

Electrical resistivity of potassium at low temperatures

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The low-temperature electrical resistivity of potassium is calculated and compared with the recent, highly accurate measurements in the range 2–20°K. Excellent agreement with experiment is obtained. It is found that explicitly including phonon drag accounts precisely for the exponential temperature dependence observed for the resistivity below 4°K. We show how to determine the screened electron-ion interaction matrix elements, crucial for obtaining quantitative agreement with experiment for a resistivity calculation in the low-temperature regime. A discussion is given showing in detail how the present calculated results for potassium are in quantitative agreement with the general theory of the low-temperature resistivity of the alkali metals.

I. INTRODUCTION

One of the long standing challenges of the theory of metals has been to calculate quantitatively the temperature-dependent part of the dc electrical resistivity $\rho(T)$ of the alkali metals. Of particular interest is the low-temperature regime, where $\rho(T)$ exhibits an extremely strong temperature dependence, increasing by orders of magnitude over a relatively short range of temperature. A detailed account of the early calculations of $\rho(T)$, up to about 1960, is given in the excellent treatise by Ziman.¹ Bailyn² has clearly demonstrated that these early calculations of $\rho(T)$ could not hope to obtain quantitative agreement with experiment because they did not take account of the highly anisotropic phonon spectrum that is characteristic of the alkali metals. Within the last two years, improved calculations^{3–5} of $\rho(T)$ in the very-low-temperature regime have been reported for potassium, which took proper account of the anisotropy of the phonon spectrum. Although these recent calculations constituted a very significant improvement over previous work, quantitative agreement with experiment^{6,7} was still lacking. Moreover, the calculations failed to account for the observed⁷ exponential decay of $\rho(T)$ below 4°K.

In this paper, we present the details of a calculation of $\rho(T)$ which leads to excellent quantitative agreement with the recent low-temperature measurements.^{6,7} The discrepancy between theory and experiment never exceeds 10% over the temperature range 2–20°K. The quality of this agreement can be appreciated by noting that over this temperature range, $\rho(T)$ increases by *five* orders of magnitude.

The principal new features of our calculation are the explicit inclusion of phonon drag,⁸ which explains the exponential temperature dependence of $\rho(T)$ below 4°K, and our analysis of the screened, electron-ion interaction matrix element. As explained in the preceding paper,⁹ the calculation we

present here is valid only for an alkali metal because of the breakdown of Matthiessen's rule for the polyvalent and noble metals. The reason we chose potassium as the alkali metal to study in detail is that only for potassium are there detailed low-temperature measurements^{6,7} having the high accuracy needed for a careful quantitative comparison between theory and experiment. Moreover, unlike Na and Li, K does not undergo a low-temperature martensitic phase transformation, which would prevent a clear interpretation of the data.

In Sec. II, we present the expression for the resistivity, including the phonon-drag contribution. In Sec. III, we show how to determine the screened, electron-ion interaction matrix elements. The results of our calculation are presented in Sec. IV and discussed in Sec. V. The summary follows in Sec. VI.

II. EXPRESSION FOR $\rho(T)$

The derivation of the expression for $\rho(T)$ for an alkali metal, obtained from the variational solution of the Boltzmann equation, is discussed in detail in the preceding paper,⁹ and we quote here the result. We separate $\rho(T)$ as follows

$$\rho(T) = \rho_{\text{eq}}(T) - \rho_{\text{e}}(T) = \rho_{\text{eq}}(T)[1 - X(T)], \quad (2.1)$$

where $\rho_{\text{eq}}(T)$ is the resistivity without phonon drag and $\rho_{\text{e}}(T)$ is the contribution due to phonon drag, conveniently written as $X(T) \equiv \rho_{\text{e}}(T)/\rho_{\text{eq}}(T)$. The explicit expression for $\rho_{\text{eq}}(T)$, valid only for the alkali metals, is given by

$$\rho_{\text{eq}}(T) = \frac{m^2}{12\pi^3 e^2 \hbar^3 n^3} \int_0^{2k_F} dk k^3 v^2(k) S(k), \quad (2.2)$$

$$S(k) = (4\pi)^{-1} \int d\Omega_{\mathbf{k}} s(\mathbf{k}), \quad (2.3)$$

where n is the density of electrons, m is the electron mass, k_F is the Fermi momentum, $v(k)$ is the screened, electron-ion interaction matrix element, and $s(\mathbf{k})$ is the structure factor.

For the structure factor,¹⁰ we use the one-pho-

non harmonic approximation

$$s(\vec{k}) = \frac{\hbar^2}{Mk_B T} \sum_{\lambda} \frac{[\vec{k} \cdot \hat{\xi}_{\lambda}(\vec{q})]^2}{(e^{\hbar\omega_{\lambda}(\vec{q})/k_B T} - 1)(1 - e^{-\hbar\omega_{\lambda}(\vec{q})/k_B T})}, \quad (2.4)$$

where M is the ionic mass and $\omega_{\lambda}(\vec{q})$ and $\hat{\xi}_{\lambda}(\vec{q})$ are the frequency and polarization vector, respectively, of a phonon of branch λ and wave vector \vec{q} . The vector \vec{q} is just the scattering vector \vec{k} "folded back" into the first Brillouin zone,

$$\vec{q} = \vec{k} - \vec{G}_n, \quad (2.5)$$

where \vec{G}_n is one of the reciprocal-lattice vectors. For a given \vec{k} , there is a unique \vec{q} and \vec{G}_n that satisfy (2.5).

The explicit expression for the phonon-drag contribution to $\rho(T)$ is given by $X(T) = P_{1L}^2/P_{11}P_{LL}$, where

$$P_{11} = C \int d^3k k^{-1} s(\vec{k}) v^2(k) k^2, \\ P_{1L} = C \int d^3k k^{-1} s(\vec{k}) v^2(k) \vec{k} \cdot \vec{q}, \quad (2.6) \\ P_{LL} = C \int d^3k k^{-1} s(\vec{k}) v^2(k) q^2.$$

Here, C is a known constant and the integration is taken over the sphere $|\vec{k}| \leq 2k_F$. Note that P_{11} is proportional to the triple integral for $\rho_{\text{eq}}(T)$ given in (2.2) and (2.3).

To obtain $s(\vec{k})$, we use the well-known Born-Von Karman analysis,¹¹ according to which $\omega_{\lambda}(\vec{q})$ and $\hat{\xi}_{\lambda}(\vec{q})$ are the eigenvalues and eigenvectors, respectively, of the dynamical matrix whose matrix elements are given in terms of the interatomic force constants. The interatomic force constants out to five nearest neighbors have been measured¹² for potassium at 9°K by means of neutron scattering.¹³ The Born-Von Karman method ensures that the very important periodicity properties of the phonon frequencies and polarization vectors are automatically satisfied

$$\omega_{\lambda}(\vec{k}) = \omega_{\lambda}(\vec{q} + \vec{G}_n) = \omega_{\lambda}(\vec{q}), \\ \hat{\xi}_{\lambda}(\vec{k}) = \hat{\xi}_{\lambda}(\vec{q} + \vec{G}_n) = \hat{\xi}_{\lambda}(\vec{q}). \quad (2.7)$$

Hence, for any given \vec{k} , we can calculate exactly the phonon frequency and polarization vector even for umklapp processes, without having to know which \vec{G}_n is needed to reduce \vec{k} into the first Brillouin zone. The computational advantages of the Born-Von Karman method can hardly be overestimated.

III. ELECTRON-ION INTERACTION MATRIX ELEMENTS

We now turn to the determination of the form factor $v(k)$ for the matrix elements of the screened electron-ion interaction. This is a vital feature of the calculation. On the one hand, the values of

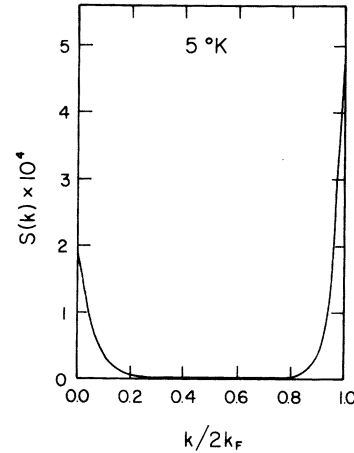


FIG. 1. Angular average of the structure factor for potassium at 5°K.

$\rho_{\text{calc}}(T)$ are very sensitive¹⁴ to $v(k)$ and, on the other hand, $v(k)$ is not reliably known. The resolution of this problem lies in exploiting the peculiar low-temperature behavior of $s(\vec{k})$. To illustrate this idea, let us consider $S(k)$, the angular average of the structure factor. We plot in Fig. 1, the curve for $S(k)$ for potassium at $T = 5^\circ\text{K}$. The most important feature of this curve is that $S(k)$ is negligible¹⁵ except near $k = 0$ and near $k = 2k_F$. Therefore, for each of the integrals appearing in the expression for ρ_{eq} and for ρ_g , Eqs. (2.2) and (2.6), the integrand contributes significantly only near $k = 0$ and near $k = 2k_F$. Therefore, it is sufficient to know $v(k)$ only near $k = 0$ and near $k = 2k_F$.

Near $k = 0$, $v(k)$ is known exactly.¹ It is proportional to the Fourier transform of a screened, point-ion Coulomb potential. For k near $2k_F$, $v(k)$ is almost constant over the very short range of integration for which $S(k)$ is non-negligible. Although one may certainly *not* ignore the k dependence of $v(k)$, even over so short a range of integration, $v(k)$ can be adequately represented by *any* one-parameter model potential, providing one adjusts the parameter to give the correct magnitude of ρ_{calc} at some fixed low temperature. This is the only adjustable parameter in our calculation of $\rho(T)$.

We shall write $v(k)$ in a form similar to the model potential of Harrison¹⁶:

$$v(k) = \frac{-(4\pi e^2/k^2) + \beta}{(1 - \Sigma)\epsilon(k)\Omega_0}, \quad k \leq 2k_F \quad (3.1)$$

where Ω_0 is the atomic volume and $1 - \Sigma$ is the normalization factor which we shall discuss presently. For the screening function $\epsilon(k)$, we use the Hartree dielectric function,

$$\epsilon(k) = 1 + \frac{1}{\pi k_F a_0 x^2} \left[\frac{1-x^2}{4x} \ln \left| \frac{1+x}{1-x} \right| + \frac{1}{2} \right], \quad (3.2)$$

where $x = k/2k_F$ and a_0 is the Bohr radius. The first term in the numerator of Eq. (3.1) is the Fourier transform of a (monovalent) point-ion Coulomb potential and the second term β represents deviations from this simple potential. These deviations arise from the effective repulsive potential that is central to the pseudopotential formulation, from corrections to the point-ion Coulomb potential in the interior of the ion core, from the approximate and local nature of the Hartree dielectric function, and from other smaller effects. Although the expression for the form factor $v(k)$ is written as a function of a single variable k , it includes in full the nonlocality of the pseudopotential. This point is explained in detail in the recent review article by Wisser and Greenfield.¹⁷

The form factor $v(k)$ differs from the Harrison¹⁶ model potential in two respects. First of all, Harrison calculated β from first principles. It is now known that this procedure is not sufficiently accurate for a resistivity calculation. We determine β by making a fit to the low-temperature resistivity data. Second, our choice for $v(k)$ contains the normalization factor $1 - \Sigma$. The scattering amplitude theorem of Austin, Heine, and Sham¹⁸ shows that it is the true wave function and not the pseudo-wave-function which must be normalized to unity. This requires inserting the normalization factor¹⁸ $1 - \Sigma_c |\langle \vec{k}_F | c \rangle|^2$, where $|\vec{k}_F\rangle$ is a normalized plane wave of wave number k_F and $|c\rangle$ is an ionic core wave function. We designate by Σ the sum over all core wave functions. The core wave functions have been calculated by Herman and Skillman.¹⁹ Performing the indicated integrals, we obtain for potassium the value $\Sigma = 0.154$.

There had previously been some question²⁰ about the need to include an orthogonalization correction to $\epsilon(k)$, which would lead to a correction factor in $v(k)$ for small k . However, Shaw and Harrison²¹ subsequently proved that the corrections due to the orthogonalization hole and the depletion hole exactly cancel, leading to the result that the pseudopotential form factor equals precisely $-\frac{2}{3}E_F$ at $k=0$. Since the pseudopotential form factor is defined as the matrix element between plane waves normalized to unity, rather than the correct normalization, we must multiply by the appropriate normalization factor to obtain

$$v(0) = -\frac{2}{3}E_F / (1 - \Sigma). \quad (3.3)$$

We emphasize that although (3.2) is an approximate expression for $\epsilon(k)$, this expression gives the correct limiting form for small k . Therefore, Eq. (3.1) for $v(k)$ is exact in the small- k region. In the high- k region, any small error introduced by using the approximate expression (3.2) for $\epsilon(k)$ is auto-

matically compensated for since β is determined by fitting to experiment.

It remains to determine the parameter β . The best over-all fit to the low-temperature resistivity data is obtained with $\beta = 45.85$, in units of $Ry a_0^3$, where a_0 is the Bohr radius. A change in β of 0.1, corresponding to a change in $v(2k_F)$ of only 0.0002 Ry, would alter $\rho_{\text{calc}}(T)$ by about 4%. This is the maximum change in $\rho_{\text{calc}}(T)$ that is compatible with the data, as we shall see in Sec. IV. We plot in Fig. 2 the curve for $v(k)$. Note that $v(k)$ is very nearly zero near $2k_F$, which explains the observed sensitivity of $\rho_{\text{calc}}(T)$ to quite small changes in $v(2k_F)$.

We close the section by emphasizing that the expression (3.1) for $v(k)$ is appropriate to calculate $\rho(T)$ only at low temperatures. Equation (3.1) provides extremely accurate values of $v(k)$ for $k \approx 2k_F$, which is the range of k that enters the umklapp resistivity integrals for the low temperatures considered here ($T < 20^\circ\text{K}$). However, for $T > 20^\circ\text{K}$, $S(k)$ rises less steeply near $2k_F$, with the result that a larger range of k enters the umklapp integrals and (3.1) is no longer sufficiently accurate. We have previously shown¹¹ what form factor to use for the calculation of $\rho(T)$ for potassium above 20°K , where the range $k_F \lesssim k \lesssim 1.8k_F$ is most important. To calculate $\rho(T)$ from absolute zero to the melting point, one needs a form factor which reduces to (3.1) for $k \approx 2k_F$ and which is also very accurate in the range $k_F \lesssim k \lesssim 1.8k_F$. In a subsequent paper, we shall show how to construct such a form factor and shall demonstrate that it leads to values of $\rho_{\text{calc}}(T)$ which are in excellent agreement with $\rho_{\text{expt}}(T)$ over the entire temperature range in the solid.

IV. CALCULATIONS AND COMPARISON WITH EXPERIMENT

In making a comparison with experiment, we are fortunate in having available the recent, highly accurate data of Guban⁷ and of Ekin and Maxfield⁸ for $\rho(T)$ for potassium. The data of Ekin and Maxfield cover the range below 20°K , whereas the data of

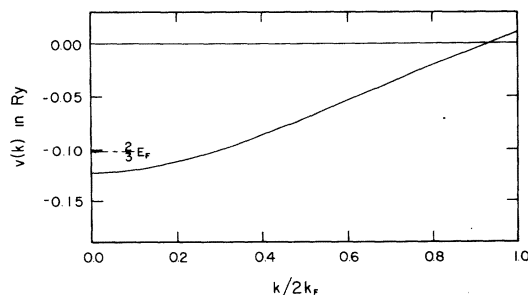


FIG. 2. Form factor $v(k)$ for potassium, including the normalization factor $1 - \Sigma$.

TABLE I. Comparison between the calculated and experimental values of $\rho(T)$ for potassium. Also given are ρ_N and ρ_U , the normal scattering and umklapp scattering contributions to $\rho(T)$, respectively, as well as their ratio (ρ_U/ρ_N) and the correction due to phonon drag (X). Finally, in the last column is presented the ratio of the calculated resistivity to ρ_U .

T (°K)	ρ_{expt} (nΩ cm)	$\frac{\rho_{\text{calc}} - \rho_{\text{expt}}}{\rho_{\text{expt}}}$		ρ_N (nΩ cm)	ρ_U (nΩ cm)	$\frac{\rho_U}{\rho_N}$	X	$\frac{\rho_{\text{calc}}}{\rho_U}$
2.0	0.001 11 (0.001 6)	-3%	(-33%)	0.001 62	0.000 82	0.50	0.56	1.31
2.2	0.002 53 (0.003 5)	+7%	(-23%)	0.002 67	0.002 07	0.78	0.43	1.31
2.4	0.005 41 (0.006 5)	+8%	(-10%)	0.004 23	0.004 53	1.07	0.33	1.30
2.6	0.010 5 (0.011 3)	+9%	(+1%)	0.006 48	0.008 82	1.36	0.26	1.29
2.8	0.019 0 (0.019 8)	+6%	(+2%)	0.009 62	0.015 7	1.63	0.21	1.28
3.0	0.031 9 (0.032 5)	+4%	(+2%)	0.013 9	0.026 0	1.86	0.17	1.28
3.2	0.050 6 (0.051 0)	+2%	(+1%)	0.019 7	0.040 5	2.06	0.14	1.27
3.4	0.076 7 (0.076 4)	0%	(0%)	0.027 3	0.060 2	2.21	0.12	1.27
3.6	0.111 (0.111)	-2%	(-2%)	0.037 0	0.085 9	2.33	0.11	1.27
3.8	0.156 (0.155)	-4%	(-4%)	0.049 2	0.118	2.41	0.10	1.27
4.0	0.211 (0.208)	-5%	(-4%)	0.064 3	0.159	2.47	0.10	1.27
4.5	0.403	-6%		0.118	0.298	2.53	0.09	1.28
5.0	0.695	-7%		0.198	0.501	2.53	0.08	1.29
5.5	1.10	-8%		0.312	0.779	2.50	0.08	1.30
6.0	1.65	-9%		0.466	1.14	2.45	0.07	1.31
7.0	3.18	-10%		0.911	2.16	2.37	0.07	1.32
8.0	5.42	-10%		1.57	3.63	2.31	0.06	1.34
9.0	8.43	-9%		2.48	5.65	2.28	0.06	1.35
10.0	12.3	-8%		3.68	8.29	2.25	0.05	1.36
12.0	22.7	-4%		7.14	15.8	2.21	0.05	1.38
14.0	37.0	0%		12.3	26.6	2.17	0.05	1.39
16.0	55.7	+3%		19.3	40.8	2.12	0.04	1.41
18.0	78.6	+5%		28.3	58.3	2.06	0.04	1.42
20.0	106.	+7%		39.0	78.8	2.02	0.04	1.43

Gugan cover the range below 4.2 °K. Potassium is in fact the only alkali metal for which highly accurate data exist in the low-temperature regime.

The need for accurate experiments for the very-low-temperature regime is readily made apparent by comparing the residual resistivity ρ_{imp} with $\rho(T)$. Even for the highest purity samples of potassium available, having residual resistivity ratios of over 8000, $\rho(T)$ is less than $\frac{1}{4}$ of ρ_{imp} at 4 °K and decreases rapidly (in fact, exponentially) at lower temperatures, until at 2 °K, $\rho(T) \approx 0.001\rho_{\text{imp}}$. Therefore, very small errors in the measurement of $\rho_{\text{tot}}(T) = \rho_{\text{imp}} + \rho(T)$ can lead to gross errors in the data for $\rho(T)$. This fact has been well appreciated both by Gugan and by Ekin and Maxfield, and the over-all excellent agreement between these two measurements of $\rho(T)$ for potassium in the temperature range where they overlap makes one confident of the reliability of their data. This is an important point in view of the significantly different experimental results previously reported²² for $\rho(T)$ for potassium.

For $\rho_{\text{expt}}(T)$, we quote the data of Gugan⁷ in the range 2–4 °K and the data of Ekin and Maxfield⁶ in the range 4–20 °K. The data of Ekin and Maxfield in the range 2–4 °K are also presented in

Table I in parentheses. In the range 2–4 °K, where the two sets of data overlap, the agreement between the two measurements is excellent (within 4%), except for the very lowest range where the values of Ekin and Maxfield are larger than those of Gugan. However, Gugan claims a significantly smaller estimated error in this range ($\pm 3\%$ at 2 °K for Gugan, as compared with $\pm 12\%$ at 2 °K for Ekin and Maxfield).

Our results for $\rho(T)$ of potassium for the temperature range 2–20 °K, are presented²³ in Table I. The second column lists the experimental data, with the values of Ekin and Maxfield in the range 2–4 °K given in parentheses. The third column gives the percentage discrepancy between the calculated value and experiment. We see that the agreement is excellent, with the discrepancy never exceeding 10% for any temperature (except for the two lowest-temperature values of Ekin and Maxfield). Note that $\rho(T)$ increases by five orders of magnitude over this temperature range.

In the fourth to sixth columns of Table I, we present the normal and umklapp contributions to $\rho(T)$ as well as their ratio ρ_U/ρ_N . Although ρ_U decreases exponentially below 6 °K, we see that even at 2 °K, ρ_U is by no means negligible compared to ρ_N .

In the seventh column of Table I, we present the the phonon-drag contribution to $\rho(T)$, represented by $X = \rho_g/(\rho_N + \rho_U)$. It should be emphasized how important a contribution phonon drag makes to $\rho(T)$ at very low temperatures. At 2°K, $X = 0.56$, showing that ignoring phonon drag would lead²⁴ to an error in $\rho_{\text{calc}}(T)$ of over a factor of 2. At 1.5°K, the error would be a factor of 10. It is clear from these results that no calculation of $\rho(T)$ for the alkali metals at very low temperatures can hope to achieve quantitative agreement with experiment without including explicitly the effect of phonon drag.

The values for $X(T)$ were calculated using the expression for P_{LL} given in Eq. (2.6). This expression assumes that the only important mechanism for achieving phonon equilibration is phonon-electron scattering. The justification for ignoring phonon-impurity scattering and phonon-phonon scattering as mechanisms for achieving phonon equilibration has been given in the preceding article.⁹ For $T > 5^\circ\text{K}$, we find that phonon-electron scattering alone is quite effective in achieving phonon equilibration. At $T = 5^\circ\text{K}$, we see from Table I that taking into account only phonon-electron scattering leads to an $X(T)$ that alters $\rho_{\text{eq}}(T)$ by only 8%, and at higher temperatures the effect is even smaller. Including phonon-phonon scattering will further reduce $X(T)$, and thus change $\rho_{\text{eq}}(T)$ by even less. Therefore, taking into account only phonon-electron scattering in the expression for $X(T)$, as we have done, is perfectly adequate for the calculation of $\rho(T)$.

V. DISCUSSION OF RESULTS

There are several aspects of these results that merit discussion. Moreover, we wish in this sec-

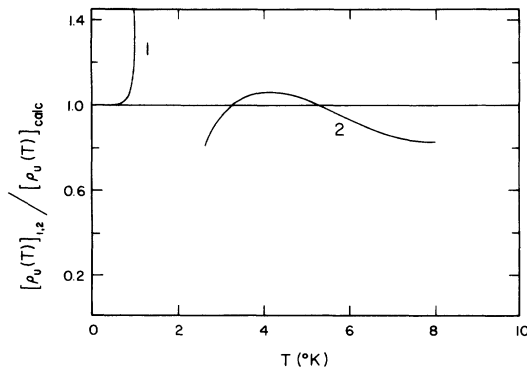


FIG. 3. Curves labeled 1 and 2 are, respectively, the ratios $\rho_U(T)/[\rho_U(T)]_1$ and $\rho_U(T)/[\rho_U(T)]_2$ plotted as a function of temperature. The quantities $[\rho_U(T)]_1$ and $[\rho_U(T)]_2$ are defined in Eq. (5.1) and $\rho_U(T)$ is the calculated umklapp scattering contribution to the resistivity given in Table I.

tion to explore quantitatively for potassium some of the predictions of the preceding paper⁹ which apply to all the alkali metals. We first consider the two analytical expressions derived in the preceding paper⁹ for the umklapp scattering contribution $\rho_U(T)$, each valid over a specific temperature range. Let us designate by $[\rho_U(T)]_1$ and $[\rho_U(T)]_2$ the expression valid in the lower-temperature and higher-temperature ranges, respectively. Then, Eqs. (3.8) and (3.9) of the preceding paper become

$$\begin{aligned} [\rho_U(T)]_1 &= AT^2 e^{-\hbar\omega_0/k_B T}, \\ [\rho_U(T)]_2 &= BT^{7/2} e^{-\hbar\omega_1/k_B T}, \end{aligned} \quad (5.1)$$

where the quantities $\hbar\omega_0/k_B$ and $\hbar\omega_1/k_B$ equal 10 and 8°K, respectively, for potassium, as obtained from the velocity of sound, and are given in Table I of Ref. 9. In Fig. 3, we plot as a function of temperature the ratios $[\rho_U(T)]_{\text{calc}}/[\rho_U(T)]_1$ and $[\rho_U(T)]_{\text{calc}}/[\rho_U(T)]_2$. A horizontal line in Fig. 3 corresponds to a ratio independent of T , implying that the analytical expression is a good description of $\rho_U(T)$ in the relevant temperature range. We see from the figure that for the range 3–6°K, $[\rho_U(T)]_2$ does not deviate from $\rho_U(T)$ by more than 6%, whereas $[\rho_U(T)]_1$ is an appropriate expression for $\rho_U(T)$ only below 1°K.

From the discussion in Appendix B of the preceding paper,⁹ it follows²⁵ that the criterion for the validity of $[\rho_U(T)]_1$ for $\rho_U(T)$ is that $5k_B T/\hbar\omega_0 \ll 1$. Inserting the value⁹ for potassium, $\hbar\omega_0/k_B = 10^\circ\text{K}$, gives $T \ll 2^\circ\text{K}$ as the temperature range for which $AT^2 e^{-\hbar\omega_0/k_B T}$ should be a valid expression for $\rho_U(T)$. This result is in agreement with the calculated range of validity, $T < 1^\circ\text{K}$. Another result derived in the preceding paper⁹ was an expression for the temperature T_{max} at which $\rho(T)/T^5$ exhibits a maximum. We used $[\rho_U(T)]_2$ for $\rho_U(T)$ to obtain $T_{\text{max}} = \hbar\omega_1/1.5k_B$. Inserting the value⁹ for potassium, $\hbar\omega_1/k_B = 8^\circ\text{K}$, leads to $T_{\text{max}} = 5^\circ\text{K}$, in agreement with experiment.⁶

Both Ekin and Maxfield⁶ and Guban⁷ analyzed their results for $\rho_U(T)$ in terms of the expression $e^{(-\Phi/T)}$. Since this expression does not include the factor $T^{7/2}$ in front of the exponential, there is a temperature dependence²⁶ to their parameter Φ . Including the factor $T^{7/2}$ leads to a temperature-independent ω_1 in the argument of the exponential, as one would anticipate from the theory.⁹

We now turn to the normal scattering contribution $\rho_N(T)$. Of course, in the limit $T \rightarrow 0$, one obtains the standard Bloch-Grüneisen result $\rho_N(T) \propto T^5$. However, one does not reach the T^5 limit for $\rho_N(T)$ until extremely low temperatures. At higher temperatures, the Bloch-Grüneisen approximations¹ are not valid, and the true $\rho_N(T)$ exhibits deviations from T^5 behavior. On the one hand, the approximation of setting $v(k)$ equal to $v(0)$ increases

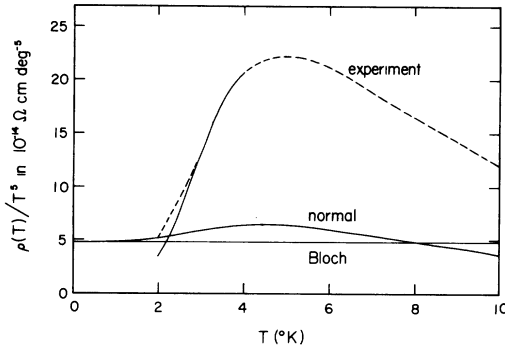


FIG. 4. Ratios $\rho_{\text{expt}}(T)/T^5$, $\rho_N(T)/T^5$, and $\rho_{\text{Bloch}}(T)/T^5$ are plotted as a function of temperature. For $\rho_{\text{expt}}(T)$, the solid and dashed lines represent the data of Gugan and of Ekin and Maxfield, respectively. The ratio $\rho_{\text{Bloch}}(T)/T^5$ is virtually indistinguishable from a horizontal line up to $T/\theta = 0.1$, corresponding to $T = 11^\circ\text{K}$ for potassium.

$\rho_N(T)$ above its true value because $v^2(k) < v^2(0)$. On the other hand, the approximation of a linear phonon spectrum in Eq. (2.4) decreases $\rho_N(T)$ below its true value because $[\omega_\lambda(\vec{q})]_{\text{linear}} < [\omega_\lambda(\vec{q})]_{\text{linear}}$. Therefore, the two approximations tend to cancel. Nevertheless, the second approximation is more serious since $\omega_\lambda(\vec{q})$ enters the integrand for $\rho_N(T)$ exponentially, whereas $v^2(k)$ enters linearly. Therefore, removing both approximations leads to the result that, for $T \lesssim 5^\circ\text{K}$, $\rho_N(T)/T^5$ increases with temperature, rather than remaining virtually constant, as predicted by the Bloch-Grüneisen theory. To illustrate these points, we plot²⁷ in Fig. 4 as a function of temperature the ratios $\rho_N(T)/T^5$ and $\rho_{\text{Bloch}}(T)/T^5$. The increasing values of $\rho_N(T)/T^5$ with increasing temperature up to 5°K are quite apparent.

We also plot in Fig. 4 the ratio $\rho_{\text{expt}}(T)/T^5$ in order to compare the magnitudes of $\rho_{\text{expt}}(T)$ and $\rho_N(T)$. We see that $\rho_{\text{expt}}(T) < \rho_N(T)$ below 2.2°K for the data of Gugan, and below 2.0°K for the data of Ekin and Maxfield. This is a very interesting result for the following reason. Since $\rho_V(T)$ is positive, it follows that without calculating either $\rho_V(T)$ or $\rho_g(T)$, we may be certain that phonon drag is very important at 2°K . The significance of this result lies in the fact that it depends only on a calculation of $\rho_N(T)$, which contains no free parameters. The single parameter β appearing in $v(k)$ given by Eq. (3.1) is completely screened at small k by $\epsilon(k)$ and hence does not contribute to $\rho_N(T)$. The above argument demonstrates unequivocally the necessity of including phonon drag in the calculation of $\rho(T)$ at these low temperatures.

It should be mentioned that all previously reported values for $\rho_N(T)$ failed to include the normalization factor $1 - \Sigma$ in $v(k)$. Since $v(k)$ appears

squared in $\rho(T)$ and $\Sigma = 0.154$ for potassium, ignoring Σ leads to values which must be increased by 40% to obtain the correct $\rho_N(T)$. This is an important point regarding the comparison between $\rho_N(T)$ and $\rho_{\text{expt}}(T)$ mentioned above.

Our analysis⁹ of phonon drag leads to several predictions which we shall now check quantitatively for potassium. For the temperature at which $\rho_V/\rho_N = 2$, we derived that $1 - X \approx 0.9$. Table I shows that $\rho_V/\rho_N = 2$ occurs at 3.2°K , for which $1 - X = 0.86$, in agreement with the prediction. We also derived that when $X = 0.5$, one should obtain $\rho_V/\rho_N \approx 0.6$. Again turning to Table I, $X = 0.5$ occurs at 2.1°K , for which the ratio $\rho_V/\rho_N = 0.6$, in agreement with the prediction.

A third result, which follows from Eq. (5.10) of the preceding paper,⁹ is that

$$\rho(T) \approx \gamma(T)\rho_V(T) \approx \gamma(0)\rho_V(T), \quad (5.2)$$

since $\gamma(T)$ is independent of temperature to within 10% for $T < 5^\circ\text{K}$. Because $\rho_V(T)$ decreases exponentially at these low temperatures, Eq. (5.2) explains the exponential temperature dependence of $\rho_{\text{expt}}(T)$ reported by Gugan.⁷ Moreover, if we insert into (5.2) the value $\gamma(0) = 1.3$, we obtain

$$\rho(T) \approx 1.3\rho_V(T) \quad (5.3)$$

for $T < 5^\circ\text{K}$. The values of $\rho_{\text{calc}}(T)/\rho_V(T)$ given in the last column of Table I are seen to conform to (5.3). This result is quite remarkable in that it enables one to include phonon drag into the calculation of $\rho(T)$ at low temperatures without having to calculate $X(T)$ at all. One ignores $\rho_N(T)$, calculates $\rho_V(T)$, and multiplies by 1.3.

The last column of Table I shows that the ratio $\rho_{\text{calc}}(T)/\rho_V(T)$ remains close to 1.3 up to $T = 20^\circ\text{K}$, far beyond the range of validity of (5.2). To understand why this is so, we write the exact relationship

$$\rho(T) = \rho_V(1 + \rho_N/\rho_V)(1 - X). \quad (5.4)$$

Above 5°K , $X(T) \ll 1$ and the ratio $\rho_N(T)/\rho_V(T)$ is fairly constant and significantly smaller than unity, as can be seen from Table I. As a result, above 5°K the product $(1 + \rho_N/\rho_V)(1 - X) \approx 1 + \rho_N/\rho_V - X$ does not exceed 1.3 by more than 10%. Therefore, we see that (5.3) is a good approximation for potassium from 0 up to 20°K , although for quite different reasons in the different temperature ranges.

VI. SUMMARY

In this paper, we have calculated the low-temperature electrical resistivity of potassium and compared our results with the recent, highly accurate experimental data covering the temperature range $2\text{--}20^\circ\text{K}$. Excellent agreement was obtained with experiment. This was achieved by explicitly including phonon drag, a crucial effect at very low

temperatures, and by a proper treatment of the form factor of the screened, electron-ion interaction matrix elements. We also showed how these quantitative results for potassium are in complete agreement with the general analysis of the low-temperature resistivity of the alkali metals that was presented in the preceding paper.

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- ²⁴Even for data points of Ekin and Maxfield, ignoring phonon drag would lead to a discrepancy between theory and experiment of 53% at 2°K and of 37% at 2.6°K.
- ²⁵The criterion for validity is that $\delta_{\max}/\delta_0 \ll 1$, where δ_{\max} and δ_0 are defined in Appendix B of Ref. 9. The explicit expressions for δ_{\max} and δ_0 , given in Eqs. (B2), (B10), and (B11), lead directly to the condition $5k_B T/\hbar\omega_0 \ll 1$.
- ²⁶At $T \approx 3^\circ\text{K}$, they each obtained $\Phi \approx 23^\circ\text{K}$. The large difference between their value of $\Phi = 23^\circ\text{K}$ and our value of $\hbar\omega_1/k_B = 8^\circ\text{K}$ is entirely accounted for by the high power of T appearing in the prefactor, $T^{7/2}$. Guban (Ref. 7) himself pointed out that inserting a single power of T into the prefactor to the exponential has the effect of reducing Φ by about 2.8°K.
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