Tricritical systems with long-range interactions

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Two elastic spin models with nonclassical tricritical points are discussed. The singular parts of the free energies near the tricritical points have homogeneity properties which must be formulated in terms of curvilinear coordinates, although the linear variables of Griffiths or Riedel may be used more generally than recently noted. One of our models, though not an example of the recent theory of constrained systems, exhibits the same tricritical behavior as these systems. This similarity is explained on the basis of a Curie-Weiss-like four-spin interaction occurring in our models as well as in the constrained spin model of Rudnick, Bergman, and Imry. A more general equation of state for tricritical systems with long-range interactions is suggested.

I. INTRODUCTION

A tricritical point (TCP) is a special critical point in the space of thermodynamic states at which a λ line (second-order transition) turns into a triple line (first-order transition). The critical behavior in the neighborhood of these points is quite different from that near normal critical points and hence a great deal of recent attention, both experimental and theoretical, has been devoted to their study.[1]

Although a number of models with a TCP have been constructed, so far as we know the Baker-Essam model[2,3] is the only example of a Hamiltonian model for which the exact equation of state near the TCP is known.[4,5] The singular part of the free energy is a homogeneous function[6-8] only in terms of certain curvilinear coordinates.[4] The tricritical exponents are nonclassical, being simply related to the λ line exponents via Fisher's relations for renormalized exponents.[9] The Baker-Essam model is an example for a theory of constrained systems[5,10] which applies to systems having a line of "ideal" second-order phase transitions with an infinite specific heat. Imposing a certain macroscopic constraint on the ideal system both induces a TCP, characterized by the ideal exponents, and causes the usual Fisher renormalization along the λ line. Identical exponents and exponent relations have also been found in the "constrained" Ising model of Rudnick, Bergman, and Imry.[11] Thus a certain class of tricritical scaling systems with similar properties exists and one may ask whether this type of TCP occurs even more generally.

In the present paper we introduce two models, A and B, one of which (A), though not an example of the theory of constrained systems, exhibits the same tricritical behavior as these systems. In particular, we find for both models that the existence of homogeneity properties of the free energies near the TCP depends on a proper choice of thermodynamic variables. The appropriate scaling variables are the same curvilinear coordinates as used in the Baker-Essam model and in the constrained systems, although the linear variables of Griffiths[7] or Riedel[8] turn out to be applicable more generally than previously noted.[4,10]

Our models are based on the elastic spin model solved by Jasnow and Wagner,[12] for which magnetothermomechanics[13-15] (MTM) is exact. The essential feature of the models—and generally of MTM—is the existence of a Curie-Weiss-like four-spin interaction which induces a TCP, provided that the specific heat of the rigid spin system is finite (model A) or a special type of spin exchange is assumed (model B). It is just this long-range interaction which seems to explain the similarity between the tricritical behavior of model A and that of the constrained systems, since this interaction also induces the TCP in the constrained model of Rudnick *et al.* We conjecture that other tricritical systems exist where long-range interactions play an important role, and we suggest a general equation of state for such systems.

The outline of this paper is as follows: In Sec. II our models are discussed qualitatively. Their tricritical and scaling behavior is analyzed in Sec. III, and alternative choices of scaling variables are considered. In Sec. IV we compare the equations of state for our models with those of the Baker-Essam model and the constrained systems, and discuss the reason for the similarity of their tricritical behavior. A more general set of phenomenological tricritical equations is given. Our results are summarized in Sec. V.

II. TWO ELASTIC SPIN MODELS WITH A TCP

The models A and B introduced below are based on the Jasnow-Wagner model,[12] a two-dimensional compressible Ising model with a horizontal nearest-neighbor lattice potential $v(x)$ and a fixed vertical separation between neighboring rows. The essential feature of the Jasnow-Wagner lattice is the constraint that each column of atoms moves as a "rigid rod", i.e., each atom along a column must have the same horizontal coordinate. This constraint leads to a Gibbs free energy per spin, G, which is identical to that of magnetothermomechanics[13-15] (MTM):

$$G(T, P, H) = \min_x [Px + v(x) + f(T, H; J(x))], \quad (2.1)$$

where x satisfies the equation of state

$$0 = P + v'(x) + \frac{\partial f(T, H; J(x))}{\partial x}. \quad (2.2)$$

The value of x which minimizes the right-hand side of (2.1) for a given temperature T, pressure P,[16] and magnetic field H,[17] is identical with the mean horizontal lattice spacing $(\partial G/\partial P)_{T,H}$. $f(T, H; J(x))$ is the free energy per spin of the rigid Ising model with a horizontal exchange constant $J(x)$ and a uniform horizontal lattice spacing x.

A qualitative discussion of (2.1) and (2.2) shows[12] that for realistic lattice potentials $v(x)$ and exchange constants $J(x)$ a first-order phase transition occurs for $H=0$ and for all $P>0$, due to a lattice instability brought on by the divergent Ising specific heat. However, a TCP may exist if the rigid spin system has a finite specific heat[18,19] (model A), or if a special x dependence of $J(x)$ is chosen (model B).

A. Model A

As a particular example,[20] where the rigid spin system has a finite specific heat, we consider a classical nearest-neighbor Heisenberg spin system on a three-dimensional Jasnow-Wagner lattice with rigid planes instead of rigid columns. Obviously the macroscopic lattice constraint again strongly suppresses lattice fluctuations, and we expect that it also ensures the validity of MTM, independent of the type of spin system. Indeed, a discussion of the Laplace method applied to model A (Appendix A) and the comparison of the effective Jasnow-Wagner spin Hamiltonian with the MTM Hamiltonian (Appendix D) strongly supports the above expectation.

Assuming the validity of MTM, we start from (2.1) and (2.2), where now

$$f(T, H; J(x)) = f_a(T, H; J(x)) + f_s(\tau(T, x), H) \quad (2.3)$$

is the rigid Heisenberg free energy divided into an analytic and singular part with

$$\tau(T, x) = \tau_0(T)\{T - T_c^0[J(x)]\}. \quad (2.4)$$

The functions $J(x)$, $\tau_0(T)$ and the rigid Heisenberg transition temperature $T_c^0[J]$ are assumed to be analytic functions with $dT_c^0/dJ > 0$, $J>0$, $dJ/dx<0$. For $H=0$, we assume $f_s(\tau, 0) \sim A_\pm |\tau|^{2-\alpha_H}$ with $\alpha_H < 0$, $A_\pm > 0$, which gives an upward-pointing cusp of the specific heat.[21] As in Ref. 12, the lattice potential $v(x)$ can be quite arbitrary.

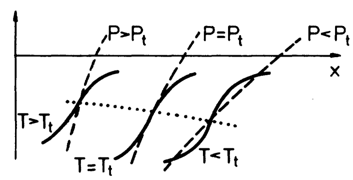

FIG. 1. Schematic plot of the curves $P + v'(x)$ (dashed lines) and $-\partial f(T, 0; J(x))/\partial x$ (full lines) for model A as functions of x for several values of P and T. The curves $-\partial f/\partial x$ have their maximal slope at the critical spacing $x_c(T)$ (dotted line). For $T=T_t$ and $P=P_t$ the curves intersect tangentially at the tricritical lattice spacing $x_t = x_c(T_t)$. The slope of the dashed curve for $P<P_t$ has been excessively flattened to show the three intersections more clearly.

The qualitative behavior of model A can be understood[12] by discussing the possible solutions $x(T, P, H)$ of (2.2), which are given by the intersections of the curves $P + v'(x)$ and $-\partial f/\partial x$ shown schematically in Fig. 1. For $H=0$ and fixed T, $-\partial f/\partial x$ is singular at a critical spacing $x_c(T)$, which is uniquely determined by inverting

$$T = T_c^0[J(x)]. \quad (2.5)$$

Since the slope of $-\partial f/\partial x$ is maximal but finite at $x_c(T)$ $(\alpha_H < 0)$, the number of possible intersections in the neighborhood of $x_c(T)$ depends on the slope of $P + v'(x_c)$. For small P, three intersections occur near $x_c(T)$ when

$$v''(x_c) < -\frac{\partial^2 f(T, 0; J(x_c(T)))}{\partial x_c^2},$$

which implies a first-order transition. For sufficiently large P, the curves intersect at small x, where $v'(x)$ becomes arbitrarily steep. This yields a unique intersection and hence a continuous phase transition at $x_c(T)$ for $H=0$. The TCP, where the transition changes from first to second order, is defined by the requirement of a tangential intersection at $x_c(T)$. Thus the tricritical temperature T_t and pressure P_t are the solutions of

$$P + v'(x_c(T)) = -\frac{\partial f(T, 0; J(x_c(T)))}{\partial x_c} \quad \text{(intersection)}, \quad (2.6)$$

$$v''(x_c(T)) = -\frac{\partial^2 f(T, 0; J(x_c(T)))}{\partial x_c^2} \quad \text{(tangential intersection)}. \quad (2.7)$$

The qualitative behavior of model A is further illustrated in Figs. 2 and 3.

The existence of the first-order surfaces ("wings") for $H \neq 0$ can be understood from a graphical solution of (2.2) similar to that shown in Fig. 1.[22] Clearly, for $P < P_t$ and sufficiently small H

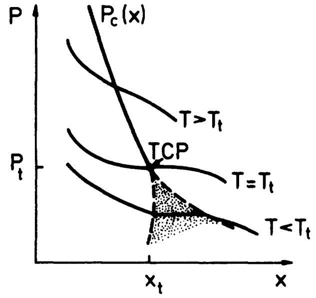

FIG. 2. $P-x$ phase diagram of model A for $H=0$ with isotherms of various temperatures (qualitatively). The isothermal compressibility $\kappa \sim -(\partial x/\partial P)_T = (v'' + \partial^2 f/\partial x^2)^{-1}$ diverges at the TCP. The boundaries $x_1(P)$ and $x_2(P)$ (dashed lines) of the coexistence region (dotted) meet the second-order line $P_c(x)$ tangentially at the TCP with $[x_1(P) - x_2(P)] \sim (P_t - P)^{-1/\alpha_H}$ as follows from scaling according to (3.11) and (3.14).

$\neq 0$, a first-order transition will still occur. As H and T increase, the maximum slope of $-\partial f/\partial x$ decreases; thus the jump of the mean lattice spacing becomes smaller and ultimately vanishes for $P = P_w(H)$ and $T = T_w(H)$, which defines the wing critical lines. The location of these lines is determined by requiring that $P + v'(x)$ intersects tangentially with $-\partial f(T, H;\ J(x))/\partial x$ at the point where $-\partial f/\partial x$ has its maximal slope:

$$P + v'(x) = -\frac{\partial f(T, H;\ J(x))}{\partial x} \quad \text{(intersection)}, \tag{2.8}$$

$$v''(x) = -\frac{\partial^2 f(T, H;\ J(x))}{\partial x^2} \quad \text{(tangential intersection)}, \tag{2.9}$$

$$0 = \frac{\partial^3 f(T, H;\ J(x))}{\partial x^3} \quad \text{(maximal slope at the tangential intersection)}. \tag{2.10}$$

Owing to the symmetry of f with respect to H, one obtains two symmetric lines of critical points in the T-P-H space (Fig. 3). The critical behavior along these lines will be classical because of the analyticity of $f(T, H;\ J)$ for $H \neq 0$.

B. Model B

In this version of the Jasnow-Wagner model we choose the x dependence of $J(x)$ such that in a certain pressure region the Ising spin system is decoupled from the lattice and thus displays the normal rigid-lattice phase transition. For values of pressure outside this region, however, a first-order transition will occur for the same reason as in the original Jasnow-Wagner model.

As a simple example[23] we consider a function $J(x)$ of the type shown in Fig. 4(a). Near x_t we assume

$$J(x) = J(x_t) - c_0 (x - x_t)^n \Theta(x - x_t), \tag{2.11}$$

where

$$\Theta(y) = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{for } y < 0 \end{cases} \tag{2.12}$$

is the step function, $c_0 > 0$, and, for simplicity, $n > 2$. The spin system is taken to be the three-dimensional Ising model with a free energy (2.3) where $f_s(\tau, 0) \sim B_\pm |\tau|^{2-\alpha_I}$, $\alpha_I > 0$, $B_\pm < 0$. The qualitative behavior of this model can be understood similarly as that of model A by discussing the graphical solutions of (2.2) [Fig. 4(b)]. The tricritical temperature T_t is

$$T_t = T_c^0 [J(x_t)] \tag{2.13}$$

and P_t is given by

$$P_t = -v'(x_t), \tag{2.14}$$

since $\partial f/\partial x = 0$ when $x = x_t$.

III. SCALING FORM OF THE FREE ENERGY NEAR THE TCP

Throughout the following analysis of critical and scaling behavior we shall use the field variables T, P, H. In this field space the λ line and its smooth continuation below the TCP plays the role of a natural reference line, although it has no direct physical significance in the first-order region below the TCP. This line, denoted by $T_c(P)$, is obtained by solving

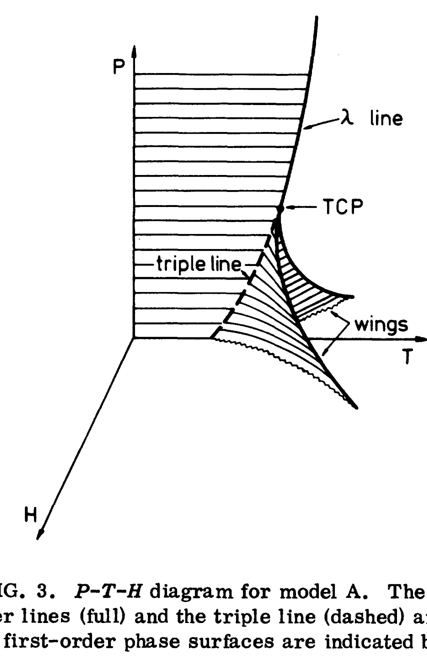

FIG. 3. P-T-H diagram for model A. The second-order lines (full) and the triple line (dashed) are shown. The first-order phase surfaces are indicated by parallel markings.

$$P+v'(x_c(T))=-\frac{\partial f(T,0;\,J(x_c(T)))}{\partial x_c} \tag{3.1}$$

for T as a function of P. Equation (3.1) follows from (2.2) by inserting the critical spacing $x=x_c(T)$ defined by (2.5). For model A, $T_c(P)$ is analytic [Fig. 5(a)], whereas for model B the particular choice (2.11) of $J(x)$ leads to a singularity of $T_c(P)$ at the TCP [Fig. 5(b)].

Expanding the right-hand side of (2.1) in terms of the variables

$$\lambda=P-P_t\,,\quad g=T-T_c(P),\quad \zeta=H, \tag{3.2}$$

we find for model A

$$G(T,P,H)=G_a(T,P,H)+G_s(\lambda,g,\zeta), \tag{3.3}$$

where G_a is an analytic function and the singular part G_s has the following leading term (Appendix B):

$$G_s(\lambda,g,\zeta)=\min_{\tau}\left[-a_1g\tau+\tfrac{1}{2}a_2\lambda\tau^2+f_s(\tau,\zeta)\right], \tag{3.4}$$

with $\tau=\tau(\lambda,g,\zeta)$ satisfying

$$0=-a_1g+a_2\lambda\tau+\frac{\partial f_s(\tau,\zeta)}{\partial\tau}\,. \tag{3.5}$$

For model B we obtain in leading order (Appendix B)

$$G(T,P,H)=G_1(g,P)+G_s(\lambda,g,\zeta), \tag{3.6}$$

with

$$G_1(g,P)=G(T_c(P),P,0)+g\left(\frac{\partial f}{\partial T}\right)_{T=T_c(P),\;H=0,\;x=x_c(T_c(P))}$$

and (3.7)

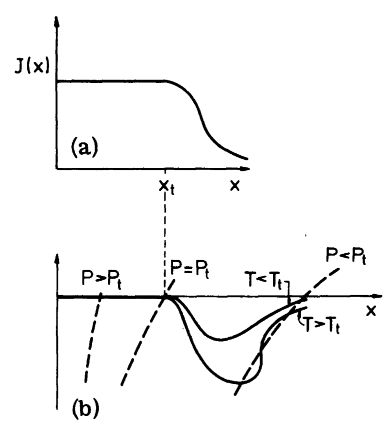

FIG. 4.(a) Horizontal spin exchange constant $J(x)$ for model B. $J(x)$ is constant for $x<x_t$. (b) Schematic plot of the curves $P+v'(x)$ (dashed lines) and $-\partial f(T,0;J(x))/\partial x$ (full lines) as functions of x for model B. For $x\leq x_t$, the curves $-\partial f/\partial x=-(\partial f/\partial J)J'(x)$ coincide for all T, since $J'(x)=0$ for $x\leq x_t$.

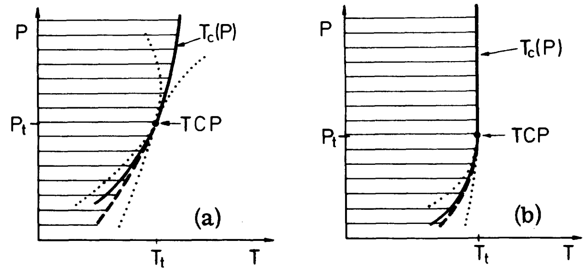

FIG. 5. (a) P-T phase diagram for model A and (b) for model B. The triple line (dashed), crossover lines (dotted), and the λ line and its smooth continuation below the TCP (full line) are shown. For model B, $T_c(P)-T_t\sim(P_t-P)^n\Theta(P_t-P)$ near the TCP.

$$G_s(\lambda,g,\zeta)=\min_{\tau}\left[\frac{b}{2}(-\lambda)^{2(n-1)}\Theta(-\lambda)\left(\frac{\partial f_s}{\partial\tau}\right)^2+f_s(\tau,\zeta)\right], \tag{3.8}$$

where Θ is the step function and $\tau=\tau(\lambda,g,\zeta)$ is a solution of

$$0=\tau-g+b(-\lambda)^{2(n-1)}\Theta(-\lambda)\,\frac{\partial f_s(\tau,\zeta)}{\partial\tau}\,. \tag{3.9}$$

The constants a_1, a_2, and b are positive.

Above the TCP ($\lambda>0$) near the second-order line $T_c(P)$ ($g\to 0$), (3.5) and (3.9) simply reduce to $g\sim\tau$, and the leading singular part of G_s becomes

$$G_s\sim f_s(g,\zeta) \tag{3.10}$$

for both models. Thus the critical behavior along the λ line is governed by the underlying rigid spin system.

The singular parts of the free energies (3.4) and (3.8) satisfy the tricritical scaling relation

$$G_s(\lambda,g,\zeta)=L^{\phi(2-\alpha_t)}G_s\left(\frac{\lambda}{L},\frac{g}{L^{\phi}},\frac{\zeta}{L^{\phi\Delta_t}}\right), \tag{3.11}$$

with $L>0$, provided that the free energy $f_s(\tau,\zeta)$ is a homogeneous function. To show this we assume for any $L>0$

$$f_s(\tau,\zeta)=L^{2-\alpha}f_s\left(\frac{\tau}{L},\frac{\zeta}{L^{\Delta}}\right) \tag{3.12}$$

and first verify that $\tau(\lambda,g,\zeta)$ scales. For model A this follows from substituting

$$\tau(L^a\lambda,L^bg,L^c\zeta)=L\tau(\lambda,g,\zeta) \tag{3.13}$$

into (3.5) and using (3.12) with $\alpha=\alpha_H$, $\Delta=\Delta_H$. We obtain the exponents $a=-\alpha_H$, $b=1-\alpha_H$, $c=\Delta_H$. Now inserting (3.13) with these exponents into (3.4), we recover Eq. (3.11), where

$$\alpha_t=\frac{-\alpha_H}{1-\alpha_H},\quad \Delta_t=\frac{\Delta}{1-\alpha_H},\quad \phi=\frac{1}{\alpha_t}\,, \tag{3.14}$$

with $\alpha_t>0$ ($\alpha_H<0$). Similarly, for model B, using (3.13) and (3.12) in (3.9) and (3.8) with $\alpha=\alpha_I>0$,

$\Delta = \Delta_I$, we obtain (3.11) with exponents

$$\alpha_t = \alpha_I\,, \quad \Delta_t = \Delta_I\,, \quad \phi = \frac{2(n-1)}{\alpha_I}\,. \qquad (3.15)$$

Hence the free energies (3.4) and (3.8) satisfy the tricritical scaling law (3.11) with exponents that are simply related to the λ line exponents, though apparently unrelated to the classical wing exponents. The asymptotic shape of the triple line and the wing lines follows from scaling arguments,[6] and the relationships among tricritical exponents implied by scaling are summarized in a paper of Griffiths.[7]

The following anomalous behavior of model B should be pointed out. As mentioned above, the phase line $T_c(P)$ is nonanalytic at the TCP due to the singular nature of $J(x)$. This singularity appears both in G_s and G_1 (3.7) and generates, in addition to the tricritical phase structure, a mathematical singularity in the plane $P = P_t = \text{const}$. Clearly this is an artifact of model B, and we have ignored G_1 in the above discussion, since it contains only this type of singularity. If G_1 is also considered, tricritical scaling of the full singular free energy is not obeyed. However, even in this case the order parameter $\psi(\lambda, g, \zeta) = -\partial G_s(\lambda, g, \zeta)/\partial\zeta$ is a homogeneous function of λ, g, ζ.

In conclusion, the set of variables (3.2) with the "floating" variable $g = T - T_c(P)$, as used previously by several authors,[4,10,24–26] indeed seems to give a natural description for a certain class of tricritical models. There may still be, however, the possibility of using the linear variables of Griffiths,[7]

$$\lambda = P - P_t\,, \quad g_1 = T - T_1(P)\,, \quad \zeta = H\,, \qquad (3.16)$$

where $T_1(P)$ is the line tangent to $T_c(P)$ at the TCP. The question as to whether the variables (3.16) and (3.2) are equivalent with respect to scaling has been recently discussed.[4,10] This discussion, however, is incomplete and can be considerably simplified (Appendix C), yielding the following results.[27]

(1) If G_s is a homogeneous function of λ, g_1, ζ, then it is also a homogeneous function of λ, g, ζ.

(2) If G_s is a homogeneous function of λ, g, ζ, then it may or may not be a homogeneous function of λ, g_1, ζ. More specifically, assuming $T_c(P) \sim T_1(P) + A|\lambda|^{\phi_1}$(Fig. 6):

(a) If $\phi_1 = \phi$, G_s is also a homogeneous function of λ, g_1, ζ.

(b) If $\phi_1 > \phi$, there exists a neighborhood of the TCP, where G_s is still a homogeneous function of λ, g_1, ζ with the same exponents [Fig. 6(a)].[28]

(c) If $\phi_1 < \phi$, no such neighborhood exists [Fig. 6(b)].

Thus the linear variables (3.16) can be used more generally than previously noted,[4,10] since only in case 2(c) are the variables (3.2) and (3.16) inequivalent with respect to scaling. Due to the analyticity of $T_c(P)$ ($\phi_1 = 2$), this is just the case for model A and the constrained systems if $\alpha_t < \frac{1}{2}$. For model B, case 2(c) applies also ($\phi_1 = n$), whereas for the Landau model[29,30] one has $\phi = \phi_1 = 2$ [case 2(a)]. All cases may occur in Kortman's phenomenological model[25] or in the more general equations (4.11) in Sec. IV. As a final remark we note that if $T_c(P)$ is analytic, the variables (3.2) as well as (3.16) are a simple example of the scaling fields g_1, g_2, g_3 of Wegner and Riedel,[31] since (3.2) and (3.16) are analytic functions of the experimental fields $T - T_t$, $P - P_t$, H.

IV. TRICRITICAL EQUATION OF STATE IN SYSTEMS WITH LONG-RANGE INTERACTIONS

In this section we compare the tricritical behavior of our models with that of the constrained systems of Imry, Bergman, and Entin-Wohlman.[5] As an example of their theory we consider the three-dimensional Baker-Essam model,[2–4] where the relation $P = F/\langle a\rangle^2$ between pressure P, force F, and mean lattice spacing $\langle a\rangle$ is interpreted as a macroscopic constraint. For constant F the transition at $T = T_c^0(F)$ remains of the rigid Ising type, whereas for given $P > 0$ a renormalized second-order transition occurs at $T = T_c(P)$, which turns into first-order for $P < 0$. Near the TCP [$P_t = 0$, $T_t = T_c(0)$, $H = 0$] the relation connecting the "constrained" and "unconstrained" system is[32]

$$g = A\tau + B\lambda\left(\frac{-\,\partial f_s(\tau, \zeta)}{\partial\tau}\right)_\zeta\,, \qquad (4.1)$$

where f_s is the singular part of the rigid Ising free energy, $\tau(T, F) = T - T_c^0(F)$, $g = T - T_c(P)$, $\lambda = P$, $\zeta = H$, and A, B are positive constants. Using (4.1) it can be shown that the leading singular part $G_s(\lambda, g, \zeta)$ of the total free energy obeys the tricritical scaling law (3.11) with $\alpha_t = \alpha_I$, $\Delta_t = \Delta_I$,

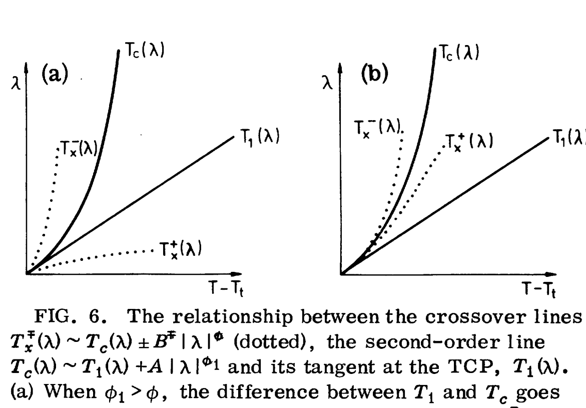

FIG. 6. The relationship between the crossover lines $T_x^\mp(\lambda) \sim T_c(\lambda) \pm B^\mp|\lambda|^\phi$ (dotted), the second-order line $T_c(\lambda) \sim T_1(\lambda) + A|\lambda|^{\phi_1}$ and its tangent at the TCP, $T_1(\lambda)$. (a) When $\phi_1 > \phi$, the difference between T_1 and T_c goes to zero more quickly than the distance between $T_x^\mp$ and T_c (as $\lambda \to 0$), thus T_1 and T_c are equivalent reference lines. (b) When $\phi_1 < \phi$, $T_x^\mp$ lie asymptotically closer to the scaling reference line T_c than T_1, thus T_1 cannot replace T_c as a reference line.

and $\phi = 1/\alpha_I$, provided that $f_s(\tau, \zeta)$ is a homogeneous function of τ and ζ with normal critical exponents α_I and Δ_I.

To simplify the comparison of the Baker-Essam model with our models and to provide a simple way for generalizing the tricritical equations of these models, we turn to the parametric scaling representation of Josephson and of Schofield.[25,33] Thus we first describe $f_s(\tau, \zeta)$ via the order parameter

$$\psi = r^{\beta_I} \psi_I(\theta) \tag{4.2}$$

using the "polar" coordinates r and θ ["distance" from and "angle" around the normal critical point of $f_s(\tau, \zeta)$], which are defined by the relations

$$\tau = r\tilde{\tau}(\theta^2), \tag{4.3a}$$

$$\zeta = r^{\Delta_I} \tilde{\zeta}(\theta) . \tag{4.3b}$$

The specific form of the θ-dependent functions is of minor interest and is briefly discussed in Appendix E. Now inserting (4.3a) into (4.1) and writing $-\partial f_s/\partial\tau = r^{1-\alpha_I} S_I(\theta^2)$, we obtain from (4.1), (4.3b), and (4.2) the Baker-Essam tricritical equation of state in terms of "cylindrical" coordinates λ, θ, and $R \equiv r^{1-\alpha_I}$ (Fig. 7)[34]:

$$\begin{aligned} g &= \lambda R B S_I(\theta^2) + R^{1/(1-\alpha_I)} A \tilde{\tau}(\theta^2), \\ \zeta &= R^{\Delta_I/(1-\alpha_I)} \tilde{\zeta}(\theta), \\ \psi &= R^{\beta_I/(1-\alpha_I)} \psi_I(\theta) . \end{aligned} \tag{4.4}$$

For the constrained systems the tricritical equations are the same, except that λ is replaced by λ^k, with k odd positive.[35] The order parameter ψ is related to $f_s(\tau, \zeta)$ and $G_s(\lambda, g, \zeta)$ according to

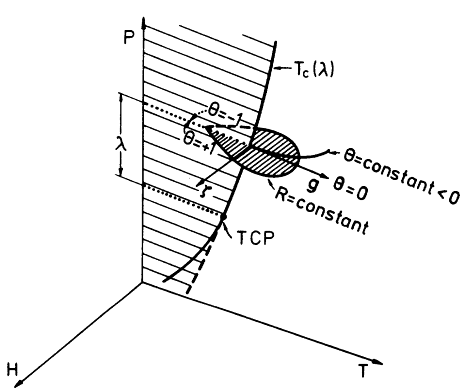

FIG. 7. "Cylindrical" coordinate system used in Sec. IV. Curves for constant Schofield parameters R and θ in the plane λ = const. > 0 are shown. The local Cartesian-coordinate system refers to the in-plane scaling variables g, ζ. By varying θ from +1 to −1 one turns from the "positive" ($H = +0$) to the "negative" ($H = -0$) sheet of the first-order surface (parallel markings). For $\lambda < 0$, the first-order surface ($\theta = \pm 1$) extends to the triple line (dashed), and the corresponding curves R = const. and θ = const. are more complicated.

$$\begin{aligned} \psi(\lambda, g, \zeta) &= -\left(\frac{\partial f_s(\tau(\lambda, g, \zeta), \zeta)}{\partial \zeta} \right)_\tau \\ &= -\left(\frac{\partial G_s(\lambda, g, \zeta)}{\partial \zeta} \right)_{\lambda, g} , \end{aligned} \tag{4.5}$$

which also holds for our models.[36]

For model A the corresponding equations are similarly obtained by introducing R and θ according to $\tau = R\tilde{\tau}(\theta^2)$, $\zeta = R^{\Delta_H}\tilde{\zeta}(\theta)$, and using $\partial f_s/\partial\tau = R^{1-\alpha_H} S_H(\theta^2)$ in (3.5):

$$\begin{aligned} g &= \lambda R \frac{a_2}{a_1} \tilde{\tau}(\theta^2) + R^{1-\alpha_H} \frac{1}{a_1} S_H(\theta^2) , \\ \zeta &= R^{\Delta_H} \tilde{\zeta}(\theta), \\ \psi &= R^{\beta_H} \psi_H(\theta) \end{aligned} \tag{4.6}$$

(the subscript H refers to the Heisenberg model). For model B we use (4.2) and (4.3) and obtain from (3.9), with $-\partial f_s/\partial\tau = r^{1-\alpha_I} S_I(\theta^2)$ and $R \equiv r^{1-\alpha_I}$,

$$\begin{aligned} g &= -(-\lambda)^{2(n-1)} \Theta(-\lambda) R b S_I(\theta^2) + R^{1/(1-\alpha_I)} \tilde{\tau}(\theta^2), \\ \zeta &= R^{\Delta_I/(1-\alpha_I)} \tilde{\zeta}(\theta), \\ \psi &= R^{\beta_I/(1-\alpha_I)} \psi_I(\theta) . \end{aligned} \tag{4.7}$$

The above parametric representation clearly shows the formal similarity of these TCP's. In particular, (4.4) and (4.6) have an identical structure apart from unimportant differences in the θ-dependent functions (Appendix E). This becomes more apparent when the Ising exponents appearing in (4.4) are rewritten in terms of the renormalized exponents $\overline{\alpha}$, $\overline{\beta}$, $\overline{\Delta}$ defined by

$$\overline{\alpha} = \frac{-\alpha_I}{1-\alpha_I} , \quad \overline{\beta} = \frac{\beta_I}{1-\alpha_I} , \quad \overline{\Delta} = \frac{\Delta_I}{1-\alpha_I} . \tag{4.8}$$

Equation (4.4) then has the form of (4.6) where the Heisenberg exponents $\alpha_H < 0$, β_H, Δ_H are replaced by $\overline{\alpha} < 0$, $\overline{\beta}$, $\overline{\Delta}$.

We thus reach the interesting conclusion that the macroscopic Hamiltonian constraint in model A—the requirement of rigid planes—combined with a spin system of finite specific heat leads to the same tricritical behavior as found in the thermodynamic theory of constrained systems. This result is surprising, since model A cannot be viewed as an example of this theory. Indeed, in model A there is no underlying transition corresponding to the "ideal" transition in the constrained system, as, for example, the Ising transition in the Baker-Essam force ensemble. Furthermore, the finite specific heat along the λ line ($\alpha_H < 0$) is not induced by the Jasnow-Wagner lattice constraint.

The similarity of the tricritical equations suggests that the existence of a macroscopic constraint, though different for each system, leads to a common mechanism governing the tricritical behavior. In fact, the essential property which induces the

TCP both in model A and in the model of Rudnick *et al.*,[11] an example of the constraint theory, is the presence of a Curie-Weiss-like interaction between pairs of spins (Appendix D).[37] The effective spin Hamiltonian of both models (and also of MTM) is essentially of the form

$$H_{\rm eff} = c_1 H_0 + (c_2/N)(H_0)^2 , \tag{4.9}$$

where $H_0 = \sum_{\langle i,k \rangle} \vec{S}_i \cdot \vec{S}_k$ is a nearest-neighbor spin interaction and N is the number of spins coupled through the four-spin term $(H_0)^2$. The coupling constants c_1 and c_2 depend on the secondary field variable (pressure in the case of model A). In the model of Rudnick *et al.* the four-spin term ("quartic pairing term") is formally introduced by a transformation of variables, whereas in model A—as well as in B and generally in MTM—it is an effective interaction arising from the spin-lattice coupling (Appendix D). In all cases this four-spin term induces the first-order transition below the TCP, and its relative strength c_2/c_1 determines the magnitude of the first-order jump in various thermodynamic quantities. Hence the nature of the instability in the model of Rudnick *et al.*, and thus presumably in the Baker-Essam model and in the constrained systems, is identical to that of the well-known MTM first-order transition.

Of course, imposing a macroscopic constraint is only one possible way of generating such long-range interactions. There are probably other systems where short-range and long-range interactions combine to produce a similar TCP, though the presence of a constraint is not apparent. We summarize the essential properties of this class of tricritical systems as follows.

(1) Short-range and long-range interactions are present.

(a) The short-range interaction leads to nonclassical critical exponents along the λ line. The specific heat is finite along the λ line.

(b) The interactions combine to produce a TCP with exponents that are simply related to the λ line exponents through the dimension independent relations of Fisher's renormalization.[9] In particular, the specific heat diverges at the TCP ($\alpha_t > 0$). The crossover exponent is $\phi = 1/\alpha_t$.

(c) The instability at the triple line and on the wings is induced by the long-range interaction. The wing exponents are classical.

(2) Tricritical scaling holds in terms of the "floating" variable set $g = T - T_c(\lambda)$, $\lambda = P - P_t$, $\zeta = H$, provided that normal critical scaling along the λ line is fulfilled.

(3) The explicit λ dependence in the equation of state as well as the function $T_c(\lambda)$ are analytic at $\lambda = 0$.

Evidently, due to case 1(b), the above tricritical properties represent only a small class of tricritical systems. We expect in general, that the presence of long-range forces need not lead to such special exponent relations. A simple example of this is the Landau tricritical model,[29,30] which has the parametric equation of state[38]

$$\begin{aligned} g &= \lambda R \tilde{g}_1(\theta^2) + R^2 \tilde{g}_2(\theta^2), \\ \zeta &= \lambda R^{3/2} \tilde{\zeta}_1(\theta) + R^{5/2} \tilde{\zeta}_2(\theta), \\ \psi &= R^{1/2} \tilde{\psi}_1(\theta) , \end{aligned} \tag{4.10}$$

as can be verified by inserting (4.10) into $\zeta = Ag\psi + B\lambda\psi^3 + \psi^5$. If we now look at (4.4), (4.6), and (4.10) in a phenomenological way and use these equations as a guide, we are immediately led to the following generalizations:

$$\begin{aligned} g &= \lambda^k R g_1(\theta^2) + R^{1+\epsilon} g_2(\theta^2), \\ \zeta &= \lambda^m R^{\Delta} \zeta_1(\theta) + R^{\Delta+\mu} \zeta_2(\theta), \\ \psi &= \lambda^n R^{\beta} \psi_1(\theta) + R^{\beta+\nu} \psi_2(\theta), \end{aligned} \tag{4.11}$$

with Greek exponents positive and integers k, m, n nonnegative with k odd. These tricritical equations have the following properties (Appendix E): The normal critical exponents ($\lambda > 0$) and tricritical exponents ($\lambda = 0$) are Δ, β and $\Delta_t = (\Delta + \mu)/(1+\epsilon)$, $\beta_t = (\beta + \nu)/(1+\epsilon)$, respectively. Normal critical scaling is implied by (4.11), and tricritical homogeneity of $\psi(\lambda, g, \zeta)$ holds if $\mu = m\epsilon/k$ and $\nu = n\epsilon/k$, leading to a crossover exponent $\phi = (1+\epsilon)k/\epsilon > 1$. Since the parametric representation (4.11) is analytic for $R > 0$, the wing critical behavior should be classical.

The tricritical equations of the constrained systems and model A are recovered from (4.11) if $m = n = \zeta_2 = \psi_2 = 0$. Similarly, the Landau model (4.10) is obtained if $k = m = 1$, $n = \psi_2 = 0$, and Kortman's equation of state[25] is obtained for $k = m = 1$, $\mu = \epsilon$, $n = \psi_2 = 0$.

We conjecture that the asymptotic phenomenological equations (4.11) describe a general class of tricritical scaling systems, where some sort of long-range interactions play an important role as suggested by the classical wing behavior. On the other hand we suspect that such systems again have similar artificial features as the Baker-Essam and the Jasnow-Wagner model or are subject to special macroscopic constraints.

V. SUMMARY

We have examined two elastic spin models A and B which exhibit nonclassical tricritical points. Both models have singular parts of the free energies which obey tricritical scaling. However, model B has an additional singular part due to the nonanalytic x dependence of the exchange constant $J(x)$. For both models tricritical scaling must be formulated in terms of the same curvilinear coordinates as in the constrained systems of Imry *et al.*[4,5,10] Thus these types of variables, which are simple examples of the g fields of Wegner and Riedel,[31] seem indeed to be appropriate for a certain class of tricritical scaling systems. If, however, the crossover exponent is sufficiently small, the linear variables of Griffiths[7] or Riedel[8] can equivalently be used.

The tricritical exponents of our models, though unrelated to the classical wing exponents, are simply related to the normal critical exponents. This relationship [see (3.14) and (3.15)] does not explicitly depend on the dimensionality of the system which enters only through the values of the normal critical exponents. Thus the exponent relations do not agree with those obtained from the theory of Bausch.[26] The exponents differ also from those of Kortman's phenomenological model[25] and of the nonscaling[39] model of Riedel and Wegner.[40]

The comparison of model A and the theory of constrained systems shows that the structure of their tricritical equations is identical, although model A is not an example of this theory. This similarity can be explained on the basis of a Curie-Weiss-like interaction between pairs of spins, occurring both in model A and in the model of Rudnick, Bergman, and Imry,[11] an example of the constraint theory. A comparison of the effective spin Hamiltonians reveals the magnetothermomechanical nature of the first-order instability in the model of Rudnick *et al.* and thus presumably in the Baker-Essam model and in the contrained systems.

Finally, a more general set of tricritical equations has been suggested which, besides model A and the constrained systems, also contains the mean field and Kortman's phenomenological model. We conjecture that these equations represent a general class of tricritical scaling systems where long-range interactions play an important role.

Note added in proof. Within the Wilson theory, F. J. Wegner [J. Phys. C (to be published)] has very recently considered the effect of long-range four-spin interactions of the type discussed above. For the case of a finite specific heat of the rigid spin system, his analysis leads to the possibility of a tricritical point with the same exponent relations as for model A [see Eq. (3.32) of Wegner and Eq. (3.14) above]. Since our thermodynamic analysis of model A applies generally to magnetothermomechanics in case of $\alpha < 0$, the resulting tricritical equation of state (3.5) or (4.6) should also describe Wegner's "modified magnetothermomechanical" model.

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APPENDIX A: LAPLACE METHOD FOR JASNOW-WAGNER LATTICES

To make apparent which assumptions are required to obtain the MTM result (2.1) and (2.2) for model A, and generally for spin systems on Jasnow-Wagner lattices, we discuss briefly the evaluation of the partition function by using the Laplace method.[41]

We consider a d-dimensional Jasnow-Wagner lattice with M vertical "rigid" planes ($d=3$) or columns ($d=2$) and N Ising or classical Heisenberg spins $\vec{S}_{k,m}$ in each plane or column ($m=1, 2, \ldots, M$; $k=1, 2, \ldots, N$). The first plane is held fixed at $x_1=0$, and an external force NP is applied to the Mth plane. The Hamiltonian in zero magnetic field is given by $H=H_L+H_s$ with a lattice part

$$H_L = N\left(Px_M + \sum_{m=1}^{M-1} v(x_{m+1}-x_m)\right), \tag{A1}$$

where $v(x)$ is the horizontal nearest-neighbor lattice potential, and with a spin part

$$H_s = \sum_{m=1}^{M-1} J(x_{m+1}-x_m) \sum_{k=1}^{N} \vec{S}_{k,m+1}\cdot\vec{S}_{k,m} + \sum_{m=1} \hat{H}^{(m)}, \tag{A2}$$

where x_m is the horizontal position of the mth plane and $\hat{H}^{(m)}$ is the spin interaction within this plane. By using the iteration procedure of Jasnow and Wagner[12] and performing the trace over the spins, the partition function can be written as

$$Z_M(T,P) = \int_0^\infty d\xi_1 \cdots \int_0^\infty d\xi_{M-1} \times \exp[-\beta NMG_M(\xi_1, \xi_2, \ldots, \xi_{M-1})], \tag{A3}$$

with

$$G_M(\xi_1, \xi_2, \ldots, \xi_{M-1}) = \frac{1}{M}\sum_{m=1}^{M-1}[P\xi_m + v(\xi_m)] + f_M(T; \xi_1, \ldots, \xi_{M-1}), \tag{A4}$$

where f_M is the free energy per spin of the rigid-spin system with separations ξ_m between the planes (columns) m and $m+1$.

We first consider (A3) for fixed M and N. Since the form of the integrand suggests using the Laplace method, we look for the extrema of G_M at fixed T and P:

$$\frac{\partial G_M}{\partial \xi_m^0} = \frac{1}{M}[P + v'(\xi_m^0)] + \frac{\partial f_M(T; \xi_1^0, \ldots, \xi_{M-1}^0)}{\partial \xi_m^0} = 0 \quad (m=1, 2, \ldots, M-1). \tag{A5}$$

We consider a realistic lattice potential $v(x)$ with $v''(x)\to\infty$ as $x\to 0$. Hence the solutions ξ_m^0 of (A5) become arbitrarily small for sufficiently high P

and any finite M, as follows from a qualitative plot similar to that given in Fig. 1. Since the matrix

$$\frac{\partial^2 G_M}{\partial \xi_i \partial \xi_k} = \frac{1}{M} v''(\xi_k)\delta_{ik} + \frac{\partial^2 f_M(T; \xi_1, \ldots, \xi_{M-1})}{\partial \xi_i \partial \xi_k} \quad \text{(A6)}$$

is positive definite for sufficiently small ξ_m (and thus G_M is convex in this region), G_M has a unique minimum for sufficiently high P and all temperatures T (and for an arbitrary magnetic field). The location of this minimum is given by the uniform solution of (A5)

$$\xi_1^0 = \xi_2^0 = \cdots = \xi_{M-1}^0 = y_M(T, P)\ , \quad \text{(A7)}$$

where y_M satisfies

$$0 = \frac{M-1}{M}[P + v'(y_M)] + \frac{\partial f_M(T; J(y_M))}{\partial y_M} \quad \text{(A8)}$$

with $f_M(T; J(y_M)) \equiv f_M(T; y_M, \ldots, y_M)$. We note that so far no specific properties of the spin system have been used.

The assumption that the uniform solution (A7) gives the absolute minimum of G_M for all pressures, seems to be correct at least in the case of the two-dimensional Ising model (outside the first-order coexistence region) in the limit $N, M \to \infty$. Indeed, if the Laplace method is applicable to (A3) for fixed M and $N \to \infty$, this assumption leads directly to the Gibbs free energy per spin

$$G_M(T, P) = \lim_{N\to\infty}\left(-k_B T \frac{\ln Z_M}{NM}\right) = \min_{y_M}\left(\frac{M-1}{M}[P y_M + v(y_M)] + f_M(T; J(y_M))\right), \quad \text{(A9)}$$

where $y_M(T, P)$ is a solution of (A8). By now taking the limit $M \to \infty$, the MTM result (2.1), which is rigorous for the two-dimensional Ising model,[12] is recovered. Note that the applicability of the Laplace method in case of a finite number of integrals should not require analyticity of G_M as $N \to \infty$, but should only depend on the convexity property of G_M (see Ref. 41).

For the three-dimensional Ising model the same result is expected,[12] since the dimensionality of the spin system enters only through the form of the rigid free energy, which should be qualitative similar to that of the two-dimensional case. Indeed, we expect that only general qualitative features of the rigid free energy f_M, in particular of the matrix $\partial^2 f_M/\partial\xi_i\partial\xi_k$, are required to establish that the uniform solution (A7) gives the absolute minimum of the function G_M (A4) in the thermodynamic limit. Since the over-all functional dependence of the rigid three-dimensional Heisenberg free energy on $J(\xi_m)$ should be qualitatively similar to that of the two- or three-dimensional Ising model, we expect the MTM result (2.1), (2.2) to be valid also for model A.

An analysis of the effective spin Hamiltonians for general spin systems on Jasnow-Wagner lattices is given in Appendix D. These Hamiltonians turn out to have the same form—independent of the type of spin system—which is essentially the same as in the case of MTM. This provides strong support for the above qualitative discussion.

APPENDIX B: SINGULAR PART OF THE FREE ENERGIES OF MODELS A AND B

In the following expansions of the free energy

$$G(T, P, H) = \min_x [Px + v(x) + f_a(T, H; J(x)) + f_s(\tau(T, x), H)] \quad \text{(B1)}$$

we consider only a small neighborhood of the TCP and therefore retain only the lowest-order terms. Besides $g = T - T_c(P)$ and $\lambda = P - P_t$ we use the notation $x_c[P] = x_c(T_c(P))$, $\Delta x = x - x_c[P]$, and $x_t = x_c(T_t)$. For simplicity we ignore the analytic function $\tau_0(T)$ in (2.4).

For $H=0$, $f_s(\tau(T, x), 0)$ in (B1) is singular along the line $\tau(T, x) = 0$ or $T = T_c^0[J(x)]$ in the T-x plane. The natural appearance of the "floating" variable $\tau(T, x)$ in (B1) suggests using the corresponding "floating" variable $g = T - T_c(P)$ in the T-P plane. Thus expanding the first three terms in the Gibbs free energy (B1) and the corresponding terms in the equation of state (2.2) around $T = T_c(P)$ and $H = 0$ for fixed P and using (3.1), we obtain

$$G(T, P, H) = G(T_c(P), P, 0) + g\frac{\partial f_a}{\partial T} + \tilde{G}(g, P, H), \quad \text{(B2)}$$

$$\tilde{G}(g, P, H) = \min_x [h_1(P) g\Delta x + \tfrac{1}{2}h_2(P)\Delta x^2 + f_s(\tau(T, x), H)], \quad \text{(B3)}$$

and

$$0 = h_1(P)g + h_2(P)\Delta x + \frac{\partial f_s(\tau, H)}{\partial \tau}\,\frac{\partial \tau}{\partial x}\ , \quad \text{(B4)}$$

with $h_1(P) = \partial^2 f_a/\partial T\partial x$, $h_2(P) = v'' + \partial^2 f_a/\partial x^2$ taken at $T = T_c(P)$, $H = 0$, $x = x_c[P]$. In order to eliminate Δx in favor of τ, we expand $\tau(T, x) = T - T_c^0[J(x)]$ around $x_c[P]$ using $T_c^0[J(x_c[P])] = T_c(P)$. Thus

$$\tau = g + h_3(P)\Delta x\ , \quad \text{(B5)}$$

where $h_3(P) = -(dT_c^0/dx)_{x=x_c[P]}$. From (B4) and (B5) we obtain

$$\Delta x = \frac{1}{h_1 h_3 - h_2}\left(h_1\tau + h_3\frac{\partial f_s(\tau, H)}{\partial\tau}\right)\ . \quad \text{(B6)}$$

The final step in obtaining Eqs. (3.3)–(3.9) is to expand $h_1(P)$, $h_2(P)$, and $h_3(P)$ around $P = P_t$.

A. Model A

According to (2.7) and the analyticity of $T_c(P)$, $h_2(P)$ reduces to

$$h_2(P)=a\lambda\ , \tag{B7}$$

with

$$a=\frac{d}{dP}\left(v''(x_c[P])+\frac{\partial^2 f_a(T_c(P),0;J(x_c[P]))}{\partial x_c^2}\right)_{P=P_t}.$$

Inserting (B6) into (B3) and replacing $h_1(P)$ and $h_3(P)$ by $h_1(P_t)$ and $h_3(P_t)$, respectively, yields the following leading term of $\tilde{G}$:

$$G_s(\lambda,g,H)=\min_\tau[-a_1 g\tau+\tfrac{1}{2}a_2\lambda\tau^2+f_s(\tau,H)], \tag{B8}$$

where, according to (B3), $\tau=\tau(\lambda,g,H)$ satisfies

$$0=-a_1 g+a_2\lambda\tau+\frac{\partial f_s(\tau,H)}{\partial\tau}\ , \tag{B9}$$

with $a_1=-h_1(P_t)/h_3(P_t)$, $a_2=a/[h_3(P_t)]^2$. For $\lambda<0$ and sufficiently small $|H|$, the solution $\tau(\lambda,g,H)$ of (B9) is triple valued. This follows from $a_1>0$, $a_2>0$ because of $h_1(P_t)<0$, $h_3(P_t)>0$, $a>0$ and from $\partial f_s(\tau,0)/\partial\tau=c_\pm|\tau|^{1-\alpha_H}\operatorname{sgn}(\tau)$ with $c_\pm>0$ (upward-pointing cusp of the specific heat).

B. Model B

Both $h_1(P)$ and $h_3(P)$ are proportional to $J'(x_c[P])$, which is zero for $P\geq P_t$; $h_2(P)$ can be replaced by $v''(x_t)$ since $\partial^2 f_a/\partial x^2=0$ at the TCP. Inserting (B6) into (B5) yields, in lowest order,

$$g=\tau+\frac{[h_3(P)]^2}{v''(x_t)}\ \frac{\partial f_s(\tau,H)}{\partial\tau}\ . \tag{B10}$$

Now substituting (B6) into (B3) and using (B10) to sort out the lowest-order singular terms of $\tilde{G}$, one obtains

$$G_s(\lambda,g,H)=\min_\tau\left[\frac{1}{2}\ \frac{h_3^2}{v''}\left(\frac{\partial f_s}{\partial\tau}\right)^2+f_s(\tau,H)\right], \tag{B11}$$

where $\tau=\tau(\lambda,g,H)$ satisfies (B10). For $P<P_t$ and sufficiently small $|H|$, $\tau(\lambda,g,H)$ is triple valued, since $h_3^2/v''>0$ and $\partial f_s(\tau,0)/\partial\tau=-c_\pm|\tau|^{1-\alpha_I}\operatorname{sgn}(\tau)$ with $c_\pm>0$.

Finally, in order to calculate $J'(x_c[P])$ we subtract (2.14) from (3.1) and expand $v'(x_c[P])$ around x_t to linear order:

$$0=\lambda+v''(x_t)(x_c[P]-x_t)+\frac{\partial f_a}{\partial J}J'(x_c[P])\ . \tag{B12}$$

Since according to (2.11)

$$J'(x)=-nc_0(x-x_t)^{n-1}\Theta(x-x_t)\ , \tag{B13}$$

with $n-1>1$, we find from (B12), in lowest order, $x_c[P]-x_t=-\lambda/v''(x_t)$ and thus, by inserting this into (B13),

$$J'(x_c[P])=-\frac{nc_0}{(v'')^{n-1}}(-\lambda)^{n-1}\Theta(-\lambda). \tag{B14}$$

APPENDIX C: EQUIVALENCE OF TRICRITICAL SCALING VARIABLES

In order to verify case (1) in Sec. III, we assume homogeneity of the singular part of the free energy in terms of the variables (3.16):

$$G_s\left(\frac{\lambda}{L},\frac{g_1}{L^\phi}\right)=\frac{1}{L^{\phi(2-\alpha_t)}}G_s(\lambda,g_1) \tag{C1}$$

where $L>0$. The ζ dependence has been dropped, since it is irrelevant to our discussion. Equation (C1) implies that since $T_c(P)$ is a line of singularities, it is a crossover line, i.e., $T_c(P)-T_1(P)\sim C|\lambda|^\phi$, which yields the transformation $g_1\sim g+C|\lambda|^\phi$. Inserting this into (C1) we see that G_s is also a homogeneous function of g and λ with crossover exponent ϕ.

To prove case (2) in Sec. III, we assume the scaling form

$$G_s\left(\frac{\lambda}{L},\frac{g}{L^\phi}\right)=\frac{1}{L^{\phi(2-\alpha_t)}}G_s(\lambda,g) \tag{C2}$$

for any $L>0$. In this case $T_c(P)$ is a reference line, and its shape is *not* determined by scaling properties. Thus we have $T_c(P)-T_1(P)\sim A|\lambda|^{\phi_1}$, where the exponent ϕ_1 need not be equal to the crossover exponent ϕ. This yields $g\sim g_1-A|\lambda|^{\phi_1}$ or

$$\frac{g}{L^\phi}\sim\frac{g_1}{L^\phi}-A\left|\frac{\lambda}{L}\right|^{\phi_1}L^{\phi_1-\phi}\ . \tag{C3}$$

Thus

$$G_s\left(\frac{\lambda}{L},\frac{g_1}{L^\phi}-A\left|\frac{\lambda}{L}\right|^{\phi_1}L^{\phi_1-\phi}\right)=\frac{1}{L^{\phi(2-\alpha_t)}}G_s(\lambda,g_1-A|\lambda|^{\phi_1})\ . \tag{C4}$$

Evidently G_s is not a homogeneous function of g_1 and λ unless $\phi_1=\phi$.

However, the scaling property (C2) is expected to be an asymptotic property which holds only in the limit $\lambda\to0$, $g\to0$, $L\to0$ with $\lambda/L\to c_1$, $g/L^\phi\to c_2$, where $|c_1|<\infty$, $|c_2|<\infty$. Assuming $\phi_1>\phi$ and going to this limit[28] in (C4), the terms proportional to A can be neglected. Thus in a neighborhood of $g=0$ and $\lambda=0$, G_s becomes a scaling function of g_1 and λ with crossover exponent ϕ. On the other hand, if $\phi_1<\phi$ and taking the above limit in (C3) with $c_1\neq0$, we see that since c_2 is bounded and $L^{\phi_1-\phi}$ diverges, both terms of the right-hand side of (C3) must be retained. Hence the inhomogeneity of G_s as a function of g_1 and λ remains.

APPENDIX D: CURIE-WEISS FOUR-SPIN INTERACTION

A. Model A

For the sake of simplicity we consider the three-dimensional Jasnow-Wagner lattice with classical Heisenberg spins $\vec{S}_{k,m}$ in the harmonic approximation $v(x)=v_0+\frac{1}{2}v_2(x-a_0)^2$ and with the interplanar horizontal nearest-neighbor exchange constant $J(x)=-J_0+J_1(x-a_0)$. The Hamiltonian in zero magnetic field is given by

$$H=\sum_{m=1}^{N}(-J_0H^{(m)}+J_1H^{(m)}\xi_m+N^2P\xi_m+\tfrac{1}{2}N^2v_2\xi_m^2), \tag{D1}$$

where $\xi_m = x_m - a_0$, $H^{(m)} = \sum_{k=1}^{N^2} \vec{S}_{k,m} \cdot \vec{S}_{k,m+1}$ and m indexes the rigid planes in the $N\times N\times N$ lattice. The in-plane spin interaction has been dropped for simplicity. The partition function is

$$Z(T,P) \propto \operatorname*{Tr}_{\{\vec{S}\}}\left(\prod_{m=1}^{N} e^{\beta J_0 H^{(m)}} \int_{-\infty}^{+\infty} d\xi_m \exp\{-\beta[\tfrac{1}{2}N^2 v_2 \xi_m^2 + (J_1 H^{(m)} + N^2 P)\xi_m]\}\right) . \tag{D2}$$

By completing the square in the exponent, the integrals are easily performed, yielding

$$Z(T,P) \propto \operatorname*{Tr}_{\{\vec{S}\}} e^{-\beta H_{\text{eff}}} , \tag{D3}$$

with

$$H_{\text{eff}} = -\sum_{m=1}^{N} [(a_1 + b_1 P) H^{(m)} + (c_1/N^2)(H^{(m)})^2], \tag{D4}$$

where $a_1 = J_0$, $b_1 = J_1/v_2$, $c_1 = J_1^2/2v_2$. The second term is a Curie-Weiss interaction between all pairs of horizontally coupled nearest-neighbor spins in two neighboring rigid planes. The result (D4) clearly holds also in two dimensions and for Ising spins.

B. Magnetothermomechanics

Following Wagner and Swift,[42] we start from the Helmholtz free energy of a cubic lattice with volume V:

$$F(T,V) = f_L(V) + f_s(T,V) . \tag{D5}$$

In (D5) the lattice free energy is assumed to be

$$f_L(V) = \frac{(V-V_0)^2}{2V_0\kappa} \tag{D6}$$

(V_0 is the reference volume, κ is a constant isothermal compressibility), and the spin free energy is

$$f_s(T,V) = -k_B T \ln\left(\operatorname*{Tr}_{\{\vec{S}\}} e^{-\beta J(V) H_0}\right) , \tag{D7}$$

where $H_0 = \sum_{\langle ik\rangle} \vec{S}_i \cdot \vec{S}_k$ is a nearest-neighbor Ising or classical Heisenberg spin Hamiltonian and $J(V) = -J_0 + J_1(V-V_0)/V_0$. Thus the partition function in the pressure ensemble is given by

$$Z(T,P) \propto \operatorname*{Tr}_{\{\vec{S}\}}\left(e^{\beta J_0 H_0} \int_0^{\infty} dV \exp\{-\beta[f_L(V) + (J_1 H_0 + P V_0)(V-V_0)/V_0]\}\right) , \tag{D8}$$

yielding (D3) with

$$H_{\text{eff}} = -(a_2 + b_2 P) H_0 - (c_2/N)(H_0)^2 , \tag{D9}$$

where $a_2 = J_0$, $b_2 = \kappa J_1$, $c_2 = (\kappa N/2V_0) J_1^2$, and N is the total number of spins. The second term in (D9) is a Curie-Weiss interaction between all pairs of neighboring spins. Thus the MTM Hamiltonian (D9) and the Jasnow-Wagner Hamiltonian (D4) have the same types of interactions.

C. Model of Rudnick, Bergman, and Imry (Ref. 11)

For completeness and the purpose of comparison with Secs. A and B of this appendix, we include the model of Rudnick *et al.* and reformulate it in a space representation with discrete Ising spins σ_i. The Hamiltonian is taken as

$$H(\lambda) = -(a+\lambda) H_0 , \tag{D10}$$

where $H_0 = \sum_{\langle ik\rangle} \sigma_i \sigma_k$ is the nearest-neighbor Ising Hamiltonian and λ is some thermodynamic variable. Defining the function

$$Z(T,\theta) = \int_{-\infty}^{\infty} d\lambda\, Z(\lambda) \exp\left(\frac{\beta N \lambda^2}{2\theta}\right) , \tag{D11}$$

with $Z(\lambda) = \mathrm{Tr}_{\{\sigma\}}\, e^{-\beta H(\lambda)}$ and $\theta < 0$, one obtains $Z(T,\theta) \propto \mathrm{Tr}_{\{\sigma\}}\, e^{-\beta H_{\text{eff}}}$, with

$$H_{\text{eff}} = -a H_0 + (\theta/N)(H_0)^2 . \tag{D12}$$

The effective Hamiltonians (D9) and (D12) are of the same type if $\theta < 0$, revealing the MTM nature of the first-order instability in the model of Rudnick *et al.*

APPENDIX E: PARAMETRIC REPRESENTATION

A. θ dependence

Without losing generality one may define the "polar" coordinates r and θ used in Sec. IV by the relations $\tau = r\tilde{\tau}(\theta^2)$ and $\zeta = r^{\Delta}\tilde{\zeta}(\theta)$, with

$$\tilde{\tau}(\theta^2) = 1 - a\theta^2 , \tag{E1}$$

$$\tilde{\zeta}(\theta) = b\theta(1-\theta^2) , \tag{E2}$$

where $a > 1$, $b > 0$, $-1 \leq \theta \leq +1$. Within the simple "linear model" of Schofield, Litster, and Ho,[43] the scaling equation of state $\psi = r^{\beta}\tilde{\psi}(\theta)$ is specified by the linear ansatz

$$\tilde{\psi}(\theta) = C\theta , \tag{E3}$$

with $C > 0$. For the singular part of the entropy one obtains for this model $-\partial f_s(\tau,\zeta)/\partial\tau = r^{1-\alpha}\tilde{S}(\theta^2)$ with $\alpha = 2 - \Delta - \beta$ and

$$\tilde{S}(\theta^2) = s_0(1 - s_1\theta^2) , \tag{E4}$$

where s_0 and s_1 are constants ($s_1 > 1$). Note that (E1) and (E4) are of the same form.

For the three-dimensional Ising and Heisenberg model $\tilde{\psi}(\theta)$ and $\tilde{S}(\theta^2)$ are unknown polynomials which are odd and even in θ, respectively; in addition, $\tilde{\psi}(\theta)$ must vanish only for $\theta = 0$, and $\tilde{S}(\theta^2)$ must vanish only at two values, $\pm\theta_0$. Thus the functions $\tilde{\tau}(\theta^2)$ and $S(\theta^2)$ used in (4.4), (4.6), and (4.7) are expected to be qualitatively similar. A simple example for the mean field functions in (4.10) is

$$\tilde{g}_1(\theta^2) = 1 - a_1\theta^2 \quad , \quad \tilde{g}_2(\theta^2) = 1 - a_2\theta^2 ;$$
$$\tilde{\zeta}_1(\theta) = b_1\theta(1-\theta^2) \; , \quad \tilde{\zeta}_2(\theta) = b_2\theta(1-\theta^2)^2 ; \qquad \text{(E5)}$$
$$\tilde{\psi}_1(\theta) = c_1\theta \; ,$$

with $a_1, \; a_2 > 1; \; b_1, \; b_2, \; c_1 > 0$.

B. Tricritical equations

In the following discussion of the tricritical equations (4.11),

$$g = \lambda^k R g_1(\theta^2) + R^{1+\epsilon} g_2(\theta^2) , \qquad \text{(E6a)}$$

$$\zeta = \lambda^m R^{\Delta} \zeta_1(\theta) + R^{\Delta+\mu} \zeta_2(\theta) , \qquad \text{(E6b)}$$

$$\psi = \lambda^n R^{\beta} \psi_1(\theta) + R^{\beta+\nu} \psi_2(\theta) , \qquad \text{(E6c)}$$

we consider the general case where the Greek exponents are positive and the θ-dependent functions do not vanish identically. An analogous discussion holds when some of the exponents or θ-dependent functions vanish as in the case of model A, the constrained systems, the mean field, and Kortman's model.

Case (a) $\lambda > 0$. Near the second-order line ($R \to 0$), (E6) reduces to the usual Schofield scaling equations with critical exponents β and Δ, where the functions g_1, ζ_1, and ψ_1 have the same qualitative properties as (E1), (E2), and (E3), respectively.

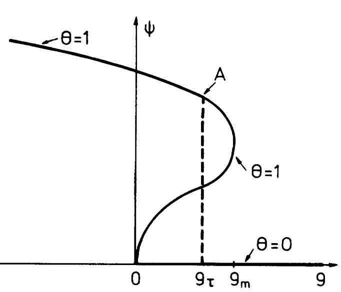

FIG. 8. Order parameter $\psi(\lambda, g, \zeta)$ as a function of g for fixed $\lambda < 0$ and $\zeta = 0$. The abscissa for $g \geq 0$ corresponds to $\theta = 0$; the curved line which has an infinite slope at $g = g_m$, corresponds to $\theta = +1$, as obtained from Eqs. (E7). The equilibrium value of ψ is positive for $g < g_T$, jumps at $A(g = g_T)$, and is zero for $g > g_T$.

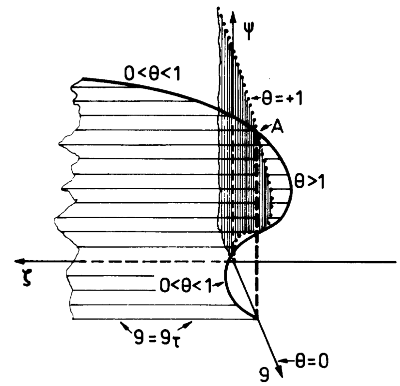

FIG. 9. Order parameter $\psi(\lambda, g, \zeta)$ for fixed $\lambda < 0$ as a function of ζ for $g = g_T$ (full line). The plane $g = g_T$ = constant is indicated by parallel markings. The heavy dashed line indicates the jump of the order parameter. The loop (full line) and the dashed line are the boundaries of two equal areas. The dotted line in the plane $\zeta = 0$ (dark markings) corresponds to the curved line in Fig. 8.

Case (b) $\lambda = 0$. In this case (E6) again reduces to Schofield equations with tricritical exponents $\beta_t = (\beta + \nu)/(1+\epsilon)$ and $\Delta_t = (\Delta + \mu)/(1+\epsilon)$ and with functions g_2, ζ_2, and ψ_2 similar to g_1, ζ_1, and ψ_1. Since $G_s(\lambda, g, \zeta)$ and thus $\psi(\lambda, g, \zeta)$ are expected to be analytic at $\lambda = 0$ if $g > 0$, the explicit λ dependence in (E6) must be analytic, i.e., k, m, n must be non-negative integers; for the same reason the second-order line $T_c(\lambda)$ must be analytic at $\lambda = 0$ [case (3) in Sec. IV].

Case (c) $\lambda < 0$. Clearly at least one of the integers k, m, n must be odd,[44] otherwise the phase structure would be identical for $\lambda > 0$ and $\lambda < 0$. We consider the case where k and m are odd and n is even. As follows from the symmetry properties of the θ-dependent functions [see cases (a) and (b) above], the first-order phase structure is symmetric with respect to the plane $\zeta = 0$ ($\theta = 0, \pm 1$). Therefore we may confine our discussion to non-negative values of θ.

First we show the existence of the triple line in the plane $\zeta = 0$ by considering $\psi(\lambda, g, +0)$ for fixed $\lambda < 0$. For $\theta = 0$, $\psi = 0$, since $\psi_1(0) = \psi_2(0) = 0$. For $\theta = +1$ we obtain from (E6a) and (E6c)

$$g = A_1 R - A_2 R^{1+\epsilon} , \qquad \text{(E7a)}$$

$$\psi = A_3 R^{\beta} + A_4 R^{\beta+\nu} , \qquad \text{(E7b)}$$

with positive constants A_i since $g_1(1)$, $g_2(1) < 0$, and $\psi_1(1)$, $\psi_2(1) > 0$. As shown in Fig. 8, this yields a continuous function $\psi(\lambda, g, +0)$ which, however, is triple valued for $0 < g < g_m(\lambda)$. In order to obtain a uniquely defined order parameter ψ corresponding to the minimum value of $G_s(\lambda, g, 0)$ for given λ and g, a jump in ψ must occur at some value $g_T(\lambda)$ with

$0 \leq g_\tau(\lambda) \leq g_m(\lambda)$. By varying λ, a line $g = g_\tau(\lambda)$ of first-order transitions in the plane $\zeta = 0$ is generated.[45] We expect that the location of this first-order line can alternatively be obtained by a Maxwell construction on the triple-valued function $\psi(\lambda, g_\tau(\lambda), \zeta)$. Indeed, the triple-valued nature of $\psi(\lambda, g, \zeta)$ (viewed either as a function of g or of ζ) should remain in a neighborhood of $\zeta = 0$ due to the analyticity of the parametric representation (E6) for $R > 0$, and loops may occur as in the special case of Kortman's equations (Fig. 9). Note that in order to obtain continuous loops, one must allow for $\theta > 1$ leading to nonstable segments similar as in the normal critical Schofield representation.

To discuss the existence of the wings we consider small $\zeta > 0$ ($\theta > 0$) and fixed $\lambda < 0$. Since $\psi(\lambda, g, \zeta)$ is expected to be triple valued (see above), the minimum requirement on $G_s(\lambda, g, \zeta)$ will again yield a first-order transition, but now for $\zeta > 0$. As λ increases, this transition must disappear, since there is no singularity for $\lambda = 0$, $g > 0$ [see case (b) above]. Due to the analyticity of the θ-dependent functions, this disappearance of the jump is expected to occur continuously, which suggests the existence of second-order lines terminating the wing surfaces. The critical exponents at these lines should again be classical due to the above-mentioned analyticity for $R > 0$.

A conclusive discussion of the first-order structure, of course, requires a more detailed specification of the θ-dependent parts, which is beyond the scope of this paper.

Finally, we note that $\psi(\lambda, g, \zeta)$ as obtained from (E6) need not be a homogeneous function. The tricritical scaling law $\psi(\lambda, g, \zeta) = L^{\phi\beta t}\psi(\lambda/L, g/L^{\phi}, \zeta/L^{\phi\Delta t})$ is only satisfied when μ and ν are chosen as $\mu = m\epsilon/\kappa$ and $\nu = n\epsilon/k$, leading to a crossover exponent $\phi = (1+\epsilon)\kappa/\epsilon$ as can be easily verified in (E6).

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[18]We are indebted to Dr. H. Horner for pointing this out to us.

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[35]This follows from the analyticity of the constraint equation (6) in Ref. 5, and from the requirement that the quantity π defined in Eq. (21) of Ref. 5 changes sign at the TCP.

[36]For models A and B, the relation (4.5) follows from (3.4), (3.5) and (3.8), (3.9), respectively. For the constrained systems (4.5) holds if the constraint equation does not depend on temperature explicitly, which is the case for the Baker-Essam model.

[37]We are indebted to Professor H. Wagner for pointing out the existence of this four-spin interaction in MTM.

[38]A simple example for the θ-dependent functions in (4.10) is given in Appendix E.

[39]The tricritical equation of state, though unknown, cannot obey homogeneity due to the appearance of logarithmic factors (Ref. 31).

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[44]In particular k must be odd. Otherwise no jump of $\psi(\lambda, g, 0)$ can occur, as can be shown from (D7).

[45]To insure that only one jump can occur requires a more detailed specification of the θ-dependent functions, which is beyond the scope of this paper.