefly review these CFTs here.

A. AFM-coupled chains

For AFM-coupled spin-1/2 $\mathrm{su}(2)_k$ chains with nearest-neighbor couplings, the bulk is described by the minimal model $\mathcal{M}(k+2,k+1)$ with central charge $c=(k^2+3k-4)/(k^2+3k+2)$ [6,7,30]. These models possess scaling fields $\phi_{(r,s)}$ for $0\leqslant r\leqslant k$ and $0\leqslant s\leqslant k+1$, with the identification $\phi_{(r,s)}\equiv\phi_{(k-r+1,k-s+2)}$. We denote the scaling dimension of this field by $h_{(r,s)}$. The boundary CFT of these models is easily described in the context of the RSOS model [33,34]. The field $\phi_{(1,s)}$ with $s=2j+1$ arises at the end of a system for which the end site of the chain is fixed to $x_L=\chi_j$ and the penultimate site is unconstrained. The field $\phi_{(r,1)}$ with $r=2j+1$ arises for a chain with both the last and penultimate sites fixed with $(x_{L-1},x_L)=(\chi_j,\chi_{j+\frac{1}{2}})$ or $(x_{L-1},x_L)=(\chi_{j+\frac{1}{2}},\chi_j)$. The connection between boundary conditions and scaling fields will play a crucial role in Sec. IV since these dictate the RG flows of the boundary degrees of freedom [19,20,35–37].

These rules also allow for a complete description of critical spectra. Since the fusion of the first two anyons is *not* included in the mapping to the RSOS model, J_1 must be incorporated as a boundary condition. The boundary conditions dictate that $x_1=\chi$. Since $J_1<0$, Eq. (2) will force $x_2=I$. From the boundary CFT given above, this gives

$$\mathcal{S}_{0,L}=\int dt\,\phi_{(1,1)}(t). \tag{11}$$

For the right-hand side of a chain with $L\gg 1$ and even, the possible fusion products of the chain correspond to the anyons χ_j with $j=0,1,2,\ldots$ with $j\leqslant k/2$. At criticality, the penultimate site of the chain is unconstrained. Thus the possible scaling fields are given by

$$\mathcal{S}_{0,R}=\int dt\,\phi_{(1,1)},\quad \int dt\,\phi_{(1,3)},\ldots. \tag{12}$$

The fusion product of the chain is a conserved quantity [not changed by Eq. (2)]. The critical spectrum of a chain is given by, after a trivial rescaling and overall shift, the scaling dimension of all fields in the fusion product of $\mathcal{S}_{0,L}$ and $\mathcal{S}_{0,R}$ and all their descendants. Since $S_{0,L}\propto I$, the spectra is simply given by the scaling dimensions of the fields appearing in $\mathcal{S}_{0,R}$ and their descendants. The field $\phi_{(1,1)}$ is I with scaling dimension 0, while its descendants have scaling dimension $h=2,3,4,\ldots$. The descendants of all the other fields have scaling dimension $h_{(1,s)}+1,h_{(1,s)}+2,\ldots$.

We now compare these predictions with our numerical results for the Ising and Fibonacci chains. For the Ising case, the relevant CFT is $\mathcal{M}(4,3)$ and the field $\phi_{(1,3)}$ (denoted ψ) has scaling dimension $h_{(1,3)}=\frac{1}{2}$. The spectrum in Fig. 3(a) has been calculated by mapping Eq. (6) to a p-wave superconductor and diagonalizing the resultant Hamiltonian for a chain with $L=100$ sites. The spectrum has been plotted as a function of Δ where $J_n=J(1-(-1)^n\Delta)$; thus $\Delta=0$ is the critical point. The corresponding many-body spectrum in Fig. 3(b) is obtained by allowing each of the fermionic modes ω_k shown in Fig. 3(a) to be either occupied or unoccupied. This many-body spectrum is in excellent

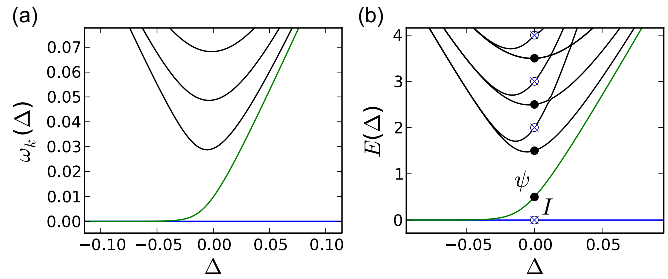

FIG. 3. (Color online) (a) Single-particle spectrum and (b) the many-body spectrum of the coupled Ising anyons described by Eq. (6) for ($L=100$). In both graphs, the energy has been rescaled so the first excited state matches the scaling dimension of ψ (1/2). The state corresponding to I (at $E=0$) and those corresponding to its descendants (at $E=2,3,\ldots$) are marked by a circle with a cross. The states corresponding to ψ and its descendants (at $E=\frac{1}{2}+1$, $\frac{1}{2}+2,\ldots$) are indicated by a filled dot.

agreement with the prediction of Eq. (12). Similarly, for the Fibonacci chain the relevant CFT is $\mathcal{M}(4,5)$, the tricritical Ising model [30]. The field $\phi_{(1,3)}$ (denoted by ε') has scaling dimension $h_{(1,3)}=\frac{3}{5}$ [see Fig. 4(a)]. We note that this also explains the field assignments made in [6].

An alternative way of viewing Eq. (12) is to note that each scaling field of the minimal model $\mathcal{M}(k+2,k+1)$ corresponds to one of the anyons χ_j of $\mathrm{su}(2)_k$. In [9], it was found that the field $\phi_{(r,s)}$ maps to the anyon χ_j for $s=2j+1$. For instance, the corresponding fields in the two theories have identical fusion rules. This notion of connecting the fields of the CFT with corresponding anyons is at the heart of the notion of topological symmetry [9–11]. A field in a CFT is topologically trivial if it represents I; all other fields are topologically nontrivial. Tacit to this argument is the physically reasonable assumption that the boundary field of $\mathcal{S}_{0,R}$ corresponds to the fusion product of the chain (we should note that this is not the

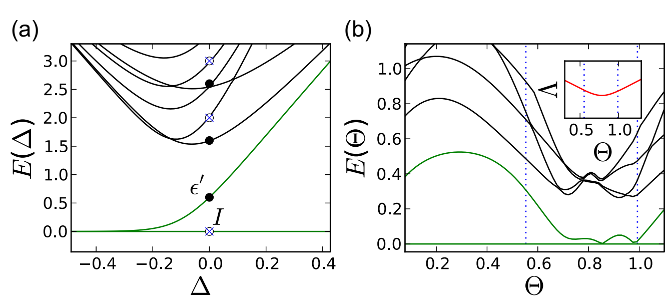

FIG. 4. (Color online) Low-energy spectrum of the AFM "golden chain" for (a) $L=18$ Fibonacci anyons with an unconstrained fusions product. The energies have been rescaled to match the dimension of the corresponding primary fields. The states corresponding to I ($E=0$) and its descendants ($E=2,3,\ldots$) are indicated by a circle with cross. The states corresponding to ε' and its descendents are marked by a filled dot. (b) The low-energy spectrum of the Majumdar-Ghosh chain [Eq. (9)] is gapped between Θ_1 and Θ_2 (indicated by the dotted vertical lines). (b) *Inset* is a plot of Λ in the same range of Θ and is consistent with the gapped phase being topologically nontrivial.

case if there is a constraint on the penultimate site). However, this argument does not fix the values of r appearing in Eq. (12).

B. FM-coupled chains

Chains with FM coupling are described by the $\mathbb{Z}_k$ parafermion coset su(2)$_k$/u(1) with central charge $c = 2(k-1)/(k+2)$ [7,15]. We note that the FM- and AFM-coupled Ising chains are equivalent, while for $k > 2$ they are described by distinct CFTs. For simplicity, we will focus on the case for which k is odd (the case k is even is a straightforward extension). The parafermion coset possesses fields ψ_m^j with $j = 0,1,\ldots,(k-1)/2$ and $m = 0,1,\ldots,k-1$ [9]. The coset construction reveals that the connection between the anyons and these fields is dictated by the value of j with $\chi_j \leftrightarrow \psi_m^j$. Even though we are not aware of general boundary CFT rules relating the FM RSOS model to these fields besides the case $k = 3$ [38,39], the identification between anyons and fields allows us to conclude that the scaling fields at the ends of the chain are

$$\mathcal{S}_{0,R} = \int dt\, \psi_m^0, \quad \int dt\, \psi_{m'}^1, \ldots \tag{13}$$

corresponding to fusion products $x_L = \chi_j$ for $j = 0,1,\ldots \leqslant k/2$. This argument does not constrain the values of $m,m',\ldots$ however. Furthermore, we cannot rule out the possibility that superpositions of fields with the same scaling dimension will appear in Eq. (13).

As discussed in the AFM case, the connection between the boundary CFT and the boundary conditions dictates the RG flows of $\mathcal{S}_R$ when a perturbation is applied to the system. Unfortunately, we are only aware of the results for $k = 3$ which have been worked out in the context of the three-state Potts model [38]. We now show that, as in the AFM case, for $k = 3$ the topologically trivial fields (ψ_m^0) correspond to boundary conditions in which the penultimate site of the chain is fixed, whereas a lack of constraint on this site leads to non-trivial fields (ψ_m^j with $j \neq 0,k/2$).

In the three-state Potts model, each lattice site can take on three possible "spin" directions. The boundary fields ψ_m^0 arise from fixing the last site of the Potts chain to *one* of three possible spin states [38]. All the other fields are associated with boundary conditions in which the last spin can take on at least two possible values. Although there is not a simple mapping between the Potts model and the golden chain, the three possible spin positions in the former must correspond to some combination of the three possible boundary conditions for the golden chain, viz. $(x_{L-1},x_L) = (\tau,I),(\tau,\tau)$, and (I,τ). Crucially, we see that for a given fusion product, fixing the penultimate site of the chain constrains the system to one state and therefore is topologically trivial. On the other hand, for the case $x_L = \tau$, freedom on the penultimate site gives rise to two possible end states and thus is described by a topologically nontrivial end mode ψ_m^j. This correspondence is natural, since any chain which fuses to the identity necessarily has no freedom on the penultimate site and thus we expect this general rule to hold for $k > 3$. However, a full discussion of this point is beyond the scope of this work.

C. Majumdar-Ghosh chain

We briefly remark on the phases of the Majumdar-Ghosh chain which are of interest here [10]. This system exhibits extended regions of Θ for which the system is critical. There exists a gapped phase for $\Theta \in (\Theta_1,\Theta_2)$, where $\Theta_1 = 0.18\pi$ and $\Theta_2 = 0.32\pi$. At the end points of this interval, the system is critical and described by the three-state Potts model [see Fig. 4(b)]. In Sec. IV, we will address whether this gapped phase is topologically trivial or nontrivial.

IV. TOPOLOGICAL INSULATING PHASES

A topological insulating phase of a chain is characterized by nontrivial end modes, χ_j with $j \neq 0$. The topological phase hosts end states and the nearly degenerate ground state is split by their residual interaction which should go as $\sim e^{-\Delta L}$ [31,32]. The possible fusion products of the chain are described by the fixed point

$$\mathcal{S}_{0,R} \to \mathcal{S}_R^* \tag{14}$$

of the RG as a gap generating bulk perturbation is introduced. The fixed points $\mathcal{S}_R^*$ reflect the possible low-energy fusion products of the end modes. Here we only consider perturbations which are left/right symmetric and thus the end modes are identical. In all the cases considered in Sec. III, it was seen that topologically trivial scaling fields are associated with boundary conditions which constrain the penultimate degree of freedom x_{L-1}, whereas nontrivial fields are associated with freedom at this site. This directly informs the RG flow in Eq. (14): perturbations which tend to fix x_{L-1} will lead to a topologically trivial phase, whereas those which relax x_{L-1} will correspond to a set of topologically nontrivial fields and thus topologically nontrivial end modes. We make this notion more precise when considering the various types of chains below.

A. AFM-coupled chains

The CFT $\mathcal{M}(k+2,k+1)$ has relevant fields of momentum $K = 0,\ \pi$ [30]. Thus staggered couplings of the form $J_n = J(1-(-1)^n\Delta)$ will open a gap for $\Delta \neq 0$. In the extreme limit $\Delta = -1$ [see Fig. 1(a)], the chain is composed of dimers (which fuse to I) and two isolated anyons at each end of the chain. We expect an end state will persist even when tuning away from this special point as long as the bulk gap does not close [31]. Thus we expect that $\Delta < 0$ represents a topologically nontrivial phase, while for $\Delta > 0$ the system is trivial.

For a perturbation which relaxes the constraint on the penultimate degree of freedom, the fixed point will contain those fields in $S_{0,R}$ which are relevant. In particular, the only relevant fields $\phi_{(1,s)}$ are $\phi_{(1,1)}$ (scaling dimension 0) and $\phi_{(1,3)}$ [scaling dimension $h_{(1,3)} = k/(k+2)$] and thus the fixed points are

$$\mathcal{S}_R^* = \int dt\, \phi_{(1,1)}, \quad \int dt\, \phi_{(1,3)}. \tag{15}$$

All other fields are irrelevant. For example, $h_{(1,5)} = 2(2k+1)/(k+2) > 1$. Since the last bond on the chain has the form $J_{L-1} = J(1+\Delta)$, we expect that taking $\Delta < 0$ relaxes the conditions on x_{L-1}. This prediction is consistent with both boundary RG flows for the Ising and tricritical Ising models [30,35,36].

Thus $\Delta > 0$ will tend to constrain the x_{L-1} degree of freedom and, according to the boundary CFT rules given in Sec. III, will lead to a fixed point

$$\mathcal{S}_R^* = c_r \int dt\, \phi_{(r,1)}, \tag{16}$$

which is a superposition of trivial fields. Again, these predictions are in agreement with the boundary CFTs of the tricritical Ising model [30,35,36].

The fixed points in Eqs. (15) and (16) have a simple interpretation. Equation (15) corresponds to the fusion products of two χ end modes, i.e., $\chi \times \chi = I + \chi_1$. On the other hand, any linear superposition of topologically trivial fields is simply a proxy for the identity anyon I, and thus Eq. (16) indicates a topologically trivial phase (while in principle bulk perturbations may induce gap flows, this cannot occur in this case [40]). This confirms the physical argument given above. This is also seen to be consistent with Fig. 3(b) in which there is a nearly degenerate ground state for $\Delta < 0$ corresponding to $x_L = I,\psi$. A similar degeneracy exists for the golden chain Fig. 4(a).

For the simple case of a staggered configuration of J_n, it is certainly reasonable that $\Delta < 0$ will lead to relaxation of the penultimate degree of freedom x_{L-1}. It would be nice to have a more systematic means of ascertaining this. Indeed, in more complicated situations in which there is considerable frustration as in the case of the Majumdar-Ghosh chain, it may be very difficult to assess the effect of altering a parameter in the Hamiltonian. While we cannot give a full resolution of this problem, we explore a simple means of assessing this in the case of a chain of Fibonacci anyons.

We consider the effect of the Hamiltonian on the degree of freedom at x_{L-1}. First, we fix $x_{L-2} = x_L = \tau$, thus allowing $x_{L-1} = I$ or τ. We now consider the quantity $\Lambda = |\epsilon_\tau - \epsilon_I|$, where ϵ_τ is the energy of a short segment of the chain with $x_{L-1} = \tau$ and ϵ_I is the energy of a segment with $x_{L-1} = I$. We then calculate the quantity

$$\nu = \mathrm{sgn}\left(\frac{d\Lambda}{d\Delta}\right)\bigg|_{\Delta=0}. \tag{17}$$

If $\nu = 1$, this suggests that as one moves away from the critical point, taking $\Delta > 0$ will tend to constrain the site x_{L-1}. As an example, consider the segment $(x_{L-3},x_{L-2},x_{L-1},x_L) = (\tau,\tau,\tau,\tau)$ for ϵ_τ and let ϵ_I be the energy of the segment (τ,τ,I,τ). From Eq. (7) we have

$$\Lambda = |J||1 + \Delta|, \tag{18}$$

giving $\nu = 1$. This method is not foolproof and may lead to ambiguous results if different sequences have different ν. Since the behavior of interest involves the low-energy part of the spectrum, the sequence should have significant overlap with the ground state. The sequence should also be long enough to include any frustration effects which occur in the Hamiltonian. This approach is imperfect, but nicely complements numerically involved approaches such as entanglement entropy or complete diagonalization.

B. FM-coupled chains

As mentioned above, the robust criticality of anyonic chains stems from topological symmetry. In particular, work has shown that only topologically trivial fields are allowed bulk perturbations [6,9]. It was shown for broad classes of chains (including those studied here) that the only relevant uniform topologically trivial perturbation is the identity [6,9].

We have studied the FM case of Eq. (7). There are two topologically trivial fields with nonzero momentum, ψ_1^0 and ψ_2^0 with momenta $K = 2\pi/3,\ 4\pi/3$, respectively. Furthermore, these fields are relevant (they have a scaling dimension of 2/3). While in principle it should be possible to open a gap with a period 3 texture of J_n, we do not find that these perturbations give rise to an appreciable gap. The reason for this is unclear and warrants further study.

We now discuss the features of topological insulating phases for $k > 3$, k odd. A key difference between $\mathcal{M}(k+2, k+1)$ and the $\mathbb{Z}_k$ parafermion theory leads to an important difference in the nature of the topological insulating phases. In particular, all fields ψ_m^j with $j(j+1) < k+2$ are relevant, and thus these operators are expected to appear in the fixed point $\mathcal{S}_R^*$ corresponding to a topological phase [30]. Since two χ anyons can only fuse to χ_1, such a phase would correspond to end modes χ_j with $j > 1/2$. This is in stark contrast to the AFM case for which only χ and I end modes are possible.

This crucial difference has a straightforward explanation based on energetics. All the topological phases which are possible for a chain arise from strongly fusing some number of χ anyons at the end of the chain and isolating this collection from the rest of the chain. In the AFM case, an even number of χ anyons would lead to a total fusion product I, whereas an odd number of anyons would fuse to a single χ. In contrast, in the FM case the χ_1 fusion channel is favored for neighboring anyons, and thus we expect that a collection of anyons will tend to fuse to χ_j with j as large as possible. This raises the question of why it is not possible to isolate a single χ at the end of the chain such as through dimerization as in Fig. 1(a). Again, this follows from the structure of the CFT. The allowed perturbations ψ_m^0 have momenta $K = 2\pi m/k$ and thus for k odd there is no field of momentum $K = \pi$.

C. Majumdar-Ghosh chain

The end points of the interval $\Theta \in (\Theta_1,\Theta_2)$ are both described by the critical $\mathbb{Z}_3$ parafermion theory. We now turn to the question of the nature of the insulating phase extant for $\Theta_1 < \Theta < \Theta_2$ for the Majumdar-Ghosh chain [Eq. (9)]. In accordance with the principles described in Sec. IV A, we consider the segments $(x_{L-4},\ldots,x_L) = (I,\tau,\tau,I,\tau)$ and (I,τ,τ,τ,τ) and calculate the index Λ. The result of this calculation is shown in the inset of Fig. 4(b). In particular, we find that Λ *decreases* as one enters the topological phase from either end. Indeed, we have found that all possible five term sequences have $d\Lambda/d\Theta < 0$ at $\Theta = \Theta_1$, lending strong support to our conclusion that this phase is topological [41]. Indeed, the exact diagonalization shown in Fig. 4(b) shows that there is a near ground state degeneracy which shrinks as one enters the gapped phase in either direction, offering strong evidence for the existence of τ end modes at the ends of the chain.

V. TOPOLOGICAL QUANTUM COMPUTATION WITH EMERGENT ANYONS

We have studied the conditions under which topological phases may arise in a linear chain as well as the nature of these phases. In addition to appearing at the ends of a chain, protected anyon modes will also occur at domain walls separating topologically trivial and nontrivial phases. As argued in the Introduction, these emergent anyons are likely to be considerably easier to manipulate than physical anyons since large-scale motion may be accomplished by simply tuning local couplings. Indeed, finding protocols which minimize the number of moving anyons has been a priority in the field of TQC [28].

In [27], a T-junction setup [shown in Fig. 1(c)] was investigated in the context of Majorana fermions. Here we consider this setup in the context of generic anyons. Rather than focusing on the very real challenges with constructing such a chain, we consider the pressing question of whether the exchange of these emergent anyons will give rise to nontrivial unitary transformations in the degenerate ground state subspace. This question was considered in [27] and it was shown that for the case of Majorana fermions, the exchange does indeed lead to a "rotation" in the ground state subspace. This is an important consideration, since it is not obvious *a priori* that such exchanges are not sterile. Given the result in [27] and the fact that braiding is a topological property of the system, we expect that this will hold for all anyonic theories.

We now sketch a proof that this is indeed the case, making full use of the results of our analysis in Sec. IV. Our general argument shows that these exchange schemes will result in nontrivial braiding properties for a wide class of anyonic theories. We make use of the fact that these exchanges necessarily obey the so-called hexagon identity [1,2,4]. This identity is analogous to the famous Yang-Baxter relations, though the details of the identity are not required here [1,2]. For any given domain wall, the quantity $\mathcal{S}_R^*$ appearing in Eq. (14) establishes the identity of the fields appearing at the ends of a topologically non-trivial strip [see Fig. 1(c)]. But this knowledge uniquely fixes the fusion rules and the F matrix describing the various ways of fusing the emergent anyons [30]. For a wide class of anyons (including those studied here), the hexagon identity with specified F matrices completely constrains the so-called R matrices which dictate the "rotation" matrices which in turn constrain the braiding matrices B [1,42]. In particular, the possibility that these exchanges are "sterile" (i.e., B is the identity matrix) is thus ruled out. This argument follows for any R matrix that enacts the exchange of an anyon and thus does not require that the anyon be "elementary." In conclusion, we have shown that the exchange of two emergent anyons satisfies the hexagon identity and thus leads to the usual R matrix with the attendant braiding properties.

VI. CONCLUSIONS

We have studied the topological insulating phases of anyonic chains, focusing on the experimentally relevant cases of Ising anyons (equivalent to Majorana fermions) and Fibonacci anyons. Boundary CFT has been shown to be a powerful tool in the study of these systems. The physics studied here suggests a TQC scheme in which emergent anyons are braided and manipulated, requiring only the fine-tuning of the couplings in an otherwise fixed array of physical quasiparticles. Our approach has allowed us to fully characterize the properties of the AFM chains with nearest-neighbor coupling, finding topological phases expected from a simple dimerization picture. Richer effects are shown to arise for FM-coupled chains which are predicted to have topological insulating phases with anyonic end modes which differ from χ. As an application, we have studied a particular region of phase space of the Majumdar-Ghosh chain and identified the gapped phase as being a topological insulator with protected τ end modes. An interesting future avenue for study is the FM chains. In particular, it is not clear why there is not a strong gap for a FM golden chain subject to a periodic perturbation. If similar problems persist for $k > 3$, three anyon fusion terms in the Hamiltonian could be a strategy for opening a gap since they do in the case of the Fibonacci Majumdar-Ghosh chain. Furthermore, the exploration of insulating phases in higher spin analogs of the chains considered here would be of interest as well.

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