Excitation of high-frequency cyclotron waves in a semi-infinite metal in the far infrared: Ordinary waves

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An approximate theory of Azbel'-Kaner cyclotron resonance in the retardation regime is derived and compared with experiment for the ordinary polarization in potassium and sodium at 29.69 cm^{-1}. Although the theory is not rigorous, it accounts in a quantitative way for both the attenuation of the subharmonics due to retardation effects, and the absorption on the high-field side of each subharmonic. This absorption is shown to be related to high-frequency cyclotron waves excited in the surface region. The waves in the surface region differ from those in the bulk because of the damping and scattering of the waves by electrons that strike the surface. The theory leaves only the mass and the relaxation rate as adjustable parameters, relieving much of the ambiguity that existed in previous determinations of electron mass and relaxation rate in potassium and sodium by far-infrared cyclotron resonance.

I. INTRODUCTION

The physical origin of cyclotron resonance in conducting solids is quite well understood as a resonance between incident electromagnetic radiation and the circulation of the electron about the magnetic field. The two physically interesting parameters that are measured at resonance are the charge-to-mass ratio e/m^* and the electron scattering rate. Although, in principle, the measurement and its interpretation is quite straightforward, the experimental manifestation of cyclotron resonance depends rather drastically on which electron system we are looking at, and theoretical treatments require a detailed solution of Maxwell's equation for the electromagnetic radiation in the solid and at the solid surface. For instance, cyclotron resonance in lightly doped semiconductors may or may not occur at ω_c, the cyclotron frequency, depending on whether or not the frequency of the incident radiation, ω, is much greater than or less than the plasma frequency ω_p. In either case, however, the solution to Maxwell's equation is found by using a conductivity which is strictly local and exact solutions can be obtained, thereby, greatly facilitating the interpretation of experimental results in terms of effective masses and relaxation rates.[1]

Cyclotron resonance in metals, where $\omega_p \gg \omega$, manifests itself by a series of resonances periodic in $1/H$, where H is the magnetic field. Each resonance is given by $\omega = n\omega_c$, where n is any integer. Unlike the semiconductor case the solution of Maxwell's equation for the electric fields in the solid requires a nonlocal conductivity and there exists no exact solution of the electrodynamics for a semi-infinite metal in a magnetic field parallel to its surface, even for the simplest case of a spherical Fermi surface. Approximate solutions have been obtained, however, the most noteworthy, the original work of Azbel' and Kaner[2] on cyclotron resonance in metals. An exact solution of the problem formulated by Azbel' and Kaner was obtained by Hartman and Luttinger,[3] but the problem itself is still approximate since the form of Maxwell's equation that is solved assumes that the electron leaves the skin depth in a time vanishingly small compared to the period of the incident field. Within the same approximation Chambers[4] solved the problem considering the possibility that only a very small fraction of the electrons in the metal may be resonant at a particular value of magnetic field. These theories have worked reasonably well to explain cyclotron resonance in real metals at microwave frequencies.

Recently, however, experimental techniques have been developed that enable cyclotron resonance to be observed in metals at frequencies as high as 84.32 cm^{-1},[5,6] the primary motivation being to study electron-phonon interactions by driving the metal at frequencies close to the Debye peak in the phonon density of states.[7–10] The above-mentioned theories do not properly describe the cyclotron-resonance line shape in this regime. The failure appears to be due to the fact that the incident radiation is changing phase so fast that the electron cannot escape the skin depth without having averaged the incident field over an appreciable fraction of its period. To extract meaningful values for the electron mass and relaxation rate at these frequencies it is clear that modification of the existing theories is required so that the proper line shape can be obtained and thence used to extract the mass and relaxation rate.

Drew[11] has generated a theory of cyclotron resonance in this frequency regime using a variational form for the surface impedance.[12] The key assumption is the same used in the Chambers[4] theory of microwave cyclotron resonance, namely, that the

number of resonant electrons is few. This theory has been used successfully to account for line-shape changes in Ga by Kamgar, Henningsen, and Koch,[13] and Henningsen.[14] Despite the fact that the assumption of relatively few resonating electrons is not valid in potassium, attempts were made to use this theory to understand the line shape observed for potassium at frequencies between 29 and 58 cm^{-1}.[6] Although it proved useful in extracting a relaxation rate, the failure to account for the complete line shape leaves a certain amount of uncertainty with regard to the actual values of electron mass and relaxation rate at these frequencies. Since then experiments have been performed in Na (these will be reported in detail at a later date),[15] which reveal the same discrepancies as seen in K and point to the incompleteness of the theory of cyclotron resonance in real metals in the far infrared.

This paper presents another theory of cyclotron resonance in the retardation regime. It differs from Drew's[11] calculation by considering the possibility of real electromagnetic wave propagation, which has been seen to be intimately connected with Azbel'-Kaner cyclotron resonance in metals like K and Na, where all bulk electrons have the same electron mass.[16–19] We consider in this paper only resonances for the incident field parallel to the applied magnetic field. In this polarization only ordinary waves are excited. The perpendicular polarization, which is slightly more complicated owing to the presence of the Hall field and two types of wave propagation, will be discussed in a subsequent publication. Although the theory is not rigorous, it reproduces in some detail the line shape observed for the ordinary polarization in both K and Na, leaving only the mass and relaxation rate as adjustable parameters.

II. THEORY

We consider a semi-infinite metal in a uniform magnetic field parallel to the sample surface (Fig. 1). The metal has a simple spherical Fermi surface and all the electrons can be characterized by a single mass m^* and relaxation rate $1/\tau$. The surface impedance is found by solving Maxwell's equations for the electric field distribution in the metal. For $\vec{E}$ parallel to $\vec{H}_0$, the applied field, we have

$$\frac{\partial^2 E(x)}{\partial x^2} = i\omega\mu_0 J(x) \quad , \tag{1}$$

where x is measured from the surface, ω is the frequency of the incident radiation, and μ_0 is the permeability of free space (mks units). E and J are the components of the electric field and current along the field. None other need be considered in this polarization.

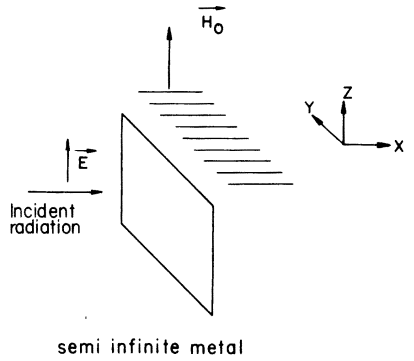

FIG. 1. Semi-infinite metal in an applied field.

J is of course related to E by the nonlocal conductivity σ,

$$J(x) = \int_0^\infty \sigma(x, x')\, E(x')\, dx' \quad , \tag{2}$$

and Eq. (1) becomes

$$\frac{\partial^2 E(x)}{\partial x^2} = i\omega\mu_0 \int_0^\infty \sigma(x, x')\, E(x')\, dx' \quad . \tag{3}$$

(Throughout σ stands for σ_{zz}, the only relevant component of the conductivity tensor.) The solution for $E(x)$ immediately yields the surface impedance by[20]

$$Z = -i\omega\mu_0 E(0^+)/E'(0^+) \quad . \tag{4}$$

In the infinite medium we have $\sigma(x, x') = \sigma(x - x')$ and the lower limit in the integration in (3) extended to $-\infty$. In this case (3) can be readily solved by Fourier-transform techniques to yield the infinite-medium dispersion relations for the high-frequency cyclotron waves with ordinary polarization.[16–19] In particular, if

$$E(x) = Ae^{-ikx} \quad , \tag{5}$$

then k must satisfy

$$-k^2 = i\omega\mu_0\sigma(k) \quad , \tag{6}$$

where

$$\sigma(k) = \frac{3}{2}\frac{Ne^2}{m^*\omega_c}\int_0^\pi d\theta \sin\theta\cos^2\theta \times \sum_{n=-\infty} \frac{\mathcal{J}_n(kR_c\sin\theta)^2}{i(\omega/\omega_c - n) + 1/\omega_c\tau} \quad . \tag{7}$$

In (7), N is the electron density, ω_c the cyclotron frequency equal to eB/m^*, e the electron charge, R_c the maximum cyclotron radius equal to V_F/ω_c, and $\mathcal{J}_n$ are nth order Bessel functions of the first kind. The solution of (6) for the allowed k values using (7) for $\sigma(k)$ is given by Walsh and Platzman,[16,17] Platzman, Walsh, and E-Ni Foo,[18] Dunifer, Schmidt, and Walsh,[19] and E-Ni Foo.[21]

In the presence of a surface, the solution of (3) is more difficult. Mathematically speaking, in the presence of the surface, $\sigma(x,x') \neq \sigma(x-x')$, and the reduction of (3) by Fourier transform to a simple algebraic equation for possible waves, as in (6), is no longer possible. Physically, if we are within $2R_c$ of the surface, electrons on trajectories such as B in Fig. 2 strike the surface and do not contribute to the resonant conductivity. In fact, as one approaches the surface, the fraction of electrons that contribute to the conductivity in a resonant manner becomes vanishingly small. The fraction of resonant electrons is given by

$$\frac{N_r}{N} = \begin{cases} \frac{3}{2}[(x/2R_c) - \frac{1}{3}(x/2R_c)^3], & x \leq 2R_c \\ 1, & x \geq 2R_c \end{cases} \tag{8}$$

and shown in Fig. 3. Stated a third way, the propagating modes of the infinite medium do not exist as such in the surface region defined by $0 < x < 2R_c$. They are damped and scattered by the presence of the surface. (It should be made clear that it is of little importance whether the electrons that strike the surface are diffusely or specularly scattered; neither are resonant at $\omega = n\omega_c$.)

Solutions of (3) and (4) for the resonant part of the surface impedance have been obtained in a number of physically interesting situations. The original work of Azbel' and Kaner[2] found an approximate solution in the limit that $R_c/\delta \gg n^2$, where δ is the skin depth and n is the subharmonic index, $n \approx \omega/\omega_c$. Hartman and Luttinger[3] found an exact solution in this limit. Chambers[4] considered the case where the number of resonant electrons is small even in the infinite medium. This is tantamount to assuming no cyclotron wave propagation in the bulk. All of the above solutions assume that $R_c/\delta \gg n^2$; that is to say, the infinite-k solution to the nonlocal conductivity dominates the surface impedance.

The opposite limit $R_c/\delta \lesssim n^2$, the so-called retardation regime, is always obtained in far-infrared cyclotron resonance in metals. The resonant contribution to Z in the case where the number of bulk resonant electrons is small compared with the nonresonant electrons has been obtained by Drew[11] by means of a variational[12] approach. This corresponds to the Chambers limit at microwave frequencies where $R_c/\delta \gg n^2$. More complicated Fermi surfaces have been considered by Henningsen[14] using this approach. Cyclotron resonance in the retardation regime, $R_c/\delta \leq n^2$, for the case where real wave propagation can occur in the bulk is the case under consideration here.

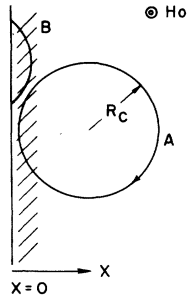

FIG. 2. Resonant and nonresonant trajectories near the surface.

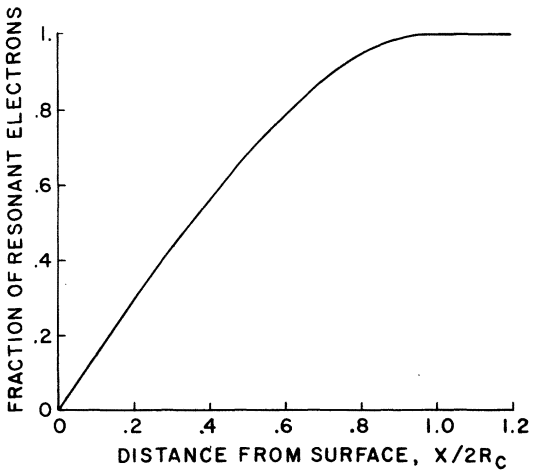

FIG. 3. Fraction of resonant electrons versus distance from the surface for a spherical Fermi surface.

The first approximation we make is that the electric field can be separated into a nonresonant field that decays rapidly as we leave the surface and a resonant field which is excited near cyclotron resonance or at a subharmonic resonance,

$$E(x) = E^0(x) + E^r(x),$$

where $E^0(x)$ is the nonresonant field and $E^r(x)$ is the resonant contribution to the field. That one can make such a separation is not obvious and may in fact not be mathematically correct. However, such a separation does seem reasonable on physical grounds. Experimentally one observes small fractional changes in surface impedance near resonance in K and Na in the far-infrared,[5,15] despite the fact that all *bulk* electrons resonate at the same frequency. This implies that the resonant electric field is small compared to the nonresonant field at the *surface*. This does not, however, mean that at distances large compared to the skin depth the resonant field remains small compared to the nonresonant field. In fact, if the resonant fields were much smaller than the nonresonant field at all distances from the surface, the variational approach formulated by Drew[11] would have been successful in explaining the cyclotron-resonance line shape in potassium.[5]

Further, we separate the nonlocal conductivity

into resonant and nonresonant parts,

$$\sigma(x,x') = \sigma^0(x,x') + \sigma_n^r(x,x') .$$

In order to be more precise, we must first specify that we are concerned with say the nth subharmonic. Then the resonant part is that part that is singular at $\omega = n\omega_c$ and the nonresonant part is not. We make this point for the following reason: $\sigma^0(x,x')$ obviously includes the contribution from all electrons that strike the surface, but those electrons that do not strike the surface also make contributions to the conductivity that are not singular at $\omega = n\omega_c$ and are included in $\sigma^0(x,x')$.

To be more specific, we write down the Fourier transform of the conductivity from all electrons that do not hit the surface (see Appendix B, Ref. 5):

$$\sigma^r(k,k') = \frac{3}{2}\frac{Ne^2}{m\omega_c}\int_0^\pi d\theta \sin\theta\cos^2\theta \sum_n \frac{g_n(kR_c\sin\theta)^2}{i(\omega/\omega_c - n) + 1/\omega_c\tau}\,\delta(k-k')$$

$$-\frac{3}{2}\frac{Ne^2}{m\omega_c}\int_0^\pi d\theta \sin\theta\cos^2\theta \sum_n \frac{g_n(kR_c\sin\theta)g_n(k'R_c\sin\theta)}{i(\omega/\omega_c - n) + 1/\omega_c\tau}\,\frac{\sin(k'-k)R_c\sin\theta}{\pi(k'-k)} . \tag{9}$$

The conductivity is a sum over terms each resonant at a different subharmonic n. $\sigma_n^r(k,k')$ and its corresponding transform $\sigma_n^r(x,x')$ is the nth term in the sum. The remaining terms are included in $\sigma(x,x')$ along with the conductivity caused by electrons that hit the surface.

Having defined how we partition the fields and nonlocal conductivity into resonant and nonresonant parts, we can rewrite the wave equation (3) as follows:

$$\frac{\partial^2 E^0(x)}{\partial x^2} + \frac{\partial^2 E^r(x)}{\partial x^2} = i\omega\mu_0\int_0^\infty dx'[\sigma^0(x,x') + \sigma_n^r(x,x')] \times [E^0(x') + E^r(x')] . \tag{10}$$

We separate this equation into two by isolating those terms that are singular at $\omega = n\omega_c$ from those that are not. In fact we assume that the solution of Eq. (10) can be found by simultaneously satisfying the following two equations:

$$\frac{\partial^2 E^0(x)}{\partial x^2} = i\omega\mu_0\int_0^\infty dx'\,\sigma^0(x,x')E^0(x') , \tag{11}$$

$$\frac{\partial^2 E^r(x)}{\partial x^2} - i\omega\mu_0\int_0^\infty dx'\,\sigma(x,x')E^r(x') = i\omega\mu_0\int_0^\infty dx'\,\sigma_n^r(x,x')E^0(x') . \tag{12}$$

Clearly if (11) and (12) are both satisfied, then so will (10). If our separation of the electric field into a nonresonant and resonant part is justified, then (11) is an adequate prescription of it. If we were to include any other terms, $E^0(x)$ would be singular at $\omega = n\omega_c$, which we have explicitly assumed is not the case. If the nonresonant field $E^0(x)$ is given by (11), then (12) follows directly. In summary the separation of Eq. (10) into Eqs. (11) and (12) follows from our assumption of the separability of conductivity and electric field into nonresonance and resonant parts.

The solution of (11) simply gives us the spatial dependence of the nonresonant field. We assume for simplicity that it is an exponential,

$$E^0(x) = Ae^{-\alpha x} , \tag{13}$$

where α is a complex constant given by the $H_0 = 0$ surface impedance Z_0,

$$\alpha = i\omega\mu_0/Z_0 . \tag{14}$$

This approximation is least appropriate in the anomalous-skin-effect regime, i.e., at microwave frequencies and at low temperatures. At far-infrared frequencies, however, we are returning to the classical limit and the exponential approximation is more satisfactory.[20]

The second equation, (12), is a wavelike equation for $E^r(x)$, driven by a resonance current

$$J_n(x) = \int_0^\infty dx'\,\sigma_n^r(x,x')E^0(x') . \tag{15}$$

The resonant current is produced by the resonant nonlocal conductivity and the nonresonant exponential field. On the left-hand side of (12) σ is the total conductivity.

We find an approximate solution to (12) in the following manner: To facilitate a Fourier transform of (12) we first extend the metal to $(-\infty)$ but keep a diffusely scattering boundary at $x = 0$. No current is carried across the boundary; i.e., $\sigma(x,x') = 0$ for $x > 0$, $x' < 0$ or $x < 0$, $x' > 0$. The driving current $J_n(x)$ in (15) exists only for $x > 0$ so this does not alter the exact solution for $E^r(x)$. Then $\sigma(-x,-x')$ may be taken equal to $\sigma(x,x')$. [This is true for the ordinary polarization where we consider only one component of the conductivity tensor $\sigma_{zz}(x,x')$. In a magnetic field the system is invariant with respect to a twofold rotation about the z

axis sending $x \to -x$, $x' \to -x'$. Therefore, $\sigma_{zz}(x,x') = \sigma_{zz}(-x,-x')$.]

Equation (12) is rewritten as

$$\frac{\partial^2 E^r(x)}{\partial x^2} - i\omega\mu_0 \int_{-\infty}^{\infty} dx'\, \sigma(x,x') E^r(x') = i\omega\mu_0 J_n(x) \tag{16}$$

and its Fourier transform

$$-k^2 \mathcal{E}^r(k) - i\omega\mu_0 \int_{-\infty}^{+\infty} dk'\, \sigma(k,k')\, \mathcal{E}^r(k') = i\omega\mu_0 J_n(k) \quad . \tag{17}$$

$J_n(k)$ can be found exactly:

$$J_n(k) = \frac{j_n(k)}{i(\omega/\omega_c - n) + 1/\omega_c\tau} \quad , \tag{18}$$

where

$$j_n(k) = \frac{3}{4\pi} \frac{Ne^2}{m^*\omega_c} \frac{1}{(\alpha - ik)} \times \int_0^{\pi} d\theta \sin\theta \cos^2\theta\, \mathcal{J}_n(-i\alpha R) e^{-\alpha R}\, \mathcal{J}_n(kR) e^{ikR} . \tag{19}$$

Finally we reduce (17) to an algebraic equation by using an approximation due to Chambers.[22] In the large-k limit Chambers argues that, to good approximation, we find an effective nonlocal conductivity $\sigma_{\text{eff}}(k)$ such that

$$\int_{-\infty}^{+\infty} dk'\, \sigma(k,k')\, \mathcal{E}^r(k') \approx \sigma_{\text{eff}}(k)\, \mathcal{E}^r(k) \quad . \tag{20}$$

This is tantamount to replacing the surface region or the region near the diffuse boundary in Eq. (16) by a uniform region with nonlocal conductivity $\sigma_{\text{eff}}(k)$. In effect, the lack of translational invariance is ignored by removing the spatial perturbation of the surface. In place of the surface or diffuse boundary one writes down a nonlocal conductivity that treats in some approximate way the fact that some electrons strike the surface while others do not. *Equation* (17) *is thus reduced to finding the electromagnetic wave* $\mathcal{E}^r(k)$ *that is excited in an infinite medium by the resonant current* $J_n(k)$ *where the conductivity is not the bulk conductivity but a conductivity approximating the surface region.*

This should be a good approximation in the large-kR_c limit. The spatial perturbation of the conductivity, due to the surface, extends out to $2R_c$ (Fig. 3). For a wavelike solution to exist at all, then, we must have wavelengths sufficiently short that $kR_c \gg 1$. That is to say, the spatial variations that can be described by an effective conductivity must have a length scale much shorter than the length scale of the variation in the conductivity, $2R_c$. The wavelength of the relevant excitations may be taken as $1/\delta$, where δ is the skin depth. In the far-infrared, in the alkali metals, in magnetic fields of ~ 100 kG, $R_c/\delta \sim kR_c > 10$. Chambers originally used the approximation (20) for $kR_c \to \infty$, although it is not exact even in this limit. In these calculations we have used this approximation at large but no infinite values of kR_c.

The form of $\sigma_{\text{eff}}(k)$ that is suggested by Chambers[22] is

$$\sigma_{\text{eff}}(k) = \tfrac{1}{2}\left[\sigma^{\infty}_{H=0}(k) + \sigma^{\infty}_{H}(k)\right] \quad . \tag{21}$$

Chambers[22] argues that at a point x in space, Fig. 4(a), the large-k conductivity is produced by two types of orbits indicated by A and B. Those orbits that are moving through the point x at an angle to the phase front see a rapidly fluctuating field and contribute little current at x, whereas the orbits nearly tangential to the phase front make a larger contribution. If we place a surface to the left of x and sufficiently close that it cuts off the extremal orbit B, it is clear that one-half of the resonant conductivity is removed. Since the resonance in the conductivity is produced by electrons returning to the point x with phase information, Chambers further argues that the contribution to the large-k conductivity by those electrons that do strike the surface is approximately one-half the infinite medium conductivity in the absence of the field, hence the form of $\sigma_{\text{eff}}(k)$ given in (21).

For $\sigma^{\infty}_{H}(k)$, we take Eq. (7). For $\sigma^{\infty}_{H=0}$ we take the usual form for $\sigma(k,\omega)_{H=0}$ given by the Boltzmann equation[22]:

$$\sigma^{\infty}_{H=0}(k) = -i\frac{3}{4}\frac{Ne^2}{m^*\omega}\left(\frac{[(kR)^2-1]}{|(kR)|^3}\ln\left|\frac{1+kR}{1-kR}\right| + \frac{2}{(kR)^2}\right) + \frac{3\pi}{4}\frac{Ne^2}{m^*\omega}\frac{[(kR)^2-1]}{|(kR)|^3} H(kR^{-1}) \quad , \tag{22}$$

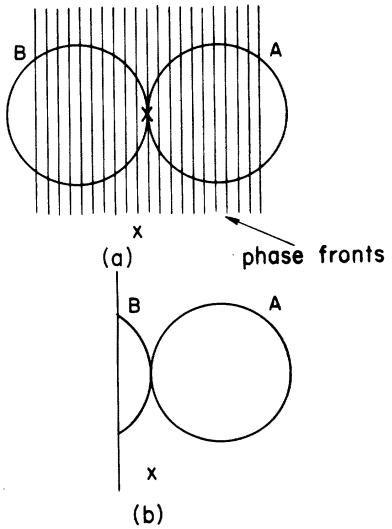

FIG. 4. (a) Two types of orbit that contribute most strongly to the conductivity at x in the short-wavelength limit. (b) Near the surface one of these collides with the surface; the other does not.

where $R = v_F/\omega$. For $\sigma^{\infty}_{H=0}(k)$ we have taken the $\omega\tau \to \infty$ limit, since the finite scattering rate is of little consequence for the background, nonresonant, conductivity.

Using (20) and (21), Eq. (13) becomes

$$[-k^2 - i\omega\mu_0\sigma_{eff}(k)]\mathcal{E}^r(k) = i\omega\mu_0 J_n(k) \tag{23}$$

and

$$\mathcal{E}^r(k) = \frac{i\omega\mu_0 J_n(k)}{k^2 + i\omega\mu_0\sigma_{eff}(k)} \quad . \tag{24}$$

The total electric field at $x = 0^+$ is

$$E(0^+) = E^0(0^+) + E^r(0^+) = E^0(0^+) + \int_{-\infty}^{+\infty} dk\, \mathcal{E}^r(k)$$

or

$$E(0^+) = E^0(0^+) - i\omega\mu_0 \int_{-\infty}^{+\infty} \frac{dk\, J_n(k)}{k^2 + i\omega\mu_0\sigma_{eff}(k)} \quad . \tag{25}$$

Correspondingly, the derivative at the surface is given by

$$E'(0^+) = E^{0\prime}(0^+) + i\omega\mu_0 \int_{-\infty}^{+\infty} \frac{dk\,(+ik) J_n(k)}{k^2 + i\omega\mu_0\sigma_{eff}(k)} \quad . \tag{26}$$

Using (4) for the surface impedance, we have

$$Z = -i\omega\mu_0 \left[\left(1 - i\omega\mu_0 \int_0^{\infty} \frac{dk\,[J_n(k) + J_n(-k)]}{k^2 + i\omega\mu_0\sigma_{eff}(k)} \right) \Big/ \left(-\alpha + i\omega\mu_0 \int_0^{\infty} \frac{dk\,(ik)[J_n(k) - J_n(-k)]}{k^2 + i\omega\mu_0\sigma_{eff}(k)} \right) \right] \quad . \tag{27}$$

If we assume, as is the case, that the resonant fields near the surface are small compared with the nonresonant fields, we can write for the change in Z at the nth resonance

$$\frac{\delta Z}{Z} = -i\omega\mu_0 \int_0^{\infty} \frac{dk\,[J_n(k) + J_n(-k)]}{k^2 + i\omega\mu_0\sigma_{eff}(k)} + i\omega\mu_0 \int_0^{\infty} dk \left(\frac{ik}{\alpha}\right) \frac{J_n(k) - J_n(-k)}{k^2 + i\omega\mu_0\sigma_{eff}(k)} \quad . \tag{28}$$

It should be made clear, in order to assume that the resonant fields at the surface remain small compared with the nonresonant fields, we cannot allow arbitrarily large $\omega_c\tau$. It appears unlikely, however, that in the far-infrared values of $\omega_c\tau$ will be found that invalidate this assumption. At least for K and Na, the examples used here, the values of $\omega_c\tau$ measured experimentally allow us to make this assumption.

Before using (28) to calculate numerically the resonant line shape for potassium and sodium, we would like to relate (28) to the variational solution found by Drew.[11] If we assume that the number of resonant electrons is small compared with the number of nonresonant electrons even in the bulk, then we may take for $\sigma_{eff}(k)$

$$\sigma_{eff}(k) = \sigma^{\infty}_{H=0}(k) \quad . \tag{29}$$

That is to say, we ignore all wave propagation effects. Within the spirit of our approximation, that the nonresonant field was exponential, we can replace

$$i\omega\mu_0\sigma^{\infty}_{H=0}(k) = \alpha^2 \quad . \tag{30}$$

Making this substitution in (28) and transforming back into real space, we obtain

$$\delta Z = -Z_0^2 \int_0^{\infty} dx \int_0^{\infty} dx'\, e^{-\alpha x}\sigma_r(x, x')e^{-\alpha x'} \quad , \tag{31}$$

which is Drew's result.[11]

It is also interesting to point out that (24) is similar to the result one obtains if one assumes the pseudospecular boundary condition. Then we have[22]

$$Z = \frac{i\omega\mu_0}{\pi} \int_{-\infty}^{+\infty} \frac{dk}{k^2 + i\omega\mu_0\sigma^{\infty}_H(k)} \quad . \tag{32}$$

The differences between (32) and (28) are instructive. Replacing $\sigma^{\infty}_H(k)$ in (32) by $\sigma_{eff}(k)$ accounts for the fact that the waves that are excited near the surface are only vestigially related to the waves that can in fact propagate at distances greater than $2R_c$. They are less dispersive and are damped by the presence of electrons scattered by the surface. Also substituting the large-k limit of $\sigma_{eff}(k)$ into (32) one obtains precisely the Azbel'-Kaner result.[22,23] It can be seen then that (28) differs from the usual Azbel'-Kaner result in the following way: Substituting the full $\sigma_{eff}(k)$, rather than its infinite-k limit, makes the solution sensitive to the finite-k behavior of the dispersion relations. Second, we weigh the waves excited at resonance by the current $J_n(k)$, which will have the effect of diminishing the strength off the resonance due to retardation effects. This manifests itself by the rapid decay of the subharmonic structure.

III. COMPARISON WITH EXPERIMENT

To numerically compute $\delta Z/Z_0$ from (28), we are forced to make two more approximations. First we write down an analytic approximation to $j_n(k)$, which is valid for $k > 2n/R_c$:

$$j_n(k) \approx \frac{3}{4\pi} \frac{Ne^2}{m^*\omega_c} \frac{1}{(\alpha - ik)} \frac{e^{i\pi/4}}{(2\pi)^{1/2}(\alpha k)^{1/2}R_c} \times \frac{e^{-n^2/2\alpha R_c - in^2/2kR_c}}{(2.08 + n^2/2\alpha R_c + in^2/2kR_c)^{3/2}} \quad . \tag{33}$$

This approximation is derived in the Appendix and compared with the exact expression, Eq. (19).

Since (33) is only valid for $k > 2n/R_c$, we truncate the integration in (28) and impose the lower limit $k = 2n/R_c$ rather than carrying the integration all the way to zero. The assumption that the dominant contribution to the resonance came from the high-k part of (28) is, however, implicit in the use of (20) to solve the integral equation (17). (It is generally correct in discussing the surface impedance of a semi-infinite metal that the short-wavelength behavior at the surface is most relevant. This is quite different from transmission or two-sided experiments, which are intrinsically sensitive to the long-wavelength behavior in the bulk.[18,24])

The only adjustable parameters we have to fit the line shape are the mass m^* and relaxation rate $1/\tau$, which we leave as fitting parameters. The skin depth $1/\alpha$ we calculate from the $H = 0$ surface impedance Z_0, which can be calculated by either Dingle[25] expansions or by using a two-exponential variational calculation provided by Baraff.[26] We use the latter.

In Fig. 5 we show the results of the calculation for K at 29.69 cm^{-1}. At the top of the figure is shown the infinite-medium dispersion relations

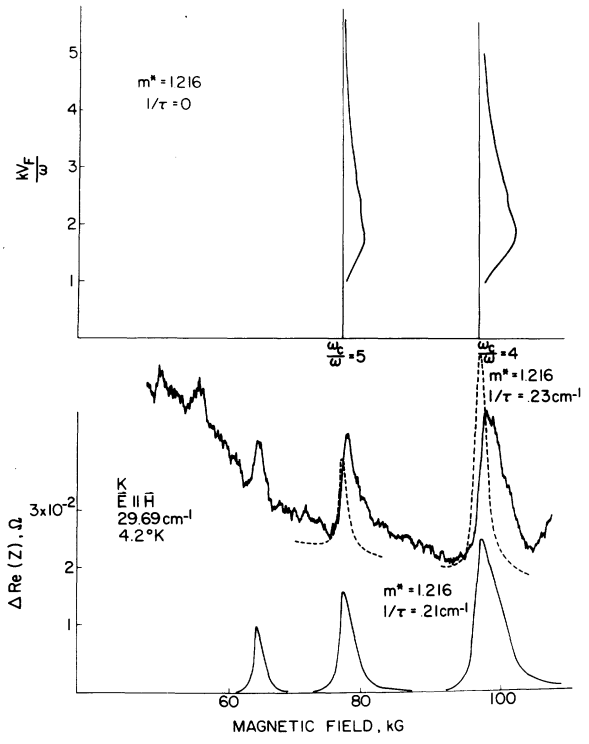

FIG. 5. Cyclotron resonance in potassium for $\vec{E} \parallel \vec{H}$ at 29.69 cm^{-1}. Top figure shows infinite medium dispersion relations for high-frequency cyclotron waves. Solid line at middle is experimental trace. Result of Eq. (24) on lower figure. Dashed curve, variational result from Ref. 5.

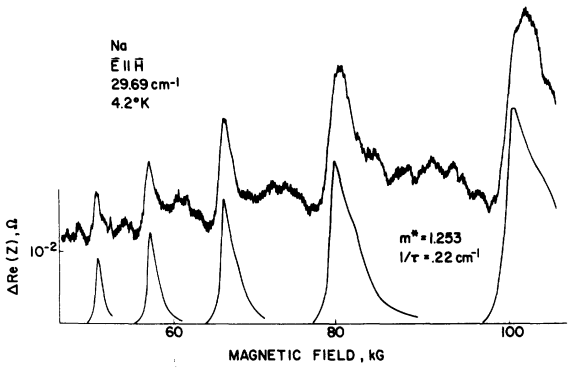

FIG. 6. Cyclotron resonance in sodium for $\vec{E} \parallel \vec{H}$ at 29.69 cm^{-1}. Top figure experiment; lower figure calculation.

which are discussed in Ref. 5. The solid line at the middle of the figure is the experimental result. (The reader should ignore the base-line drift. In particular the upward motion above 100 kG is not real.) The calculation given by (24) with the above-mentioned approximation for $j_n(k)$ and truncation of the integral is shown in the lower part. The dashed curve superimposed on the experimental trace is the result of the earlier calculation[5] based on the variational result.[11] It fails in two regards. There is no absorption on the high-field side of cyclotron resonance as is seen experimentally. Second, although it gives a reduction in the subharmonic structure as one goes to lower fields, the rate of decay is too fast compared with experiment. In comparison, the results of the present calculation give the observed line shape, especially with regard to the absorption on the high-field side of each subharmonic, and also give the experimentally observed rate of decay of the subharmonic structure. We have fit the lower curve to experiment with a mass of 1.216 and a relaxation rate of $1/\tau = 0.21$ cm^{-1}. In a previous publication[5] we fit just the leading edge of the subharmonic resonance with the variational calculation due to Drew.[11] This required a 0.5% larger mass and substantially the same relaxation rate. It appears that the more exact calculation gives the line position and shape with the microwave value of the mass[24] and a relaxation rate dominated by electron-phonon scattering.[5]

A similar calculation is shown for Na[17] in Fig. 6. We have fit the leading edge and position to a mass value of 1.253 and a relaxation rate of $1/\tau = 0.22$ cm^{-1}. As can be seen, the rate of decay of the subharmonic structure and the absorption on the high-field side of each cyclotron resonance are reproduced in a quantitative way. There appears to be a departure from experiment, however, on the low-field side of resonance. It is not clear what this rather modest failure is due to. Since it is notice-

TABLE I. Cyclotron resonance in metals.

N_r/N \ R_c/δ	$>n^2$	$<n^2$
≈ 1	Azbel-Kaner limit	
$\ll 1$	Chambers limit	Variational calculation (Drew)

ably stronger for Na than K, it may signal a failure of the approximation of the nonresonant field by an exponential. This approximation is worse in Na than K. Although less likely, it may be some failure in our expression for $\sigma_H^\infty(k)$, which is used in $\sigma_{eff}(k)$, due to departures from the one-electron solution to the conductivity caused by Fermi-liquid effects.

SUMMARY

Although this derivation of Azbel'-Kaner cyclotron resonance in the retardation regime is hardly rigorous, the assumptions used to arrive at the final answer appear to be physically quite reasonable. The relationship of this approach to previous solutions of the Azbel'-Kaner problem is best seen by referring to Table I. We can characterize the type of resonance seen by two parameters. The first is the fraction of resonant electrons, N_r/N, in the infinite medium. For a large fraction of resonant electrons we must allow the possibility of real wave propagation in the bulk; for a small fraction, no wave propagation occurs and the resonance may be described by resonances in the single-particle currents, near the surface. Second, we must determine whether the resonance is retarded or not. As discussed previously,[5,11,13,14] if the electron escapes the skin depth before the incident field changes phase, we may consider the usual cyclotron resonance,[2-4] but if this does not hold, the resonance is retarded.[5,11,13,14] For circular pieces of Fermi surface the resonance is retarded or not as $R_c/\delta < n^2$ or $R_c/\delta > n^2$, respectively. The theory constructed here describes resonance for $N_r/N \simeq 1$, in the retardation regime $R_c/\delta < n^2$. In the retardation regime it is apparent that cyclotron resonance may be described as the excitation of high-frequency cyclotron waves by resonant currents generated by the dominant, rapidly decaying nonresonant field. The high-frequency cyclotron waves near the surface are related to the bulk waves, but have reduced dispersion and an increased damping rate produced by the electrons that strike the surface. The retardation effect manifests itself by reducing the strength of the current exciting the high-frequency cyclotron wave as one goes to high subharmonics.

In a future publication we consider the problem for the extraordinary polarization, $\vec{E}$ incident perpendicular to $\vec{H}_0$. The problem is slightly more complicated owing to the presence of the Hall field and the possibility of longitudinal as well as transverse wave propagation. The problem, however, should be amenable to the same basic approach used here.

ACKNOWLEDGMENTS

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APPENDIX

We wish to find an approximate but analytic form for the resonant current, Eq. (19),

$$j_n(k) = \frac{3}{4\pi}\frac{Ne^2}{m^*\omega_c}\frac{1}{\alpha - ik}\int_0^\pi d\theta \sin\theta \cos^2\theta \times g_n(-i\alpha R)e^{-\alpha R} g_n(kR)e^{ikR} \quad , \tag{A1}$$

where $R = R_c \sin\theta$. By using the method of stationary phase[27] one can find an asymptotic form for $g_n(z)$ for large z:

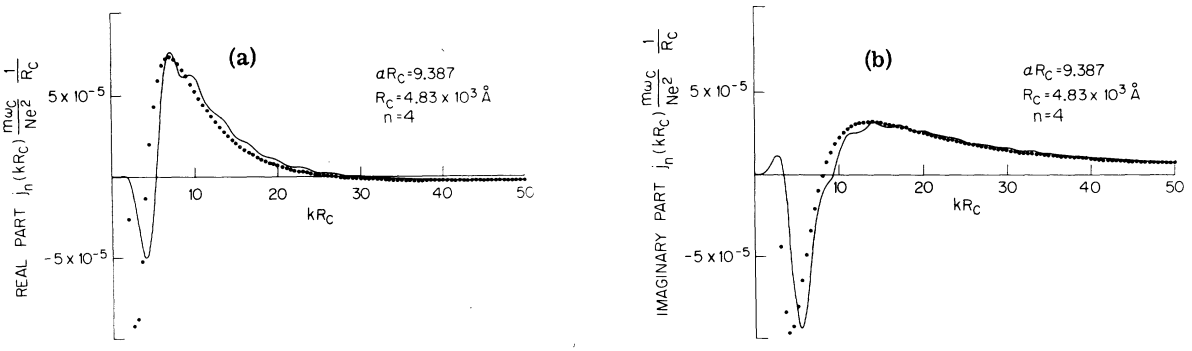

FIG. 7. (a) Real resonant current for real skin depth. Solid curve Eq. (19); filled circles, approximation given by Eq. (33). (b) Imaginary resonant current for real skin depth. Solid curve Eq. (19); filled circles, approximation given by Eq. (33).

$$g_n(z) \sim \frac{1}{2\pi}[e^{iz}e^{-in\pi/2}(2\pi/z)^{1/2}e^{-i\pi/4}e^{in^2/2z} + e^{-iz}e^{in\pi/2}(2\pi/z)^{1/2}e^{i\pi/4}e^{-in^2/2z}] \quad . \quad \text{(A2)}$$

In the integral (A1) the Bessel function $g_n(z)$ is multiplied by $e^{\pm iz}$.

$$g_n(z)e^{iz} \sim (1/2\pi)[e^{2iz}e^{-in\pi/2}(2\pi/z)^{1/2}e^{-i\pi/4}e^{in^2/2z} + e^{in\pi/2}(2\pi/z)^{1/2}e^{i\pi/4}e^{-in^2/2z}] \quad . \quad \text{(A3)}$$

The first term is rapidly oscillating for large z and hence makes no contribution in the integral (A1). A similar argument for $g_n(i\alpha R)e^{-\alpha R}$ gives the following for the important part of the integrand:

$$g_n(-i\alpha R)e^{-\alpha R}g_n(kR)e^{ikR} \sim \frac{1}{2\pi}\frac{1}{(\alpha R)^{1/2}}\frac{e^{i\pi/4}}{R_c\sin\theta}e^{-n^2/2\alpha R}e^{-in^2/2kR} \quad . \quad \text{(A4)}$$

The integral then becomes

$$j_n(k) = \frac{3}{4\pi}\frac{Ne^2}{m^*\omega_c}\frac{1}{(\alpha - ik)}\frac{1}{(2\pi)}\frac{1}{R_c}\frac{1}{(\alpha k)^{1/2}} \times \int_0^{\pi} d\theta \cos^2\theta\, e^{-n^2/2\alpha R}e^{-in^2/2kR} \quad . \quad \text{(A5)}$$

By substituting $v = 1/\sin\theta - 1$ we can rewrite the integral in (A5) as

$$2\int_0^{\infty}\frac{dv}{(v+1)^3}(v^2+2v)^{1/2}e^{(-n^2/2\alpha R_c - in^2/2kR_c)v} \quad . \quad \text{(A6)}$$

The important region of the integral is $v \lesssim 1$ and we can approximate quite accurately

$$\frac{(v^2+2v)^{1/2}}{(v+1)^3} \approx \sqrt{2}\,v e^{-2.08v} \quad . \quad \text{(A7)}$$

The integral can now be quickly done giving for $j_n(k)$ the following form:

$$j_n(k) = \frac{3}{4\pi}\frac{Ne^2}{m^*\omega_c}\frac{1}{(\alpha - ik)}\frac{e^{i\pi/4}}{(2\pi)^{1/2}(\alpha k)^{1/2}}\frac{1}{R_c} \times \frac{e^{-n^2/2\alpha R_c}e^{-in^2/2kR_c}}{(2.08 + n^2/2\alpha R_c + in^2/2kR_c)^{3/2}} \quad , \quad \text{(A8)}$$

which is used in Eq. (33) in the main text.

A comparison between the approximate result and the exact integral is shown in Fig. 7. A real $\alpha = 1.94\times10^7\ \mathrm{m}^{-1}$ has been assumed for the purposes of this figure. As can be seen the approximation is quite good for the range $kR_c > 2n$, for both real and imaginary parts.

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