Critical exponents of the random field hierarchical model

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We study the one-dimensional Dyson hierarchical model in the presence of a random field. This is a long range model where the interaction scales with the distance in a power-law-like form, $J(r) \sim r^{-\rho}$, and we can explore mean-field and non-mean-field behavior by changing ρ. We analyze the model at $T = 0$ and we numerically compute the non-mean-field critical exponents for Gaussian disorder. We also compute an analytic expression for the critical exponent δ, and give an interesting relation between the critical exponents of the disordered model and the ones of the pure model, which seems to break down in the non-mean-field region. We finally compare our results for the critical exponents with the expected ones in D-dimensional short range models and with the ones of the straightforward one-dimensional long range model.

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I. INTRODUCTION

The critical behavior of models in the presence of a quenched random field has aroused a lot of interest since the pioneering work of Imry and Ma [1] because of its very interesting nature. In fact, while in a short range system with Ising spins a simple domain-wall argument suggests that a low temperature ordered phase can survive just if $D > 2$, the exact value of the lower critical dimension has been a debated issue for a long time. In particular, it was unclear whether or not a phase transition occurred in three dimensions until Imbrie demonstrated it does, and $D_L^c = 2$ [2,3].

This work solved a problem but left open other questions. In fact, it was not clear why other approaches such as perturbation theory [4,5] and the Parisi-Sourlas supersymmetric (SUSY) argument [6] predicted the wrong result, $D_L^c = 3$. In order to discuss this point, it is convenient to consider the modified hyperscaling relations that hold in the presence of a random field (RF), which are different from the usual ones. Let us first consider a system without random fields with volume V. While, at $T = \infty$, the entropy density is proportional to $\ln 2$ (since spins are of the Ising type, $s = \pm 1$), a nonanalytic term, proportional to ξ^{-D}, appears as T decreases to the critical temperature T_c. This term comes from the V/ξ^D flipping clusters, where ξ is the correlation length, and gives the most relevant contribution to the nonanalytic part of the free energy density. Thus, given the critical exponents ν and α, which may be defined from $\xi \sim t^{-\nu}$ and $\partial_t^2 f \sim t^{-\alpha}$, the usual hyperscaling relation $2 - \alpha = D\nu$ follows from the critical behavior of the free energy density $f \sim t^{D\nu}$, where $t \sim (T - T_c)/T_c$. Now let us go back to random field models, where the above hyperscaling law has to be generalized including a third critical exponent, θ [7,8]:

$$2 - \alpha = \nu(D - \theta). \tag{1}$$

In fact, the energy density of correlated clusters at the critical point is much more important than the entropy density in the presence of a random field [9] leading to a nonzero exponent θ. Moreover, thermal fluctuations turn out to be less important than the sample-to-sample ones [6].

The exponent θ is equal to 2 according to perturbation theory and the SUSY approach [4–6], while it is known to decrease to one in the limit $D \to 2^+$ [9]. While its behavior with D is still unknown, a simple result can be obtained if we approximate the magnetization m by χh, where h is the effective field acting on a correlated cluster and χ is the susceptibility. In fact, let us consider the average energy of such a cluster $u \sim \overline{mh}$. We can notice that $\overline{h^2}$ is proportional to ξ^{-D} because the largest contribution to the effective field comes from the average of ξ^D independent and identically distributed (i.i.d.) random fields. This leads to $u \sim \chi\overline{h^2} \sim t^{-\gamma+D\nu}$ since $\chi \sim t^{-\gamma}$. Thus $\theta = \gamma/\nu = 2 - \eta$, where η is defined form the connected correlation function as $C_{\text{conn}}(r) \sim r^{-(D-2+\eta)}$. This relation was proposed by Schwartz [10–12] and other authors [13–15]. In any event, another relation may be found with a scaling theory at $T = 0$ [9], that is, $\theta = 2 + \eta - \overline{\eta}$, where $\overline{\eta}$ is defined from the disconnected correlation function as $C_{\text{disc}} \sim r^{-(D-4+\overline{\eta})}$. This relation is based on the usual scaling assumptions and does not require approximations such as the former one, which is consistent with the latter if and only if $\overline{\eta} = 2\eta$. This relation has been tested in $D = 2 + \epsilon$ dimensions at the first order in ϵ [9], and numerically all simulations give a small value for $2\eta - \overline{\eta}$. The most impressive and recent one in the three-dimensional random field Ising model (RFIM) [16] states that $2\eta - \overline{\eta} \sim 10^{-3}$. In $D = 4$, numerical studies [17,18] lead to $2\eta - \overline{\eta} \sim -0.01 \pm 0.05$ while in $D = 5$, from the critical exponents computed in Ref. [19], it is difficult to obtain an estimation of $2\eta - \overline{\eta}$. Thus, summing up, it is unclear whether or not θ is an independent exponent. A nonperturbative functional renormalization group approach [20–22] suggests that the relation $\overline{\eta} = 2\eta$ is not true, in general. In particular, this approach leads to the result that for dimensions greater than $D \approx 5.1$, $\theta = 2$ and $\overline{\eta} = \eta$ [23], as can be found in Refs. [4–6].

In this paper we compute the critical exponents of the Dyson hierarchical version of the random field problem. The hierarchical model (HM) is a one-dimensional model with a long range interaction invented by Dyson [24] where the coupling between spins mimics a power-law interaction strength of the form $J(r) = r^{-\rho}$. In a general one-dimensional long range model ρ controls the distance from the mean-field behavior: Increasing ρ, the system becomes less and less mean field. This is qualitatively similar to exploring different dimensions in a D-dimensional short range system, where the critical behavior may or may not be of the mean-field type, and this feature has motivated many studies on disordered versions of the long range model [25–32]. It must be noted that in these models the integer parameter D is replaced by a continuous parameter ρ, and a mapping between them has been proposed in Ref. [27] and recently revisited in Refs. [30,31]. This mapping is believed to hold in the whole mean-field region and near the upper critical dimension, but seems to break down near the lower critical dimension [33]. We will call this model RFHM.

We decided to study the hierarchical model instead of the long range counterpart because the two models are very similar, even if they are not thought to be in the same universality class for values of ρ near the lower critical dimensions [33], that is, $\rho = 2$ in the pure model and $\rho = 3/2$ in the random field case (the hierarchical model does not have a phase transition when $\rho \geqslant 2$ [24], while the long range model has a Kosterlitz-Thouless transition for $\rho = 2$ [34–36]). Moreover, the equilibrium distribution of the magnetization $P(M)$ of the hierarchical model can be exactly computed at every temperature in a polynomial time. This iteration equation is not spoiled by the disorder induced by a random field. The time complexity of this algorithm is $O(N^2)$, where N is the size of the system, even at $T = 0$. Moreover, the model can be studied at $T = 0$ using a recently developed algorithm [37] whose time complexity is $O(N \log N)$ and that computes the ground state magnetization and energy of a disordered sample. Thus, it allows one to analyze big systems and provide accurate statistics in a reasonable computation time. We used this algorithm to compute the critical exponents at $T = 0$.

This paper is organized as follows. In the first section we introduce the main features of the hierarchical model, both in its pure version and its random field version. In the second section we explain how we computed the critical exponent ν and plot the curve $1/\nu(\rho)$. In the third section we compute other critical exponents and note an interesting relation between critical exponents of the RFHM and the ones of the pure hierarchical model, which is somehow reminiscent of the phenomenology of the D-dimensional short range models. In the last section we draw the conclusions: We compare our critical exponents with the ones of the RFIM in three and four dimensions, using the results obtained in Refs. [16,17], and with the critical exponents of the one-dimensional long range model studied in Refs. [31,32].

II. THE HIERARCHICAL MODEL

The hierarchical model is a one-dimensional model defined by [24]

$$H_n(s_1,\dots,s_N) = -\sum_{p=1}^{n}\left(\frac{c}{4}\right)^p \sum_{r=1}^{2^{n-p}} S_{pr}^2, \tag{2}$$

where c is a coupling constant, $N = 2^n$ is the total number of spins, and S_{pr} is the sum of all the spins contained in the rth p-level block:

$$S_{pr} = \sum_{i=(r-1)2^p+1}^{r2^p} s_i, \quad r = 1,\dots,2^{n-p}. \tag{3}$$

Spins are organized in a hierarchy of levels, indexed by p, whose physical meaning is that spins at the same level interact with each other through the same coupling. This model resembles a one-dimensional chain where the coupling between spins is given by a power-law interaction strength decreasing with the intersite distance $J(r) = r^{-\rho}$, where $c = 2^{2-\rho}$. It was invented by Dyson [24] to study phase transitions in one-dimensional models with long range interactions and has been studied since then [38–42]. Later on, it was realized [43,44] that it could be very useful in the study of the renormalization group (RG) theory developed by Wilson [45]. More recently it has been studied in the field of quenched disordered models [37,46–51], as well as for the Anderson localization problem [52,53].

A. Pure model

In a model without random fields, we will consider the interval $\rho \in [1,2]$. In fact, for $\rho < 1$, the free energy corresponding to Eq. (2) is not defined in the thermodynamic limit; the limit $\rho \to 1^+$ corresponds to the limit $D \to \infty$ in short range models. On the other hand, for $\rho \geqslant 2$, there is no phase transition [24]. One of the ways to verify this statement is to see that the singular part of the cost of a bubble in a magnetized phase is of order $L^{2-\rho}$ and so bubbles have $O(1)$ cost for $\rho > 2$. The nontrivial critical region, where critical exponents differ from their mean-field values, is $\rho \in (3/2,2)$ [39–41]. This may be seen from the hierarchical structure of the Hamiltonian in Eq. (2), which allows an exact realization of the block spin transformation [54,55].

Let us write the partition function,

$$\mathcal{Z}_N = \int ds_1 \cdots ds_{2^n} \exp\left\{-\beta H_n(s_1,\dots,s_N) + \sum_{i=1}^{2^n} f(s_i)\right\},$$

where $P(s) = \exp\{f(s)\}$ is a weight function on each spin s, for example, the Ising weight $\delta(s^2 - 1)$. After the renormalization transformation,

$$\frac{s_i + s_{i+1}}{2} = \gamma s'_{(i+1)/2}, \quad \frac{s_i - s_{i+1}}{2} = t'_{(i+1)/2}, \tag{4}$$

$\mathcal{Z}_N$ may be rewritten in terms of the new effective spins $\{s'_i\}_{i=1,\dots,2^{n-1}}$ through the integration over the other 2^{n-1} variables $\{t'_i\}_{i=1,\dots,2^{n-1}}$. The number of degrees of freedom has been halved, and the price that has been paid is the introduction of a new weight function $P'(s')$, given in terms of the old one by

$$P'(s') = e^{4\beta J\gamma^2 s'^2}\int dt' P(\gamma s' + t')P(\gamma s' - t'), \tag{5}$$

where $J = c/4 = 2^{-\rho}$. It is worth noticing that this equation has the same form of the approximate recursion formula derived by Wilson [43–45,56]. In this sense, the hierarchical model is a model for which Wilson's formula is exact. Let us also mention that a part form the new weight function in Eq. (5), the new Hamiltonian, has the hierarchical structure of the old one and a new coupling constant: If $K = \beta J$, K' is given by

$$K' = 4J\gamma^2 K, \tag{6}$$

and it depends on γ. This coefficient can be determined as follows: At the critical point, where the interaction between clusters of spins does not change with the scale at which we observe the system, we impose

$$\gamma = 2^{\rho/2-1}. \tag{7}$$

We can use this result to evaluate the η index, defined as $C_{\text{conn}}(r) = \langle s_i s_{i+r}\rangle - \langle s_i\rangle\langle s_{i+r}\rangle \sim r^{-(D-2+\eta)}$. In fact, suppose that the above transformation is iterated n times,

$$\frac{\sum_i^{2^n} s_i}{2^n} = \gamma^n s_i^{(n)},$$

where $s_i^{(n)}$ is the renormalized spin after n transformations. γ absorbs the diverging part of the right-hand side of the last equation, making $s_i^{(n)}$ a finite quantity. Thus we obtain

$$m = \frac{\left\langle \sum_i^{2^n} s_i\right\rangle}{2^n} \propto N^{\rho/2-1}, \tag{8}$$

and since $C_{\text{conn}}(r) \sim m^2$ near T_c, the connected susceptibility scales as

$$\chi_{\text{conn}} = \sum_r C_{\text{conn}}(r) \propto N\gamma^{2n} \propto N^{\rho-1}. \tag{9}$$

In a one-dimensional finite size system it may also be expressed as

$$\chi_{\text{conn},L}(T=T_c) = \int_L \frac{dr}{r^{D-2+\eta}} \propto L^{2-\eta}, \quad D=1, \tag{10}$$

where $L = N$ is the size of the system, and thus comparing Eqs. (9) and (10) we obtain $\eta(\rho) = 3-\rho$. This relation holds both in the mean-field and in the non-mean-field regions, the reason being that we performed an exact RG transformation and computed an exact value for γ at T_c.

From Eqs. (5) and (7) we can also calculate ν. Therefore, in this case, we must make an ansatz on the form of $P(s)$ and then we must study its stability. If $P(s)$ is a normal distribution $\mathcal{N}(0,1)$, the new weight function is still a normal distribution whose variance $(\Sigma')^2$ is given by

$$\frac{1}{2(\Sigma')^2} = \frac{1}{c\Sigma^2} - \beta, \quad \Sigma = 1. \tag{11}$$

The only unstable fixed point of the recursion equation (11) is found by imposing $\Sigma' = 1$, i.e., when the system is invariant under RG transformations. This leads to an evaluation of the critical temperature $\beta_c = \frac{2-c}{2c}$ for this particular choice of the weight function and to the fixed point value $O^* = 1/2$ for the operator $O = 1/(2\Sigma)$. The exponent ν can be extracted from the evolution of a small perturbation from this fixed point value, that is, starting with $O = 1/2+\delta$ and calculating δ'. From Eq. (11) we have

$$\frac{1}{2} + \delta' = 2^{\rho-1}\left(\frac{1}{2}+\delta\right) - \frac{2-c}{2c},$$

and thus $\delta' = 2^{\rho-1}\delta$, leading to $\nu(\rho) = 1/(\rho-1)$. While the expression for η that we computed before holds whatever ρ, this expression for ν is valid when the Gaussian ansatz is stable. Many perturbative analyses have been done [39–41] and it has been found that the mean-field region is $\rho \in (1,3/2)$. An easy way to grab the upper critical value of ρ is to use hyperscaling relations, for example, $2-\alpha = \nu D$, with $D=1$, since they are valid just in the non-mean-field region [57,58] up until the upper critical dimension, where they are satisfied by classical indices,

$$2 = \frac{1}{\rho_c^u - 1} \Longrightarrow \rho_c^u = \frac{3}{2}, \tag{12}$$

where we used that the mean-field value of α is zero.

B. Random field model

We now consider the case in which there are uncorrelated random fields whose variance is h^2. The effect of the random fields is to weaken the ordered phase and thus it is natural to expect that it survives just in regions where T and h are small. We will consider the interval $\rho \in (1,3/2)$. In fact, a simple domain wall argument suggests that the singular part of the cost of a bubble is of order $L^{2-\rho} - h^2 L^{1/2}$, and thus a low temperature–low disorder magnetized phase cannot survive when $\rho > 3/2$. The nontrivial critical region, where critical exponents differ from their mean-field values, is $\rho \in (4/3,3/2)$ [51], as we will discuss below.

In order to deal with the disorder we replicate the partition function

$$\mathcal{Z}_N^m = \int \prod_\alpha \prod_{i=1}^{2^n} ds_i^\alpha \exp\left\{\sum_{i\alpha} f\left(s_i^\alpha\right) - \beta H_n\left(s_1^\alpha,\ldots,s_N^\alpha\right) + \beta\sum_i h_i \sum_\alpha s_i^\alpha\right\},$$

where α runs over the m replicas and $\overline{h_i h_j} = h^2\delta_{ij}$. The next step is to average over the disorder, assuming it is Gaussian,

$$\overline{\mathcal{Z}_N^m} = \int \prod_\alpha \prod_{i=1}^{2^n} ds_i^\alpha \exp\left\{\sum_{i\alpha} f\left(s_i^\alpha\right) - \beta H_n\left(s_1^\alpha,\ldots,s_N^\alpha\right) + \frac{h^2\beta^2}{2}\sum_{i,\alpha\beta} s_i^\alpha s_i^\beta\right\},$$

and make the same RG transformation as before, Eq. (4). Again, the partition function may be written in terms of the new effective spins $\{s_i'\}_{i=1,\ldots,2^{n-1}}$, if we introduce a new weight function,

$$P'(\{s_\alpha'\}) = e^{4\beta J\gamma^2\sum_\alpha (s_\alpha')^2} \int \prod_\alpha dt_\alpha' \prod_\alpha \times P(\gamma s_\alpha' + t_\alpha')P(\gamma s_\alpha' - t_\alpha')e^{\beta^2 h^2 \sum_{\alpha\beta} t_\alpha' t_\beta'}. \tag{13}$$

The new coupling constant K' and the new variance $(h')^2$ are given by

$$K' = 4J\gamma^2 K, \quad (h')^2 = 2\gamma^2 h^2. \tag{14}$$

At the ferromagnetic critical point T/J is invariant under RG transformations, leading to Eq. (7), and this gives $(h')^2 = 2^{\rho-1}h^2$. It means that this fixed point is always unstable with respect to the addition of a random field. The local

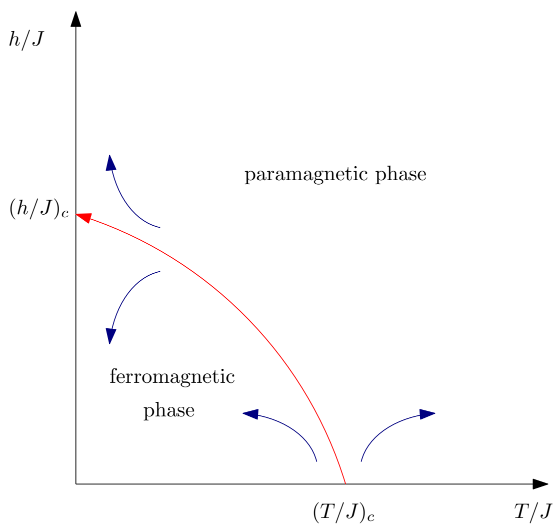

FIG. 1. (Color online) Schematic plot of the phase diagram of a ferromagnetic model in the presence of a random field. In the case of the $D = 1$ hierarchical model, this phase diagram holds for $\rho \in (1,3/2)$. At $h = 0$, the ratio T/J is invariant at the critical point. Incidentally, β is just a parameter, and thus at the critical point J is invariant by itself. This leads to Eq. (7). For $h > 0$, the renormalization group flow departs from this fixed point toward a second fixed point on the vertical axis. At this fixed point neither J nor h is invariant, as they both get renormalized [see Eq. (14)]. In any event, their ratio h/J is invariant, and this leads to Eq. (15). The red line is the critical line, separating a ferromagnetic phase from a paramagnetic phase. Blue arrows are sketches of the renormalization group flow near the critical line.

renormalization group flow departs from there versus regions of higher disorder and thus at the relevant RF-fixed point one expects that h/J is invariant, even if neither J nor h are. This may only happen if $T = 0$. Moreover, the RG invariance of h/J implies that the value γ at this fixed point is given by

$$\gamma = 2^{\rho-3/2}. \tag{15}$$

A detailed description of the renormalization group flow in this model can be found in Ref. [59], while a pictorial representation of this flow is given in Fig. 1. This relation leads to

$$m^{(2)} = \frac{1}{2^{2n}} \overline{\left(\left\langle \sum_i^{2^n} s_i \right\rangle\right)^2} \propto N^{2\rho-3}, \tag{16}$$

which can be used to evaluate η and $\bar{\eta}$. This last critical exponent is defined from the disconnected correlation function $C_{\rm disc}(r) = \overline{\langle s_i\rangle\langle s_{i+r}\rangle}$ as $C_{\rm disc}(r) \sim r^{-(D-4+\bar{\eta})}$ for $r \gg 1$. Thus

$$\chi_{\rm disc} = \sum_r C_{\rm disc}(r) \propto N m^{(2)} \propto N^{2\rho-2}. \tag{17}$$

In a one-dimensional system it can also be expressed as

$$\chi_{{\rm disc},L}(h = h_c) = \int_L \frac{dr}{r^{D-4+\bar{\eta}}} \sim L^{4-\bar{\eta}}, \tag{18}$$

and thus comparing Eqs. (17) and (18) we get $\overline{\eta}(\rho) = 6 - 2\rho$. In random field models η is still defined from the connected correlation function as $C_{\rm conn}(r) = \overline{\langle s_i s_{i+r}\rangle - \langle s_i\rangle\langle s_{i+r}\rangle}$ and thus at the critical point $\chi_{{\rm conn},L}$ scales with N as in Eq. (10). This equation may be now compared to the other definition of the connected susceptibility $\chi_{\rm conn} = T dm/dH$, where H is a uniform field that gets renormalized according to $H' = 2\gamma H$ and $T \sim J$ in the region $T/J \ll 1$. Equation (15) implies that $J' = 2^{\rho-1}J$ at the zero temperature fixed point and after n steps of the renormalization procedure, we get $H \sim (2\gamma)^{-n} \sim N^{1/2-\rho}$, while the coupling constant scales as $J \sim N^{1-\rho}$. This leads to

$$\chi_{\rm conn} = T\frac{dm}{dH} \sim J\frac{d}{dH}(m^{(2)})^{1/2} \sim N^{\rho-1}. \tag{19}$$

From Eq. (10) we get $\eta(\rho) = 3 - \rho$, in the same way as in the pure model. Therefore $\eta = \bar{\eta}/2$ [51], leading to $\theta(\rho) = \rho - 1$ for every ρ. The same result for θ can be obtained by looking at the RG exponent associated with J. In fact, the scaling theory at $T = 0$, developed by Bray and Moore [9], assumes that J gets renormalized even at the critical point, but that the ratio h/J is invariant, and θ appears in this calculation as the RG exponent associated with J: $J' = b^\theta J$, where b is the rescaling factor.

If we make a Gaussian ansatz for the weight function, Eq. (13) implies that the mean-field value of ν is given by $1/(\rho - 1)$ [37,51], as in the pure case. The non-mean-field region is given by $\rho \in (4/3,3/2)$ and again hyperscaling relations, e.g., Eq. (1), may be used to compute the upper critical value of ρ. In fact, they are valid only in non-mean-field regions and are satisfied at the upper critical dimension by classical indices,

$$2 = \frac{2 - \rho_c^u}{\rho_c^u - 1} \Longrightarrow \rho_c^u = \frac{4}{3}, \tag{20}$$

where again we used that here $D = 1$ and that the mean-field value of α is zero.

III. CALCULATION OF ν

The hierarchical structure of the Hamiltonian implies that

$$H_n(s_1, \ldots, s_{2^n}) = H_{n-1}^{(L)}(s_1, \ldots, s_{2^{n-1}}) + H_{n-1}^{(R)}(s_{2^{n-1}+1}, \ldots, s_{2^n}) + \Delta_{L,R}^n, \tag{21}$$

where $N = 2^n$ is the number of spins,

$$\Delta_{L,R}^{(n)} = -J_n \left(\sum_{i=1}^{2^n} s_i\right)^2 \tag{22}$$

is the interaction term, and $J_n = (c/4)^n = (1/2^\rho)^n$. This relation is not spoiled by a random field; the only difference with respect to the pure case is in the starting condition. In fact, in the pure case, $P_0(s) = P_0^{\rm pure}(s) = 1/2[\delta(s - 1) + \delta(s + 1)]$ while in the presence of disorder $P_0(s) = P_0^{\rm pure}(s)\exp[\beta h s]/2\cosh(\beta h)$, where h is the random field acting on s. This equation may be used to find a recursion equation for the probability $P_l(M)$ that a system of 2^l sites has magnetization given by M. $P_l(M)$ is defined by

$$P_l(M) \sim \sum_{\{s_i\}_{i=1,\ldots,2^l}} \delta\left(M - \sum_i^{2^l} s_i\right) e^{-\beta H_l(s_1,\ldots,s_{2^l})}, \tag{23}$$

where the symbol $\sim$ means that a normalization factor is understood, and the recursion equation obeyed by the probabilities reads

$$P_l(M) \sim e^{\beta J_l M^2} \sum_{S=-2^{l-1}}^{2^{l-1}} P_{l-1}(S)P_{l-1}(M-S). \qquad (24)$$

The sum is done over all the possible magnetizations of the smaller systems, that is, $S = \{-2^{l-1}, -2^{l-1}+2, -2^{l-1}+4, \ldots, 2^{l-1}\}$. This equation may be used to calculate the moments of the distribution $P_l(M)$ up until the last level of the interaction; it holds whatever T and h, and thus it is suitable to study the transition at $T > 0$ both by fixing the temperature and tuning the variance of the random field and vice versa. So, the critical exponents of the RFHM can be already obtained at $T = 0$, and thus we decided to study the phase transition at $T = 0$ using the algorithm that Monthus and Garel have recently developed [37]. This algorithm finds the ground state of a RFHM sample in a linear time, a part from logarithmic corrections, and it is faster than the $T = 0$ limit of Eq. (24), whose time complexity is quadratic. It is based on the observation that in the presence of an external uniform field H, the energy of a configuration C is linear in H, $E(C) = -M_C H + a_C$, and that as H grows, the ground state magnetization also grows. This property is called the "no-passing rule" [60,61] and, in other words, the collection of ground states for different values of H is ordered in magnetization: The magnetization of the ground state is a nondecreasing function of H. The magnetic field at which there is a change in the ground state corresponds to a collective rearrangement of spins. This phenomenon is called equilibrium avalanche. Given a configuration made by 2^l sites, we can divide it in its left and right part, each one containing 2^{l-1} sites. The relation between a_{C_L}, a_{C_R}, and a_C has been computed in Ref. [37],

$$a_C = a_{C_L} + a_{C_R} - J_l M_C^2, \quad M_C = M_L + M_R, \qquad (25)$$

and thus, given the ground state list of the left and of the right part, we can construct the ground state list of the total system versus H. These lists contain no more than 2^l configurations [37] and thus this algorithm takes a $O(N)$ time. We checked for several samples that the ground state magnetizations obtained by the $T = 0$ limit of Eqs. (24) and (25) are the same, and since we have to average over many disordered samples, Eq. (25) is the best option to compute the critical exponents.

For each ρ we studied around $\mathcal{N} = 80\,000$ samples at different values of h (typically we studied 15 or more values of h). For each sample made by $N = 2^n = 2^{21}$ sites we extracted $\mathcal{N}_k = 2^{n-k}$ ground state magnetizations of systems whose size is $N_k = 2^k$. We considered $k \in (6,21)$. In fact, the hierarchical structure of the Hamiltonian makes it possible to divide a sample in two subsamples and calculate their magnetizations before considering the coupling between spins of the two different parts. Each subsample can be further divided and this procedure can be repeated until the single spins. So, at the end, we have many more samples to average over for small sizes than we have for bigger sizes. For each h and k we have then randomly picked up $n_k/2$ samples and averaged their squared ground state magnetization, repeating this procedure

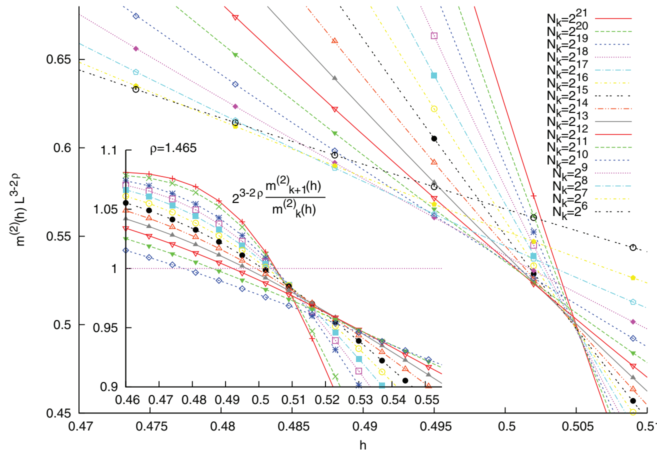

FIG. 2. (Color online) Plot of the curves $O_k(h)$ vs h, where $O_k(h)$ is defined in Eq. (26). h_c is given by the point where different curves cross. Inset: We plotted $O_{k+1}(h)/O_k(h)$ in order to show that as k grows, this ratio goes to one as it should, being $O_k(h)$, a size-invariant quantity. In this plot $\rho = 1.465$.

M times. We found that $M \sim 50$ was already large enough. We used these data each time to compute the observable

$$O_k(h) = L^{2y} m_k^{(2)}(h), \quad y = \tfrac{3}{2} - \rho, \qquad (26)$$

where $m^{(2)}$ has been defined in Eq. (16). This quantity is size invariant at the critical point [see Eq. (15)], and thus the intersections of these curves, for different values of k, give the critical value of h [62] (see Fig. 2). The derivative $\partial O_{k+1}/\partial O_k$ at the critical point leads to ν_k, the values of ν at the level k, in a way that will be exposed later [see Eqs. (29) and (28)]. The errors over ν_k have been calculated from the standard deviation of the M instances of these quantities, and the asymptotic behavior of the ν_k has been studied to get $\nu = \lim_{k\to\infty} \nu_k$ (see Fig. 3).

The asymptotic values of ν may be computed from finite size scaling [57,63–65] as follows. In a system of linear size L, a size-invariant quantity at the critical point has the form

$$\begin{aligned} O(L,t) &= f(tL^{1/\nu}, L^{-\omega}) \\ &= O_c + f_1' t L^{1/\nu} + f_2' L^{-\omega} + f_{12}'' t L^{1/\nu-\omega} + \cdots, \end{aligned} \qquad (27)$$

where t is the rescaled difference from the critical temperature $t = (\beta - \beta_c)/\beta_c$, L is the size of the system, ω is the correction-to-scaling exponent, ν is defined from $\xi \sim t^{-\nu}$, and O_c is the critical value of O. In the $T = 0$ transition of the random field model, the only relevant variable is the rescaled difference from the critical variance, so Eq. (27) is still valid if we replace t with $t = (h - h_c)/h_c$. In Eq. (27) nonlinear terms in t can be neglected since we assumed to be near the critical point. A scale-invariant quantity has the property to remain constant under RG transformations at the critical point and thus, in different size systems, O_c is a universal value that does not scale with L. The value of h at which $O(L,t)$ and $O(L',t)$ crosses is defined by h_L^* and goes to h_c as L grows.

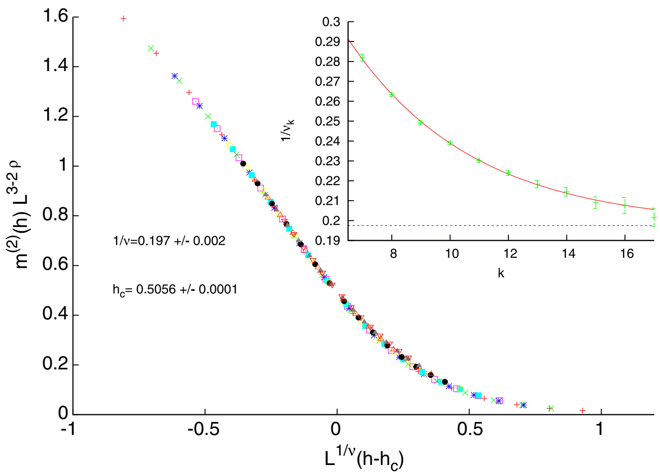

FIG. 3. (Color online) Collapse of the curves $O_k(h)$ vs $L^{1/\nu}(h-h_c)$ for $\rho = 1.465$. Good collapse occurs just for large enough systems, and thus here we have taken k from 13 to 21. Inset: Plot of the curve ν_k vs k, computed as in Eq. (29), to get the asymptotic value of ν. The errors over ν_k's have been computed with the *bootstrap* method, as explained in the text. For this particular value of ρ we found $1/\nu = 0.197(2)$ [see Eq. (28)]. The red line is just as a guide for the eye.

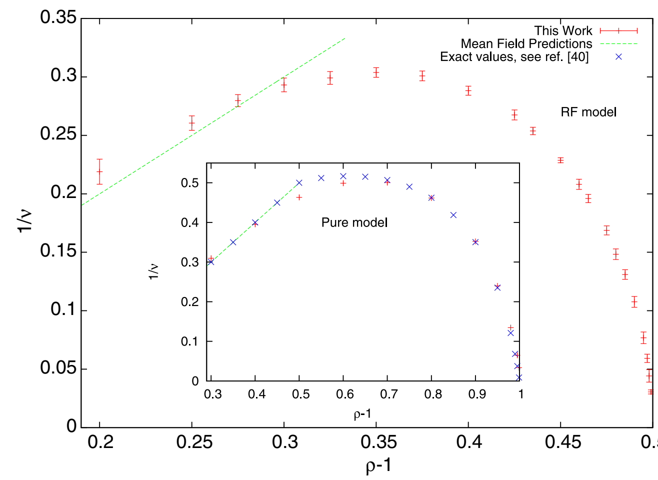

FIG. 4. (Color online) Inverse of the critical exponent ν as a function of ρ in the RFHM. The red points stand for the values of $1/\nu$ computed in this work, while the green lines are the known mean-field values. In the disordered case the non-mean-field region ($\rho > 4/3$) was unexplored, and here $1/\nu$ has been computed using the algorithm developed in Ref. [37]. Inset: Inverse of the critical exponent ν as a function of ρ in the pure hierarchical model. Blue points have been computed by Kim and Thompson [40]. Here we confront these values with the ones we computed using Eqs. (24) and (29) to show that our method works well almost in the whole non-mean-field region, apart from the extremes.

Thus we have

$$\frac{\partial O_{L'}}{\partial O_L} = \frac{\partial O_{L'}}{\partial h}\frac{\partial h}{\partial O_L} = \frac{f_1' L'^{1/\nu} + f_{12}'' L'^{1/\nu-\omega}}{f_1' L^{1/\nu} + f_{12}'' L^{1/\nu-\omega}},$$

and taking logs on both sides we obtain

$$\log_b\left(\frac{\partial O_{L'}}{\partial O_L}\right)\Bigg|_{h_L^*} = \frac{1}{\nu} + AL^{-\omega}, \tag{28}$$

where b is defined as $L' = bL$ and is equal to 2, and A a constant. The left-hand side of this equation gives ν_k,

$$\frac{1}{\nu_k} = \log_b\left(\frac{\partial O_{k+1}}{\partial O_k}\right)\Bigg|_{h_k^*}, \tag{29}$$

where we recall that $k = \log_2(N_k)$. We used these equations to compute ν, with the scale-invariant quantity $O_k(t)$ defined by Eq. (26), where $k = 6,\ldots,17$. We studied even bigger samples, until $k = 21$, but the error bars for $k > 17$ are usually too big to be significative. In each of the M extractions of data, we computed the asymptotic value of the quantity in Eq. (29), let us call it ν^S, where $S = 1,\ldots,M$, and we computed ν and its error as the mean and standard deviation of the histogram of the ν^S. The procedure here illustrated is called *bootstrap* [57,64].

In Fig. 4 we show the values of $1/\nu$ computed at various ρ's. We also plotted the results we obtained for the pure model, which can be compared with the ones computed in Ref. [40]. The pure model can be studied using Eqs. (24) and (29). In the pure case, as well as for the disordered case, we studied systems up to 2^{17} spins, and the asymptotic critical exponents have been computed using Eq. (29) (see also the inset of Fig. 3). The pure model has been studied using Eq. (24) and we see that Eq. (29) works quite well in the non-mean-field region, apart from the limits $\rho \to 3/2$ and $\rho \to 2$, where we have seen that logarithmic corrections have to be taken into account. Thus, apart from the regions where $\rho \to 4/3$ and $\rho \to 3/2$, we expect that the values we obtained are very accurate.

IV. RELATIONS BETWEEN CRITICAL EXPONENTS

In this section we discuss all the other critical exponents of the RFHM. The exponent δ, defined from the vanishing of the magnetization in presence of a magnetic field H, $m \sim H^{1/\delta}$, has an analytical expression, shown in Eq. (35). All the other critical exponents depend only on ν. We first review the critical exponents of the pure hierarchical model. The hyperscaling relation $\alpha = 2 - D\nu$ can be used to compute the non-mean-field value of α, where α is defined as $\partial_t^2 f \sim t^{-\alpha}$. Since $D = 1$, we have

$$\alpha(\rho) = 2 - \nu(\rho). \tag{30}$$

While this relation gives only the non-mean-field value of α, which is zero otherwise, the other scaling relation $\gamma = \nu(2-\eta)$ is more general and it is also valid in the mean-field region. Thus, the exponent γ, defined from the divergence of susceptibility at the critical temperature $\chi \sim t^{-\gamma}$, is given by

$$\gamma(\rho) = \nu(\rho)(\rho - 1), \tag{31}$$

whatever ρ and leads to $\gamma = 1$ in the mean-field region. In general, δ can be computed from η using the relation $\delta = (D+2-\eta)/(D-2+\eta)$ (see, for example, Refs. [57,63,64]) and thus we have the following analytical expression:

$$\delta(\rho) = \frac{\rho}{2-\rho}. \tag{32}$$

We can also estimate β from the relation $\gamma = \beta(\delta - 1)$, defined from $m \sim t^{\beta}$, which turns out to be equal to

$$\beta(\rho) = \left(1 - \frac{\rho}{2}\right)\nu(\rho). \tag{33}$$

These critical exponents have been computed by Kim and Thompson [40], who tabulated very accurate estimations of the values of $\nu(\rho)$ (see the inset of Fig. 4) in the non-mean-field region and also gave the analytic expression for $\delta(\rho)$ in Eq. (32).

In the RFHM the hyperscaling law is modified according to $\alpha = 2 - (D - \theta)\nu$, where θ has been defined in Eq. (1) and we have $\theta(\rho) = 2 + \eta(\rho) - \overline{\eta}(\rho) = \rho - 1$. Thus, while the non-mean-field critical exponent α is given by

$$\alpha(\rho) = 2 - (2 - \rho)\nu(\rho), \tag{34}$$

Eq. (31), which governs the behavior of γ, is still valid. In the presence of a random field δ can be expressed in terms of η and θ using the relation $\delta = (D - 2\eta + \overline{\eta})/(D - 4 + \overline{\eta})$ (see, for example, Refs. [57,63,64]), and thus we get

$$\delta(\rho) = \frac{1}{3 - 2\rho} \tag{35}$$

in the non-mean-field region. At last, using again the scaling relation $\gamma = \beta(\delta - 1)$, we obtain

$$\beta(\rho) = \left(\tfrac{3}{2} - \rho\right)\nu(\rho). \tag{36}$$

This equation is consistent with Eq. (16) because in a finite size system $m^{(2)}(L) \sim L^{-2\beta/\nu}$ at the critical point [see also Eq. (26) and the inset of Fig. 3].

At this point, we may notice that Eq. (35) reduces to Eq. (32) if we replace its argument ρ by $2 - 1/\rho$, i.e.,

$$\delta_{\mathrm{RF}}(2 - 1/\rho) = \delta_{\mathrm{pure}}(\rho). \tag{37}$$

We can ask if a similar replacement also works for the other critical exponents. This would lead to the *dimensional-reduction* rule

$$\rho_{\mathrm{RF}} \to \frac{1}{2 - \rho_{\mathrm{RF}}}. \tag{38}$$

Thus, we can compare the respective exponents $\gamma(\rho)$'s obtained through Eq. (31). First of all, it is easy to see that $\gamma_{\mathrm{pure}}(3/2) = \gamma_{\mathrm{RF}}(4/3) = 1$, as follows from Eq. (31). As we move from the respective mean-field thresholds, our data are not very good (see Fig. 4). Anyway, we can use perturbative results obtained in Refs. [40,51], which give

$$\begin{aligned}\nu_{\mathrm{pure}}\left(\rho = \tfrac{3}{2} + \epsilon\right) &= 2 - \tfrac{4}{3}\epsilon + O(\epsilon^2),\\ \nu_{\mathrm{RF}}\left(\rho = \tfrac{4}{3} + \epsilon\right) &= 3 + O(\epsilon^2).\end{aligned} \tag{39}$$

These expansions may be used to compare $\gamma_{\mathrm{RF}}(2 - 1/\rho)$ and $\gamma_{\mathrm{pure}}(\rho)$ in $\rho = 3/2 + \epsilon$. It turns out that they are both equal to $1 + (4/3)\epsilon$. Thus Eq. (38) also works for γ's in perturbation theory, at least at the first order in ϵ. In the non-mean-field regions, our data suggest that this relation breaks down. For example, at $\rho_{\mathrm{RF}} = 1.45$, which would correspond to $\rho_{\mathrm{pure}} \approx 1.818$, a spline interpolation of the data obtained in Ref. [40] gives $\gamma_{\mathrm{pure}}(1.818) = 0.548(1)$ while according to Eq. (38) we should get $\gamma_{\mathrm{RF}}(1.45) = 0.507(3)$. Similar discrepancies are also found for other ρ_{RF}'s in the non-mean-field region, where we expect that our results are very accurate (see Fig. 4): The mapping between the random field model and the pure model described above seems to break down somewhere below the mean-field threshold. As our data are not accurate in this limit, we are not able to detect the point where this breaking occurs. Moreover, the relation noted here, given in Eq. (38), corresponds to the famous one, $D_{\mathrm{RF}} \to D_{\mathrm{RF}} - 2$, which is found for short range D-dimensional models. This may be seen using the usual mapping between these models and long range one-dimensional models [27], [30]. Thus, the breakdown of this relation is similar to the dimensional-reduction breaking of the D-dimensional short range models [4–6].

V. SUMMARY AND CONCLUSIONS

In this paper we computed the critical exponent ν of the RFHM for many non-mean-field values of ρ. Then, we found an analytic expression for the critical exponent $\delta(\rho)$, Eq. (35). This expression and Eq. (32) lead to the relations (37) and (38) between the critical exponents of the RFHM and of the pure hierarchical model. This relation turns out to hold for δ whatever ρ, while it is not exactly satisfied by the other critical exponents in the non-mean-field region. While this result suggests that some features of the dimensional-reduction breaking can be found also in the problem we studied, a comparison between a short range model in D dimensions and a one-dimensional long range models is not always possible. In fact, as was accurately studied in Ref. [30] and discussed in Ref. [31], it is very difficult to compare these two models when D is much smaller than the upper critical dimension, that is, 6 for the RFIM. Thus we do not expect to have a satisfying agreement between the RFIM results in $D = 3$ and the corresponding ones in the RFHM if we use the usual relation [27,30] between ρ and D. In fact, we do not. In Ref. [16] Martin-Mayor and Fytas computed $\overline{\eta} = 1.0268(1)$ and $\nu = 1.34(11)$ for the RFIM in $D = 3$. The ρ corresponding to $D = 3$ is $\rho = (2 - \overline{\eta}_{\mathrm{SR}}/2)/D + 1 = 1.49\,550(2)$ [31], where SR stands for short range. What we should compare is $\nu_{\mathrm{LR}}[\rho = 1.49\,550(2)] = 14(3)$, where LR stands for long range, and $D\nu_{\mathrm{SR}}(D) = 4.0(3)$ for $D = 3$, and it is clear that they do not agree. The situation improves in $D = 4$, where it is possible to estimate $\nu = 0.82(6)$, leading to $D\nu_{\mathrm{SR}}(D) = 3.3(2)$, from the results given in Ref. [18], and $\overline{\eta} = 0.45(17)$. The ρ corresponding to $D = 4$ is $\rho = 1.444(21)$ at which we get $\nu_{\mathrm{LR}}[\rho = 1.444(21)] = 4.2(8)$.

Another interesting comparison can be done with the critical exponents of the long range model evaluated in Refs. [31,32]. In $\rho = 1.25$ our result $1/\nu = 0.260(6)$ has to be compared with the long range value $1/\nu = 0.262 \pm 0.035$ obtained in Ref. [32]. Thus they are found to be in good agreement as expected because $\rho = 1.25$ belongs to the mean-field region, where we expect $1/\nu = 0.25$. In $\rho = 1.4$ we found $1/\nu = 0.288(4)$, which we can compare with the estimations given in Ref. [31] $[1/\nu = 0.316(9)]$ and in

Ref. [32] [$1/\nu = 0.29(3)$]. Even if our result seems to be compatible with the one given in Ref. [32], it is not with the one found in Ref. [31]. However, we did not expect to find agreement between the critical exponents of the RFHM and of the long range model in the non-mean-field region, since they are not thought to be in the same universality class [33].

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[1] Y. Imry and S. K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[2] J. Z. Imbrie, Phys. Rev. Lett. **53**, 1747 (1984).
[3] M. Aizenman and J. Wehr, Phys. Rev. Lett. **62**, 2503 (1989).
[4] A. Aharony, Y. Imry, and S. K. Ma, Phys. Rev. Lett. **37**, 1364 (1976).
[5] A. P. Young, J. Phys. C **10**, L257 (1977).
[6] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[7] D. S. Fisher, Phys. Rev. Lett. **56**, 416 (1986).
[8] G. Grinstein, Phys. Rev. Lett. **37**, 944 (1976).
[9] A. J. Bray and M. A. Moore, J. Phys. C **18**, L927 (1985).
[10] M. Schwartz, J. Phys. C **18**, 135 (1985).
[11] M. Schwartz and A. Soffer, Phys. Rev. B **33**, 2059 (1986).
[12] M. Schwartz, M. Gofman, and T. Nattermann, Physica A **178**, 6 (1991).
[13] K. Eichhorn and K. Binder, J. Phys.: Condens. Matter **8**, 5209 (1996).
[14] R. L. C. Vink, T. Fischer, and K. Binder, Phys. Rev. E **82**, 051134 (2010).
[15] R. L. C. Vink, K. Binder, and H. Löwen, J. Phys.: Condens. Matter **20**, 404222 (2008).
[16] N. G. Fytas and V. Martin-Mayor, Phys. Rev. Lett. **110**, 227201 (2013).
[17] A. K. Hartmann, Phys. Rev. B **65**, 174427 (2002).
[18] A. A. Middleton, arXiv:cond-mat/0208182.
[19] B. Ahrens and A. K. Hartmann, Phys. Rev. B **83**, 014205 (2011).
[20] G. Tarjus and M. Tissier, Phys. Rev. Lett. **93**, 267008 (2004); Phys. Rev. B **78**, 024203 (2008).
[21] M. Tissier and G. Tarjus, Phys. Rev. Lett. **96**, 087202 (2006); Phys. Rev. B **78**, 024204 (2008).
[22] M. Tissier and G. Tarjus, Phys. Rev. Lett. **107**, 041601 (2011); Phys. Rev. B **85**, 104202 (2012).
[23] I. Balog, G. Tarjus, and M. Tissier, arXiv:1304.4146.
[24] F. J. Dyson, Commun. Math. Phys. **12**, 91 (1969).
[25] H. G. Katzgraber and A. P. Young, Phys. Rev. B **67**, 134410 (2003).
[26] H. G. Katzgraber, D. Larson, and A. P. Young, Phys. Rev. Lett. **102**, 177205 (2009).
[27] D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, Phys. Rev. B **81**, 064415 (2010).
[28] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **101**, 107203 (2008).
[29] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **103**, 267201 (2009).
[30] R. A. Banos, L. A. Fernandez, V. Martin-Mayor, and A. P. Young, Phys. Rev. B **86**, 134416 (2012).
[31] L. Leuzzi and G. Parisi, Phys. Rev. B **88**, 224204 (2013).
[32] A. K. Hartmann, arXiv:1307.3987.
[33] M. C. Angelini, Ph.D. thesis, Universitá La Sapienza, 2013.
[34] J. Frohlich and T. Spencer, Commun. Math. Phys. **84**, 87 (1982).
[35] P. W. Anderson and G. Yuval, J. Phys. C **4**, 607 (1971).
[36] E. Luijten and H. Messingfeld, Phys. Rev. Lett. **86**, 5305 (2001).
[37] C. Monthus and T. Garel, J. Stat. Mech. (2011) P07010.
[38] P. M. Bleher and J. G. Sinai, Commun. Math. Phys. **33**, 23 (1973).
[39] P. Collet and J. P. Eckmann, Commun. Math. Phys. **55**, 67 (1977).
[40] D. Kim and C. J. Thompson, J. Phys. A **10**, 1579 (1977).
[41] P. M. Bleher and P. Major, Ann. Probab. **15**, 431 (1987).
[42] G. Jona-Lasinio, Phys. Rep. **352**, 439 (2001).
[43] G. A. Baker, Phys. Rev. B **5**, 2622 (1972).
[44] G. Felder, Commun. Math. Phys. **111**, 101 (1987).
[45] J. Kogut and K. G. Wilson, Phys. Rep. **12**, 75 (1974).
[46] S. Franz, T. Jörg, and G. Parisi, J. Stat. Mech. (2009) P02002.
[47] M. Castellana, A. Decelle, S. Franz, M. Mezard, and G. Parisi, Phys. Rev. Lett. **104**, 127206 (2010).
[48] M. Castellana and G. Parisi, Phys. Rev. E **83**, 041134 (2011).
[49] M. Castellana, Europhys. Lett. **95**, 47014 (2011).
[50] M. C. Angelini, G. Parisi, and F. Ricci-Tersenghi, Phys. Rev. B **87**, 134201 (2013).
[51] G. J. Rodgers and A. J. Bray, J. Phys. A: Math. Gen. **21**, 2177 (1988).
[52] F. L. Metz, L. Leuzzi, and G. Parisi, Phys. Rev. B **89**, 064201 (2014).
[53] C. Monthus and T. Garel, J. Stat. Mech. (2011) P05005.
[54] L. P. Kadanoff, Physics **2**, 263 (1966).
[55] L. P. Kadanoff, W. Götze, D. Hamblen, R. Hecht, E. A. S. Lewis, V. V. Palciauskas, M. Rayl, J. Swift, D. Aspnes, and J. Kane, Rev. Mod. Phys. **39**, 395 (1967).
[56] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).
[57] G. Parisi, *Statistical Field Theory* (Perseus Books, New York, 1998).
[58] A. Coniglio, Physica A **281**, 129 (2000).
[59] A. Decelle, G. Parisi, and J. Rocchi, Phys. Rev. E **89**, 032132 (2014).
[60] C. Frontera and E. Vives, Comput. Phys. Commun. **147**, 455 (2002).
[61] Y. Liu and K. A. Dahmen, Phys. Rev. E **76**, 031106 (2007).
[62] With respect to the model studied in Ref. [37], the couplings J_k have been rescaled with the factor $1 - 2/2^{\rho}$, thus their critical variance is equal to ours times this factor. This has been done in order to reduce the spread of the region in h where crosses between curves occur.
[63] J. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, Cambridge, U.K., 1996).
[64] D. Amit and V. Martin-Mayor, *Field Theory, The Renormalization Group, and Critical Phenomena: Graphs to Computers* (World Scientific, Singapore, 2005).
[65] Y. Meurice, J. Phys. A **40**, R39 (2007).