Ising-model, spin-spin correlations on the hypercubical lattices*

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The first seven terms of the high-temperature series expansion for the true range of correlation are derived for general dimension on the family of hypercubical lattices. Analysis of this series shows that a cross-over phenomenon occurs at high dimension so that for $d = \infty$ one obtains for the correlation-length critical index $\nu = 1$ instead of the classical value $\nu = 1/2$. This behavior is also illustrated by a more tractable, analogous, random-walk problem. The behavior of the critical temperature as a function of dimension is discussed. All present evidence is consistent with an essential singularity at $d = 4$.

I. INTRODUCTION AND SUMMARY

One of the more dramatic predictions of the renormalization-group approach of Wilson[1] has been the existence of a distinguished dimension, 4, above which the classical Bragg-Williams approximation, in some sense, holds. In this paper we seek to investigate by high-temperature series-expansion techniques some aspects of this high-dimensional region for the ferromagnetic nearest-neighbor spin-$\frac{1}{2}$ Ising model on a d-dimensional hypercubical lattice. (The plane square and simple cubic lattices are the $d=2, 3$ members of this family of lattices.)

To this end we have derived the first seven terms for general dimension of the expansion for the true range of the spin-spin correlations. We do not find an expansion in powers of d^{-1} for the behavior of the temperature at which the spin-spin correlation length becomes infinite, in contrast to the results of Fisher and Gaunt[2] on the susceptibility. We trace this behavior to a cross-over phenomenon. The critical region for the spin-spin correlation range is only $(1-T_c/T)\approx O(d^{-1})$. For temperatures above this range, the spin-spin correlation length behaves as though its critical index were $\nu=1$ instead of the expected value, $\nu=\frac{1}{2}$. This cross-over phenomenon may explain the failure of Abe[3] to obtain an expansion for γ in powers of d^{-1}.

To further clarify the nature of this behavior we consider a more tractable, closely analogous, random-walk problem. Basically one needs to consider the d-dimensional random walks which begin and end in a $(d-1)$-dimensional hyperplane. The results of this calculation confirm in detail the above observations on the Ising problem. We also give bounds on the Ising-model coefficients in terms of the coefficients to the random walk, and hence an upper bound on the critical temperature.

Finally, we consider the behavior of $T_c(d)$. We show that in the renormalization-group approach, this function is expected to be regular for real d greater than 4 and to have an essential singularity, having all derivatives finite, at $d=4$. We analyze the series data of Fisher and Gaunt[2] for this function and find that they are not inconsistent with the renormalization-group predictions in this regard. A more definite result would require additional data.

II. TRUE RANGE OF CORRELATION

In order to calculate the true range of correlation, we follow Fisher and Burford.[4] First, we define the true (inverse) range of correlation to be

$$\kappa_{\vec{e}} = -\lim_{r\to\infty} \limsup_{N\to\infty}\{(1/r)\ln|\Gamma_N(r\vec{e})|\}, \qquad (2.1)$$

where $\vec{e}$ is a unit vector in the direction $\vec{r}$ and $\Gamma_N(\vec{r})$ is the spin-spin correlation function between a spin at the origin and a spin at $\vec{r}$ in a lattice of N spins. Fisher and Burford[4] give the following series expansion for $\kappa_{\vec{e}}$ (we select κ_x along the first coordinate axis for study):

$$\omega_x = e^{-\kappa_x a} = v\sum_{n=0}^{\infty} w_n v^n, \qquad (2.2)$$

where a is the lattice spacing in the x direction and $v=\tanh(J/kT)$. The w_n are the coefficients of N^0 in w_n $(L=1; N)$, where

$w_n(L;N)$ is the number of configurations of $L+n$ lines on a lattice of N sites ($L>n$, and $N>Ln^{d-1}$) in which (a) each lattice bond is used at most once, (b) an even number of lines meet at each site, and (c) a chain of bonds reaches around the lattice in the direction $\vec{e}$, and passes through the origin. (2.3)

Here the lattice considered is made up of L,$(d-1)$-dimensional layers. A chain of L bonds will just reach from the top to the bottom and close if we impose "skew-periodic" boundary conditions, as required by Fisher and Burford.[4] [We thank Prof. Fisher for making his unpublished derivation of Eq. (2.3) available to us.] Using these rules, we have derived for general d, by enumerating the different allowed configurations, the following expression for hypercubical lattices;

$$\begin{aligned}\omega_x = v[&1+2(d-1)v+2(d-1)(2d-3)v^2\\&+2(d-1)(2d-3)^2v^3+2(d-1)(2d-3)^3v^4\\&+2(d-1)[(2d-3)^4+2(d-2)(2d-2)]v^5\\&+2(d-1)[(2d-3)^5\\&\qquad+2(d-2)(12d^2-30d+11)]v^6+\cdots,\end{aligned} \tag{2.4}$$

which exactly checks the results of Fisher and Burford[4] for $d=2, 3$. It would appear that Moore[5] has calculated, but not published, these terms for $d=4$. We have taken the number of self-avoiding walks in d dimensions from the work of Fisher and Gaunt.[2]

We are in a position to make some observations concerning the structure of (2.4) for large d. First, to examine the expected divergence in the range of correlation, we consider

$$(1-\omega_x)^{-1} \approx (\kappa_x a)^{-1} \propto (1-T_c/T)^{-\nu}, \tag{2.5}$$

when the true range of correlation is large. The critical index ν is defined by (2.5). Rewriting (2.4) in the form (2.5), we find

$$\begin{aligned}(1-\omega_x)^{-1} = 1+v+(2d-1)v^2&+(4d^2-6d+3)v^3+(8d^3-20d^2+20d-7)v^4+(16d^4-56d^3+84d^2-60d+17)v^5\\&+(32d^5-144d^4+296d^3-340d^2+214d-57)v^6\\&+(64d^6-352d^5+928d^4-1496d^3+1484d^2-782d+155)v+\cdots\end{aligned} \tag{2.6}$$

which, for large d, has the structure

$$(1-\omega_x)^{-1} \approx 1+\frac{1}{2d}f(2dv). \tag{2.7}$$

More precisely, since the contributions to leading order in d to a fixed w_n come exclusively from the number of self-avoiding random walks in $d-1$ dimensions, and as for each retrograde step in the x direction there must be a corresponding forward step which reduces the contribution by a factor d^{-2}, we can show that the coefficients of

$$G(v) = [(1-\omega_x)^{-1}-1]/v = \sum_{n=1}^{\infty} g_n v^n \tag{2.8}$$

have the structure, for $d \gg n$, $n \geq 3$,

$$g_n = (2d)^n\left(1-\frac{2n-1}{2d}+\frac{\frac{5}{2}n^2-\frac{13}{2}+7}{4d^2}+\cdots\right). \tag{2.9}$$

Instead of analyzing (2.6) directly by Padé approximants which Fisher and Burford[4] had found to be unsatisfactory, we have rather preferred to analyze $G(v)$.

First, we note that an expansion, for the critical value v_c at which $G(v)$ becomes singular, in powers of d^{-1} is not available; Fisher and Gaunt[2] found such an expansion from the susceptibility. The reason here is that $[\ln g_n(d^{-1})]$ is not formally proportional to n. We will return to the expected behavior of g_n, $n \gg d$, in Sec. III. Fisher and Gaunt[2] found that $v_c = 1/(2d-1)+O(d^{-3})$. If we use the Fisher and Gaunt value for v_c then it is easy to show that for $0 \leq 2dv < 1$ that in the limit as $d\to\infty$ we have, term by term

$$G(v) = \frac{1}{1-2dv}+O(d^{-1}). \tag{2.10}$$

Thus, if we adjust the exchange integral, J, to keep the total interaction energy per lattice site fixed, $J=J_0/2d$, then we have, for fixed $kT > J_0$,

$$G = \frac{1}{1-J_0/kT}+O(d^{-1}) \tag{2.11}$$

which implies, for $d=\infty$, the value $\nu=1$, instead of the expected value of $\nu=\frac{1}{2}$.

There is however, more to be said than (2.11) about the index ν. From structure (2.7), the range of correlation does not get large until $f=2dvG$ gets large compared to $2d$, or by (2.11) until $(1-T_c/T)\approx O(d^{-1})$. This region is beyond the range of validity of (2.11). Thus, as far as d large, but not infinite, is concerned, the true critical region only occurs for T very close to T_c, and we have cross-over behavior as $d\to\infty$.

We have analyzed $G(v)$ by forming Padé approximants to $[d\ln G(v)/dv]$. These Padés are expected to have a pole at $v=v_c$ with residue $-\nu$. The [2/3] also has the property of being exact for $d=1$ and $d=2$ as

$$\begin{aligned}\frac{d\ln G_1(v)}{dv} &= \frac{1}{1-v},\\ \frac{d\ln G_2(v)}{dv} &= \frac{3+v+v^2}{1-v-3v^2-v^3},\end{aligned} \tag{2.12}$$

by the exactly known solutions for those problems.[4] We find, comparing with the results of Fisher and Gaunt,[2] that the pole of the [2/3] is correctly located to at worst within a 2% error for all values of d. For d large, using the notation $\xi = 1/(2d)$ and $x = 2dv$, we compute, to order ξ,

$$[2/3] = \frac{5-2x+\xi(-70+34x-25x^2)}{5-7x+2x^2+\xi(-65+112x-68x^2+27x^3)}, \tag{2.13}$$

which vanishes at $x = 1+2\xi+O(\xi^2)$ or $v_c = [2(d-1)]^{-1}$ instead of $v_c = (2d-1)^{-1}$, and has a residue of $-1+O(\xi^2)$. Note that if we set $\xi = 0$, (2.13) reduces to $(1-x)^{-1}$ as expected.

For ν we obtain from the [2/3] Padé approximants the results given in Table I.

These results are consistent with our picture that the critical region shrinks as d increases and, instead of $\nu \simeq 0.64$ $(d=3)$,[6] $\nu \simeq 0.536$ or 0.5 $(d=4)$,[1,5] and $\nu \simeq 0.5$ $(d>4)$,[1] we see progressively more of the perturbing influence of $d=\infty$ behavior on our short series results. Our results on the size of the critical region in high dimension may also explain the failure of Abe[3] to obtain an expansion for γ in powers of d^{-1}.

III. RELATED RANDOM WALKS

In this section we show that bounds on the coefficients w_n in Eq. (2.2) may be obtained in terms of the number of self-avoiding random walks on the d-dimensional lattice which begin and end in the same $(d-1)$-dimensional layer. The fundamental result needed is Fisher's[7] inequality

$$\Gamma_N(r\vec{e}) \le C_N(r\vec{e}), \tag{3.1}$$

where $\Gamma_N(r\vec{e})$ is as in Eq. (2.1) and $C_N(r\vec{e})$ is the generating function for the self-avoiding walks between the origin and $r\vec{e}$. As in Sec. II we will select for study $\vec{e}$ parallel to the first coordinate axis. From the definition (2.1) and from (3.1) we obtain

$$\omega_x = e^{-\kappa_x a} \le \lim_{r\to\infty} \lim_{N\to\infty} \sup\{[C_N(r\vec{e}_x)]^{a/r}\}. \tag{3.2}$$

If we increase the number of configurations

TABLE I. $\nu(d)$ based on [2/3].

d	ν
1	1.000 (exact)
2	1.000 (exact)
3	0.778
4	0.688
5	0.692
6	0.728
10	0.870
20	0.969
50	0.998

counted in $C_N(r\vec{e})$, as each makes a positive contribution, we will continue to have a bound. We will therefore include the following: (i) all the self-avoiding random walks which begin at the origin and end in the $L = r/a$ layer (ii.) Since for the purposes of deriving a series expansion we are concerned with those walks which are only a finite number, n, of steps longer than L, it is convenient to classify them by the number of breaks in the chain of steps leading directly from the origin to the Lth layer. For m breaks there will be $\binom{L}{m}$ ways in which these breaks can be positioned. At each break we can put a self-avoiding random walk which begins and ends in the same $(d-1)$-dimensional layer. This procedure may over count, as if the two random walks start close to each other they may intersect or even overlap. We explicitly include them even so.

Now, if we take the Lth root and the limit as $L\to\infty$ (formally equivalent to setting $L=1$)[8] we see since $\binom{1}{m}=0$, $m>1$, that the only contribution comes from the case with a single break in the main chain of steps. Thus we have found

$$\omega_x \le v\sum_{n=0}^{\infty} b_n v^n = B(v), \tag{3.3}$$

where b_n is the number of self-avoiding walks in d dimensions which begin and end in the same $(d-1)$-dimensional layer. We remark that b_n and w_n are identical for $n=0$, 1, 2, and 3, and the two leading powers of d in the polynomials in d, b_n and w_n, are equal for all values of n.

We will now estimate the asymptotic behavior of the b_n. First for ease of presentation we will compute this behavior for unrestricted random walks which begin and end in the same layer. It is well known, if we assign a weight x to steps to the right and x^{-1} to steps to the left, and similarly y and y^{-1}, etc., in other directions, that the number of n step random walks is given by

$$(x+x^{-1}+\cdots+z+z^{-1})^n. \tag{3.4}$$

We may select those for which the number of right and left steps is equal by the Cauchy formula

$$\frac{1}{2\pi i}\oint \frac{dx}{x^{n+1}} = \begin{cases} 1 & n=0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.5}$$

where the contour of integration encircles the origin. Thus, the number of unrestricted n-step walks, u_n, which begin and end in the same $(d-1)$-dimensional layer, is

$$u_n(d) = \frac{1}{2\pi i}\oint (x+x^{-1}+\cdots+z+z^{-1})^n \frac{dx}{x} \tag{3.6}$$

or, letting $x=e^{i\varphi}$, and the other weights be unity, we have

$$u_n(d)=\frac{1}{2\pi}\int_{-\pi}^{\pi}[2\cos\varphi+2(d-1)]^n\,d\varphi\ . \qquad (3.7)$$

This integral can be evaluated explicitly, as

$$u_n(d)=\sum_{j=0}^{[\frac{1}{2}n]}\frac{n!}{(j!)^2(n-2j)!}[2(d-1)]^{n-2j}, \qquad (3.8)$$

but it is more informative for our purposes to give an approximate, asymptotic evaluation by means of the saddle-point method. The integrand has a single peak at $\varphi=0$. We find, by direct calculation that

$$u_n(d)\approx\left(\frac{d}{2\pi n}\right)^{1/2}(2d)^n\ , \qquad (3.9)$$

where $n\gg d$ is required for the validity of this result. It is to be noted that the function

$$U(v)=\sum_{n=0}^{\infty}u_n(d)\,v^{n+1} \qquad (3.10)$$

mimics the behavior found in Sec. II for $\omega_x(v)$. That is, if we take the limit of $U(x/2d)$ for $0\leqslant x\leqslant 1$, we see from (3.8) that

$$\lim_{d\to\infty}2d\,U(x/2d)=x(1-x)^{-1} \qquad (3.11)$$

yet, if we take the limit $x\to 1$ before the limit $d\to\infty$ we see from (3.9), if we use the known relation[9]

$$\sum_n n^{\lambda}(w/w_0)^n\propto(1-w/w_0)^{-1-\lambda} \qquad (3.12)$$

as $w\to w_0$, that

$$U(v)\propto(1-2dv)^{-1/2} \qquad (3.13)$$

as $v^{-1}\to 2d$. This exponent corresponds to the expected value of $\nu=\frac{1}{2}$, where as (3.11) corresponds to $\nu=1$. We presume that the behavior of the true range of correlation for high dimension parallels that of the U function.

By use of matrix methods[10] and perturbation theory, we can do an analogous calculation for random walks in d dimensions with no immediate reversals which begin and end in the same $(d-1)$-dimensional layer. In this case we compute

$$l_n\approx 2(d-1)(2d-1)^{n-1}\left(\frac{(d-1)}{2\pi(n-1)}\right)^{1/2}\ , \qquad (3.14)$$

which displays a shifted critical point $[v_c=(2d-1)^{-1}$ instead of $v_c=(2d)^{-1}]$, but otherwise the same essential features as were seen in the $U(v)$ function.

IV. CRITICAL TEMPERATURE AS A FUNCTION OF DIMENSION

As pointed out by Fisher and Gaunt[2] one may study by series techniques for general dimension, the behavior of the critical temperature as a function of dimension on the hypercubical family of lattices. In this section we first discuss what behavior is to be expected from the renormalization-group approach, and then how this expectation compares with the actual series data.

The renormalization-group approach of Wilson[11,12] has been extremely successful in calculating many of the thermodynamic properties of critical phenomena. One of its more dramatic predictions has been that in dimensions higher than 4, the Bragg-Williams approximation should be valid.[12,13] Much of the underlying behavior of the approach is exemplified by an approximate set of nonlinear integral recursion relations. These recursion relations can be thought of either as an approximation to the usual Ising model[12] or (slightly modified) as the exact solution[14] to an Ising model with a certain long-range non-translationally-invariant interaction. The Hamiltonian for this system [of size $(2^L)^d$ spins] is

$$H=J\sum_{l=0}^{Ld-1}2^{-l\sigma/d}\sum_{m=1}^{2^{Ld-1-l}}s_{ml}^2+mH2^{dL/2}\hat{s}_{1,Ld-1}$$
$$-\tfrac{1}{2}J\left(\frac{1-2^{-L(d+\sigma)}}{1-2^{-1-\sigma/d}}\right)\sum_j\nu_j^2 \qquad (4.1)$$

where

$$s_{m,l+1}=(\hat{s}_{2m-1,l}-\hat{s}_{2m,l})/\sqrt{2}\,,\ m=1,\ldots,2^{Ld-2-l}$$
$$\hat{s}_{m,l+1}=(\hat{s}_{2m-1,l}+\hat{s}_{2m,l})/\sqrt{2}\,,\ l=1,\ldots,Ld-1$$
$$\hat{s}_{j,-1}=\nu_j\,,\ j=1,\ldots,2^{Ld} \qquad (4.2)$$

and the spin weights usually considered are $\exp(a\nu_j^2-0.1\nu_j^4)$. For this model, the effective spin-spin interaction decays in a stair-step fashion and behaves roughly like $1/r^{d+\sigma}$. By renormalization-group techniques one can compute, for example, for $d=3$, $\sigma=1.94$ (this value corresponds to $\eta=0.06$) that the standard critical indices have the values $\gamma=\gamma'=1.256$, $\beta=0.3429$, and $\delta=4.66$. For $d>2\sigma$, we find $\gamma=\gamma'=1.000, \beta=0.500$, $\delta=3$ which are the Bragg-Williams values. The model obeys the identities[15]

$$\eta=2-\sigma\,,$$
$$\delta=\max\left(3,\frac{d+\sigma}{d-\sigma}\right),$$
$$\beta=\gamma/(\delta-1)\,. \qquad (4.3)$$

Now, in order to compute $T_c(d)$, we follow

Wilson[12] and expand the fundamental $Q_l(y)$ function[14] in powers of y^2 as

$$Q_l(y) \approx r_l y^2 + u_l y^4 + \cdots . \qquad (4.4)$$

Then the approximate recursion relations, neglecting terms of the order u_l^3 are

$$r_{l+1} = 2^{\sigma/d}(r_l + 3u_l q_l - 9u_l^2 q_l^3), \qquad (4.5)$$

$$u_{l+1} = 2^{2\sigma/d-1}(u_l - 9u_l^2 q_l^2), \qquad (4.6)$$

where $q_l = (K + r_l)^{-1}$ and $K = J/kT$. The equation for the critical temperature is $\lim_{l\to\infty} r_l = 0$. For the range of dimension, $d > 2\sigma$, which we are now treating, we see from (4.6) that the u_l tend to zero automatically as the prefactor is less than one. We can therefore use recursion relation (4.5) to compute the critical-point requirement for finite l. Thus to first order in u_l,

$$r_l = -3 \sum_{n=l}^{\infty} 2^{-(n-l)\sigma/d} u_n q_n . \qquad (4.7)$$

If we now examine (4.6), we may give an approximate solution. We use the form $(\epsilon = 2\sigma/d - 1)$,

$$u_l = \frac{\alpha 2^{\epsilon l}}{\beta - 2^{\epsilon l}} \qquad (4.8)$$

suggested by the solution to the analogous differential equation

$$\frac{du_l}{dl} = -au - bu^2 . \qquad (4.9)$$

The result is

$$u_l = \frac{[-2^{-\epsilon}K^2(1-2^{-|\epsilon|})/9\,\mathrm{sgn}(\epsilon)]2^{\epsilon l}}{1 - [2^{-\epsilon}(1-2^{-|\epsilon|})K^2/9u_0\,\mathrm{sgn}(\epsilon)] - 2^{\epsilon l}}, \qquad (4.10)$$

where $\mathrm{sgn}(\epsilon)$ is the algebraic sign of ϵ. Since u_0 is necessarily positive, this expression is valid for all real ϵ and reduces to

$$u_l = \frac{u_0}{1 + 9u_0 l K^{-2}} \qquad (4.11)$$

for $\epsilon = 0$ in conformity with Wilson's[12] results. We have approximated q_l by K^{-1} in this solution. If we use the same approximation, we have by (4.7)

$$r_0 K_c = -3 \sum_{n=0}^{\infty} 2^{-n\sigma/d} u_n \qquad (4.12)$$

which gives an approximate equation for the critical temperature in terms of d by way of (4.10). For $d \neq 2\sigma$ we predict perfectly smooth, regular behavior for $T_c(d)$. For $d = 2\sigma$, we obtain a finite continuous value, and a little computation shows that all left- and right-hand derivatives exist at $d = 2\sigma$. The left-hand $(d < 2\sigma)$ derivatives are related to the ϵ expansion.[13] Nevertheless, inspection of (4.10) and (4.12) show that $d = 2\sigma$ is a limit point of poles which get closer and closer to $\epsilon = 0$ as we contemplate higher and higher values of l. The poles approach from the plus and minus imaginary ϵ directions. Therefore $d = 2\sigma$ is an essential singularity of the $T_c(d)$ function for this model. We speculate that with $\sigma = 2$, the short-range equivalent value, the same may be true of the Ising model—that is, $d = 4$ is an essential singularity of $T_c(d)$ for the hypercubical nearest-neighbor spin-$\frac{1}{2}$ ferromagnetic Ising model.

We give a mathematical example, which illustrates a possible behavior for the power series of a function with a "smooth" singularity when expanded about another point. We use

$$F(\xi) = \int_0^\infty \frac{e^{-t^{-1}}dt}{1 + (1-\xi)t} = F_0 + F_1\xi + F_2\xi^2 + \cdots \qquad (4.13)$$

which has its singularity at $\xi = 1$ and has all right-hand derivatives there. The coefficients are easily calculated in terms of exponential integrals. Thus we get

$$F(\xi) \approx 0.219\,383\,934 + 0.070\,888\,427\xi + 0.032\,084\,887\xi^2 + 0.016\,910\,823\xi^3 + 0.009\,757\,982\xi^4 + 0.005\,987\,308\xi^5 + \cdots \qquad (4.14)$$

which yields the ratios

0.323, 0.453, 0.527, 0.577, 0.614 (4.15)

which increase rapidly and then flatten out well below unity.

We have available, to investigate this problem, the series expansion of Fisher and Gaunt[2]

$$kT_c/J = q - 1 - \tfrac{4}{3}q^{-1} - \tfrac{13}{3}q^{-2} - 21\tfrac{34}{45}q^{-3} - 133\tfrac{14}{15}q^{-4} - \cdots \qquad (4.16)$$

where $q = 2d$. The ratios of successive terms are rapidly increasing, and the constant sign locates the nearest singularity at positive real d. Originally it was supposed that the series was asymptotic, but it may be that it is convergent and indicating a singularity at a distinguished dimension. For ease of comparison we give the ratios, normalized so that $d_D = 4$ corresponds to a ratio of unity.

0.125, 0.167, 0.469, 0.628, 0.770. (4.17)

These ratios are in the rapidly increasing phase, but showing some signs of slowing down. It seems likely that the next few terms would reveal if the ratios are stabilizing at a finite value (assuming it is near $d=4$) or simply continuing to grow.

A Padé analysis [actually of the series in $(2d-1)^{-1}$] predicts a weak singularity at around $d \simeq 3\frac{1}{2}$. This result is not inconsistent with our expectations of a singularity processing all derivatives at $d=4$. Internal evidence[16] on error estimation suggests the available Padé approximants have converged to one part in 10^4 at $d=8$, one part in 10^3 at $d=6$, and one part in 10^2 at $d=5$. Actually, at $d=5$ they agree to about one part in 10^3 with the direct analysis of Fisher and Gaunt[2] based on the usual susceptability series and with our Padé reanalysis of their data. Our reanalysis yields $v_c = 0.113\,54 \pm 0.000\,02$ and $\gamma = 1.039 \pm 0.003$. The error estimates here are of the traditional (optimistic) self-consistency sort. If one forces the renormalization-group result, $\gamma = 1.000$ as an assumption into the analysis, then the larger error quoted previously would be appropriate.

It would seem that available series evidence on the existence of a distinguished dimension at which $T_c(d)$ has a singularity is not inconsistent with our expectations based on renormalization-group theory. A stronger statement will have to await more data.

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