Absence of a gap in the Gaffnian state

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We study the Gaffnian trial wave function proposed to describe fractional quantum Hall correlations at Bose filling factor $\nu = 2/3$ and Fermi filling $\nu = 2/5$. A family of Hamiltonians interpolating between a hard-core interaction for which the physics is known and a projector whose ground state is the Gaffnian is studied in detail. We give evidence for the absence of a gap by using large-scale exact diagonalizations in the spherical geometry. This is in agreement with recent arguments based on the fact that this wave function is constructed from a nonunitary conformal field theory. By using the cylinder geometry, we discuss in detail the nature of the underlying minimal model and we show the appearance of heterotic conformal towers in the edge energy spectrum where left and right movers are generated by distinct primary operators.

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I. INTRODUCTION

Coulomb interactions between electrons in the lowest Landau level (LLL) lead to the appearance of the fractional quantum Hall effect (FQHE). Since the kinetic energy is quenched by the magnetic field, the nature of phases is entirely dictated by the interactions and is not amenable to standard many-body treatments. Hence the use of trial wave functions has been the key to unlocking the FQHE physics. The most experimentally prominent quantum Hall fraction at LLL filling factor $\nu = 1/3$ has been explained by Laughlin by means of an explicit first-quantized wave function [1]. Some successful wave functions have been introduced to describe other fractions in the picture of "composite fermions" (CF) [2]. Another line of attack uses conformal field theory (CFT) to produce candidate wave functions by computing expectation values of CFT vertex operators. The prime example in this family is the Moore-Read "Pfaffian" which is built from an Ising-like theory [3]. It is a promising candidate [4,5] to describe the FQHE observed in the second Landau level at filling $\nu = 5/2$. To assess the relevance of a given FQHE wave function, there are a handful of tools at our disposal. If the function has an explicit analytical expression one may compute observables by Monte Carlo evaluation of expectation values. This is the case of the CF wave functions. It is also possible to test these predictions with exact diagonalization (ED) of systems with small numbers of particles that can be treated without *ad hoc* approximations. Some of the trial wave functions have also a simple property: They are exact ground states of some local positive-definite quantum Hamiltonians. This last case is relevant to wave functions that have special vanishing properties when two or more particles come close. The important quantities that should be extracted from a given trial state are the charge and statistics of the quasiparticles, the edge mode characteristics with their scaling exponents. In principle they can be measured by using various experimental techniques. For example, tunneling measurements give access to scaling exponents, interferometry may lead to the braiding statistics of quasiparticles, noise measurements can give the charge, and thermal transport can also be used to find exponents.

In the CFT approach it is immediately clear what are the physically relevant theories from which one can construct a trial wave function. The Laughlin state can be derived from a simple compact boson theory with $U(1)$ symmetry. The Pfaffian state is obtained by adding an extra Ising CFT on top of a similar boson theory. It can be seen as the first member of a set of so-called Read-Rezayi [6] states that involve Z_k parafermions. The Z_k parafermions belong to the series of $\mathcal{W}_k$ minimal models. Such models are indexed by two integers $\mathcal{W}_k(p,p')$ and the parafermions correspond to the subset $\mathcal{W}_k(k+1,k+2)$. There is evidence that they play a role in some FQHE states. Indeed, the state observed at filling $\nu = 2 + 2/5$ in some samples has been suggested to be related by particle-hole symmetry in the second Landau level to the $\nu = 3/5$ fermionic Z_3 parafermionic state. In the realm of the bosonic FQHE, which may be relevant to ultracold atomic gases [7,8], the sequence of states with filling $\nu = k/2$ for $k = 1,2,3,4,\ldots$ may be the ground state after melting of the vortex lattice. These W_k models do not exhaust the list [9,10] of CFTs. One also can consider the family of so-called minimal models $\mathcal{M}(p,p')$, with p,p' integers, to build FQHE wave functions. While the Pfaffian state is related to $\mathcal{M}(4,3)$, which is the same as Z_2 Ising fermions, the other states are different. The state at $\nu = 2/5$ constructed from $\mathcal{M}(5,3)$ was proposed by Simon *et al.* in Ref. [11], where it was called the "Gaffnian." It has certainly some desirable properties like good overlap with the true ground state for Coulomb interactions. Several studies have been devoted to its properties [12–19]. However, there is a potential problem which has been raised originally by Read [20–22]. One expects that the CFT from which one constructs the FQHE trial wave function should be *unitary* if the bulk state has a gap. This severely restricts minimal models to $p = p' + 1$. In the plasma language [23] it means that there is screening of charge. Studies of Gaffnian quasiparticles have also raised doubts [24]. The Pfaffian and the Z_k parafermions are all based on a unitary CFT, hence qualify for describing incompressible FQHE states. The other states should be gapless, hence describe critical points presumably in between several types of FQHE phases.

There is also another way of classifying the models based on the family of Jack polynomials [25–28]. These are multivariate symmetric polynomials that are known to play a role in several integrable systems. It has been shown that they can be characterized by their vanishing properties when two or more coordinates become equal. Some of the wave functions mentioned above, i.e., the Laughlin, Moore-Read, and parafermion wave functions, are all Jack polynomials. Generally speaking they are indexed by a real parameter and a partition of an integer. Bernevig and Haldane have proposed new trial FQHE states for filling factors $\nu = k/r$, k and r being integers. They can be derived from a CFT which is the minimal model [29,30] $\mathcal{W}_k(k+1,k+r)$. These models are unitary only if $r=2$ and this leads back to the Read-Rezayi family of states. We are thus facing again a set of wave functions that should describe critical states. So it is natural to ask if standard tools used in the study of FQHE physics are able to directly prove the criticality of these states. The original study of the Gaffnian state for example is rather inconclusive due to the small number of particles reached in exact diagonalization studies as pointed out by the authors themselves. Subsequent investigations have pointed out possible problems in the entanglement spectrum and also that there are level crossings when interpolating between the Coulomb interaction and the special hard-core interaction for which the Gaffnian is the exact ground state.

In this paper we concentrate on the Gaffnian state in order to obtain direct evidence for criticality. We use exact diagonalizations in the spherical geometry for the bosonic formulation and we are able to reach larger system sizes than previously considered. The Hamiltonian we consider is a one-parameter family interpolating between the pure hard-core two-body δ function interaction and the special three-body interaction involving derivatives of δ functions for which the Gaffnian is the exact ground state. We start from the pure hard-core limiting case where there is ample evidence that the physics is gapped with a description in terms of CF appropriate to bosonic systems. We observe then a definite scaling behavior of the gap of neutral collective excitations vs system size which serves as a reference law when we tune the Hamiltonian towards the Gaffnian limit. Provided we focus on the right angular momentum sector we observe a change in the scaling law which is fully consistent with a critical system. The sphere geometry has no boundary and is well suited to get gap estimates. To get additional insight in the CFT properties we also study the Gaffnian in the cylinder geometry with open boundary conditions. It is known that this is efficient to capture edge state physics when adding a shallow confining potential. Of course we expect that in the thermodynamic limit there should be no distinction between bulk and edge modes for a critical system. However, it needs not be so for finite size systems. Indeed, we show that there are well-defined edge modes that can be classified according to the underlying CFT. As in the case of the Pfaffian [31] the edge modes form conformal towers of the Virasoro algebra. These special sets of states are generated by primary operators of the CFT. In the Gaffnian case we find a new property: the towers are "heterotic," i.e., the left and right moving modes are not generated by the same operator. It is a very special combination of primaries that can explain the edge spectrum. While these properties are found for all cylinder diameters, they become manifest when the cylinder radius goes to zero. Then the problem can be exactly solved for a large number of particles in the subspace generated by quasiholes.

In Sec. II we discuss the vanishing properties of quantum Hall wave functions and their classifications. The Gaffnian state is formulated explicitly. Section III is devoted to CFT aspects of the Gaffnian and we give elements of CFT concerning conformal towers in the cylinder geometry and discuss edge mode properties. In Sec. V we give gap estimates for a family of Hamiltonians and show evidence for criticality of the Gaffnian state. Finally, Sec. VI contains our conclusions.

II. THE GAFFNIAN WAVE FUNCTION

We first discuss FQHE wave functions in the planar disk geometry. In this case, the symmetric gauge is appropriate and the lowest Landau level wave functions are of the form

$$\Psi(z_1,\ldots,z_N) = P(z_1,\ldots,z_N)e^{-\sum_i |z_i|^2/4}, \tag{1}$$

where P is a polynomial which is symmetric for bosons with coordinates $z_1,\ldots,z_N$ and antisymmetric for fermions. In this paper we focus on spin-polarized states only. The magnetic length is set to unity. The angular momentum L_z with respect to the axis perpendicular to the plane is a conserved quantity. The Laughlin state for filling factor $\nu = 1/m$ is given by

$$P(z_1,\ldots,z_N) = \prod_{i<j}(z_i - z_j)^m. \tag{2}$$

It is also very convenient from a theoretical point of view to put the particles on the surface of a sphere with a radial magnetic field as if it were created by a magnetic monopole in the center. This geometry has no boundaries and a finite area. The sphere has full rotation symmetry so states can be classified by their total angular momentum L in addition to the component L_z: They form multiplets of definite L. If the number of flux quanta piercing the sphere is N_ϕ then the LLL has degeneracy $N_\phi + 1$ and a (unnormalized) basis can be taken as

$$\Phi_S^M = u^{S+M}v^{S-M}, \quad M = -S,\ldots,+S, \tag{3}$$

where $2S = N_\phi$ and the spinors are given by $u = \cos(\theta/2)e^{i\phi/2}$, $v = \sin(\theta/2)e^{-i\phi/2}$ in spherical coordinates. Many-body wave functions then become polynomials in u,v variables. For example, the Laughlin state is given by

$$\Psi^{(m)} = \prod_{i<j}(u_i v_j - u_j v_i)^m. \tag{4}$$

The Laughlin wave function Eq. (2) is a model state which is not an exact ground state for electrons with Coulomb interactions. However, it is known to be the exact ground state of a special hard-core Hamiltonian. To understand this property in more detail, one first has to note that any two-body interaction Hamiltonian $\mathcal{H}_{2b}$ in the LLL can be written as

$$\hat{\mathcal{H}}_{2b} = \sum_{i<j}\sum_k V_k \hat{\mathcal{P}}_{ij}^{(k)}, \tag{5}$$

where $\hat{\mathcal{P}}_{ij}^{(m)}$ is the projector onto relative angular momentum m for the pair of particles i,j, m is a positive integer, and the real numbers V_m are the so-called Haldane pseudopotentials that are defined by the choice of the Hamiltonian. In the Laughlin state [Eq. (2)] each pair of particles has a common factor $(z_i - z_j)^m$ and thus at least relative angular momentum m. We construct a special Hamiltonian with the following recipe: We take all pseudopotentials with $k > m$ equal to zero. Then the Laughlin state has exactly zero energy and if we ask for minimum angular momentum it is unique. For example, fermions at $\nu = 1/3$ have an exact Laughlin ground state for the hard-core Hamiltonian $\sum_{i<j} V_1 \hat{\mathcal{P}}^{(1)}_{ij}$.

There is a generalization [32,33] of this line of reasoning by considering now arbitrary k-body interactions instead of two-body interactions. We first discuss the three-body case $k = 3$. Similarly we can define relative angular momentum for three particles. For bosons the minimum value is then zero since all three bosons may be in the same quantum state while for fermions it is three. Consider now the projectors onto definite values of the three-body relative angular momentum $\hat{\mathcal{P}}^{(m)}_{ijk}$. They can be used to define hard-core Hamiltonians by projecting out wave function components with relative momenta less than some values. The simplest case corresponds to the projection of the smallest value. This is the Hamiltonian:

$$\hat{\mathcal{H}}_{\rm Pf} = \sum_{i<j<k} \hat{\mathcal{P}}^{(m)}_{ijk}, \quad m = 0\,(\text{bosons}), \quad m = 3\,(\text{fermions}). \tag{6}$$

If we ask for the zero-energy ground state with minimal angular momentum there is a unique state which is called the Moore-Read Pfaffian state. In planar coordinates the Bose Pfaffian can written as

$$\Psi^{\rm Bose}_{\rm Pf} = \mathcal{S}\left[\prod_{i_1<j_1}(z_{i_1} - z_{j_1})^2 \prod_{i_2<j_2}(z_{i_2} - z_{j_2})^2\right]. \tag{7}$$

In this formula we distribute the particles into two packets of equal size, the respective indices being $i_1, j_1, \ldots$ in one packet and $i_2, j_2, \ldots$ in the other one and we symmetrize ($\mathcal{S}$) over the choices of the two packets. The vanishing properties of this wave function are easily read-off from this formula: If two particles come close together then the function does not vanish because they may belong to distinct packets, however if three particles coincide then at least two of the them will be in the same packet and the wave function will vanish as the second power of their distance. The filling factor is $\nu = 1$ for the Bose case. It can also be written as

$$\Psi^{\rm Bose}_{\rm Pf} = {\rm Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j), \tag{8}$$

where Pf stands for the Pfaffian symbol. This latter object is defined for an arbitrary skew-symmetric $N \times N$ (N even) matrix A:

$$\text{Pf}(A) = \sum_\sigma \epsilon_\sigma A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(N-1)\sigma(N)}, \tag{9}$$

where the sum runs over all permutations of the index with N values and ϵ_σ is the signature of the permutation. This is the original definition of the Pfaffian state by Moore and Read from CFT arguments [3]. The Pfaffian appears from the correlation functions of Ising model Majorana fermions. The fermionic wave functions are obtained by multiplying by the Jastrow factor $\prod_{i<j}(z_i - z_j)$ and the filling factor is now $\nu = 1/2$.

Now we note that if we forbid relative angular momentum zero for three particles with the operator [Eq. (6)], then the next allowed momentum is two. We can then consider the special Hamiltonian that projects out these two possible three-body relative momenta:

$$\hat{\mathcal{H}}_{\rm Gf} = \sum_{i<j<k} \hat{\mathcal{P}}^{(0)}_{ijk} + \sum_{i<j<k} \hat{\mathcal{P}}^{(2)}_{ijk}. \tag{10}$$

This is the definition appropriate to the bosonic case. There is a unique zero-energy ground state provided we ask for the smallest angular momentum. The wave function is then given by the following formula:

$$\Psi_{\rm Gf} = \mathcal{S}\left[\prod_{i_1<j_1\leqslant N/2}(z_{i_1} - z_{j_1})^{2+p} \prod_{N/2<i_2<j_2}(z_{i_2} - z_{j_2})^{2+p} \times \prod_{i\leqslant N/2<j}(z_i - z_j)^{1+p} \prod_{k\leqslant N/2}\frac{1}{(z_k - z_{k+N/2})}\right]. \tag{11}$$

It has been called the Gaffnian in Ref. [11]. The Bose case corresponds to $p = 0$ and the filling factor is $\nu = 2/3$, while $p = 1$ describes a fermionic state at $\nu = 2/5$. It is also possible [34] to write the Gaffnian wave function in the following way:

$$\Psi_{\rm Gf} = \prod_{i<j}(z_i - z_j)^{2(p+1)} \times \mathcal{S}\left[\prod_{i_1<j_1}(z_{i_1} - z_{j_1}) \prod_{i_2<j_2}(z_{i_2} - z_{j_2})\,{\rm per}\,[M]\right]. \tag{12}$$

in terms of the $N/2 \times N/2$ matrix $M_{i,j} = [z_i - z_{j+N/2}]^{-1}$ and $\text{per}[M] = \sum_{\{\sigma\}}\prod_{k=1}^N M_{k,\sigma(k)}$ is the permanent of M, where the sum is over all σ permutations of N elements. The symmetrized wave function (11), as a candidate for the FQHE at $\nu = 2/5$, has been studied by Yoshioka *et al.* in Ref. [35], and one obtains a large overlap with the ED ground state for a Coulomb interaction. Furthermore, this wave function, also called "Gaffnian," has recently been studied within CFT and may support non-Abelian quasiparticle excitations.

III. CONFORMAL FIELD THEORY CONSTRUCTION OF THE GAFFNIAN

We describe the construction of the Gaffnian wave function according to the general CFT approach introduced by Moore and Read. The bulk wave function in the planar geometry is taken to be the expectation value of a set of operators of a 1 + 1D CFT:

$$\Psi(z_1,\ldots,z_N) = \langle 0|O(z_1)\cdots O(z_N)O_{bk}|0\rangle, \tag{13}$$

where the operator $O(z)$ is the product of a vertex operator for the charge sector described by a free boson $\phi_c(z)$ and a statistical field belonging to some CFT $O(z) = \psi(z)e^{i\phi_c(z)/\sqrt{\nu}}$. One needs to add also a background charge operator O_{bk} to ensure global electric neutrality and having a nonzero correlation function. Alternative constructions are possible with a different role for the neutrality [36]. In the case of the Laughlin state there is no statistical sector and we just have to compute the correlation function of exponentials of a free boson:

$$\Psi_L(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^{1/\nu}. \quad (14)$$

In the Moore-Read Pfaffian case the statistical operator $\psi(z)$ is taken to be a Majorana fermion so that there is an additional factor given by

$$\langle\psi(z_1)\cdots\psi(z_N)\rangle = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right). \quad (15)$$

This leads to the previous formula [Eq. (8)] for the Pfaffian state. Now if we consider a more general CFT state we need a field ψ with fusion relation $\psi \times \psi \sim \mathbf{1}$, conformal weight Δ_ψ, and operator product expansion

$$\psi(z)\psi(w) \sim \frac{1}{(z-w)^{2\Delta_\psi}}[\mathbf{1} + \cdots], \quad (16)$$

so that the wave function is given by

$$\Psi(z_1,\dots,z_N) = \langle\psi(z_1)\cdots\psi(z_N)\rangle \prod_{i<j}(z_i - z_j)^{2\Delta_\psi + q}. \quad (17)$$

The possible values of q are then dictated by simple statistics requirements. In the case of the Gaffnian state, authors of Ref. [11] have advocated the use of the CFT defined by the Virasoro minimal model $\mathcal{M}(5,3)$ which contains a field ψ with the correct fusion rule [Eq. (16)] and dimension $\Delta_\psi = 3/4$. This CFT also contains two other primary fields φ and σ of dimensions $\Delta_\varphi = 1/5$ and $\Delta_\sigma = -1/20$, respectively.

According to the bulk/boundary correspondence we expect that the edge theory of a CFT-derived quantum Hall state should be given by the very same CFT that is used to construct the bulk ground state wave function. A precise derivation of this correspondence has been given in Ref. [22].

If we consider the edge spectrum of a theory we expect the energies to be arranged in so-called conformal towers of states since they should form a representation of the underlying Virasoro algebra of the CFT. We briefly describe for completeness the tower structure which is explained in detail in the CFT literature [9,10]. It is based on the use of the so-called generators of the Virasoro algebra L_n with n an integer, positive or negative.

If we create a state by acting with a primary operator onto the vacuum $|\Phi\rangle = \Phi(0)|0\rangle$ then this state is annihilated by all Virasoro generators L_k with $k > 0$ and is an eigenstate of L_0 with eigenvalue Δ_Φ. Action with the other generators with $k < 0$ generates a family of states called the descendants of the primary state:

$$|\{n_i\}\rangle = L_{-n_k}\cdots L_{-n_1}|\Phi\rangle, \quad (18)$$

where the indices n_i are positive. We call the "level" of such a state the integer $n \equiv \sum_k n_k$. Not all these state are orthogonal and in a given CFT there are relations between states at a given level so that the number of possible states is not simply given by counting the partition of the level n into k integers. If we introduce $p(\Phi,n)$ as the number of independent states at level n, it is conveniently manipulated through its generating functional:

$$\chi(\Phi,q) = \sum_{n=0}^{\infty} p(\Phi,n)q^n. \quad (19)$$

This quantity is called the character and encodes the structure of the representation of the Virasoro algebra. Expansion in powers of q allows us to count the states. In the case of minimal models $\mathcal{M}(p,p')$ there is a finite number of primary operators $\Phi_{r,s}$ with $r = 1,\dots,p-1$, $s = 1,\dots,p'-1$ with the redundancy $\Phi_{r,s} \equiv \Phi_{p-r,p'-s}$. Their scaling dimensions are given by

$$h_{r,s} = \frac{(sp - rp')^2 - (p-p')^2}{4pp'}. \quad (20)$$

The Gaffnian CFT $\mathcal{M}(5,3)$ has fields $\psi = \Phi_{1,4}$, $\varphi = \Phi_{1,3}$, $\sigma = \Phi_{1,2}$ and identity operator $\mathbf{1} = \Phi_{1,1}$. To obtain the characters of a minimal model, we can use the Rocha-Caridi formula which is valid for generic minimal models, unitary or not. It is given by

$$\chi(\Phi_{r,s},q) = \frac{1}{(q)_\infty}\sum_{k=-\infty}^{k=+\infty}\left(q^{k^2pp' + k(pr - p's)} - q^{(kp'+r)(kp+s)}\right), \quad (21)$$

where we have used the symbol

$$(q)_\infty \equiv \prod_{n=1}^{\infty}(1 - q^n). \quad (22)$$

Alternatively, a fermionic representation for the characters in the peculiar case of $\mathcal{M}(5,3)$ is available from the work of Kedem *et al.* [37]:

$$\chi(\Phi_{1,1},q) = \sum_{f=0}^{\infty}\frac{q^{f(f+1)}}{(q)_{2f}}, \quad \chi(\Phi_{1,2},q) = \sum_{f=0}^{\infty}\frac{q^{f^2}}{(q)_{2f}}, \quad (23)$$

$$\chi(\Phi_{1,3},q) = \sum_{f=0}^{\infty}\frac{q^{f(f+1)}}{(q)_{2f+1}}, \quad \chi(\Phi_{1,4},q) = \sum_{f=0}^{\infty}\frac{q^{f(f+2)}}{(q)_{2f+1}}, \quad (24)$$

where

$$(q)_k \equiv \prod_{n=1}^{k}(1 - q^n). \quad (25)$$

With these character formula one can extract the degeneracy appearing in the towers. Low-lying state counting is given in the upper part of Table I for each primary field of the Gaffnian CFT. Provided the bulk is gapped, these degeneracies would be seen in the spectrum of edge modes in the disk geometry. While this is the case for the Laughlin and Pfaffian case, we show in the next section that this is not case for the Gaffnian.

IV. GAFFNIAN CONFORMAL TOWERS FROM THE CYLINDER GEOMETRY

In a generic nonchiral statistical critical model we expect energies to be related to Virasoro generators by

$$\mathcal{H} \propto L_0 + \bar{L}_0, \quad (26)$$

TABLE I. Upper part: Number of states at each level of the Virasoro algebra for the statistical part $\mathcal{M}(5,3)$ of the Gaffnian CFT for each primary operator. This gives edge state counting in the disk geometry. Lower part: We have added the boson sector to obtain the full Gaffnian state counting.

$\Phi_{1,1} = \mathbf{1}$	1 0 1 1 2 2 4 4 6 7 10 11
$\Phi_{1,2} = \sigma$	1 1 1 2 3 4 5 7 9 12 15 19
$\Phi_{1,3} = \varphi$	1 1 2 2 3 4 6 7 10 12 16 20
$\Phi_{1,4} = \psi$	1 1 1 2 2 3 4 5 7 9 11 14
$\phi_c + \mathbf{1}$	1 1 3 5 10 16 29 45 74
$\phi_c + \sigma$	1 2 4 8 15 26 44 72 115
$\phi_c + \varphi$	1 2 5 9 17 29 50 80 129
$\phi_c + \psi$	1 2 4 8 14 24 40 64 101

and momenta are expressed through the difference of the generators:

$$\mathcal{P} \propto L_0 - \bar{L}_0. \tag{27}$$

In these equations the barred Virasoro generators pertain to the antichiral copy of the algebra. If we act with these relations onto a set of descendant states [Eq. (18)], we generate a set of energies and momenta:

$$E_{n,\bar{n}} \propto n + \bar{n} + h + \bar{h}, \quad P \propto n - \bar{n}, \tag{28}$$

where $n = \sum_i n_i$ and $\bar{n} = \sum_i \bar{n}_i$ are levels in the chiral and antichiral Virasoro algebras.

In the realm of FQHE a nonchiral setup is obtained by considering the cylinder geometry which has naturally two counterpropagating edges. The Fock space of the edge theory is given by the tensor product of the two boundary Fock spaces. We use thus use periodic boundary conditions along one direction of the cylinder and impose for simplicity a hard cutoff in orbital space to define a finite-dimensional problem. In this geometry there is only one conserved momentum $\mathcal{K}$ along the periodic direction. It can be used to label many-body eigenstates.

The Gaffnian Hamiltonian in the disk geometry can be written as

$$\hat{\mathcal{H}}_{\rm Gf} = \sum_{i<j<k} \mathcal{S}\left[\nabla_i^4\, \delta^2(z_i - z_j)\, \delta^2(z_j - z_k)\right]. \tag{29}$$

This Hamiltonian when written in the cylinder basis has a unique ground state with zero momentum provided we fix the number of orbitals as

$$2\mathcal{K} + 1 = \tfrac{3}{2}N - 2, \tag{30}$$

as in the spherical geometry. If we add extra orbitals then additional zero-energy quasiholes appear. This is displayed in Fig. 1.

The set of quasihole states is exactly degenerate provided there is perfect translation invariance. If we now add a shallow confining potential along the axis of the cylinder the degeneracy is lifted and it is possible to identify the edge excitations as conformal towers in the low-energy points of the spectrum. This is possible in the Laughlin [38] and Pfaffian [31] cases. This picture is valid as long as there is a clear separation between the set of quasihole-derived states and the bulk states. In the Gaffnian case we show in the next

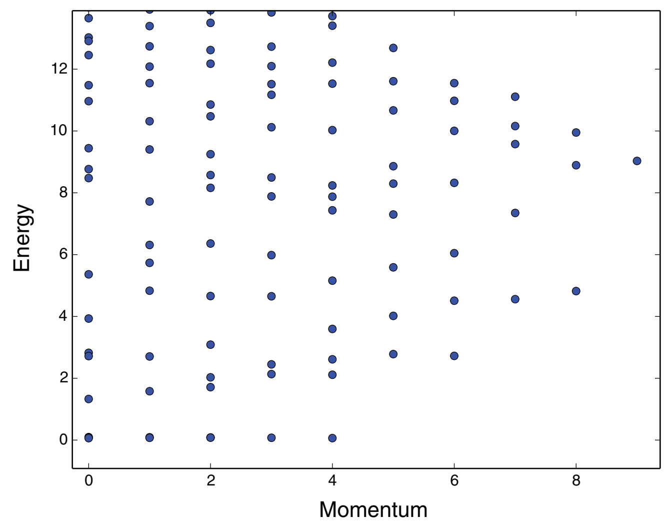

FIG. 1. (Color online) Low-lying spectrum of the special Gaffnian Hamiltonian [Eq. (29)] in the cylinder geometry for $N = 8$ bosons and 10 orbitals. Eigenstates are classified as a function of the conserved total momentum K. There is a set of zero-energy eigenstates that extend up to $K = 4$ that are Gaffnian quasiholes.

section that the bulk gap goes to zero in the thermodynamic limit. This means that the edge mode identification should be considered as an artifact: They will mix with bulk modes as soon as the system is large enough. Indeed, we observe that when the radius L of the cylinder is of order of a few magnetic lengths it is difficult to identify the edge modes. The situation becomes on the contrary very clear when going to the thin cylinder limit $L \to 0$. It is then possible to study numerically large systems provided one focuses onto the subspace generated by the quasihole states that have exactly zero energy in the absence of confining potential: One has just to solve a problem of minimum electrostatic energy under constraints. The lowest-lying states are then clearly arranged into conformal towers as can be seen in Fig. 2. The ground state in each of the sectors we find can be tentatively identified by comparing the degeneracies observed with those deduced from the CFT predictions in the lower part of Table I, which includes the charged boson mode in addition to the statistical field.

When the number of bosons is even there are two towers that alternate when increasing the total momentum: (a) and (b) in Fig. 2. The tower of type (a) is generated by the identity operator as well as φ operator. The ground state of tower (a) is thus

$$|0,a\rangle = \mathbf{1}\; e^{2in\phi_c/\sqrt{6}} e^{2in\bar{\phi}_c/\sqrt{6}}|0\rangle, \tag{31}$$

and the first excited state with the same total momentum is obtained by acting with the primary field φ:

$$|1,a\rangle = \varphi\bar{\varphi}\; e^{2in\phi_c/\sqrt{6}} e^{2in\bar{\phi}_c/\sqrt{6}}|0\rangle. \tag{32}$$

For tower (b) it involves the two other primary fields, the ground state is now

$$|0,b\rangle = \sigma\bar{\sigma}\; e^{i(2n+1)\phi_c/\sqrt{6}} e^{i(2n+1)\bar{\phi}_c/\sqrt{6}}|0\rangle, \tag{33}$$

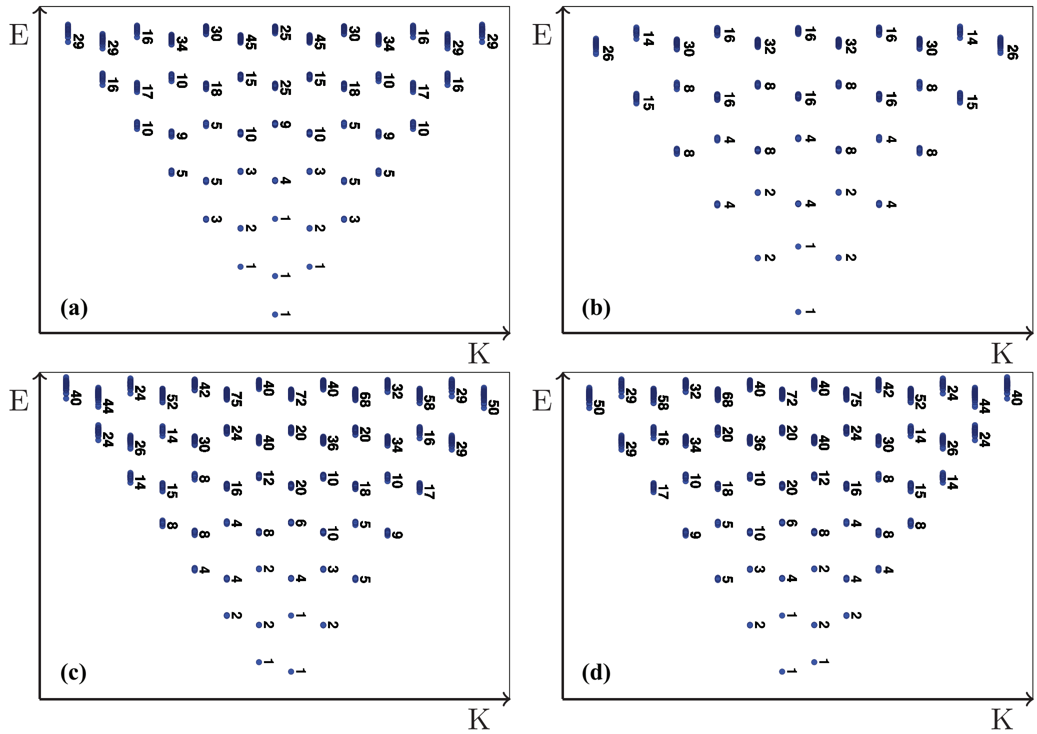

FIG. 2. (Color online) Conformal towers of the Gaffnian state. The spectra are found for even number of bosons in (a) and (b), and odd number of bosons for (c) and (d). Close to each set of quasidegenerate levels we have given the associated degeneracy. These numbers are exactly those predicted by CFT. Operator assignments of these towers are given in the text. We note that odd towers (c) and (d) involve different operators for left and right moving modes: they are "heterotic."

and the first excited state now given by

$$|1,b\rangle = \psi\bar{\psi}\; e^{i(2n+1)\phi_c/\sqrt{6}} e^{i(2n+1)\bar{\phi}_c/\sqrt{6}}|0\rangle. \tag{34}$$

These towers are analogous to those found in the Pfaffian case [31] in the sense that the left and right moving modes are generated by the same operator. In these formulas n is an integer that varies as we go from one tower to its neighbor. Increasing total momenta and hence going through several towers means that we are encountering all the topological sectors [29,39] of the Gaffnian (in general multiple times).

When there is an odd number of bosons there are also two alternating types of towers: see (c) and (d) in Fig. 2 that are mirror symmetric from each other. Hence, there is essentially only one type of conformal tower. While their almost twofold ground state degeneracy is reminiscent of the ψ tower of the Pfaffian, they are structurally different since now right and left moving modes are not generated by the same operator. To borrow terminology from string theory [40] they are aptly said to be "heterotic." In the (c) case the leftmost ground state leads to a tower based on two operators ψ for the left-handed modes and the identity for the right-handed modes. Its almost degenerate rightmost ground state generates a tower based on σ for the left modes and φ for the right modes. We have checked that these assignments hold at least up to the sixth level. This can be easily checked with the data in Table I. We have thus recovered the tower structure of edge states expected from the Gaffnian CFT. This is possible due to two facts: We use the cylinder geometry with widely separated edges for small radius and we restrict the Hilbert space to the set of quasihole states. The states that appear in these towers are bona fide quantum mechanical states with a perfectly well-defined positive scalar product since we are dealing with constrained electrostatics. Since the underlying CFT is nonunitary it means that this structure cannot persist in the true thermodynamic limit. On the cylinder geometry, we observe that when going to large cylinder radius it is no longer possible to identify clearly the descendant states. This is a hint of what goes wrong with the Gaffnian. In fact, we now give direct evidence in the next section for its critical character.

V. GAP ESTIMATES ON THE SPHERICAL GEOMETRY

To investigate the possible criticality of the Gaffnian state we revert to the spherical geometry. We focus on its bosonic formulation because this allows for larger sizes to be studied numerically. When written on the sphere the wave function [Eq. (11)] requires a definite relationship between flux and number of bosons:

$$2S = N_\phi = \tfrac{3}{2}N - 3. \tag{35}$$

This coincides with the relation for the principal or Jain series of quantum Hall fractions at filling factor $\nu = 2/3$. Indeed, it is known that bosons in the LLL with hard-core interactions form usual Abelian FQHE states at fillings $\nu = p/(p+1)$. They fit in the CF scheme by introducing ^{1}CF entities that are bound states of one boson and one vortex. These ^{1}CFs feel reduced magnetic field $2S^* = 2S - (N-1)$ and integer fillings of CF LLs describes a FQHE state as in the fermionic FQHE. With two filled levels we obtain the Bose state $\nu = 2/3$ which appears to be incompressible in numerical studies for

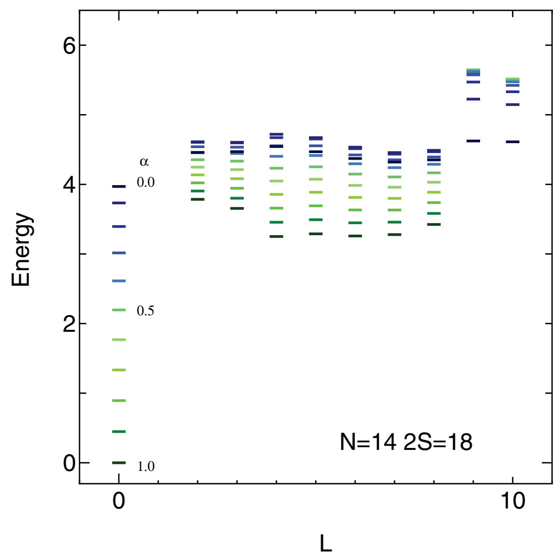

FIG. 3. (Color online) The energy spectrum of the family of Hamiltonians interpolating between the Gaffnian and the pure two-body projector. There are $N = 14$ bosons on the sphere with $N_\phi = 18$. Energy levels are classified by their total angular momentum in the spherical geometry. The two-body case is at the top of the picture and displays a collective mode with a dispersion relation similar to what is observed in smaller systems. When we switch on the three-body Hamiltonian [Eq. (10)] the dispersion relation first flattens and then acquires a hanging-chain shape as in other paired states. The value of $L_{\rm tot}$ with minimal energy changes as a function of α.

the pure contact interaction. On the sphere such a state appears as a rotationally invariant $L_{\rm tot} = 0$ state. Neutral excitations are obtained by promoting some CF from full to empty levels. This operation creates a branch of states dubbed excitons where a CF goes from the level with $L = S^* + 1$ to the first empty level $L = S^* + 2$. This branch is prominent in exact diagonalization studies and is reasonably well described by CF wave functions adapted for the Bose statistics.

In order to have some control on the gap estimate for the Gaffnian we study a family of Hamiltonians that interpolates between the special three-body interaction of Eq. (10) and the pure two-body hard-core case:

$$\hat{\mathcal{H}}_\alpha = \alpha\ \hat{\mathcal{H}}_{Gf} + (1-\alpha)\sum_{i<j}\hat{\mathcal{P}}^{(0)}_{ij}, \tag{36}$$

At fixed flux number of particles relation [Eq. (35)] one interpolates between the Gaffnian state and the CF state. The spectra for $N = 14$ bosons is displayed in Fig. 3 for various values of the α interpolating parameter. We have plotted only the lowest energy state in each L sector. The ground state is always at $L = 0$. It is only for $\alpha = 1$ that the ground state energy is zero. For other values there are pairs of particles with zero relative angular momentum. If we now try to extrapolate the finite-size gap to the thermodynamic limit, we observe the scaling in Fig. 4. As small boson systems generally have severe finite-size effects, larger boson systems are desirable for extrapolation. In fact, excitation energies with $N = 8$ bosons shows irregularity for all cases. As α increases, those with $N = 10$ bosons do still show some irregularity. On the other hand, taking into account only the three largest sizes $N = 12$, 14 and 16, we find very good regularity. For increasing values of α towards the pure Gaffnian Hamiltonian we find also a rather smooth dependence that we consider as evidence for zero gap when $\alpha = 1$.

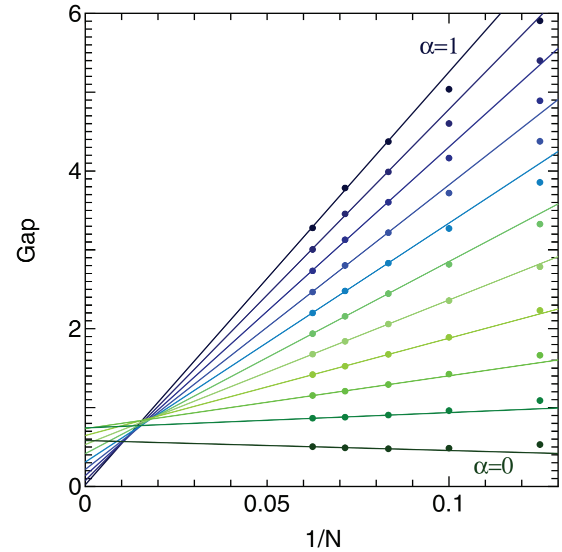

FIG. 4. (Color online) Extrapolation of gaps to the thermodynamic limit. The gaps are plotted versus inverse of the number of particles. We display excitation energies from 8 to 16 bosons while extrapolations are carried out with only the largest sizes $N = 12$, 14, and 16 bosons. The angular momentum of the state is fixed at $L_{\rm tot} = 2$. Only the special Gaffnian Hamiltonian is gapless within the precision of our method. The lowest-lying straight line corresponds to $\alpha = 0$ and the topmost line to $\alpha = 1$ with intermediate values differing by 0.1.

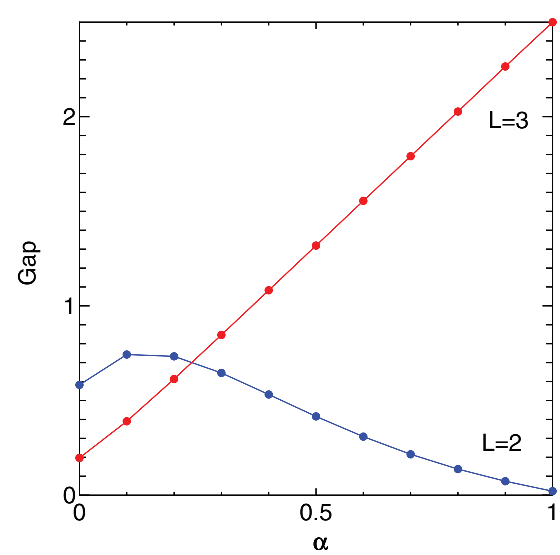

FIG. 5. (Color online) The extrapolated gaps as a function of the interpolating parameter α based on sizes $N = 12$, 14, 16. The lower (blue online) curve is obtained from $L_{\rm tot} = 2$ gaps and the upper (red online) curve is for $L_{\rm tot} = 3$. The gapped two-body case is for $\alpha = 0$ and the Gaffnian case for $\alpha = 1$. There is evidence for zero gap only at the Gaffnian point and the gap opens linearly with $|1-\alpha|$.

We present estimates for infinite-system gaps in Fig. 5. The gap opens immediately as soon as we add the two-body interaction. The gap behavior is close to linear as a function of $|1-\alpha|$ close to $\alpha = 1$. This means that the two-body interaction should be considered as a relevant perturbation with respect to the critical Gaffnian state. The gap in the $L = 3$ sector is also plotted in Fig. 5 and shows that there are in fact multiple level crossings in the excited states when approaching the CF regime for $\alpha \approx 0$.

This is similar to the observation of Ref. [24] of level crossings in the sector of charged quasiparticles.

VI. CONCLUSIONS

We have studied various aspects of the Gaffnian state which is based on a nonunitary CFT $\mathcal{M}(5,3)$. As such it is expected to describe a gapless state of matter [22]. By exact diagonalizations on large systems in the spherical geometry we have given numerical evidence for its critical nature. The pure two-body contact interaction projected onto the LLL appears to be a relevant perturbation which opens a gap even for infinitesimal coupling. The criticality of the Gaffnian we find is in agreement with other lines of attack [28,41,42].

By use of the cylinder geometry we have shown the appearance of the conformal towers expected from its underlying CFT structure. The towers are based on all primary operators of the theory and in the case of an odd number of particles they are heterotic, i.e., their left and right movers are created by different operators, at variance with the Laughlin [38] or Moore-Read case [31]. These towers appear however only in the thin-torus limit which has no thermodynamic limit. Indeed, since there are negative norm states in the representations of the Virasoro algebra, there should be drastic changes in the energy spectrum when going to the true thermodynamic limit. Here what happens is that there are states that were looking like "bulk" states in Fig. 3 that mix with "edge" states.

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