Renormalization-group calculations of exponents for critical points of higher order*

T. S. Chang, George F. Tuthill, and H. Eugene Stanley
Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 20 March 1973; revised manuscript received 12 November 1973)

Critical points of higher order can exist in complex magnetic and fluid systems. By definition, at a critical point of order $\mathcal{O}$, $\mathcal{O}$ phases become identical simultaneously. Here the Wilson renormalization-group method is generalized from ordinary critical points ($\mathcal{O} = 2$) and Gaussian tricritical points ($\mathcal{O} = 3$) to critical points of arbitrary order $\mathcal{O}$. An expansion scheme in $\epsilon_{\mathcal{O}} \equiv 2\mathcal{O}/(\mathcal{O}-1)-d$ is proposed. The nontrivial fixed points and the critical exponents for $\mathcal{O} = 3,4$ are calculated to order $\epsilon_{\mathcal{O}}$. We present the relevant scaling fields and densities for $\mathcal{O} = 3$, and, in an appendix, justify the validity of the approximate recursion relation to order $\epsilon_{\mathcal{O}}$.

The concept of a "critical point of higher order" has been introduced for complex magnetic systems[1] and for ternary and quaternary fluid mixtures.[2] At such a critical point three or more phases can become critical simultaneously.

The Wilson renormalization-group approach[3] has been extensively applied to critical points of order $\mathcal{O}=2$. Of particular utility have been the expansions, due to Wilson and Fisher,[4] in the variable $\epsilon_2 \equiv 4-d$, where d is the lattice dimensionality.

Can these results be generalized to critical points of arbitrary order $\mathcal{O}$? Recently, Riedel and Wegner[5] have treated tricritical points ($\mathcal{O}=3$) for the special case $d=3$. Since this corresponds to the Gaussian fixed point for $\mathcal{O}=3$, one might expect the existence of expansions in the variable $\epsilon_3 \equiv 3-d$, for $\mathcal{O}=3$. There also exist examples of critical points of still higher order[1,2,6] ($\mathcal{O}>3$), for which there has been no renormalization-group work done of any sort.

In this paper we give the appropriate generalization of the Wilson renormalization-group procedure[3] and propose a generalized Wilson-Fisher expansion[4] scheme for critical points of *arbitrary* order $\mathcal{O}$. Our starting point is the reduced Ising Hamiltonian H_0 of the general form[3]

$$H_l\{s(\vec{x})\} = -\int d\vec{x}\,[\tfrac{1}{2}|\nabla s(\vec{x})|^2 + Q_l\{s(\vec{x})\}]\ , \qquad (1a)$$

where, in obvious generalization of the cases $\mathcal{O}=2, 3$,

$$Q_l(s) = \sum_{k=1}^{\mathcal{O}} r_{kl}\, s^{2k}\ . \qquad (1b)$$

The approximate recursion formulas are[7]

$$Q_{l+1}(s) = -2^d \ln[I_l(2^{1-d/2}s)/I_l(0)]\ , \qquad (2a)$$

where

$$I_l(z) \equiv \int_{-\infty}^{\infty} dy\, \exp[-y^2 - \tfrac{1}{2}Q_l(z+y) - \tfrac{1}{2}Q_l(z-y)]\ . \qquad (2b)$$

The basic motivation of using the form of (1b) for Q_l is suggested by the general Landau model used by Griffiths[2(c)] to provide a phenomenological theory of critical points of higher order. It can be shown that Q_0 is an appropriate linear combination of a set of $\mathcal{O}$ symmetric densities $Q^{(k)}$ which contain even powers of s such that

$$Q_0 = \sum_{k=2}^{\mathcal{O}+1} x_k\, Q^{(k)}\ .$$

As $x_k \to 0$ for $k=2,3,\ldots,\mathcal{O}$, $\mathcal{O}$ phases become identical simultaneously. Thus the model is expected to simulate the behavior of a critical point of order $\mathcal{O}$.

It is possible to include other symmetry-breaking densities in Q_l. This, however, will require in general some modifications of the recursion formulas (2), and will be considered elsewhere. On the other hand, the inclusion of an ordering density s in Q_l is quite straightforward because such a term remains completely uncoupled from the other symmetric densities in the Hamiltonian upon iteration. It will be shown below that for (1b) and with the inclusion of an ordering density, there will be exactly $\mathcal{O}$ relevant scaling fields for a critical point of order $\mathcal{O}$. The number of relevant scaling fields can increase, of course, if other symmetry-breaking densities are included. In a separate phenomenological paper,[2(d)] we have shown that for a critical point of order $\mathcal{O}$, there are in general

$$s = n - x - d$$

relevant *scaling fields*. Here n is the total number of thermodynamic fields of the system, d is the dimension of the critical subspace, and x is the number of phases which are in equilibrium at the critical subspace but are not critical.

Substituting (1b) into (2b) and expanding the non-Gaussian terms for $r_{kl} \ll 1$, it can be shown by induction that for $k=1, 2, \ldots, \mathcal{O}$,

$$r_{k,l+1} \cong 2^{(2-d)k+d}[L_k(\{r_{jl}\}) + N_k(\{r_{jl}\})]\ , \qquad (3)$$

where L_k and N_k are linear and nonlinear functions, respectively, of the set of variables $\{r_{jl}\}$, and j ranges over $1,2,\ldots,\mathcal{O}$. From the structure of the approximate recursion relations (3) (cf. Table I) we note that to the first order of $r_{\Theta l}$, the fixed points for $(2-d)\Theta+d=0$ [or $d=2\Theta/(\Theta-1)$] must be of the form

$$Q_l(s)\cong K_l\sum_{k=1}^{\Theta} c_k s^{2k}\,, \tag{4}$$

where c_k are constants. To evaluate K_l, terms from $r_{\Theta+1,l}s^{2\Theta+2}$ to $r_{2\Theta-2,l}s^{4\Theta-4}$ are included in (1b). From (3), which now includes the additional irrelevant parameters $r_{\Theta+1,l}$ to $r_{2\Theta-2,l}$ (which are of order $r_{\Theta,l}{}^2$), and from recursion relations for these additional parameters, we find that

$$K_l=K_0/(1+AK_0c_\Theta l)\,,$$

where A is a positive constant. Therefore, $\lim_{l\to\infty}K_l=0$. Expression (4) corresponds to an Θ-well potential in the Hamiltonian.[8]

For lattice dimensions $d\neq 2\Theta/(\Theta-1)$, we generalize the Wilson-Fisher[4] expansion scheme by defining an expansion parameter

$$\epsilon_\Theta=2\Theta/(\Theta-1)-d\ . \tag{5}$$

We make the ansatz that the r_{kl} for $k=1,2,\ldots,\Theta$ are of order ϵ_Θ, and replace Eqs. (3) for $j,k=1,2,\ldots,2\Theta-2$, by the following set of differential equations:

$$\frac{dr_k(l)}{dl}=\sum_{j=1}^{2\Theta-2}e_{jk}r_j+\sum_{i,j=1}^{\Theta}f_{ijk}r_ir_j + \text{terms of order } \epsilon_\Theta^3\,, \tag{6}$$

where e_{jk} include terms up to order ϵ_Θ^1, f_{ijk} are of order ϵ_Θ^0, and $k=1,2,\ldots,2\Theta-2$. For consistency,[9] we find that it is again necessary to include in our calculations the additional irrelevant parameters $r_{\Theta+1}$ to $r_{2\Theta-2}$, which are assumed to be of order ϵ_Θ^2. To first order of ϵ_Θ, Eqs. (6) admit the nontrivial fixed-point solutions

$$r_k^*=K_\Theta c_k\epsilon_\Theta\ln 2+\text{terms of order } \epsilon_\Theta^2\ , \tag{7}$$

where K_Θ is a constant and $c_k=0$ for $k>\Theta$. Equation (7) justifies the ansatz. We find that exactly the same results can be obtained using Feynman-diagrammatic techniques and the exact recursion relations (see Appendix A). Thus, these results, are exact to the first order of ϵ_Θ. We note that $r_k^*\propto c_k$ for $k=1,2,\ldots,\Theta$, because the expansion scheme requires the nontrivial fixed point Q^* to have the same form as (4).

Linearizing the recursion relations (3) with $j,k=1,2,\ldots,2\Theta-2$, we obtain, for small changes of r_{kl}, the following set of $2\Theta-2$ linear algebraic equations:

$$\sum_{k=1}^{2\Theta-2}(B_{jk}-\lambda\delta_{jk})A_k=0\quad(j=1,2,\ldots,2\Theta-2)\ , \tag{8}$$

where B_{jk} includes terms up to order ϵ_Θ and the A_k denote changes of r_{kl}. To lowest order of ϵ_Θ, the eigenvalues of Eq. (8) are

$$\lambda_k=2^{\mu_k}+b_k\epsilon_\Theta\ln 2\quad(k=1,2,\ldots,\Theta)\ , \tag{9}$$

where $\mu_k=2(\Theta-k)/(\Theta-1)$ and b_k are constants. The eigenvalues $\lambda_{\Theta+1}$ to $\lambda_{2\Theta-2}$ are not included.

The eigenvalues may be relevant or irrelevant,[6] according as $\lambda_k>1$ or $\lambda_k<1$. Thus we must investigate the relative values of 2^{μ_k} and $b_k\epsilon_\Theta\ln 2$, particularly for $k=\Theta$, since this gives the smallest value of μ_k ($\mu_\Theta=0$).

For $\epsilon_\Theta=0$ [or $d=2\Theta/(\Theta-1)$], Eq. (7) reduces to the Gaussian fixed point for a critical point of order Θ [cf. Eq. (4)] and the relevant eigenvalues λ_k are given by 2^{μ_k} with

$$\mu_k=2(\Theta-k)/(\Theta-1)\quad(k=1,2,\ldots,\Theta)\ . \tag{10}$$

Thus far the ordering field was not included. To consider the ordering field we include an additional term $h_l s$ in $Q_l(s)$. The additional eigenvalue μ_0 corresponding to the ordering field for the Gaussian fixed point can be shown to be $(2\Theta-1)/(\Theta-1)$. These μ_k characterize the homogeneous scaling laws for the critical point, with additional logarithmic corrections[5(b)] introduced by the fact that $\mu_\Theta=0$.

For a critical point of third order (i.e., a tricritical point), we find

$$c_k=(1,\ -\tfrac{1}{3},\ \tfrac{1}{45});\quad K_3=\tfrac{1}{5}(\epsilon_3\ln 2)\ . \tag{11}$$

Starting from an initial condition of $r_{k0}=(45,\ -15,\ 1)r_0$ with $\epsilon_3\gtrsim r_0>0$, the recursions relations indicate that for $\epsilon_3>0$ the nontrivial fixed point of Eq. (11) is stable. For $\epsilon_3<0$ the Gaussian fixed point wins the competition and $\mu_k=d(1-k)+2k$, with $k=\frac{1}{2}$, 1, 2. Thus, there are three relevant scaling powers for $-1<\epsilon_3<0$, with no logarithmic corrections. At $\epsilon_3=-1$, the Gaussian fixed point for $d=4$ is recovered and there are only two relevant scaling powers with logarithmic corrections.

For $\epsilon_3>0$, $y_{k+1}\equiv\ln\lambda_k/\ln 2$ calculated from the linearized recursion relations about the nontrivial fixed point of Eq. (11) are

$$\begin{aligned} y_2&=2+O(\epsilon_3^2)\,,\\ y_3&=1+\tfrac{1}{5}\epsilon_3+O(\epsilon_3^2)\,,\\ y_4&=-2\epsilon_3+O(\epsilon_3^2)\ . \end{aligned} \tag{12}$$

The eigenvalue corresponding to the ordering density is $y_1=1+\frac{1}{2}d=\frac{1}{2}(5-\epsilon_3)$. Thus, we deduce from Eq. (12) that the singular part of the Gibbs potential scales as

$$G(l^{\bar a_1}x_1,\ l^{\bar a_2}x_2,\ l^{\bar a_3}x_3)=lG(x_1,x_2,x_3)\ , \tag{13}$$

where the x_i are the relevant scaling fields.[10] The corresponding scaling powers,[11] $\bar a_k=y_k/d$, to $O(\epsilon_3)$ are

TABLE I. Approximate recursion relations for the symmetric Hamiltonian (see Ref. 15 concerning notation).

General expression:

$$\sum_{i=1}^{N} r_i' z^{2i} = 2^d \left\{ \sum_{i=1}^{N} r_i (2^{2-d} z^2)^i + I(2^{1-d/2} z) - I(0) \right\},$$

where

$$I(z) = \sum_{i=2}^{N} r_i \sum_{k=1}^{i} \binom{2i}{2k} z^{2(i-k)} (2+2r_1)^{-k} (2k-1)!!$$
$$+ \frac{1}{2} \sum_{i=2}^{N} \sum_{j=2}^{N} \sum_{k=1}^{i} \sum_{\kappa=1}^{j} r_i r_j \binom{2i}{2k} \binom{2j}{2\kappa} z^{2(i-k+j-\kappa)} (2+2r_1)^{-(k+\kappa)}$$
$$\times [(2k-1)!!\,(2\kappa-1)!! - (2k-2\kappa+1)!!] + O(r_k^3,\ k \geqq 2).$$

To order r_3^2 ($N = 2\mathcal{O} - 2 = 4$)

$$r_1' = 2^2 \left\{ r_1 + 3 \frac{r_2}{(1+r_1)} + \frac{45}{4} \frac{r_3}{(1+r_1)^2} + \frac{105}{2} \frac{r_4}{(1+r_1)^3} - 9 \frac{r_2^2}{(1+r_1)^3} - \frac{495}{4} \frac{r_2 r_3}{(1+r_1)^4} - \frac{3375}{8} \frac{r_3^2}{(1+r_1)^5} + O(r_3^3) \right\}$$

$$r_2' = 2^{4-d} \left\{ r_2 + \frac{15}{2} \frac{r_3}{(1+r_1)} + \frac{105}{2} \frac{r_4}{(1+r_1)^2} - 9 \frac{r_2^2}{(1+r_1)^2} - \frac{315}{2} \frac{r_2 r_3}{(1+r_1)^3} - \frac{6075}{8} \frac{r_3^2}{(1+r_1)^4} + O(r_3^3) \right\}$$

$$r_3' = 2^{6-2d} \left\{ r_3 + 14 \frac{r_4}{(1+r_1)} - 45 \frac{r_2 r_3}{(1+r_1)^2} - \frac{675}{2} \frac{r_3^2}{(1+r_1)^3} + O(r_3^3) \right\}$$

$$r_4' = 2^{8-3d} \left\{ r_4 - \frac{225}{4} \frac{r_3^2}{(1+r_1)^2} + O(r_3^3) \right\}$$

$$\bar{a}_1 = \tfrac{5}{6}(1 + \tfrac{2}{15}\epsilon_3),$$
$$\bar{a}_2 = \tfrac{2}{3}(1 + \tfrac{1}{3}\epsilon_3), \qquad (14)$$
$$\bar{a}_3 = \tfrac{1}{3}(1 + \tfrac{8}{15}\epsilon_3).$$

Expressions for all tricritical exponents can be obtained directly from Eq. (14). For example, we have[11]

$$\varphi(\text{crossover exponent}) = \bar{a}_3/\bar{a}_2 = \tfrac{1}{2}(1 + \tfrac{1}{5}\epsilon_3) + O(\epsilon_3^2), \quad (15)$$

$$\gamma \equiv \gamma_u/\varphi\ (\text{susceptibility exponent}) = \alpha = -\left(\frac{1-2\bar{a}_2}{\bar{a}_2}\right) = \tfrac{1}{2}(1+\epsilon_3) + O(\epsilon_3^2), \qquad (16)$$

$\gamma_{\mathrm{st.}}$(staggered susceptibility exponent)

$$= -\left(\frac{1-2\bar{a}_1}{\bar{a}_2}\right) = 1 + O(\epsilon_3^2), \qquad (17)$$

and

$$\beta_{\mathrm{st.}} = \frac{1-\bar{a}_1}{\bar{a}_2} = \tfrac{1}{4}(1-\epsilon_3) + O(\epsilon_3^2). \qquad (18)$$

The scaling fields can be calculated by expanding the function Q_0 in the reduced Hamiltonian in terms of the eigendensities $Q^{(k)}$ and the uncoupled density s;

$$Q_0 = x_1 s + \sum_{k=2}^{2\mathcal{O}-2} x_k Q^{(k)}. \qquad (19)$$

For $\mathcal{O} = 3$ and $\epsilon_3 > 0$, the relevant scaling fields are

$$x_1 = h_0, \qquad (20a)$$
$$x_2 = \tfrac{1}{2}[(1 + \tfrac{1}{5}\bar{\epsilon}) r_{1,0} + (6 + \tfrac{29}{5}\bar{\epsilon}) r_{2,0} + (45 + 66\bar{\epsilon}) r_{3,0} + (420 + 630\bar{\epsilon}) r_{4,0}], \qquad (20b)$$
$$x_3 = \tfrac{1}{4}[\tfrac{1}{30}\bar{\epsilon}\, r_{1,0} + (1 + \tfrac{16}{5}\bar{\epsilon}) r_{2,0} + (15 + \tfrac{81}{2}\bar{\epsilon}) r_{3,0} + (210 + \tfrac{2282}{3}\bar{\epsilon}) r_{4,0}], \qquad (20c)$$

where $\bar{\epsilon} \equiv \epsilon_3 \ln 2$. The corresponding densities are

$$Q^{(1)} = s, \qquad (21a)$$
$$Q^{(2)} = s^2 - \tfrac{1}{30}\bar{\epsilon} s^4, \qquad (21b)$$
$$Q^{(3)} = (-6 + \tfrac{22}{5}\bar{\epsilon}) s^2 + s^4 - \tfrac{1}{5}\bar{\epsilon} s^6. \qquad (21c)$$

Expanding (20) about the tricritical point ($h_0 = 0$, g_t, T_t) and choosing $x_2 = x_3 = 0$ there, we obtain, to linear order, $x_2 = A_{22}(g - g_t) + A_{23}(T - T_t)$ and $x_3 = A_{33}(T - T_t)$, where g is the nonordering field and A_{ij} are constants. At the tricritical point, $x_1 = x_2 = x_3 = 0$ and three separate phases become identical simultaneously; by definition, this is a critical point of order $\mathcal{O} = 3$.

For an $\mathcal{O} = 4$ critical point, we find

$$c_k = (1,\ -0.3700,\ 0.036\,51,\ -0.000\,9651); \qquad (22)$$
$$K_4 = -(0.046\,00)(\epsilon_4 \ln 2).$$

Starting from an initial condition of $r_k = (-1036, +383.4, -37.83, +1) r_0$ with $\epsilon_4 > r_0 > 0$, the recursion relations for $\epsilon_4 > 0$ indicate that the nontrivial fixed point of (22) is stable. The calculated scaling powers $\bar{a}_i$ for $\epsilon_4 > 0$ are given in Table II.

For $\epsilon_4 < 0$, the Gaussian fixed point wins the competition and the scaling powers are given by $\mu_k = d(1-k) + 2k$, with $k = \frac{1}{2}, 1, 2,$ and 3. Thus, there are four relevant scaling powers for $-\frac{1}{3} < \epsilon_4 < 0$, with no logarithmic correction terms. At $\epsilon_4 = -\frac{1}{3}$, the Gaussian fixed point for $d = 3$ is recovered and there are only three relevant scaling fields (with logarithmic corrections).

In summary, then, we have considered the appropriate generalization to arbitrary order $\mathcal{O}$ of the Wilson renormalization group approach (and the Wilson-Fisher expansion about a particular lattice dimensionality). Except for limited work[5] for order $\mathcal{O} = 3$ (tricritical points), previous attention has been focused entirely on the case of ordinary critical points, $\mathcal{O} = 2$, even though there exist numerous systems displaying higher-order critical points.[1,2,6] Our results (some of which are summarized in Table II) demonstrate the utility of the

TABLE II. Expansion parameters $\epsilon_{\mathcal{O}}$, Gaussian eigenvalues μ_k ($k=1,2,\ldots\mathcal{O}$), fixed-point coefficients c_k ($k=1,2,\ldots,\mathcal{O}$) and $K_{\mathcal{O}}$, and the relevant scaling powers $\bar{a}_k$ (for $\epsilon_{\mathcal{O}}>0$) for critical points of arbitrary order $\mathcal{O}$.

$\mathcal{O}$	2	3	4	$\mathcal{O}$
$\epsilon_{\mathcal{O}}$	$4-d$	$3-d$	$\frac{8}{3}-d$	$\frac{2\mathcal{O}}{\mathcal{O}-1}-d$
μ_k	2 0	2 1 0	2 $\frac{4}{3}$ $\frac{2}{3}$ 0	$\frac{2(\mathcal{O}-k)}{\mathcal{O}-1}$ $k=1,2,\ldots,\mathcal{O}$
c_k	1 $-\frac{1}{4}$	1 $-\frac{1}{3}$ $+\frac{1}{45}$	1 -0.3700 0.03651 -0.0009651	1 . (signs alternate) ⋮ $c_{\mathcal{O}}$
$K_{\mathcal{O}}$	$-\frac{4}{9}\epsilon_2\ln 2$	$\frac{1}{5}\epsilon_3\ln 2$	$-0.04600\epsilon_4\ln 2$	<0 (for even $\mathcal{O}$) >0 (for odd $\mathcal{O}$)
$\bar{a}_k$	$\frac{3}{4}(1+\frac{1}{12}\epsilon_2)$ $\frac{1}{2}(1+\frac{1}{12}\epsilon_2)$	$\frac{5}{6}(1+\frac{2}{15}\epsilon_3)$ $\frac{2}{3}(1+\frac{1}{3}\epsilon_3)$ $\frac{1}{3}(1+\frac{8}{15}\epsilon_3)$	$\frac{7}{8}(1+\frac{9}{56}\epsilon_4)$ $\frac{3}{4}(1+\frac{3}{8}\epsilon_4)$ $\frac{1}{2}(1+1.061\epsilon_4)$ $\frac{1}{4}(1+1.446\epsilon_4)$	$\bar{a}_k=y_k/d$ $y_{k+1}=\frac{\ln\lambda_k}{\ln 2}$ $\bar{a}_1=\left(\frac{2\mathcal{O}-1}{2\mathcal{O}}\right)$ $\times\left(1+\frac{(\mathcal{O}-1)^2}{2\mathcal{O}(2\mathcal{O}-1)}\epsilon_{\mathcal{O}}\right)$

renormalization group methods for handling more complex systems. Although the results presented here are only to first order in the expansion variable $\epsilon_{\mathcal{O}}$, and are for spin dimension unity, similar results can be obtained for higher-order expansions in $\epsilon_{\mathcal{O}}$ using the exact recursion relations or correlation function, and these ideas can be extended to higher spin dimensions.[12]

We are indebted to Professor K. G. Wilson for advice and discussions, as well as to M. E. Fisher, R. B. Griffiths, B. D. Hassard, J. Nicoll, L. L. Liu, and E. K. Riedel. We were informed, after submission of this manuscript, that two of the three critical point exponents have been subsequently obtained, for the special case $\mathcal{O}=3$, by M. J. Stephen and J. L. McCauley, Jr. (these authors do not calculate the crossover exponent and the scaling fields, nor do they locate and consider the stability of the fixed points).

APPENDIX: JUSTIFICATION FOR USING THE APPROXIMATE RECURSION RELATION TO FIND THE FIXED POINTS TO ORDER $\epsilon_{\mathcal{O}}$

To show that the exact recursion relations reduce to the approximate recursion relations (ARR), to order $\epsilon_{\mathcal{O}}$, we follow closely the arguments of Wilson[3(b)] for $\mathcal{O}=2$. For the sake of simplicity, we restrict ourselves to a discussion of the tricritical case, $\mathcal{O}=3$.

First note that the exact recursion relations are in the form of functional integral equations for the momentum-dependent variables[15] $r_1(\vec{q})$, $r_2(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)$, $r_3(\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_6)$, $r_4(\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_8)$. An exact fixed point of these functional equations is some set of fixed functions $r_1^*(\vec{q})$, $r_2^*(\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_4)$, $r_3^*(\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_6)$, $r_4^*(\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_8)$. We wish to show that, to order ϵ_3, these reduce to the momentum-independent fixed values determined by the approximate recursion formulas (cf. Table I).

The combinatorial factors for the Feynman dia-

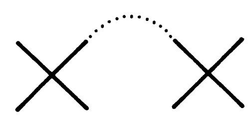

FIG. 1. The $(r_2)^2$ contribution to r_3' (Ref. 15). For a contribution to $r_3'(0,\ 0,\ 0,\ 0,\ \vec{p},\ -\vec{p})$, the momentum constraint on the connecting leg requires $|\vec{p}|=1$.

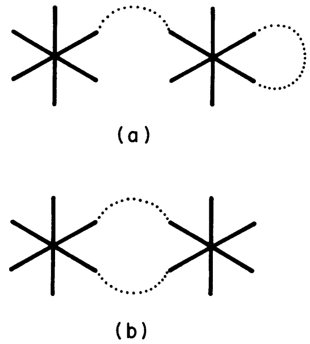

FIG. 2. The $(r_3)^2$ contributions to r_4' (Ref. 15). Diagram (a) does not contribute to $r_4(0,0,0,0,0,0,\vec{p},-\vec{p})$, while diagram (b) does.

grams leading to the exact relations are shown by Wilson to be the same as those for the corresponding terms of the approximate formulas.[13] Further, setting the momenta of the external legs of these graphs equal to zero reduces the set of separate graphs in the exact relations to precisely those of the ARR of Table I. Thus to begin we assume that the exact fixed functions can be expanded as:

$$\begin{aligned} r_1^*(\vec{q}) &= r_1^* + q^2 + O(\epsilon_3^2) \;, \\ r_2^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_4) &= r_2^* + O(\epsilon_3^2), \\ r_3^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_6) &= r_3^* + O(\epsilon_3^2) \;, \\ r_4^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_8) &\sim O(\epsilon_3^2) \;, \end{aligned} \tag{A1}$$

where r_1^*, r_2^*, r_3^* are of order ϵ_3. That is, the momentum dependence of each fixed function [aside from $r_1^*(\vec{q})$, which always has a bare q^2 term] is carried by a term which is smaller by at least one order of ϵ_3 than the corresponding momentum-independent term. Wilson's arguments[14] then show that, if these expansions are substituted into the exact functional integral equations, the resulting relations for the momentum-independent terms are just those given by the ARR, to order ϵ_3 for r_1^* and r_2^*, and to $O(\epsilon_3^2)$ for r_3^*. There is one modification in the tricritical case, however, which involves feedback from the r_4^* term.

It has been shown[14] that an irrelevant variable, say w, can affect the fixed point value [to $O(\bar{\epsilon})$] of the last relevant variable, say u, provided w^* is of order $\epsilon_{\mathcal{O}}^2$ and w enters the u' equation linearly.[15] In the $\mathcal{O}=2$ case, w corresponds to r_3 and u to r_2. In the zero-external-momentum limit, r_3 enters the r_2' equation only in the form $r_3(0,0,0,0,\vec{p},-\vec{p})$. But $r_3(0,0,0,0,\vec{p},-\vec{p})$ can be shown to be $\mathcal{O}(\epsilon_2^3)$, as follows: The only $\mathcal{O}(\epsilon_2^3)$ term contributing to $r_3'(0,0,0,0,\vec{p},-\vec{p})$ consists of two r_2 (four-leg) vertices connected once (Fig. 1). If all but two external legs have zero momentum, the internal leg can carry at most momentum $|\vec{p}|$. By the momentum constraint ($|\vec{p}_{\rm ext}| \le 1$, $|\vec{p}_{\rm int}| \ge 1$), this contribution therefore vanishes.

For the tricritical case ($\mathcal{O}=3$) we note that r_4 enters linearly into the r_3' equation in the form $r_4(0,0,0,0,0,0,\vec{p},-\vec{p})$. Now the r_4' relation contains two diagrams (Fig. 2) involving $(r_3)^2$. While Fig. 2(a) does not contribute to $r_4'(0,0,0,0,0,0,\vec{p},-\vec{p})$, Fig. 2(b) does, and r_4^* indeed enters into the r_3^* equation in an important way.

To take into account r_4, we write

$$r_4^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_8) = r_4^* + v_4^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_8) \;, \tag{A2}$$

where by definition

$$v_4^*(0, \ldots, 0) = 0 \;. \tag{A3}$$

Using this expression (A2) and that of (A1) for $r_3^*(\vec{q}_1, \ldots, \vec{q}_6)$, the exact recursion relation for $r_4^*(\vec{q}_1, \ldots, \vec{q}_8)$ reduces to

$$r_4^* = 2^{3\epsilon_3 - 1}\left(r_4^* - 225 (r_3^*)^2 \int \frac{1}{(p^2+r)^2} + \text{terms of } O(\epsilon_3^3) \right) \;, \tag{A4}$$

$$v_4^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_8) = 2^{3\epsilon_3 - 1}\left[v_4^*(\tfrac{1}{2}\vec{q}_1, \tfrac{1}{2}\vec{q}_2, \ldots, \tfrac{1}{2}\vec{q}_8) + \epsilon_3^3 F_4^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_8) \right] . \tag{A5}$$

Equation (A5) admits a convergent series solution for v_4^* in terms of F_4^*:

$$v_4^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_8) = \epsilon_3^3 \sum_{n=0}^{\infty} 2^{(3\epsilon_3 - 1)n} F_4^*(2^{-n}\vec{q}_1, 2^{-n}\vec{q}_2, \ldots, 2^{-n}\vec{q}_8) \;. \tag{A6}$$

This means that $v_4^* \sim O(\epsilon_3^3)$. Therefore, to order ϵ_3^2, the exact equation for $r_3^*(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_6)$ does not contain v_4^*. Under the momentum-integral approximations of Ref. 14, Eq. (A4) reduces to the approximate relation given in Table I. Therefore the last two equations of Table I suffice to determine r_3^*.

*Work supported by the National Science Foundation, Office of Naval Research, and the Air Force Office of Scientific Research. Work forms part of a Ph. D. thesis to be submitted by G. F. T. to the Physics Department of MIT.

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[7]Equations (2) lead to $\eta = 0$ and are thus approximate. However, independent Feynman-diagram calculations show that the results obtained in this paper are exact to the first order of ϵ_Θ. See comments after Eq. (7) and Appendix A.

[8]The reader may legitimately inquire whether there exist fixed points that are not of the form of (4). It is straightforward to show that *to first order* in $r_{\Theta l}$, the structure of the fixed points is determined by the linear part (i.e., the L_k term) of the recursion relation (3) and therefore the structure of $Q_l(s)$ (to first order in $r_{\Theta l}$) must be of the form of (4).

[9]The reason that these additional irrelevant parameters must be included in the first-order ϵ_Θ calculations can be understood easily using the Feynman diagrams. For example, the r_4 recursion relation for the ϵ_3 expansion contains $(r_3)^2$ diagrams and therefore must be considered. (See Fig. 2.)

[10]F. J. Wegner, Phys. Rev. B 5, 4529 (1972).

[11]For a discussion of the relation of the scaling powers $\bar{a}_k$ to tricritical exponents—and to geometrical considerations—see A. Hankey, H. E. Stanley, and T. S. Chang, Phys. Rev. Lett. 29, 278 (1972). See also R. B. Griffiths, Phys. Rev. B 7, 545 (1973); and E. K. Riedel, Phys. Rev. Lett. 28, 675 (1972). In particular, Hankey *et al.* defined the crossover exponent in terms of the scaling powers $\bar{a}_i$ explicitly.

[12]J. F. Nicoll, T. S. Chang, and H. E. Stanley (unpublished).

[13]Reference 3(b), Sec. VI.

[14]Reference 3(b), Sec. V.

[15]In Appendix A, Table I, and Figures 1 and 2, we replace the notation $r_{k,l} \to r_{k,l+1}$ by $r_k \to r_k'$ in order that the expressions are less cumbersome.