Susceptibility amplitudes for the two-dimensional Ising model

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The recent critical amplitude calculation of Barouch, McCoy, and Wu for the susceptibility of the Ising model on a square lattice is extended to yield amplitudes for the triangular, honeycomb, and kagomé lattices. The extension assumes the validity of the generalized law of corresponding states—as first postulated by Betts, Guttmann, and Joyce—for the two-dimensional Ising model.

Recently, Barouch, McCoy, and Wu,[1] in an intricate calculation, have determined the four amplitudes $C_{0\pm}$ and $C_{1\pm}$ of the reduced high-temperature zero-field isothermal susceptibility of the nearest-neighbor spin-$\frac{1}{2}$ Ising model on a rectangular lattice, as defined by

$$kT\chi_0(T)/m^2 = C_{0\pm}\,|\,1 - T_c/T\,|^{-7/4} + C_{1\pm}\,|\,1 - T_c/T\,|^{-3/4} + O(1) \quad , \qquad (1)$$

where the subscripts + and − refer to $T > T_c$ and $T < T_c$, respectively. It is presumably possible with the expenditure of a great deal of effort to repeat the calculations for other two-dimensional Ising lattices. It is, however, possible to calculate the leading amplitude terms C_{0+} and C_{0-} given the results of Barouch, McCoy, and Wu, using only the *generalized law of corresponding states*, as postulated by Betts, Guttmann, and Joyce.[2]

The law of corresponding states asserts that the equation of state of, say, a magnetic system is a universal equation valid for all lattices, once the temperature, field, and reduced magnetization are scaled according to $t = T/T_c - 1$, $h = \mu H/kT_c$, and $m = M(T)/M(0)$, respectively. That is, the equation of state can be written

$$m_x(t, h) = m_y(t, h) \quad , \qquad (2)$$

where x and y refer to two distinct underlying lattices. This law holds for the Weiss model, but not for the spherical model or the Ising model. For these models the *"generalized law of corresponding states"* is believed to hold, in which the singular part of the free energy per site on lattice x is related to the singular part of the free energy on lattice y by

TABLE I. Tabulation of the critical scaling parameters n_x and g_x for four common planar Ising lattices.

Lattice	g_x	n_x
Triangular	1.000 000 000 0	1.000 000 000 0
Square	1.134 568 121 2	1.299 038 105 7
Kagomé	1.260 958 918 4	1.652 973 376 3
Honeycomb	1.384 193 503 3	2.000 000 000 0

$$n_x f_x(t_x, h_x) = n_y f_y(t_y, h_y) = f(t, h) \quad , \qquad (3)$$

where the reduced field and temperatures are scaled by

$$n_x h_x = n_y h_y = h$$

and $\qquad (4)$

$$g_x t_x = g_y t_y = t \quad .$$

The field derivatives of the free energy are therefore related by

$$n_y^{l-1} \frac{\partial^l f_x}{\partial h_x^l} = n_x^{l-1} \frac{\partial^l f_y}{\partial h_y^l} \quad , \qquad (5)$$

while in zero field the derivatives behave as

$$\frac{\partial^l f_x}{\partial h_x^l} = C^+_{l,x} t_x^{-\gamma_l} \quad l \text{ even}, \quad t > 0$$
$$= C^-_{l,x} (-t_x)^{-\gamma'_l} , \qquad t < 0 \qquad (6)$$

so that combining the above equations we obtain

$$\frac{C^+_{2l,x}}{C^+_{2l,y}} = \left(\frac{n_x}{n_y}\right)^{2l-1} \left(\frac{g_x}{g_y}\right)^{-\gamma_{2l}} , \quad t > 0$$

and $\qquad (7)$

TABLE II. Tabulation of the critical amplitudes for four common planar lattices.

Lattice	C_{0+}	C_{1+}	C_{0-}	C_{1-}
Triangular	0.924 206 958 2	0.063 459 070 1	0.024 518 902 0	− 0.001 683 547 9
Square	0.962 581 732 2	0.074 988 153 8	0.025 536 971 9	− 0.001 989 410 7
Kagomé	1.018 142 230 9	0.088 152 342 9	0.027 010 973 4	− 0.002 338 652 2
Honeycomb	1.046 417 076 1	0.099 454 879 3	0.027 761 095 6	− 0.002 638 504 7

TABLE III. Tabulation of the latest available critical amplitude estimates for three common planar Ising lattices.

Lattice	C_{0+}	C_{1+}	C_{0-}
Triangular	0.92422 ± 0.00002	0.0633	0.0246 ± 0.0002
Square	0.962589 ± 0.00002	0.0742	0.0256 ± 0.0001
Honeycomb	1.04642 ± 0.00003		0.0279 ± 0.0002

$$\frac{C_{l,x}^-}{C_{l,y}^-} = \left(\frac{n_x}{n_y}\right)^{l-1} \left(\frac{g_x}{g_y}\right)^{-\gamma_l'}, \quad t<0 \quad .$$

The foregoing summarizes the discussion given by Betts, Guttmann, and Joyce. The "law" was tested using the best available data at that time. One conclusion was that the theory seemed to be valid for the two-dimensional Ising model. The more recent series work of Sykes *et al.*[3] permits this conclusion to be reaffirmed with even greater confidence. In the following we will assume the validity of the theory for the two-dimensional Ising model.

The calculation of Betts, Guttmann, and Joyce can be readily extended to yield amplitude relations for the next most singular term in the free energy. Rewriting Eq. (6) for the susceptibility ($l=2$) in the notation of Barouch, McCoy, and Wu [as in Eq. (1)], we have

$$\frac{\partial^2 f_x}{\partial h_x^2} = C_{0+,x} t_x^{-\gamma} + C_{1+,x} t_x^{1-\gamma}, \quad t>0$$

and (8)

$$\frac{\partial^2 f_x}{\partial h_x^2} = C_{0-,x} t_x^{-\gamma'} + C_{1-,x} t_x^{1-\gamma'}, \quad t<0 \quad .$$

Combining Eqs. (5) and (8) and in addition making use of Eqs. (4) and (7), we obtain

$$\frac{C_{0+,x}}{C_{0+,y}} = \left(\frac{n_x}{n_y}\right)\left(\frac{g_x}{g_y}\right)^{-\gamma}, \quad \frac{C_{1+,x}}{C_{1+,y}} = \left(\frac{n_x}{n_y}\right)\left(\frac{g_x}{g_y}\right)^{1-\gamma}, \tag{9}$$

$$\frac{C_{0-,x}}{C_{0-,y}} = \left(\frac{n_x}{n_y}\right)\left(\frac{g_x}{g_y}\right)^{-\gamma'}, \quad \frac{C_{1-,x}}{C_{1-,y}} = \left(\frac{n_x}{n_y}\right)\left(\frac{g_x}{g_y}\right)^{1-\gamma'} \quad .$$

From Eq. (8) the exact values $\gamma = \frac{7}{4} = \gamma'$ and the exact values for n_x, n_y, g_x, g_y on a two-dimensional lattice, we can extend all the results of Barouch, McCoy, and Wu to other two-dimensional lattices. The required input data is provided in Table I for the four most common two-dimensional lattices: the honeycomb, square, kagomé, and triangular lattices. This same data is given by Betts, Guttman, and Joyce, but is restated here to a higher degree of precision.

These data are combined with the following results of Barouch, McCoy, and Wu for the square lattice:

$$C_{0-} = 0.0255369719\ldots, \qquad C_{0+} = 0.9625817322,$$

$$C_{1-} = -0.0019894107\ldots, \qquad C_{1+} = 0.0749881539,$$

whence it is a straightforward application of Eq. (9) to derive these four amplitudes for the other three lattices. These results are summarized in Table II.

The high-temperature amplitudes compare well with those given recently by Sykes *et al.*,[3] which were obtained by analysis of extended series, and the low-temperature amplitudes C_{0-} also compare favorably with those obtained by Guttmann[4] from an analysis of extended low-temperature series derived by Sykes, Gaunt, and co-workers at King's College. These results are summarized in Table III.

In conclusion, by assuming the validity of the generalized law of corresponding states, first postulated by Betts, Guttmann, and Joyce,[2] we have extended the recent work of Barouch, McCoy, and Wu[1] to the other common two-dimensional lattices.

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