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Critical integer quantum Hall topology and the integrable Maryland model as a topological quantum critical point

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One-dimensional tight binding models such as the Aubry-André-Harper (AAH) model (with an on-site cosine potential) and the integrable Maryland model (with an on-site tangent potential) have been the subject of extensive theoretical research in localization studies. AAH can be directly mapped onto the two-dimensional (2D) Hofstadter model which manifests the integer quantum Hall topology on a lattice. However, such a connection needs to be made for the Maryland model (MM). Here we describe a generalized model that contains AAH and MM as the limiting cases with the MM lying precisely at a topological quantum phase transition (TQPT) point. A remarkable feature of this critical point is that the one-dimensional MM retains well defined energy gaps whereas the equivalent 2D model becomes gapless, signifying the 2D nature of the TQPT.

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The integer quantum Hall Effect (IQH) is a canonical example of a gapped bulk topological phase with no generic symmetry protection. IQH can be captured by the two-dimensional (2D) Hofstadter model [1–10], a 2D lattice tight binding model with nonzero flux per unit cell. The Hofstadter model can be mapped onto the one-dimensional (1D) Aubry-André-Harper [11,12] (AAH) model, a 1D tight binding chain with an on-site cosine potential. Aubry and André [12] identified a localization transition in the AAH model with modulation incommensurate with the lattice (corresponding to an irrational value of flux). This result led to an extensive theoretical investigation of the AAH model in the context of localization studies [12–17]. Recent experimental developments in photonic crystals [18–20] and ultracold atoms [21–23] have realized these localization phenomena in 1D quasiperiodic AAH lattices.

A completely different example of a 1D tight binding model with an on-site tangent modulation is presented by the 1D "Maryland model." The Maryland model (MM) was proposed and solved exactly by Grempel *et al.* [24–26]. MM has a one-to-one correspondence with the quantum kicked rotor problem, which has been experimentally realized in ultracold atoms [27], and has been extensively studied [28,29]. In this Rapid Communication, we provide a mathematical connection between MM and IQH. In addition, we show that MM presents an intriguing example of a topological quantum phase transition (TQPT).

The Maryland model with the period of the on-site potential incommensurate with the lattice spacing presents an example of a 1D quasicrystal (QC) for which a special "quasiperiodic" translation symmetry was recently identified [19]. A family of 1D QCs taken together (generalized AAH, Fibonacci [30]) [20,31] has been topologically classified with an equivalent IQH topology in 2D. This classification was identified by connecting different models of QCs with the same topological invariant corresponding to the real space 2D lattice with a flux [31]. An argument was made [19] and subsequently debated [32,33] that this quasiperiodic translation symmetry allows one to associate 2D IQH invariants to each 1D member of the family of QCs [19]. The fact that the MM belongs to this 1D quasicrystal symmetry class and was not associated with the IQH topology calls for an investigation of this model from another perspective. We base our arguments only on the well established connection between families of 1D tight binding models with periodic modulation and 2D IQH topology [1,11,12].

In this Rapid Communication, we take a different approach in understanding the relationship between the IQH topology and the Maryland model. We construct a family of 1D tight binding models parametrized by a phase with a general on-site modulation potential that contains AAH and MM as limiting cases. We construct the equivalent real space 2D lattice model by taking an inverse Fourier transform with respect to this phase parameter. We analyze the energy spectrum of the general 1D model as a function of the phase parameter. We identify the topological invariants for this general model by using the theory of electric polarization [34,35], which provides a natural framework to study IQH invariants. Based on this analysis we explicitly show that the Maryland model sits at the critical point of a quantum phase transition to the topologically trivial state. The criticality of the Maryland model allows us to associate topological invariants to it in a purely mathematical sense by using the limiting procedure along the deformation path in the parameter space. We show that even though the 1D gaps are preserved throughout the deformation from AAH to MM, the energy gaps in the equivalent 2D model close at the TQPT, as required by general considerations. We discuss the consequences of this result for the topological classification of 1D QC families [19,32,33].

We consider a 1D tight binding chain of size N with an on-site potential modulation $V_n(\alpha,\varphi)$,

$$H(\varphi,\alpha) = -\sum_{n=1}^{N-1} t(c_{n+1}^{\dagger}c_n + c_n^{\dagger}c_{n+1}) - \sum_{n=1}^{N} V_n(\alpha,\varphi)c_n^{\dagger}c_n,$$
$$V_n(\alpha,\varphi) = 2\lambda\left(\frac{\cos\left(2\pi nb + \varphi - \alpha\frac{\pi}{2}\right)}{1+\alpha\ \cos(2\pi nb+\varphi)}\right), \tag{1}$$

where $c_n^{\dagger}$ and c_n are creation and annihilation operators on the site $n = 1,2,\ldots,N$, and t is the amplitude of the nearest neighbor hopping. The on-site potential $V_n(\alpha,\varphi)$ is characterized by the strength λ, period $1/b$, and the phase parameter φ. The parameter α interpolates between the limiting cases AAH ($\alpha = 0$) and MM ($\alpha = \pm 1$),

$$V_n(\alpha,\varphi) = 2\lambda \begin{cases} \cos(2\pi nb + \varphi), & \text{for} \quad \alpha \to 0, \\ \left[\tan\left(\frac{2\pi nb+\varphi}{2}\right)\right]^{\alpha} & \text{for} \quad \alpha \to \pm 1. \end{cases} \tag{2}$$

This general on-site potential is a smooth function of α in the open interval $\alpha \in (-1,1)$. $V_n(\alpha,\varphi)$ has singularities at $\alpha = \pm 1$ corresponding to the integrable MM, which we approach asymptotically in a limiting sense, and we define TQPT in terms of these singularities. $V_n(\alpha,\varphi)$ is a specific example of a generic 2π periodic on-site potential $\mathcal{F}[2\pi nb + \varphi]$, where $\mathcal{F}(z)$ is an analytic function everywhere except in the limit of singular MM, where it acquires isolated poles.

2D ancestor. Taking an inverse Fourier transform with respect to φ results in a real space lattice, which is the 2D Hofstadter model with a flux b per unit cell. The same idea applies to the Hamiltonian in Eq. (1),

$$H_{2\text{D}}(\alpha) = \int_{-\pi}^{\pi} d\varphi H(\varphi,\alpha) e^{im\varphi},$$
$$c_n \equiv c_n(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_m e^{im\varphi} c_{n,m}, \tag{3}$$

where $n,m = 1,\ldots,N$. The resulting real space 2D Hamiltonian that is equivalent to $H(\varphi,\alpha)$ (up to a constant energy shift) reads

$$H_{2\text{D}}(\alpha) = \sum_{m,n} \left[t(c^{\dagger}_{n,m} c_{n+1,m} + c^{\dagger}_{n+1,m} c_{n,m}) + 2\lambda \sum_{l=0}^{\infty} [I_{nl}(\alpha) c^{\dagger}_{n,m} c_{n,m-l} + \text{H.c.}] \right], \tag{4}$$

where

$$I_{nl}(\alpha) = e^{-il(2\pi nb + \alpha\frac{\pi}{2})} \left[\frac{e^{i\pi\alpha}}{\alpha} \delta_{l,0} + (-1 + \sqrt{1-\alpha^2})^{l-1} \times \frac{[2 - \alpha^2(1 - e^{i\pi\alpha}) - 2\sqrt{1-\alpha^2}]}{2\alpha^{l+1}\sqrt{1-\alpha^2}} \right] \tag{5}$$

describes the hopping amplitude from site $m - l$ to m, i.e., a hopping of range l (the $l = 0$ term is the constant shift in the on-site energy). Note that $I_{nl}(\alpha)$ in Eq. (5) is defined in a limiting sense at the special points $\alpha = 0, \pm 1$ (AAH and MM) [36]. Figure 1 plots the absolute value of the hopping amplitude $|I_{nl}(\alpha)|$ for different values of the hopping range l as a function of α. In the limiting case of $\alpha \to 0$ (AAH) only the $l = 1$ term survives, $I_{n1}(0) = 1/2$, which corresponds to the nearest neighbor hopping of the Hofstadter model. As α increases, hopping terms of longer range l in the m direction acquire nonvanishing amplitudes. In the limiting case of $\alpha \to \pm 1$ (MM), the dual 2D lattice acquires long range hopping terms of arbitrarily large l in the m direction, all of equal unit amplitude [see Eq. (5)]. This arbitrarily long range hopping singularity is indicative of a quantum phase transition occurring at the critical points $\alpha = \pm 1$. To further elucidate the physical nature of these $\alpha = \pm 1$ critical points, we analyze the band structure and the topological invariants of Eq. (1) as a function of α.

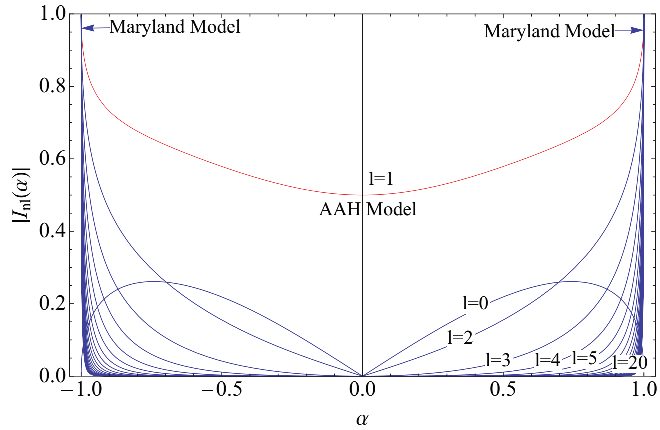

FIG. 1. (Color online) $|I_{nl}(\alpha)|$ as a function of α for different hopping range l. As $\alpha \to 0$ (AAH model), only the nearest neighbor hopping term is nonzero ($l = 1$, shown in red). Long-range hopping amplitudes increase with α, and in the limit $\alpha \to \pm 1$ (Maryland model) the hopping amplitudes of all ranges are equal to unity ($|I_{nl}(\alpha)| = 1$).

Band structure. We impose open boundary conditions on the 1D tight binding Hamiltonian $H(\varphi,\alpha)$ in Eq. (1) and numerically diagonalize it for the system size of $N = 200$ sites. It is instructive to plot the resulting energy bands as a function of the phase parameter φ, which captures the 2D band structure in the hybrid space (n,φ). We start with the case of a commensurate modulation by setting $b = 1/5$ and $\lambda = 1$. Figure 2 shows the resulting band structure as a function of the phase parameter φ for four different values of α. The case of AAH ($\alpha = 0$) (top left panel of Fig. 2) demonstrates a well defined set of gaps reflecting the robust integer quantum Hall topology of the 2D Hofstadter model. For $\alpha = 0.8$ and 0.98,

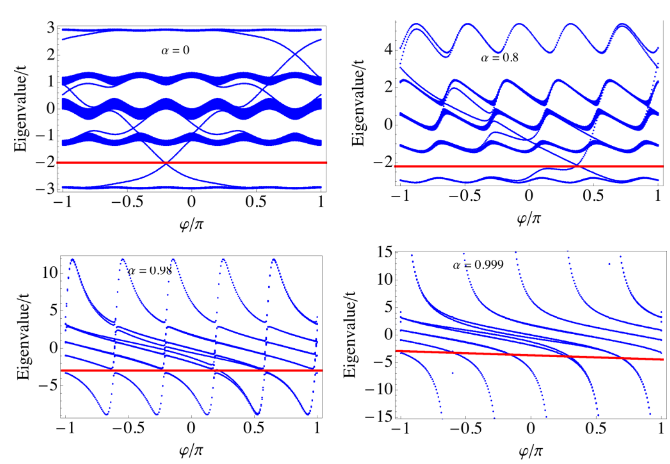

FIG. 2. (Color online) The energy spectrum plotted for $N = 200$ sites, $b = 1/5$, and $\lambda = 1$ for $\alpha = 0$ (AAH model), $\alpha = 0.8$, $\alpha = 0.98$, and $\alpha = 1.0$ (Maryland model). The red line separates empty and filled states in the spectrum. Instead of fixing the Fermi level in our numerics we fix the number of particles per site since the latter can be fixed throughout the deformation driven by the parameter α and even at $\alpha = 1$.

the band gaps gradually decrease. All band gaps close (scale to zero with the system size) precisely at the critical point $\alpha = 1$ (as explicitly shown using the exact spectrum [36,37]). The gapless nature of the 2D spectrum for the MM case ($\alpha = \pm 1$) in the hybrid space (n,φ) is explicitly confirmed using the exact analytical expression for the MM spectrum with commensurate modulation [37].

The closing of the spectral gaps coincides with the hopping range divergence in the 2D lattice and indicates a TQPT in the system as $\alpha \to \pm 1$ (i.e., at the MM point). The interesting aspect of MM is that the 2D spectrum is gapless in the reciprocal space (k_n,φ) whereas the 1D spectrum has well defined gaps for each value of φ. Here k_n is the Fourier image of the site index n. The fact that the 1D MM spectrum has gaps whereas the corresponding dual 2D spectrum is gapless makes perfect sense since the nontrivial TQPT can only exist in the 2D space. The scale invariance of the system at the transition point can also be explicitly demonstrated [36].

Chern number from polarization theory. In the following, we change α from 0 to 1 and track the change in the IQH topological invariant associated with the 2D system [Eq. (1)] in the hybrid space (n,φ). An ideal tool for this task is the polarization of the 1D chain defined in the hybrid space [34,35]. The polarization of a finite 1D insulator is given by the average charge center of the hybrid Wannier function (HWF) $[\bar{n}(\varphi)]$ of the system [38],

$$\bar{n}(\varphi) = \frac{\sum_n \langle n\rho(n,\varphi)\rangle}{\sum_n \langle \rho(n,\varphi)\rangle},$$
$$\rho(n,\varphi) = \sum_{\text{occupied states}} |n,\varphi\rangle\langle n,\varphi|, \tag{6}$$

where n is the real space site index and $|n,\varphi\rangle$ is the hybrid eigenstate of the system, and the angular brackets $\langle\cdots\rangle$ stand for the ground state expectation value given a fixed filling factor. Note that here we fix the filling factor in contrast to the typical approach of fixing the Fermi energy. This choice is equally applicable in the case of MM, $\alpha = 1$, where on-site energies on some sites are divergent. The latter can be occupied by a maximum of one electron and therefore present no problem in the definition of the filling factor.

The nonzero Chern number is reflected in a discontinuity of $\bar{n}(\varphi)$ as a function of the phase (or gauge) parameter φ. This discontinuity is a robust feature of the IQH and was recently proposed [38] as a tool to measure topological invariants directly in 2D cold atomic systems [9,10,39]. Note that the generalized 1D chain [Eq. (1)] has well defined gaps in the spectrum for any fixed φ and $|\alpha| \leqslant 1$ (including the Maryland model), which allows us to define the 1D polarization in terms of HWF centers in the whole parameter space.

In Fig. 3, we plot the shift in the HWF centers for the same values of α (for $b = 1/5$) as in Fig. 2. We fix the filling factor (particle number per site) such that the chemical potential is in the gap above the top of the lowest band in the AAH limit ($\alpha = 0$). In the limit of AAH, the HWF center as a function of φ shows a one unit cell jump corresponding to the Chern number $C = 1$, or, equivalently, a transfer of charge e by a distance of one unit cell as φ changes by a period, reflecting topological charge pumping [2]. We monitor this jump (invariant) as we deform AAH ($\alpha = 0$) to MM ($\alpha \to 1$),

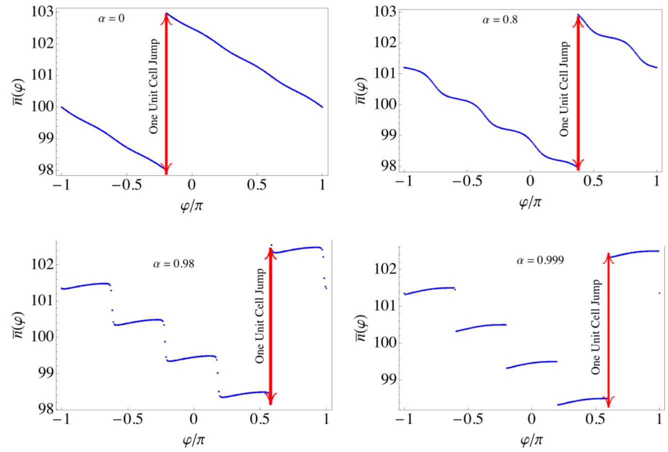

FIG. 3. (Color online) HWF centers plotted as a function of the adiabatic parameter φ for $\alpha = 0$ (AAH model), $\alpha = 0.8$, $\alpha = 0.98$, and $\alpha = 1.0$ (Maryland model).

keeping the filling factor fixed. Note that the polarization jump corresponding to the topological charge transfer survives in the MM limit $\alpha \to 1$ (see Fig. 3, bottom right). It may seem paradoxical at first that we can associate a Chern number with a gapless system. The limiting procedure $\alpha \to 1$ allows one to project on to the states that are connected to the topological band defined for $|\alpha| < 1$. Note that the topology is not robust as any infinitesimal perturbation may mix the states, thereby violating the quantization of the topological response. Such behavior is expected of a critical phase at $\alpha = 1$ on general grounds. Note the additional discontinuities appearing in HWF shift $\bar{n}(\varphi)$ in the case of MM (Fig. 3, bottom right) arise due to the divergent on-site potential, effectively breaking the system up into smaller subsystems coupled by tunneling.

Topological classification of 1D quasicrystals. Families of 1D incommensurate tight binding models manifest a special "quasiperiodic" translational invariance: An arbitrary shift in the phase $\varphi \to \varphi + \delta\varphi$ can be compensated by a shift along the chain $n \to n + \delta n_{\delta\varphi}$. Note that this is true only at irrational

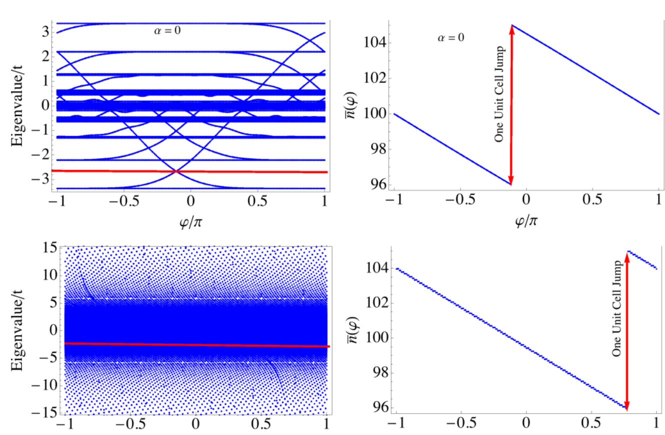

FIG. 4. (Color online) 1D quasicrystal band structure and the shift in polarization as a function of the phase φ for the AAH model (upper panel) and Maryland model (lower panel) for $b = \frac{110\,001}{1\,000\,000}$.

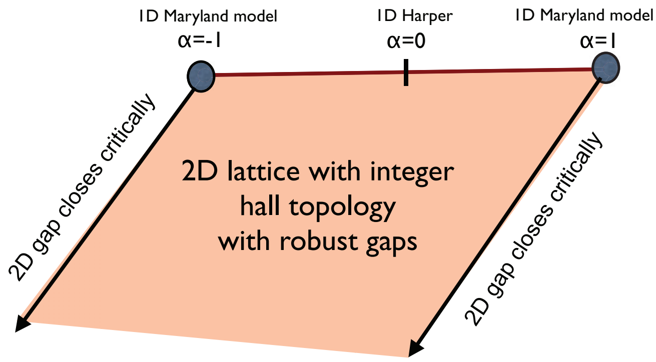

FIG. 5. (Color online) The phase diagram of Eq. (1) parametrized by $|\alpha| \leqslant 1$, the deformation parameter interpolating between the AAH and the Maryland model.

values of b since only in this case $2\pi bn$ forms a dense set mod 2π. It has been argued [19] that this quasiperiodic translational invariance allows one to assign the same Chern number to each member of the family of QCs, i.e., for each value of the phase parameter φ. This interpretation has been challenged in Ref. [32]. The quasiperiodic translation symmetry is preserved in the case of the Maryland model ($\alpha = 1$), which sits exactly at the critical point of a 2D TQPT. In Fig. 4 we plot the band structure and the change in the polarization as a function of the phase φ for the incommensurate AAH and MM. We choose the flux fraction to be a truncated Liouville constant (Liouville numbers are irrational numbers infinitely close to rational numbers). Note that the finite size Maryland model still demonstrates the presence of a nonzero Chern number in the same restricted sense as we found for the commensurate case. The constant slope of $\bar{n}(\varphi)$ in Fig. 4 manifests the constant Berry curvature (as a function of φ) (see Ref. [36] for details). The latter is a signature of the "quasiperiodic" translation invariance, as noted by Kraus *et al.* in Ref. [19]. Remarkably, the spectrum is gapped in the incommensurate (Liouville) 1D model Eq. (1) (for fixed φ) for $|\alpha| < 1$ and forms a dense set for $\alpha = \pm 1$ (rather than a continuous set), whereas the equivalent 2D model becomes gapless as we approach critical points $\alpha = \pm 1$. The details of the 1D spectrum depend on the type of the irrational number b, however, at no value does the spectrum become absolutely continuous [29]. Within the class of 1D models with quasiperiodic symmetry, the Maryland model manifests a 2D topological phase transition as a function of the deformation parameter α which can only be realized by sweeping the phase φ (see Fig. 5).

Conclusion. We have identified a topological feature of the Maryland model introduced in Refs. [24,25] in the context of Anderson localization and kicked quantum rotor studies. We show that this model represents a topological quantum phase transition point in a class of corresponding 2D lattice models with IQH topology. The criticality allows us to associate topological invariants with the Maryland model in a restricted mathematical sense at the special filling factors that are adiabatically connected to the spectral gaps in the 1D Aubry-André-Harper model. Our theory presented here establishes deep mathematical connections between 2D topological models and a family of 1D incommensurate localization models.

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