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Fermi-liquid effects in cyclotron-phase-resonance transmission

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The transmission amplitude for circularly polarized microwaves incident on a slab of interacting electron liquid is calculated assuming the presence of a uniform steady magnetic field directed normal to the faces of the slab. The amplitude is evaluated for field strengths in the neighborhood of cyclotron resonance, a situation which has recently been investigated experimentally by Phillips, Baraff, and Dunifer (PBD). In the calculations presented here, the electron liquid is assumed to be described by Fermi-liquid theory and to interact with the slab faces as though the quasiparticles were scattered diffusely. The results of the calculation indicate that the correlation-produced mode studied by Cheng, Clarke, and Mermin (CCM) should, in fairly thick slabs, produce a secondary maximum in the transmission amplitude at approximately the magnetic field strength for which the mode suffers Doppler-shifted cyclotron resonance. In thin slabs, the calculation indicates that this magnetic field strength is that for which the transmission will be most intense, in contradiction to the PBD observations that the maximum transmission occurs at a magnetic field strength much closer to cyclotron resonance. The calculation agrees with the PBD observation that the transmission is much greater on that side of cyclotron resonance for which the CCM mode can propagate than on the other side, and supports the PBD assertion that this characteristic asymmetry in the transmission spectrum is caused by the presence of electron correlations.

I. INTRODUCTION

Cyclotron phase resonance seems to be a fairly general feature of microwave-transmission experiments through metallic samples under anomalous-skin-effect conditions when these experiments are performed in the presence of a magnetic field directed *normal* to the faces of the sample.[1,2] It occurs when the strength of the magnetic field is such that a large group of carriers has a cyclotron frequency equal to the frequency of the incident microwave field. When this happens the many carriers in the group, each of which has its own individual velocity across the sample, all arrive at the emergent face of the sample in phase with each other, producing a transmitted signal which is larger than that produced at other values of magnetic field strength.

Observations of cyclotron-phase-resonance transmission through alkali metals at 116 GHz[3] have raised some intriguing questions about the role played by Fermi-liquid effects[4-6] in determining the transmission spectrum. On the one hand, a very simple model (namely, that the metallic sample can be regarded as a finite-thickness slab whose interior is filled uniformly with the interacting electron liquid) treated in a heretofore-successful heuristic manner gave a remarkably faithful description of the rather complicated data. On the other hand, there was no firm theoretical justification for some of the steps in the heuristic treatment. Although the impressive fit between the data and the calculation might be regarded as supplying the missing justification, the value assigned to one of the parameters of the model (the Landau-theory A_1 parameter)[7] differed sufficiently in sign and size from currently accepted theoretical estimates[8] to cast doubt as to whether the heuristic treatment was, in fact, valid.

Examining the validity of the heuristic treatment therefore becomes a matter of considerable interest. If the heuristic treatment of the simple physical model can be shown to be valid, then one has, for the first time, an experimental determination of the Landau parameter A_1 in a metal. If the heuristic treatment of the simple physical model is shown to be invalid, i.e., if a direct calculation based on the same model fails to reproduce the results of the heuristic calculation, then the physical model itself is defective because it cannot account for the data. In this latter case, one is left with two questions. First, what physical mechanism is needed to account for the data, and second, why was the heuristic approach seemingly so successful?

This paper, a theoretical study of the cyclotron-phase-resonance phenomenon in a finite slab of interacting Fermi liquid, presents a numerical self-consistent variational calculation of the microwave transmission through the metallic sample using the same physical model which the heuristic treatment purports to approximate. Though the results of these calculations do confirm ideas which have evolved from theoretical study of the bulk properties of Fermi liquids[9] and our earlier studies of the slab-transmission problem without correlations,[10] the calculations here support only *one* of the several important features of the heuristic treatment. On balance, one must conclude that the heuristic treatment (which does describe the data) does *not* correspond to the actual predictions of the model which it was supposed to represent, and thus the two questions posed in the last paragraph must ultimately be faced.

In the last sections of this paper we shall return to a further discussion of the heuristic treatment and of the comparison between its predictions and those of our numerical calculation here. First, however, in Sec. II we shall formulate the mathematical problem to be solved. In Sec. III we recast the problem as a variational one in which the quantity we are interested in, the transmission amplitude, is the quantity to be varied. The success of any variational calculation rests on the choice made for the trial forms representing the fields. This step, the choice of physically meaningful fields, is both the most important and the most difficult part of the entire calculation. The choice of trial fields is discussed in Sec. IV. From that point on, the calculation is relatively straightforward and we merely summarize the steps of that calculation in Appendix A. The results of our calculations appear in Sec. V. In Sec. VI we return to the heuristic treatment, "deriving" it in a way which makes clear what additional assumptions have been made, and then comparing and contrasting its predictions with those of the variational calculation. Finally, in Sec. VII we summarize and speculate about profitable directions for further theoretical investigation.

II. FORMULATION OF PROBLEM

The basic transmission problem to be studied is this: A transverse circularly polarized electromagnetic wave at frequency ω is incident on the $z=0$ face of a metallic slab which extends from $z=0$ to $z=D$. There is a reflected wave at $z<0$, a transmitted wave at $z>D$, and we are to calculate the amplitude of the transmitted wave.

Within the slab, Maxwell's wave equation relates the electric field $e(z)$ to the transverse circularly polarized current $j(z)$,

$$\left(\frac{d^2}{dz^2}+k_0^2\right)e(z)=-i\omega\mu_0 j(z), \quad 0<z<D \tag{2.1}$$

where $k_0=\omega/c$. The current $j(z)$ can be calculated from the quasiparticle distribution function $n(z,p,t)$. The relationships between j and n and the equation governing the time development of n are assumed to be those postulated in Landau's theory of the Fermi liquid. In particular, the equation governing the time development of n is the spin-independent part of the transport equation

$$\frac{\partial n}{\partial t}+\left(\frac{\partial n}{\partial \vec{r}}\cdot\frac{\partial \epsilon}{\partial \vec{p}}-\frac{\partial \epsilon}{\partial \vec{r}}\cdot\frac{\partial n}{\partial \vec{p}}\right)+q(\vec{e}+\vec{V}\times\vec{B})\cdot\frac{\partial n}{\partial \vec{p}}=\left(\frac{\partial n}{\partial t}\right)_{\text{coll}}, \tag{2.2}$$

where $\epsilon(\vec{r},\ \vec{p},\ t)$ is the energy of a quasiparticle with momentum $\vec{p}$ at position $\vec{r}$ at time t, $\vec{e}$ and $\vec{B}$ are the total electric and magnetic fields, and $\vec{V}=\vec{\nabla}_p\epsilon$ is the velocity of the quasiparticle.

The transport equation above must be supplemented by boundary conditions which describe how the quasiparticles behave at the incident and emergent faces of the slab. Although there has been considerable recent progress in understanding the interaction between charge carriers and the metallic surface,[11-13] the over-all difficulty of the transmission problem we pose is so great that we are forced to fall back on the oldest but mathematically simplest characterizations of the quasiparticle behavior at the bounding faces of the slab.

The very simplest boundary condition is to assume that each quasiparticle suffers specular reflection at the surface. In this case, the method of images used by Platzman and Buchsbaum[14] may be applied to the problem and an explicit solution for the fields, the transmission, and the surface impedance may be obtained in a rather straightforward manner. The specular boundary condition does not produce a cyclotron-phase-resonance peak in the calculated transmission when there are no Fermi-liquid effects. Since the peak is the most prominent feature of both the heuristic calculation and the data, there is no reason to consider the specular boundary condition further here.

The next-simplest boundary condition is to assume that each quasiparticle suffers diffuse reflection at the faces of the slab. This boundary condition does produce a peak in the calculated transmission (in the uncorrelated situation) when the magnetic field strength is such that the cyclotron frequency of the carriers is equal to the frequency of the incident electromagnetic field. This boundary condition provides a reasonable approach to the problem of how Fermi-liquid effects alter the cyclotron-phase-resonance phenomenon.

We have shown in an earlier paper dealing with Fermi-liquid effects in a semi-infinite medium[15] that the diffuse-scattering boundary condition can easily be incorporated into the transport equation (2.2) when the uniform ambient magnetic field is directed normal to the surface. The situation here (where a finite slab replaces the semi-infinite medium) is virtually identical to that considered earlier as far as this step is concerned, and by analogy with the earlier derivation, we can replace the transport equation and diffuse-scattering boundary condition with the following set of integral equations:

$$\psi_m(x)+\sum_{n=1} i\int_0^L K_{mn}(x-x')\,h_n\psi_n(x')\,dx' + \int_0^L K_{m1}(x-x')\,e(x')\,dx'=0\,. \tag{2.3}$$

[In (2.3) the electron mean free path l is used as the unit of length so that $L \equiv D/l$ is the dimensionless thickness of the slab and $x \equiv z/l$ is the dimensionless depth coordinate.]

The quantities $\psi_m(x)$ are proportional to angular moments of the (modified) quasiparticle distribution function[7] $\bar{g}$ and the K_{mn} are transport kernels, similar in structure to the nonlocal conductivity kernel of the free-electron gas. The exact definition of these kernels is

$$K_{mn}(x-y) \equiv \int_0^{2\pi} d\varphi \int_0^1 \frac{du}{u} Y^*_{m1}(u, \varphi) Y_{n1}(u, \varphi) e^{-a(x-y)/u} \quad (x > y) \tag{2.4a}$$

$$= (-1)^{m+n} K_{mn}(y-x) \quad (y > x), \tag{2.4b}$$

$$a \equiv 1 - i(\omega - \omega_c)\tau . \tag{2.4c}$$

The Y_{m1} are normalized spherical harmonics, and from the definition (2.4a) it follows that $K_{mn} = K_{nm}$. ω_c is the cyclotron-resonance frequency eB/m^*c, where m^* is the mass which would be measured in an Azbel-Kaner cyclotron-resonance experiment. $\tau = l/V_F$ is the mean free time for particles having a mean free path l and a Fermi speed V_F.

All of the Fermi-liquid correlation effects enter via the second term in (2.3), where the parameters h_n are defined by

$$h_n \equiv \omega\tau A_n/(1 + A_n), \tag{2.5}$$

A_n being the parameters which appear when the orbital part of the Landau-theory interaction function is expanded in spherical harmonics.

Fermi-liquid theory also gives the prescription for calculating the current j from the quasiparticle distribution function. Using that prescription, the circularly polarized transverse current j is proportional to ψ_1, the coefficient of proportionality being such that (2.1) becomes

$$\left(\frac{d^2}{dx^2} + k_0^2 l^2\right) e(x) = ib\psi_1(x), \tag{2.6}$$

$$b \equiv (\omega_p V_F/\omega c)^2 (\omega\tau)^3, \tag{2.7}$$

where ω_p is the plasma frequency.

Equation set (2.3) and (2.6) can be considered to be a finite set of equations for the finite number of unknown functions $e(x)$, $\psi_1(x), \ldots, \psi_N(x)$ if we ignore (set equal to zero) all correlation parameters A_n for $n > N$. These parameters are supposed to decrease rapidly with increasing n. For this paper we shall (as was done in the heuristic treatment) take $N = 2$, ignoring A_3 and all higher parameters.

We shall be concerned with the solution of this set of equations subject to transmission-experiment boundary conditions; namely, that an electric field of amplitude A is incident on the slab at $z = 0$ and only a transmitted wave is present beyond $z = D$. Expressed as conditions on (2.6), this means that

$$\left(1 + \frac{1}{ik_0 l}\frac{d}{dx}\right) e(x) = 2A, \quad x = 0 \tag{2.8a}$$

$$\left(1 - \frac{1}{ik_0 l}\frac{d}{dx}\right) e(x) = 0, \quad x = L . \tag{2.8b}$$

III. VARIATIONAL PRINCIPLE FOR TRANSMISSION

The form of variational principle and its derivation are quite similar to the principle and derivation used in calculating the transmission through a slab in the absence of correlation effects.[16] For this reason, we refer the reader to Sec. II of Ref. 16 (the second paper in this series of three) for the details of the arguments we shall merely summarize here.

The derivation of the variational principle proceeds in two parts. First, we combine Maxwell's equation (2.6) and the boundary conditions (2.8) into a single integral relation in which the transmission amplitude f [equal to the ratio between the field $e(L)$ at the emergent face of the slab and the field $Ae^{ik_0 lL}$ which would have been found at $x = L$ had the slab been removed] appears in the eigenvalue. The electric field $e(x)$ is denoted as $\psi_0(x)$. The integral relation is

$$\mu \int_0^L e^{ik_0 l(x-y)} \psi_1(y)\, dy = \psi_0(x) + \frac{ib}{k_0 l} \times \int_0^L \theta(y-x) \sin k_0 l(x-y) \psi_1(y)\, dy , \tag{3.1}$$

where

$$\theta(y-x) \equiv 1 \quad (y > x) \tag{3.2a}$$

$$\equiv 0 \quad (y < x) \tag{3.2b}$$

and

$$\mu = -\left(\frac{b}{2k_0 l}\right)\left(\frac{f}{1-f}\right) . \tag{3.3}$$

Equations (3.1) and (2.3) are a set of coupled homogeneous simultaneous eigenvalue equations for the three fields $\psi_0 (= e)$, ψ_1, and ψ_2. They can be written symbolically as

$$\mu \mathfrak{M}\psi - \mathfrak{L}\psi = 0, \tag{3.4}$$

where only

$$\mathfrak{M}_{01}(x-y) = e^{ik_0 l(x-y)} \tag{3.5}$$

differs from zero and where

$$\mathcal{L}_{mn}(x-y)=\delta_{mn}\delta(x-y)+\begin{pmatrix} 0 & ibG(x-y) & 0 \\ K_{11}(x-y) & ih_1K_{11}(x-y) & ih_2K_{12}(x-y) \\ K_{21}(x-y) & ih_1K_{21}(x-y) & ih_2K_{22}(x-y)\end{pmatrix}, \tag{3.6}$$

$$G(x-y)=\theta(y-x)(k_0l)^{-1}\sin k_0l(x-y). \tag{3.7}$$

In the second step, Eq. (3.4) is turned into a variational problem for the eigenvalue μ in the usual way by introducing adjoint fields Φ and writing

$$\mu=(\Phi\mathcal{L}\psi)/(\Phi\mathfrak{M}\psi). \tag{3.8}$$

The argument that first-order errors in Φ and ψ lead only to second-order errors in μ follows immediately from the equations determining ψ and Φ, namely,

$$\delta\mu/\delta\Phi_m(x)=0 \tag{3.9a}$$

and

$$\delta\mu/\delta\psi_n(y)=0. \tag{3.9b}$$

One proposes trial fields for the functions $\Phi_m(x)$ and $\psi_n(y)$, letting these trial fields depend on certain parameters. The parameters are varied to give stationary values of μ and then, with the trial-field parameters set in this way, μ is calculated from (3.8). Having obtained μ, the transmission amplitude f follows from (3.3).

IV. FORM OF TRIAL FIELDS

We are going to approach the problems of calculating μ in four stages. In the first stage, we shall set A_1 and $A_2\equiv 0$ and determine ψ_0 and ψ_1 variationally. (This part of the problem has already been solved in Ref. 16 and we omit all details here.) In the second stage, we allow A_2 to be infinitesimally small but not zero. This cannot change the fields ψ_0 and ψ_1 which were determined in the first stage, but it does allow us to determine ψ_2 variationally. In the third stage, we allow A_2 to take its finite value and calculate the changes induced in the ψ fields. Finally, in the fourth step, we also allow A_1 to be finite. In carrying out this last step, we recall the results of Ref. 17 (the first paper in this series of three), in which we showed that the effect of A_1 on the electromagnetic fields in a semi-infinite medium was so small that it could be computed by first-order perturbation theory. We are aware of no mechanism that could cause the introduction of an emergent surface (changing the semi-infinite medium to a finite slab) to alter this situation, and we shall assume that the effects of A_1 on the fields in a finite slab can also be calculated by first-order perturbation theory. But first-order perturbation theory, in a variational context, means the use of zeroth-order trial functions in the variational functional which contains the perturbation. Therefore, in stage four, we evaluate μ using the fields determined in stage three but retaining the A_1 terms in the operator $\mathcal{L}$.

Suppose temporarily that only A_1 were equal to zero. Consider the equation set (3.9a) which governs the ψ fields. Using (3.5)–(3.8), we have

$$\psi_0(x)+ib\int_0^L\hat{G}(x-y)\psi_1(y)\,dy-\mu\times\int_0^L\hat{H}(x-y)\psi_1(y)\,dy=0, \tag{4.1a}$$

$$\int_0^L K_{11}(x-y)\psi_0(y)\,dy+\psi_1(y)+ih_2\times\int_0^L K_{12}(x-y)\psi_2(y)\,dy=0, \tag{4.1b}$$

$$\int_0^L K_{21}(x-y)\psi_0(y)\,dy+\psi_2(y)+ih_2\times\int_0^L K_{22}(x-y)\psi_2(y)\,dy=0, \tag{4.1c}$$

where

$$\hat{G}(x-y)\equiv(k_0l)^{-1}\theta(y-x)\sin k_0l(x-y), \tag{4.2a}$$

$$\hat{H}(x-y)\equiv e^{ik_0l(x-y)}. \tag{4.2b}$$

We may also write out in detail the equation set (3.9b) which governs the Φ fields. When we do so and then make the substitutions $y=L-x'$ and $x=L-y'$, we discover that

$$\Phi_0(x)=\psi_1(L-x), \tag{4.3a}$$

$$\Phi_1(x)=\psi_0(L-x), \tag{4.3b}$$

$$\Phi_2(x)=ih_2\psi_2(L-x). \tag{4.3c}$$

Because of (4.3), we can regard the functional μ as depending on the three fields ψ_j rather than on the six fields Φ_i and ψ_j.

Let us use a superscript zero to denote the fields in the semi-infinite medium in the absence of correlations. These fields satisfy the $L=\infty$, $h_2=0$ limit of (4.1), namely,

$$\psi_0^0(x)+ib\int_0^\infty\hat{G}(x-y)\psi_1^0(y)\,dy=0, \tag{4.4a}$$

$$\int_0^\infty K_{11}(x-y)\psi_0^0(y)\,dy+\psi_1^0(x)=0, \tag{4.4b}$$

$$\int_0^\infty K_{21}(x-y)\psi_0^0(y)\,dy+\psi_2^0(x)=0. \tag{4.4c}$$

In obtaining (4.4) we have recognized that the transmission amplitude f (and therefore the eigenvalue μ) must decrease exponentially with increasing slab thickness when the thickness becomes large.

For the first stage of the calculation we shall take

$$\psi_0(x)=\psi_0^0(x)+Ae^{-px}\equiv\hat{\psi}_0(x), \tag{4.5a}$$

$$\psi_1(x) = \psi_1^0(x) \equiv \hat{\psi}_1(x), \tag{4.5b}$$

and shall determine A and p variationally. The piece Ae^{-px} is a simple exponential representation of the change in the electric field caused by the introduction of the emergent face of the slab. In Ref. 16 we found that the variational determination of p gave it a negative real part (so that this change in the electric field was greatest at the emergent face of the slab) and an imaginary part which was very nearly equal to $+(\omega-\omega_c)\tau$ (so that this change in the electric field is similar in structure to the field carried by electrons traveling from the emergent face of the slab back towards the incident face). This added piece thus corresponds in form to what a multiple-reflection analysis of the problem would suggest if only the first reflection at the emergent face were important.

Also note that in (4.5b) we are saying that the current ψ_1 in the finite slab is not changed from the value it would have had in the infinite medium. This assumption is actually one of the conclusions of the study carried out in Ref. 16, where we found that the variational principle rejected any substantial change in the current. It means, of course, that terminating the infinite medium by an emergent surface so as to produce a finite slab *must change the electric field in just such a way as to leave the current nearly unaltered.*

In the second stage of the calculation, we shall maintain ψ_0 and ψ_1 at their values (4.5) but shall take

$$\psi_2(x) = \psi_2^0(x) + Be^{-qx} \equiv \hat{\psi}_2(x) \tag{4.6}$$

and shall determine B and q variationally. Again, the piece Be^{-qx} is a simple exponential representation of the change in the field ψ_2 (proportional to the $l=2$, $m=1$ moment of the modified distribution function) caused by the introduction of the emergent face of the slab. We can expect that this term, like the change in ψ_0, should be largest at the emergent face of the slab, and therefore that q, like p, should have a negative real part. Indeed, the variational determination of q (numerically) will turn out to give $q=p$.

In the third stage, where A_2 is allowed to be finite, the fields will change from the values (4.5) to (4.6) and we shall have

$$\psi_0(x) = \hat{\psi}_0(x) + \delta\psi_0(x), \tag{4.7a}$$

$$\psi_1(x) = \hat{\psi}_1(x) + \delta\psi_1(x), \tag{4.7b}$$

$$\psi_2(x) = \hat{\psi}_2(x) + \delta\psi_2(x). \tag{4.7c}$$

In order to make a good choice of functions for the variational representation of the $\delta\psi_j$, it is useful to consider the formal solution which would arise if the forms (4.7) were used in Eq. (4.1).

Let us insert (4.7) into (4.1) and, in doing so, let us pretend that the functions $\hat{\psi}_j(x)$ would provide an exact solution to (4.1) if A_2 were zero instead of being (as they are) our variational approximation to the exact solution. That is, we shall proceed as though the following set of equations were rigorously (instead of approximately) satisfied by the ψ_j:

$$\hat{\psi}_0 + (ib\hat{G} - \mu_0\hat{H})\psi_1 = 0, \tag{4.8a}$$

$$K_{11}\hat{\psi}_0 + \hat{\psi}_1 = 0, \tag{4.8b}$$

$$K_{21}\hat{\psi}_0 + \hat{\psi}_2 = 0. \tag{4.8c}$$

Here μ_0 is the value of the eigenvalue μ when A_2 is zero. When A_2 is not zero, we shall have $\mu=\mu_0+\delta\mu$. If we now insert (4.7) into (4.1) and use (4.8), the equations for the $\delta\psi_j$ are

$$\delta\psi_0 + [ib\hat{G} - (\mu_0+\delta\mu)\hat{H}]\delta\psi_1 = \delta\mu\hat{H}\hat{\psi}_1, \tag{4.9a}$$

$$K_{11}\delta\psi_0 + \delta\psi_1 + ih_2K_{12}\delta\psi_2 = -ih_2K_{12}\hat{\psi}_2, \tag{4.9b}$$

$$K_{21}\delta\psi_0 + (1+ih_2K_{22})\delta\psi_2 = -ih_2K_{22}\hat{\psi}_2. \tag{4.9c}$$

The huge size of the constant b (which, in the experiments reported in Ref. 3, was of order 10^{10}) appearing in (4.9a) means that $\delta\psi_1$, the change in the current, is going to be an exceedingly small quantity. Thus, for the purposes of satisfying (4.9b) and (4.9c), we can set $\delta\psi_1$ equal to zero. This result is the exact analog of what we found in Ref. 16 and mentioned above, namely, the tendency of the *current* to remain unaltered. In Ref. 16 we found that addition of an emergent surface leaves the current nearly unaltered; here we find that addition of the A_2 correlation parameter leaves the current nearly unaltered.

Setting $\delta\psi_1=0$ in (4.9b) and (4.9c) we obtain the solution

$$\delta\psi_0 = -ih_2K_{11}^{-1}K_{12}S^{-1}\hat{\psi}_2, \tag{4.10a}$$

$$\delta\psi_2 = -(S-1)S^{-1}\hat{\psi}_2, \tag{4.10b}$$

where

$$S \equiv 1 + ih_2(K_{22} - K_{21}K_{11}^{-1}K_{12}). \tag{4.11}$$

The difficulty with this formal solution of the form

$$\delta\psi_0 = \theta_1\hat{\psi}_2, \tag{4.12a}$$

$$\delta\psi_2 = \theta_2\hat{\psi}_2 \tag{4.12b}$$

is that we cannot evaluate the operators θ_1 and θ_2. If θ_1 and θ_2 were multiples of the unit operator, we then would have

$$\delta\psi_0 = \alpha\hat{\psi}_2, \tag{4.13a}$$

$$\delta\psi_2 = \beta\hat{\psi}_2, \tag{4.13b}$$

where α and β are numbers. The error made in approximating (4.12) by (4.13) should be a function smoother than $\hat{\psi}_2$ and so we shall approximate this residual as a series of (relatively slowly varying) exponentials. We therefore take

$$\delta\psi_0 = \alpha\hat{\psi}_2 + \sum_{i=1}^{N_0} a_i e^{-p_i x}, \qquad (4.14a)$$

$$\delta\psi_1 = 0, \qquad (4.14b)$$

$$\delta\psi_2 = \beta\hat{\psi}_2 + \sum_{i=N_0+1}^{N_0+N_2} a_i e^{-p_i x}. \qquad (4.14c)$$

There are N_0 exponential terms in $\delta\psi_0$, N_2 exponential terms in $\delta\psi_2$, and we have $2(N_0+N_2+1)$ variational parameters to determine. However, the labor involved in setting the p_i parameters is so great that, as a working expedient, we shall set them beforehand and only determine the linear parameters (α, β, and the a_i) variationally. The test of the validity of this procedure is that the calculated value of μ is insensitive to the number of terms chosen (N_0 and N_2) and to the exact values chosen for the p_i parameters. (See Appendix B for further discussion of this matter.)

When the parameters in (4.14) have been set, the resulting functions $\delta\psi_j$ will be approximations to those appearing in (4.10). In this connection it is especially interesting to consider the operator S appearing in (4.10). Suppose that we wanted to evaluate the change in transmission caused by having a finite value of A_2. Equivalently, we evaluate the $\delta\mu$ appearing in (4.9a). In principle, this could be done by using the variational expression (3.8), using for Φ and ψ the fields given by (4.3), (4.7), and (4.10). The result is

$$\delta\mu = -ih_2 \int_0^L \int_0^L \hat{\psi}_2(L-x) S^{-1} \hat{\psi}_2(y)\, dx\, dy \times \left(\int_0^L \int_0^L \psi_1^0(L-x) e^{ik_0 l(x-y)} \psi_1^0(y)\, dx\, dy \right)^{-1}.$$

This shows that a particularly large value of the operator S^{-1} will be associated with a particularly large value of the transmission amplitude.

The significance of this remark lies in the intimate connection between the operator S^{-1} and the correlation-produced modes which propagate along the magnetic field. These modes, whose existence was predicted by Silin,[18] have been discussed by Cheng, Clarke, and Mermin (CCM),[9] who gave the explicit dispersion relation for that infinite-medium mode which should exist when A_3 and all higher parameters are equal to zero. It turns out that if the operator S^{-1}, defined in (4.11), is allowed to operate on an infinite-medium plane wave $e^{i(kx-\omega t)}$, the result diverges if k and ω satisfy the CCM dispersion relation. *We can infer then that an anomalously large value of S^{-1} in the finite slab will correspond to the finite-slab realization of the infinite-medium CCM mode.* Because $\delta\mu$ is proportional to S^{-1}, this should result in an enhanced transmission amplitude when the CCM mode can exist.

Having determined the fields by setting the parameters in (4.14), one uses those fields in (3.8) (including the finite values of A_1 and A_2 in the operator $\mathcal{L}$) to calculate the transmission amplitude. An outline of the way the entire calculation proceeds, stages one through four, may be found in Appendix A.

V. RESULTS AND INTERPRETATION OF CALCULATIONS

A. Uncorrelated slab

When both Fermi-liquid-theory parameters A_1 and A_2 are set equal to zero, the physical problem reduces to that of calculating the transmission of electromagnetic radiation parallel to the magnetic field in a free-electron gas with diffuse boundary conditions. The main features of the transmission spectrum have already been described in Ref. 10. [See especially Eqs. (4.1) and (4.3) of that work for an explicit formula for the transmission amplitude.] Although the derivation given in Ref. 10 applied to thick slabs, subsequent work[16] and results to be presented here verify the accuracy of that earlier solution even for slabs of fractional mean-free-path thickness.

There are two main features in the transmission amplitude as a function of ambient magnetic field

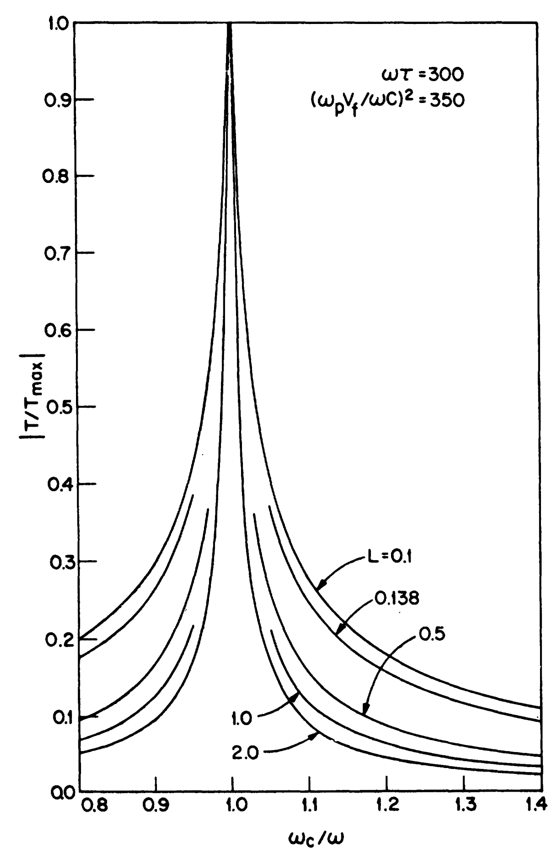


FIG. 1. Absolute value of the transmission amplitude, normalized to a maximum transmission of unity, as a function of magnetic field strength for various values of slab thickness, calculated for the free-electron gas.

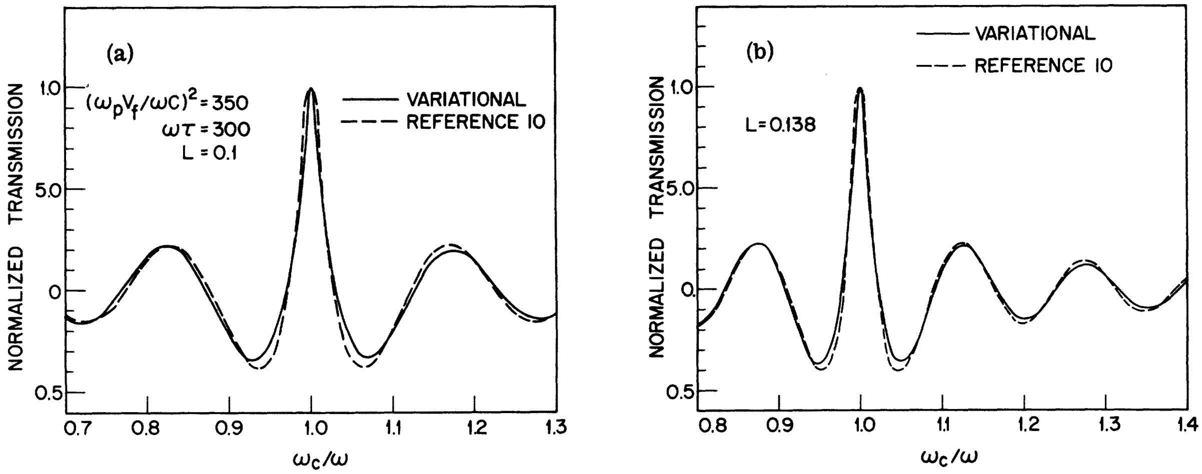


FIG. 2. Transmission amplitude, such as would be seen in a transmission experiment if the receiver were sensitive to the electric field component along a single spatial direction, as a function of magnetic field strength. For each calculation the direction of receiver sensitivity has been chosen to give a symmetric line shape by taking a phase shift $\alpha = \frac{7}{6}\pi$ and the maximum amplitude has been normalized to unity.

strength. The first is that the direction of the emergent electric field is rotated in the plane of the sample relative to the direction of the incident electric field. The amount of that rotation increases linearly with magnetic field. Therefore, a transmission experiment which measures the strength of the electric field parallel to some fixed direction will exhibit a signal which varies sinusoidally with magnetic field. [The rotation of the emergent field and the sinusoidal signal are called Gantmakher-Kaner oscillations (GKO).[19]] The amount of rotation also increases linearly with the sample thickness. Therefore ΔB, the change in magnetic field required to produce a rotation of 2π (or one complete sinusoidal period), is inversely proportional to the slab thickness.

The second main feature of the transmission spectrum is the cyclotron-phase-resonance (CPR) phenomenon itself: When the magnetic field strength is such that the cyclotron frequency of the carriers is equal to the frequency of the microwave field, the amplitude of the GKO is predicted to be largest. The envelope of the GKO was calculated[10] (in the limit of a thick slab) to have the shape of a square root of a Lorentzian. The phase of the GKO (again, in the thick-slab limit) was calculated to slip by π (relative to the linear increase) as the magnetic field is swept through the resonance value. For thinner slabs the shape of the peak in the envelope was predicted to be less pronounced.

In Fig. 1 we have evaluated the envelope of the GKO (i.e., the absolute magnitude of the transmission amplitude calculated variationally) for the parameters indicated and a series of slab thicknesses, ranging from $L = 2$ mean free paths down to $L = 0.1$ mean free path. In this figure the curves have been scaled to the same peak height so that the decrease of transmission with slab thickness is suppressed, while the broadening of the cyclotron phase resonance in very thin slabs is made apparent. The thickest slabs have such a rapid variation of phase with magnetic field that it would be impractical to calculate the sinusoidal signals variationally. We have calculated the sinusoidal signals variationally for the thinner slabs and, in Fig. 2, we compare these calculated results to those evaluated using the explicit formula of Ref. 10.

B. Case of A_2 correlations only

Let us now consider the situation in which $A_1 = 0$ but A_2 is given a finite value. The new feature to be expected in this situation is related to the infinite-medium CCM mode, a mode whose dispersion relationship $k = k(\omega, \omega_c)$ is independent of A_1. In the limit of infinitely long mean free path, CCM[9] found that the mode could propagate at values of magnetic field lying between mode turn-on at $(\omega_c/\omega) = (1 + A_2)^{-1}$, for which value $k = 0$, and mode cutoff at $(\omega_c/\omega) = \frac{5}{3}(1 + A_2)^{-1}$, for which value k becomes large enough for Doppler-shifted cyclotron resonance[20] to damp the mode. Our calculation of the mode dispersion relation assuming a finite mean free path[15] indicated that the attenuation length for the mode was *always less* than a mean free path, and that the propagation became better (attenuation length became longer) as the magnetic field advanced from mode turn-on to mode cutoff. In neither calculation was the coupling to the mode calculated, but CCM expressed the opinion that coupling to the field should be best at turn-on.

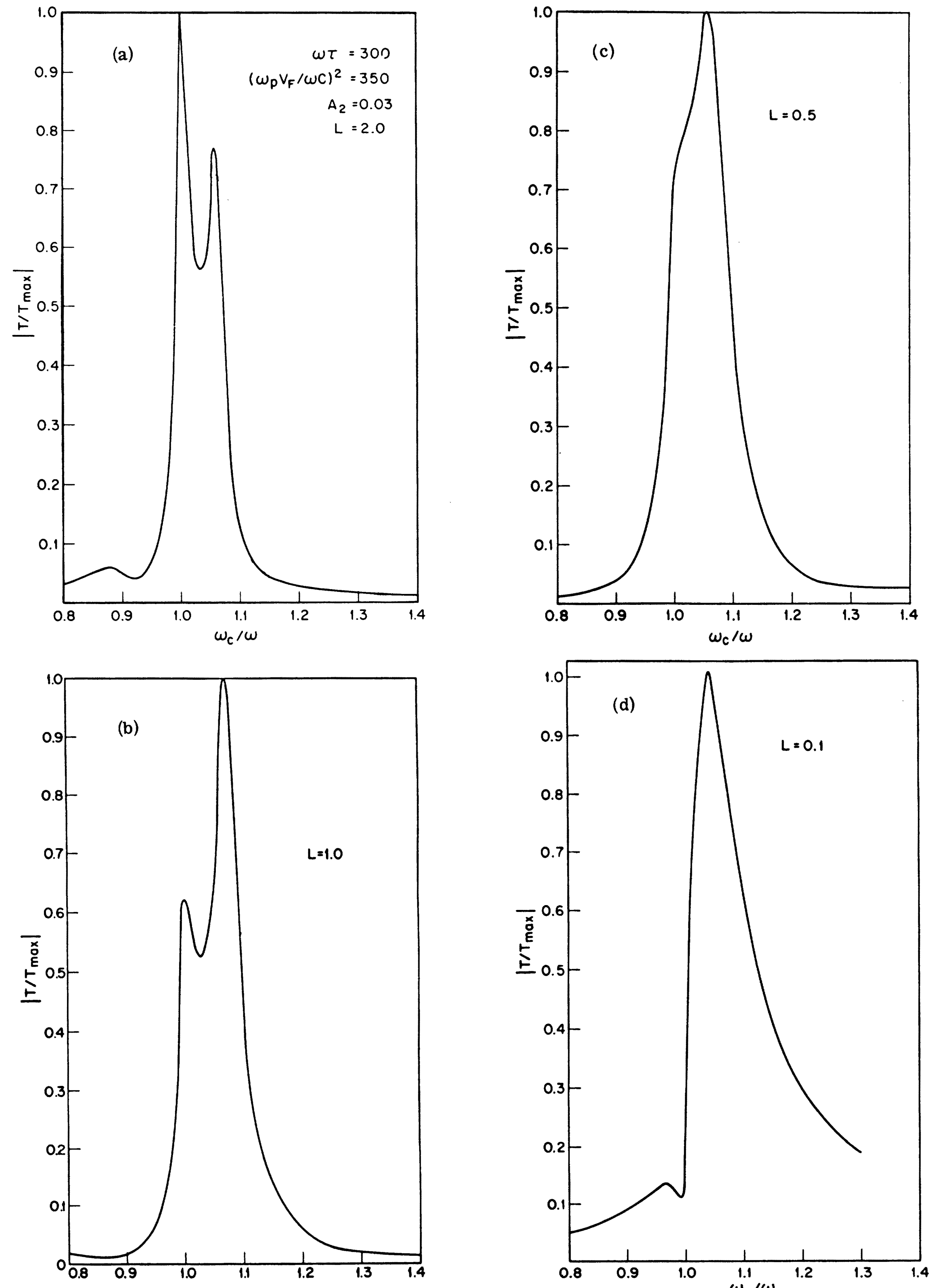


FIG. 3. Absolute value of transmission amplitude, normalized to a maximum transmission of unity, as a function of magnetic field strength for various of slab thickness, calculated for a correlated electron liquid in which $A_2 = -0.03$. Th infinite-medium CCM mode will propagate in the range $1.03 < \omega_c/\omega < 1.05$.

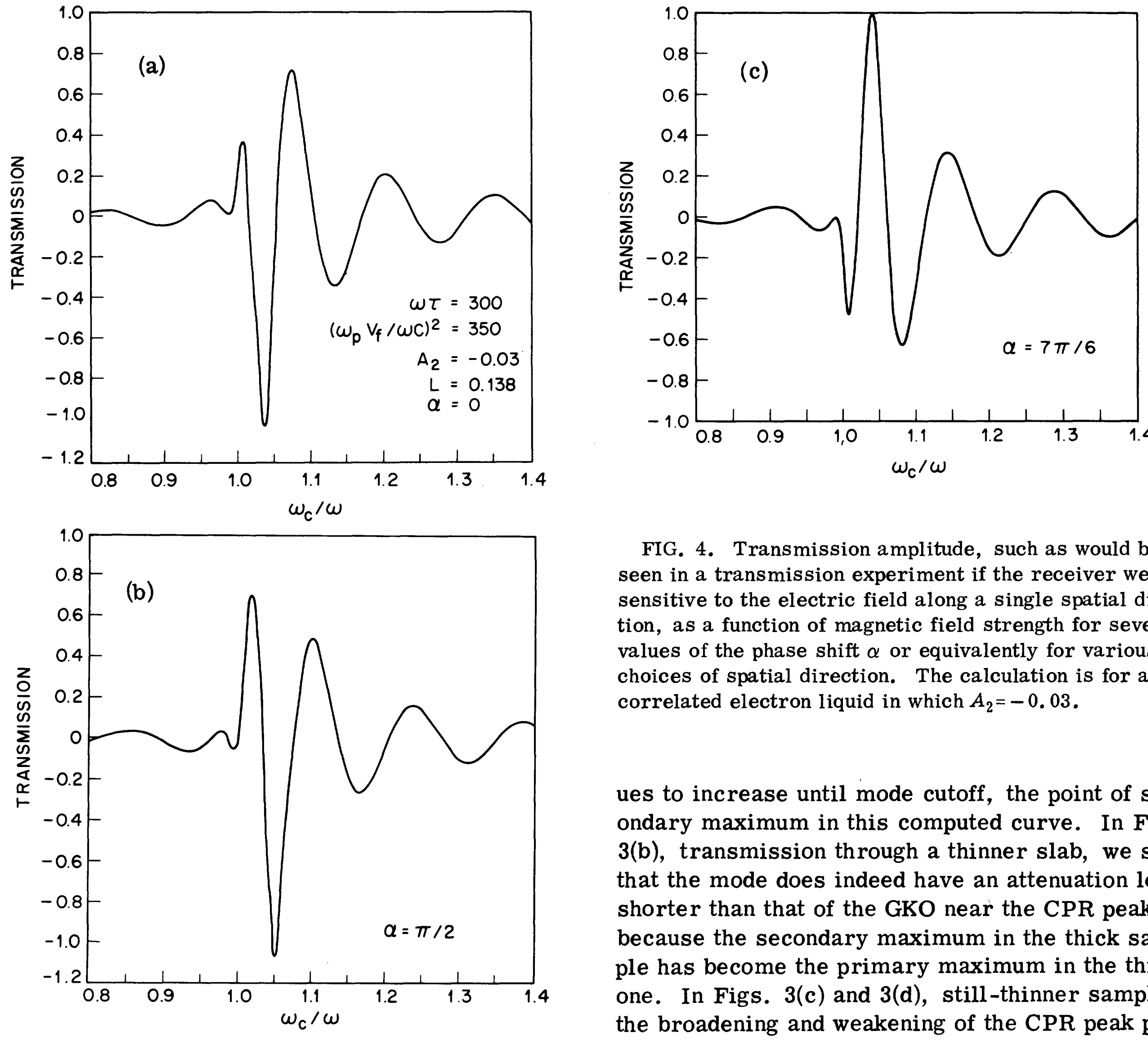
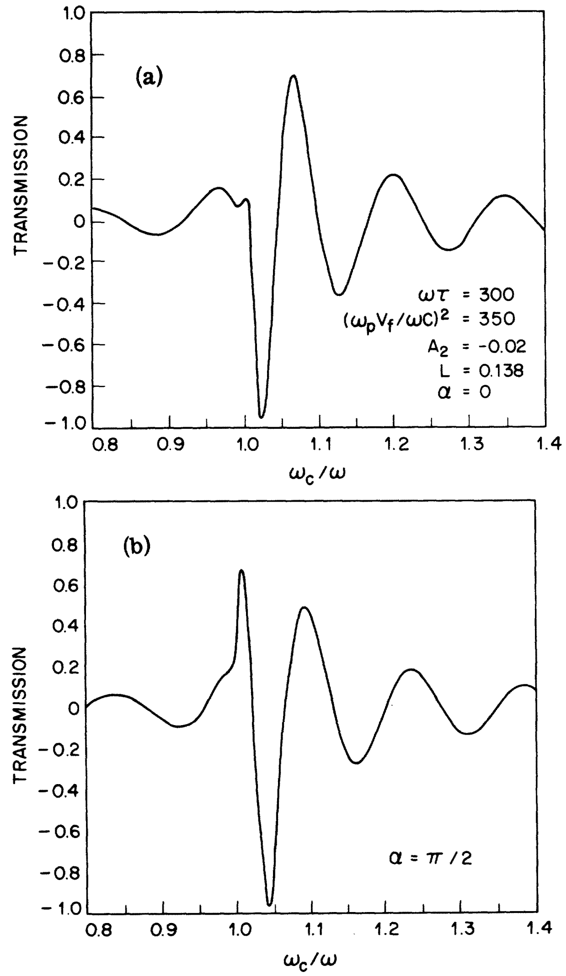


FIG. 4. Transmission amplitude, such as would be seen in a transmission experiment if the receiver were sensitive to the electric field along a single spatial direction, as a function of magnetic field strength for several values of the phase shift α or equivalently for various choices of spatial direction. The calculation is for a correlated electron liquid in which $A_2 = -0.03$.

As observed in transmission, the strength of the mode will depend on how strongly it is coupled to the field and on how well it propagates. It is therefore not evident *a priori* where, in the field range for which propagation can occur, to look for features associated with the mode. The remarks made at the end of Sec. IV suggest, however, that existence of the mode is signaled by an *increase* in transmission.

In Fig. 3 we have evaluated the envelope of the transmission (i.e., the absolute magnitude of the transmission amplitude calculated variationally) for the same parameters and slab thicknesses as in Fig. 1, but assuming that A_2 has the reasonable value -0.03. The infinite-medium mode should propagate in the range $1.03 < \omega_c/\omega < 1.05$ and indeed, in Fig. 3(a), transmission through the thickest slab, we see that, although the cyclotron-phase-resonance peak at ω_c/ω is the point of greatest transmission, the falloff in transmission is interrupted at mode turn-on. The transmission continues to increase until mode cutoff, the point of secondary maximum in this computed curve. In Fig. 3(b), transmission through a thinner slab, we see that the mode does indeed have an attenuation length shorter than that of the GKO near the CPR peak, because the secondary maximum in the thick sample has become the primary maximum in the thinner one. In Figs. 3(c) and 3(d), still-thinner samples, the broadening and weakening of the CPR peak progresses further. There is another feature present in Fig. 3 which was not expected on the basis of knowledge of the CCM mode. The new feature is a marked asymmetry in the transmission envelope, extending far beyond the regions of mode propagation. The asymmetry is especially marked in the thinner samples and is exaggerated still more if one compares the height of the envelope at equal distances either side of CPR at $\omega_c/\omega = 1$. This means that the GKO in these thin samples will have far larger amplitude on one side of CPR (the side for which mode propagation occurs) than on the other.

In Fig. 4 we have plotted the transmission spectrum at various phase shifts; i.e., we have plotted the real part of (transmission amplitude times $e^{i\alpha}$) for various phase angles α, choosing that sample thickness which corresponds to the observed GKO period.[3] Figure 4 reveals three features—a rapid phase change near CPR, the transmission peak near mode cutoff, and the marked enhancement of the GKO on the high-field side of CPR. Figure 5



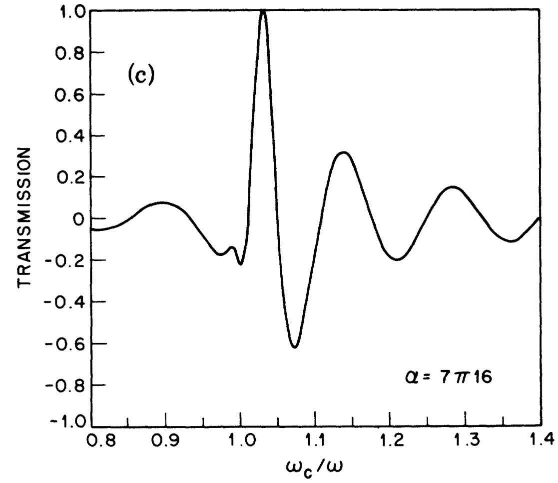


FIG. 5. Same as Fig. 4(c) except $A_2 = -0.02$.

displays the same calculation as does Fig. 4 but for a smaller (but still reasonable) value of A_2, namely, -0.02. In Figs. 6 and 7 we exhibit the envelope and sinusoidal signal for smaller values of mean free path than is used in Fig. 4, but for parameters which are otherwise identical. The amount of asymmetry, the height, and sharpness of the peak in the transmission envelope are all adversely affected by the deterioration in mean free path. This is to be expected on the basis of Eqs. (2.3) and (2.5), which show that the parameter by which correlations enter the problem is $\omega\tau A_2/(1+A_2)$. In Figs. 4, 6, and 7, this parameter is approximately 9, 3, and 0.9, respectively.

C. Effect of A_1

Finally, we consider the effect of giving A_1 a finite value. Figure 8 displays the transmission computed for parameters identical, aside from the value of A_1, to those used in Fig. 4(c). In Fig. 8(a) A_1 has been given the value 0.2, and in Fig. 8(b) A_1 has been given the value -0.2. The contrast between the effectiveness of A_1 and A_2 in altering the transmission is striking. Whereas a relatively small value of A_2 (-0.03) produced a large change in the transmission spectrum, a large value of A_1 (± 0.2) has only a small effect on the transmission. That small effect is apparent only near $\omega_c/\omega = 1$, and can be regarded as a line-shape distortion rather than a shifting of the transmission spectrum or any of the features in it.

The reason that A_1 is so relatively ineffective compared to A_2 (and to the higher A_n if we had included them) is related to the reason that A_1 does not influence the dispersion relation of the infinite-medium modes, namely, the tendency of Maxwell's equation (2.6) to exclude current from the bulk of a good conductor. [When the conductivity is good and b is a huge number, Eq. (2.6) causes ψ_0 to oscillate rapidly unless ψ_1 is exceedingly small. The result, as is well known, is that the current ψ_1 is strongly confined to the incident skin-depth region, the only place where the field ψ_0 is both large and rapidly oscillating.] One sees from (2.3) that the parameter A_n enters the physics only in the combination $h_n\psi_n(x)$. In the bulk, $\psi_1(x)$ is virtually zero, so A_1 does not enter the infinite-medium dispersion relation. In our transmission problem here, A_1 enters only because there *is* current in the incident skin depth so that $\psi_1(x)$ is not negligible at very small x. Each higher A_n parameter, on the other hand, enters the problem multiplied by a $\psi_n(x)$ which is not constrained to be small in the bulk.

The possibility that A_1 might be measured in an experiment which is independent of surface conditions requires that one work in a regime where $\psi_1(x)$ is not virtually zero in the bulk. One approach to such a regime is to raise the experimen-

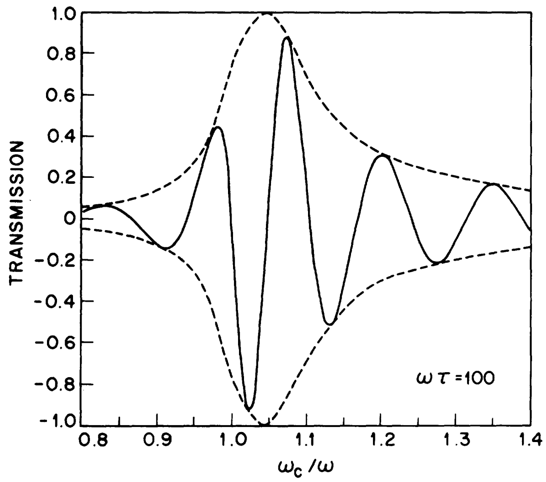


FIG. 6. Same as Fig. 4(c) except $\omega\tau = 100$.

tal frequency and to observe the high-frequency cyclotron waves, as Dunifer, Schmidt, and Walsh[21] have suggested. Their calculations of the dispersion relation for these waves suggest that A_1 might measurably affect the wave propagation at frequencies approximately double those used by Phillips, Baraff, and Dunifer,[3] but such an experiment has yet to be performed.

VI. COMPARISON BETWEEN HEURISTIC TREATMENT AND VARIATIONAL CALCULATION

The heuristic treatment is based on three simple ideas which were described and criticized in Ref. 17. A more formal (but in many respect much more satisfactory) approach to the same treatment is provided by the expression for the transmission derived by Falk, Henningsen, Skriver, and Christensen.[22] The essential result of their analysis is that the transmission is proportional to

$$T \approx \int_0^D \int_0^D f(D-z)\sigma(z, z')f(z')\,dz\,dz' , \qquad (6.1)$$

where $\sigma(z, z')$ is the nonlocal conductivity, relating the current j and the field e by

$$j(z) = \int_0^D \sigma(z, z')e(z')\,dz' ,$$

and where $f(z)$ is a function closely related to the electric field, namely,

$$f(z) = [e(z) - \rho e(D-z)]/(1-\rho^2) ,$$

$$\rho \equiv e(D)/e(0) .$$

Since ρ is such a small quantity, $f(z)$ is essentially equal to $e(z)$ everywhere except within a skin depth of the emergent face, where the functions differ in that $f(D) = 0$. The integral over the region near the emergent skin depth provides such a small contribution that we can, with excellent accuracy, replace (6.1) by

$$T \approx \int_0^D \int_0^D e(D-z)\sigma(z, z')e(z')\,dz\,dz' . \qquad (6.2)$$

If one proposes that the electric field within the slab is large only in the incident skin depth, then it would seem reasonable to replace (6.2) by

$$T \approx \left(\int_0^D e(D-z)\,dz\right)\sigma(D, 0)\left(\int_0^D e(z')\,dz'\right) . \qquad (6.3)$$

This form, simple as it is, still contains much of the essential physics. As an example, suppose we consider transmission through a fairly thick slab of free-electron gas. The large-D behavior of the conductivity is proportional to

$$\sigma(D, 0) \approx e^{-aD}/(aD)^2 . \qquad (6.4)$$

This contains the GKO (in the exponential term) and a *strong* cyclotron phase resonance (Lorentzian envelope and 2π phase slip) in the term a^{-2}. The electric field $e(z)$ is almost everywhere equal to the infinite-medium field $e_0(z)$ and we can make the replacement

$$\int_0^D e(z)\,dz \approx \int_0^D e_0(z)\,dz \approx \int_0^\infty e_0(z)\,dz .$$

But this last integral is the $k = 0$ component of the Fourier transform of $e_0(z)$. It is equal, to within a constant which depends on the normalization of the field, to[23]

$$\int_0^\infty e_0(z)\,dz = (ia/b)^{1/2} . \qquad (6.5)$$

Combining (6.4) and (6.5) gives a transmission

$$T \approx e^{-aD}/aD^2 , \qquad (6.6)$$

a form which exhibits the GKO (in the exponential term) and a *weak* cyclotron phase resonance (square root of Lorentzian envelope and π phase shift) in the term a^{-1}. This latter form, the weak cyclotron phase resonance, corresponds exactly to what is found by a detailed calculation using the two-sided Wiener-Hopf technique,[10] the multiple-reflection

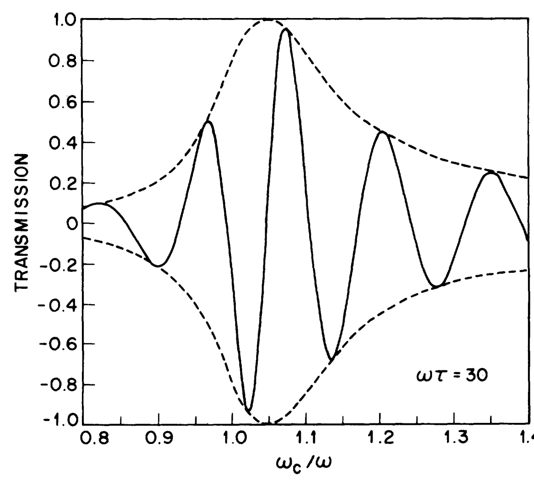


FIG. 7. Same as Fig. 4(c) except $\omega\tau = 30$.

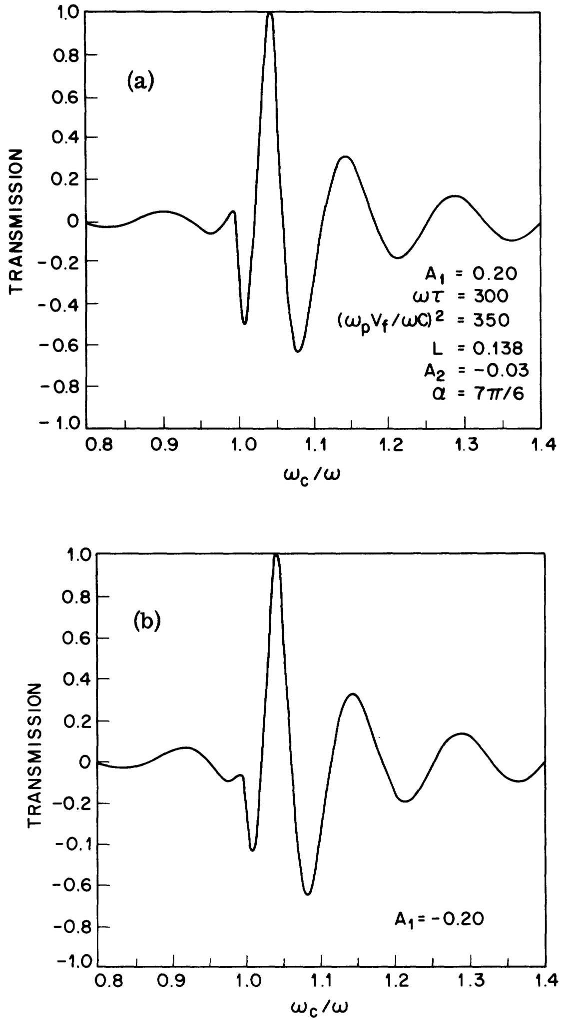


FIG. 8. Same as Fig. 4(c) except $A_1 = \pm\, 0.20$.

technique,[24] or the variational technique.[16]

In the heuristic treatment, the magnetic field dependence of the integral (6.5) is ignored, so that (6.3) is replaced simply by

$$T \approx \sigma(D,\ 0) \tag{6.7}$$

and the transmission T is taken proportional to the conductivity.

The nonlocal conductivity is a complicated object which describes how the electrons interact with themselves, with the ambient magnetic field, and, in the slab problem, with the boundary faces of the slab. This latter feature makes it impractical to calculate σ. If the medium were infinite, then there would be no problem of interaction with the boundaries and the nonlocal conductivity would be translation invariant:

$$\sigma(z,\ z') = \sigma_\infty(z - z') . \tag{6.8}$$

The quantity σ_∞, the infinite-medium conductivity, can be calculated as was explained in Ref. 3. The formula used to interpret the data in Ref. 3 was (6.7), as approximated by (6.8), namely,

$$T \approx \sigma_\infty(D) . \tag{6.9}$$

Let us summarize the steps leading from (6.2), which is firmly based on simple general considerations, to (6.9), which does seem to describe the data.

(a) The assumption that the field $e(z)$ is well enough confined to the skin-depth region that one can write

$$\int_0^D e(z) f(z)\, dz \approx f(0) \int_0^D e(z)\, dz \equiv f(0) I$$

for slowly varying functions $f(z)$.

(b) The assumption that the magnetic field dependence of the integral I is weak enough to ignore.

(c) The assumption that the nonlocal conductivity $\sigma(z,\ z')$ calculated for the slab differs negligibly from the infinite-medium conductivity $\sigma_\infty(z - z')$.

When (6.9) is evaluated numerically as discussed in Ref. 3, the features of interest are these. (i) CPR occurs at $(\omega_c/\omega) = (1 + A_1)^{-1}$. (ii) The GKO have a much larger amplitude on that side of CPR for which the CCM mode can propagate. (iii) The value of T at $(\omega_c/\omega) = (1 + A_2)^{-1}$, the turn-on field strength for the CCM mode, is reduced, giving rise to what we called the "A_2 notch."

The second and third features here are certainly prominent features in the data.[3] The first features may or may not be a feature of the data, since there is no other measurement of A_1 which could corroborate this. In any event, the experimental CPR peak occurs very close to $(\omega_c/\omega) = 1.0$, which implies an exceedingly small value for A_1 if one regards this first feature as valid.

The variational calculation, which proceeds from start to finish without utilizing the added assumptions (a)–(c), does exhibit feature (ii), the asymmetry of the GKO envelope. It is in contradiction with the heuristic treatment as far as concerns feature (iii). In the variational calculation, the region at mode propagation is one of enhanced, rather than diminished, transmission. Finally, the variational treatment fails, for finite A_2, to exhibit a strong CPR in the very thin samples. In the thicker samples where CPR is present, it occurs at $(\omega_c/\omega) = 1.0$, rather than at $(\omega_c/\omega) = (1 + A_1)^{-1}$. In addition, the A_2 mode appears as a distinct feature of the calculated transmission in the thicker samples.

VII. SUMMARY AND SPECULATIONS

The primary motive for embarking on the study reported in this series of three papers (Refs. 16, 17, and the present work) was the hope that such a study would support the heuristic treatment. If it had done so, the experiments reported in Ref. 3 would have given, for the first time, a measurement of the Landau parameter A_1 in a metal. Support for the heuristic treatment evaporated upon completion of the first phase of the work,[17] which dealt with the effect of A_1 on the fields in a semi-infinite slab. At that point, three important predictions of the model, a uniform Fermi liquid terminated by a single diffuse scattering surface, had emerged.

(a) The maximum field amplitude deep within the medium is *not* at $(\omega_c/\omega)=(1+A_1)^{-1}$.

(b) The effect of A_1 on the fields within the slab is much greater than its effect on the fields in the anomalous skin depth (and hence on the surface impedance). It is great enough to suggest that a transmission experiment in this geometry would be measurably influenced by a reasonable-sized A_1.

(c) The effect of A_1 on the fields within the medium is small enough to calculate via first-order perturbation theory.

The first of these conclusions raised doubts about the value of A_1 suggested in Ref. 3. It did, however, leave open the possibility that a full calculation (including A_1, A_2, and a finite slab) might still reproduce the data, perhaps with A_1 at some value other than the one suggested by the heuristic treatment.

The second of these conclusions made the full calculation a matter of interest, for the promise that A_1 would measurably affect the calculation left open the possibility that a value of A_1 might be extracted from the data of Ref. 3.

The third of these conclusions made the full calculation feasible, for it allowed us to consider a variational treatment in which two fields, rather than six, had to be varied.

The second phase of the work, that reported in Ref. 16, laid the mathematical groundwork upon which the final phase was built. In that work we developed the variational technique and the representation for ψ_0^0, both of which were essential for the work continued here. From that work there emerged two interesting observations.

(i) The first term of either the two-sided Wiener-Hopf method or of the multiple-reflection series,[24] both techniques presumably being expansions in the parameter e^{-L}, provided an excellent approximation to the true solution even for L as small as 0.1.

(ii) The change in physical problem from the semi-infinite medium to finite slab produced a large change in electric *field* ψ_0 but very little change in electric *current* ψ_1.

The full variational calculation reported here is at variance with the heuristic treatment. This means merely that one or more of the three added assumptions [(a)–(c) of Sec. VI] has been proved false. Unfortunately, the full variational calculation cannot be made to agree with the data of Ref. 3. In particular, the observed strong sharp transmission peak near $\omega_c/\omega=1$ does not show up in the calculation. Moreover, the shape of the calculated spectrum in that region is not similar enough to what is observed to allow a determination of A_1.

The variational calculation does account for asymmetry of the GKO amplitudes which it, like the heuristic treatment, ascribes to A_2. It also supports features of the earlier two papers and of other bulk studies. Among these shared features are the following.

(a) The role of A_1, away from cyclotron phase resonance, is to augment the GKO amplitude on the high- (low-) field side of CPR for negative (positive) sign of A_1.

(b) The change in the physical problem from non-correlated electrons to correlated quasiparticles produced large changes in the (electric) field ψ_0 and the (quadrupole moment of the distribution function) field ψ_2, but very little change in the electric current.

(c) This rigidity of the electric current, namely, the possibility of changes in ψ_0 and ψ_2 but not in ψ_1, which is the essence of the physics of the modes discussed by CCM, resulted in the finite-slab equivalent of these modes appearing in the transmission spectrum. The analysis here predicted, and the numerical work confirmed, that in this model, the mode is signaled by an increase in transmission.

An interesting new aspect of the variational calculation is its prediction that the CCM mode should appear with reasonable observable strength in transmission experiments carried out in thicker samples as a secondary maximum located at mode cutoff. The most interesting aspect of the variational calculation, however, is its *failure* to account for the observations on the thin slabs—in particular, for its failure to predict the strong sharp transmission peak at $\omega_c/\omega=1$, which is the most prominent feature of the data. This has destroyed the possibility which was proffered in Ref. 3, namely, that the transmission spectrum could be understood to be a consequence of bulk Fermi-liquid theory and reasonably simple boundary conditions on the quasiparticles.

What has gone wrong? It is hard to imagine any mechanisms which would invalidate or seriously modify the standard apparatus of bulk Fermi-liquid theory under the conditions of the experiment. Our present knowledge of how the particles interact with the boundary faces of the slab is great enough to let us recognize that the diffuse-scattering boundary condition is a gross simplification. However, we have no simple physical arguments which we can make to connect any feature of quasiparticle behavior at the boundary with any of the unexplained features in the transmission spectrum. Nor do we have sufficient mathematical power to make a frontal analytical or numerical attack on the problem seem promising. Still, it is possible to offer some suggestions which may be of use.

The transmission problem assuming specular boundary conditions and a correlated Fermi liquid has not, to our knowledge, been solved numerically. We bypassed this problem (even though its explicit analytic solution can be evaluated with perhaps 5% of the labor we expended here) because of our prejudice that the CPR peak in the data made uninteresting a model which (in the absence of correlations) could produce no peak. Once the focus of the investigation changes to the role played by boundary conditions, then the specular-boundary problem becomes interesting again as another solvable model whose results can be compared with what we have presented here.

There is yet another interesting boundary condition which should be mentioned, namely, the one suggested by Carolan and Van Gelder.[25] In their work the quantum-mechanical consequences of the boundary are noted explicitly. Briefly, the electrons which are most effective in establishing the skin-effect field are those which are in Doppler-shifted cyclotron resonance with the field. When $\omega_c = \omega$, this is the group of electrons with zero momentum normal to the surface. But if electronic wave functions vanish at the surface, and if it takes a healing distance of the order of a de Broglie wavelength for the electron density to return to its bulk value, then this group of electrons is the one with the longest healing distance. Their exclusion from the surface region *must* have a strong effect on the fields at the surface, because their presence (in the nonquantum description) is the most important factor in establishing the distribution of the field. Carolan and Van Gelder reported calculating a transmission peak even with specular-boundary conditions and no Fermi-liquid effects. One wonders what further exploration of their model as applied to thin slabs, or of extensions of their model to include Fermi-liquid effects, might reveal. Finally, it is tempting to speculate whether some phenomenological treatment of surface roughness[26] would be of as much use in explaining the data here as it was in the corresponding surface impedance problem.

It would be wrong to conclude without again mentioning the remarkable agreement between the heuristic treatment and the experimental data. Even if *no* model supports the heuristic treatment, the empirical conclusion which can be drawn from Ref. 3 is that the observed transmission T looks very much like the nonlocal conductivity $\sigma_\infty(D)$ calculated using Landau Fermi-liquid theory, provided that A_1 (whatever its true value may be) is taken almost equal to zero. Is there an underlying reason that this may be so? Certainly the idea that A_1 is relatively ineffective in altering the fields in this situation (where the large conductivity constant b acts to suppress the current) makes it plausible that there is validity in a treatment which discards A_1 at the outset, but this statement is a long way from having a model which explains the resemblance between the transmission and the conductivity.

ACKNOWLEDGMENTS

We thank Professor M. Ya. Azbel' for calling our attention to Ref. 5, in which he specifically suggested the utility of high-frequency cyclotron-resonance experiments for determining the Fermi-liquid parameters. We also thank T. G. Phillips, P. M. Platzman, T. M. Rice, J. M. Rowell, and W. M. Walsh, Jr. for useful discussions during the course of this work.

APPENDIX A: OUTLINE OF VARIATIONAL CALCULATION

Let us first establish notation for the integrals we shall encounter, all of which are of the form

$$\int_0^L \int_0^L \Phi_i(x) K(x-y) \psi_j(y)\, dx\, dy\,.$$

To accommodate the relation (4.3) which we shall use throughout, we shall define

$$(f, K, g) \equiv \int_0^L \int_0^L f(L-x) K(x-y) g(y)\, dx\, dy = (g, K, f). \tag{A1}$$

If f, K, or g is an exponential, we shall suppress the argument $(L-x)$, $(x-y)$, or y.

For the first stage of the calculation, we shall set $A_1 = A_2 = 0$, and use (4.5) and (4.3) as trial functions to evaluate μ [Eq. (3.8)] with the result

$$\mu = [(\psi_1^0, \psi_0^0 + Ae^{-p}) + (\psi_1^0, ib\hat{G}, \psi_1^0) + (\psi_0^0 + Ae^{-p}, K_{11}, \psi_0^0 + Ae^{-p}) + (\psi_0^0 + Ae^{-p}, \psi_1^0)][(\psi_1^0, \mathfrak{M}_{01}, \psi_1^0)]^{-1} \tag{A2a}$$

$$\equiv [F_0 + 2F_1(p)A + F_3(p)A^2]/G_0\,, \tag{A2b}$$

where

$$F_1 \equiv (e^{-p}, \psi_1^0) + (e^{-p}, K_{11}, \psi_0^0)\,, \tag{A3a}$$

$$F_3 \equiv (e^{-p}, K_{11}, e^{-p})\,, \tag{A3b}$$

$$F_0 = (\psi_1^0, \psi_0^0) + ib(\psi_1^0, \hat{G}, \psi_1^0) + (\psi_0^0, K_{11}, \psi_0^0) + (\psi_0^0, \psi_1^0)\,, \tag{A3c}$$

and

$$G_0 = (\psi_1^0, \mathfrak{M}_{01}, \psi_1^0)\,. \tag{A3d}$$

The two variational equations $\partial\mu/\partial A = 0$ and $\partial\mu/\partial p = 0$ become

$$A = -\frac{F_1(p)}{F_3(p)} \tag{A4a}$$

and

$$\frac{d}{dp}\left(\frac{F_1^2(p)}{F_3(p)}\right)=0. \tag{A4b}$$

This problem, defined by Eqs. (A2)–(A4), was solved in Ref. 16, and so we start our calculation with both A and p known.

For the second stage of the calculation, we maintain $A_1=0$ but let A_2 be infinitesimally small. Using (4.5), (4.6), and (4.3) as trial functions to evaluate μ [Eq. (3.8)], we have

$$\mu=[(\hat{\psi}_1,\ \hat{\psi}_0)+ib(\hat{\psi}_1,\ \hat{G},\ \hat{\psi}_1)+(\hat{\psi}_0,\ K_{11},\ \hat{\psi}_0)+(\hat{\psi}_0,\ \hat{\psi}_1)+ih_2(\hat{\psi}_0,\ K_{12},\ \psi_2^0+Be^{-q}) + ih_2(\psi_2^0+Be^{-q},\ K_{21},\ \hat{\psi}_0)+ih_2(\psi_2^0+Be^{-q},\ 1+ih_2K_{22},\ \psi_2^0+Be^{-q})]/G_0 \tag{A5a}$$

$$\equiv[f_0+2\tilde{f}_1(q)B+\tilde{f}_3(q)B^2]/G_0, \tag{A5b}$$

where

$$\tilde{f}_1/ih_2=(e^{-q},\ K_{21},\ \hat{\psi}_0)+(e^{-q},\ 1+ih_2K_{22},\ \psi_2^0)\equiv f_1,$$

$$\tilde{f}_3/ih_2=(e^{-q},\ 1+ih_2K_{22},\ e^{-q})\equiv f_3.$$

The variational equations $\partial\mu/\partial B=0$ and $\partial\mu/\partial q=0$ give

$$B=-\frac{f_1(q)}{f_3(q)} \tag{A6a}$$

and

$$\frac{d}{dq}\left(\frac{f_1^2(q)}{f_3(q)}\right)=0 \tag{A6b}$$

and, going to the limit $A_2=0$, we have

$$f_1=(e^{-q},\ \psi_2^0)+(e^{-q},\ K_{21},\ \hat{\psi}_0), \tag{A7a}$$

$$f_3=(e^{-q},\ e^{-q}). \tag{A7b}$$

In the $A_2=0$ limit the solution to (A6b) is given by $q=p$. With q known, B can be evaluated using (A6a).

Solving (A6), we have B, q, and therefore $\hat{\psi}_2$. For the third stage, we keep $A_1=0$ but let A_2 be finite, and we use (4.7) and (4.3) as trial functions to evaluate μ [Eq. (3.8)], with the result

$$\mu=[(\hat{\psi}_1^0,\ \hat{\psi}_0+\delta\psi_0)+ib(\hat{\psi}_1,\ \hat{G},\ \hat{\psi}_1)+(\hat{\psi}_0+\delta\psi_0,\ K_{11},\ \hat{\psi}_0+\delta\psi_0)+(\hat{\psi}_0+\delta\psi_0,\ \hat{\psi}_1)+ih_2(\hat{\psi}_0+\delta\psi_0,\ K_{12},\ \hat{\psi}_2+\delta\psi_2) + ih_2(\hat{\psi}_2+\delta\psi_2,\ K_{21},\ \hat{\psi}_0+\delta\psi_0)+ih_2(\hat{\psi}_2+\delta\psi_2,\ 1+ih_2K_{22},\ \hat{\psi}_2+\delta\psi_2)]/G_0 \tag{A8a}$$

$$\equiv(V_0+2V_1+V_3)/G_0, \tag{A8b}$$

where V_0 is independent of, V_1 is linear in, and V_3 is quadratic in, the fields to be varied. In particular, we have

$$V_0=(\hat{\psi}_1,\ \hat{\psi}_0)+ib(\hat{\psi}_1,\ \hat{G},\ \hat{\psi}_1)+(\hat{\psi}_0,\ K_{11},\ \hat{\psi}_0)+(\hat{\psi}_0,\ \hat{\psi}_1) + 2ih_2(\hat{\psi}_0,\ K_{12},\ \hat{\psi}_2)+ih_2(\hat{\psi}_2,\ 1+ih_2K_{22},\ \hat{\psi}_2).$$

The first four terms here are the same as those appearing in (A2a) and, because of (A4a), their sum is $F_0-F_1^2/F_3$. The last two terms here can be added together to give

$$ih_2(\hat{\psi}_2,\ -1+ih_2K_{22},\ \hat{\psi}_2)+2ih_2(\hat{\psi}_2,\ \hat{\psi}_2+K_{12}\hat{\psi}_0).$$

If $\hat{\psi}_0$, $\hat{\psi}_1$, and $\hat{\psi}_2$ actually satisfied (4.8) instead of being merely our variational approximation to the actual solution, then $\hat{\psi}_2+K_{12}\hat{\psi}_0$ would be zero. To the extent that our variational approximation is a good one, this term will be small. Since we are not particularly interested in improving our variational approximation for $\hat{\psi}_0$ and $\hat{\psi}_2$ but are interested in $\delta\psi_0$ and $\delta\psi_2$ as the changes in ψ_0 and ψ_2 induced by A_2, we drop this small term and are left with

$$V_0=F_0-F_1^2/F_3+ih_2(\hat{\psi}_2,\ -1+ih_2K_{22},\ \hat{\psi}_2). \tag{A9a}$$

Furthermore, from (A8), we have

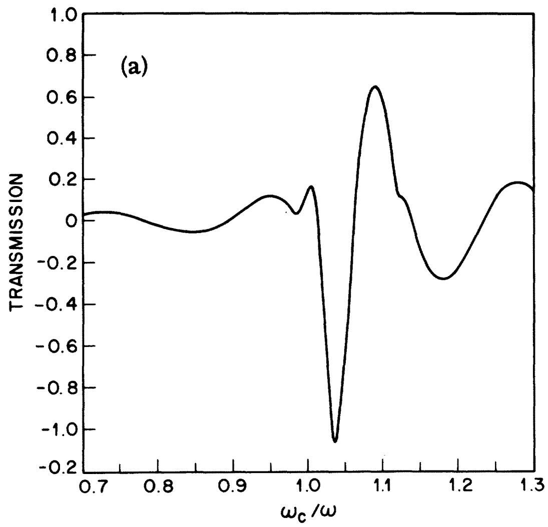
$$V_1=(\delta\psi_0,\ \hat{\psi}_1+K_{11}\hat{\psi}_0)+ih_2(\delta\psi_0,\ K_{12},\ \hat{\psi}_2) + ih_2[(\delta\psi_2,\ \hat{\psi}_2+K_{12}\hat{\psi}_0)+ih_2(\delta\psi_2,\ K_{22},\ \hat{\psi}_2)].$$

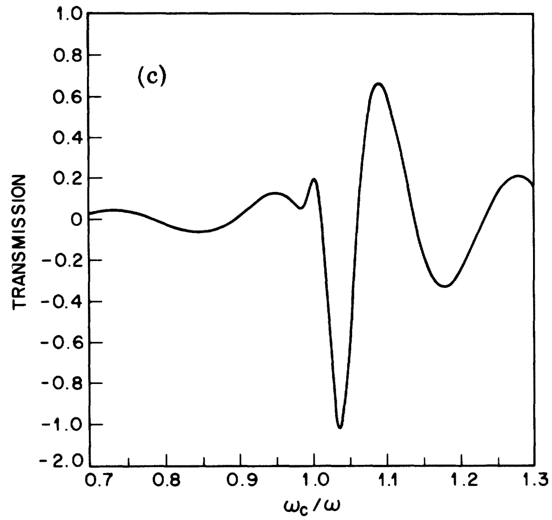
Again, because we are not interested in further improving the quality of our variational solution to (4.8), we drop those terms in V_1 which would have vanished if the $\hat{\psi}_j$ were indeed the exact solutions to (4.8). This leaves us with

$$V_1=ih_2(\delta\psi_0,\ K_{12},\ \hat{\psi}_2)+(ih_2)^2(\delta\psi_2,\ K_{22},\ \hat{\psi}_2). \tag{A9b}$$

Finally, we have

$$V_3=(\delta\psi_0,\ K_{11},\ \delta\psi_0)+2ih_2(\delta\psi_0,\ K_{12},\ \delta\psi_2) + ih_2(\delta\psi_2,\ 1+ih_2K_{22},\ \delta\psi_2). \tag{A9c}$$





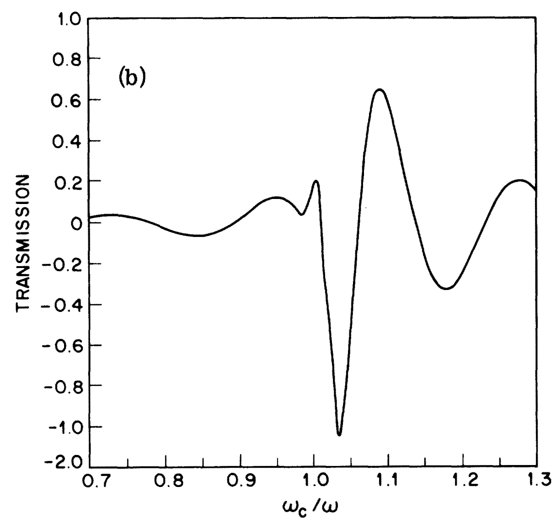


FIG. 9. Transmission amplitude as a function of magnetic field as calculated for three different sets of p_i parameters. The physical parameters are the same as those in Fig. 4 except that $L=0.1$. Note the over-all similarity of the three curves here and the sharp feature near $\omega_c/\omega=1.12$ which is present in Fig. 9(a) but absent in Figs. 9(b) and 9(c). Values of the p_i parameters are given in Table I.

Let us use the forms (4.14) in (A9). Changing the notation of (A14) slightly so that α, β, and the a_i are regarded as the set of linear variational parameters b_i, while $\hat{\psi}_2$ and $e^{-p_i x}$ are regarded as a set of given functions $u_i(x)$, would allow us to write (4.14) compactly as

$$\delta\psi_0=\sum b_i\mu_i(x)\,,\quad i\leq N_0+1$$

$$\delta\psi_2=\sum b_i\mu_i(x)\,,\quad i>N_0+1$$

which would give us

$$V_1=\sum Y_i b_i\,, \qquad \text{(A10a)}$$

$$V_3=\sum\sum M_{ij}b_i b_j\,, \qquad \text{(A10b)}$$

where

$$Y_i\equiv ih_2(u_i,\,K_{12},\,\hat{\psi}_2)\qquad (i\leq N_0+1) \qquad \text{(A11a)}$$

$$\equiv(ih_2)^2(u_i,\,K_{22},\,\hat{\psi}_2)\qquad (i>N_0+1)\,, \qquad \text{(A11b)}$$

$$M_{ij}=(u_i,\,K_{11},\,u_j)\qquad (i\leq N_0+1,\ j\leq N_0+1)$$

$$=ih_2(u_i,\,K_{12},\,u_j)\qquad (i\leq N_0+1,\ j\leq N_0+1 \text{ or} \qquad \text{(A12a)}$$

$$j\leq N_0+1,\ i>N_0+1) \qquad \text{(A12b)}$$

$$=ih_2(u_i,\,1+ih_2K_{22},\,u_j)\ (i>N_0+1,\ j>N_0+1). \qquad \text{(A12c)}$$

The variational equations $\delta\mu/\delta b_i=0$ give

$$\sum_j M_{ij}b_j=-Y_i\,, \qquad \text{(A13)}$$

which may be solved for the b_i. Using that solution gives $V_1=-V_3$ and, as a result,

$$\mu=(V_0-V_3)/G_0\,. \qquad \text{(A14)}$$

Finally, we let A_1 be nonzero. The fields just determined are used in the variational principle. The change in μ which results is

$$\delta\mu=ih_1[(\psi_0,\,K_{11},\,\psi_1)+ih_2(\psi_2,\,K_{21},\,\psi_1)]G_0$$

$$=ih_1\left[\left(\hat{\psi}_0+\sum^{N_0+1}b_iu_i,\,K_{11},\,\psi_1^0\right)+ih_2\left(\hat{\psi}_2+\sum_{N_0+2}b_iu_i,\,K_{21},\,\psi_1^0\right)\right]\Big/G_0\,. \qquad \text{(A15)}$$

TABLE I. Real part of p_i.

Parameters used for	i	Figure 3(d)	4, 5, and 8	9(a)	9(b)	9(c)	All others
	1	20	17	25	20	20	
$\delta\psi_0$	2	10	6	18	10	10	$N_0=0$
	3	−10.5	−6	11	−10.5	−10.5	
	4	−21	−17	4	1.5	−21	
	5	19	16	25.5	19	19	
$\delta\psi_2$	6	9	7	18.5	9	9	$N_2=0$
	7	−11.5	−7	11.5	−11.5	−11.5	
	8	−22	−16	4.5	−1.5	−22	

Our final value for μ is then

$$\mu=(V_0-V_3)/G_0+\delta\mu\,. \tag{A16}$$

APPENDIX B: COMMENTS ON CERTAIN NUMERICAL ASPECTS OF CALCULATION

It is clear that numerical representations of the fields ψ_0^0, ψ_1^0, and ψ_2^0 are needed if the various integrals appearing in Appendix A are to be evaluated. For ψ_0^0, the most convenient representation is that developed in the appendices of Ref. 16, namely, one of the form

$$\psi_0^0(x)=\int_0^\infty \psi(u)e^{-(a+u)x}\,du\,. \tag{B1}$$

The explicit form for ψ_0^0 is given in that reference.

Although the representation (B1) is not valid in the skin-depth region, the error introduced by using this representation can be virtually eliminated by making use of Eq. (4.4) in such a way that values of $\psi_0^0(x)$ near $x=0$ do not appear in any of the terms to be evaluated. Examples of this technique appear repeatedly in Sec. III of Ref. 16.

Most of the appearances of ψ_1^0 in the integrals appearing in Appendix A here can also be eliminated by use of Eq. (4.4). The two which cannot are those found in (A15). To handle these terms, we recognize that the current is so sharply concentrated near the surface of a semi-infinite medium that, for all functions $f(x)$ which vary slowly on the scale of the skin depth, we can expand $f(x)$ near $x=0$ and retain only the lowest-order terms in the integral,

$$\begin{aligned}\int_0^L f(x)\psi_1^0(x)\,dx &\approx \int_0^\infty f(x)\psi_1^0(x)\,dx\\ &\approx \int_0^\infty [f(0)+xf'(0)]\,\psi_1^0(x)\,dx\\ &= f(0)\int_0^\infty \psi_1^0(x)\,dx+f'(0)\\ &\qquad\times\int_0^\infty x\psi_1^0(x)\,dx\,.\end{aligned} \tag{B2}$$

For the special choice $f(x)=e^{\pm ik_0lx}$, we have the exact evaluation [cf. Eq. (3.9), Ref. 16]

$$\int_0^\infty e^{\pm ik_0lx}\psi_1^0(x)\,dx=-\left(\frac{k_0l}{b}\right)(Z\mp 1)\,, \tag{B3}$$

which is of the form (B2). Comparing (B2) and (B3), we have

$$\int_0^\infty \psi_1^0(x)\,dx=(-k_0lZ/b)\,, \tag{B4a}$$

$$\int_0^\infty x\psi_1^0(x)\,dx=1/ib\,. \tag{B4b}$$

Thus, (B2) becomes

$$\int_0^L f(x)\psi_1^0(x)\,dx=-(k_0lZ/b)[f(0)-f'(0)/(ik_0lZ)]\,, \tag{B5}$$

where Z is the dimensionless surface admittance. This quantity is so large that we may drop the second term in (B5) for functions which vary slowly on the scale of the skin depth, which is the case for those in (A15). Thus, for example,

$$\begin{aligned}(\psi_0,\,K_{11},\,\psi_1^0)&=\int_0^L\int_0^L \psi_0(L-x)K_{11}(x-y)\psi_1^0(y)\,dx\,dy\\ &=-(k_0lZ/b)\int_0^L \psi_0(L-x)K_{11}(x)\,dx\,.\end{aligned} \tag{B6}$$

The remaining integrals involve the fields ψ_2^0 and ψ_0^0, which are related. Because of the definition of the kernels (2.4), we have

$$K_{21}(x-y)=\sqrt{5}\left(a\int_x^\infty dx'K_{11}(x'-y)-\theta(y-x)\right).$$

Then, using (4.4b) and (4.4c), we have

$$\begin{aligned}\psi_2^0(x)&=-\int_0^\infty K_{21}(x-y)\psi_0^0(y)\,dy\\ &=\sqrt{5}\int_x^\infty [a\psi_1^0(y)+\psi_0^0(y)]\,dy\,.\end{aligned}$$

Considering the relative sizes of ψ_1^0 and ψ_0^0, we drop the first term here and obtain

$$\begin{aligned}\psi_2^0(x)&=\sqrt{5}\int_x^\infty \psi_0^0(y)\,dy\\ &=\sqrt{5}\int_0^\infty (a+u)^{-1}\psi(u)e^{-(a+u)x}\,du\,.\end{aligned} \tag{B7}$$

Finally, we must comment on the choice of N_0 and N_2, the number of exponential terms taken in the representation of $\delta\psi_0$ and $\delta\psi_2$ according to (4.14), and on the choice of the decay constants p_i appearing in the same equation.

We found that the results of the computation were qualitatively similar and relatively insensitive to the choice of the p_i values when we took N_0 and N_2 equal to 2, 3, or to 4. We chose the forward-propagating terms as

$$p_j = \gamma_j + i\text{Im}a$$

and the backward-propagating terms as

$$p_j = -(\gamma_j + i\text{Im}a),$$

where the γ_j were real numbers, greater than unity, which described the decay lengths of the terms. We distributed these γ_j in a range between 1 and an upper limit set by the requirement that the term be slowly varying relative to ψ_2^0. This upper limit decreased as the slab became thicker, and so for the thickest slab, we were able to dispense with the exponential terms (i.e., took N_0 and N_2 equal zero) altogether. The values of γ_j, set at the beginning of a calculation, were held fixed while the magnetic field was swept through a range (i.e., while a transmission spectrum was calculated). Occasionally a sharp feature of no obvious physical origin would appear in the spectrum [see, for example, Fig. 9(a)], but the position and strength of the feature was sensitive to the choice made for one of the γ_i. Changing that γ_i slightly would remove the feature [compare Figs. 9(a)–9(c)], leaving the rest of the spectrum essentially unaltered. The obvious procedure (ignore those runs in which a fortuitous choice of γ_i has introduced a sharp feature) is probably the correct one, because the variational technique asks that the parameters of the calculation be chosen in the way that makes the transmission least sensitive to their exact value.

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