Theories of subharmonic gap structures in superconducting junctions

L.-E. Hasselberg*
Department of Theoretical Physics, The Royal Institute of Technology, Stockholm, Sweden

M. T. Levinsen†
Physics Laboratory I, H.C. Ørsted Institute, University of Copenhagen, Denmark

M. R. Samuelsen
Physics Laboratory I, Technical University of Denmark, DK-2800 Lyngby, Denmark

(Received 2 October 1973)

The two theories of subharmonic gap structures in superconducting junctions, multiparticle tunneling and self-coupling due to an electromagnetic field set up by the ac Josephson current, are analyzed when microwaves are applied. Both theories give the same location in voltage for the microwave-induced satellites and the same microwave-power dependence for the subharmonic gap structure and the satellites. Therefore other properties than these are to be considered in order to distinguish between the two theories. We suggest that self-coupling is the main cause of the subharmonic gap structure.

I. INTRODUCTION

Subharmonic gap structure has been observed by several workers in tunneling in thin-film superconducting junctions[1–3] and in superconducting point-contact junctions.[4,5] In both cases the structure appears as small bumps or peaks in the current-voltage characteristic at, for identical superconductors, voltages $V=2\Delta/me$, where Δ is the gap parameter for the superconductors, $-e$ the electronic charge, and m is a positive integer. Usually the structure appears with a steadily decreasing magnitude as m increases. $m=1$ corresponds to the ordinary gap structure. Structure has been reported up to as high a value as $m=12$.[2]

Also changes in the subharmonic gap structure due to applied microwaves have been observed.[3,5,6] It turns out that the microwaves induce satellites around the subharmonic gap structures at voltages V given by

$$eV=(2\Delta+n\hbar\omega)/m \;, \qquad (1.1)$$

where $2\pi\hbar$ is Planck's constant, $\omega/2\pi$ is the frequency of the applied microwave field, and n is an integer. $m=1$ corresponds to ordinary microwave-assisted tunneling.[7]

The experiment of Longacre and Shapiro[5] on point contacts also suggested a rule for the power dependence of the magnitude of the satellites, namely, that the magnitude of the nth satellite around the mth subharmonic is proportional to

$$J_n^2(m\alpha) \qquad (1.2)$$

where J_n is the ordinary Bessel function of order n and

$$\alpha=eV^{\mathrm{rf}}/\hbar\omega \;. \qquad (1.3)$$

V^{rf} is the microwave voltage amplitude across the junction.

Recent measurements on both thin-film junctions and point-contact junctions[8] have confirmed both the location of the satellite structure, Eq. (1.1), and the microwave power dependence, Eq. (1.2), over a wide range of α values for various m and n values.

It is the purpose of this paper to analyze the existing theories of subharmonic gap structures in the presence of microwave fields in order to find which of them are in accordance with the observed location Eq. (1.1) and power dependence Eq. (1.2) of the satellites.

In Sec. II multiparticle tunneling[9,10] as an explanation of the subharmonic gap structure is investigated. It is found to yield both the observed location Eq. (1.1) and the observed power dependence Eq. (1.2). In Sec. III a self-coupling mechanism[11,12] is investigated, where the ac Josephson current induces an rf voltage across the junction and this rf voltage yields microwave-assisted tunneling. With a proper form for the self-coupling this yields the structure of Eq. (1.1) with the observed power dependence of Eq. (1.2) but only for *odd m*. The structure with *even m*, however, can be explained by invoking pair breaking due to the self-induced rf voltage. This yields structure at the observed locations Eq. (1.1) and also the observed power dependence of Eq. (1.2). An *odd-m* series can also be obtained by *pair breaking* if the process results in one quasiparticle at each side of the junction. This supplements the odd-*m tunneling* series. In Sec. IV a discussion of the two theories is given where properties other than those of Eqs. (1.1) and (1.2) are considered in order to find which of the mechanisms are responsible for the subharmonic gap structure. The conclusion (Sec. V) is that self-coupling is the main cause of the subharmonic gap structure.

II. MULTIPARTICLE TUNNELING

Multiparticle tunneling can give the structure at the observed voltages Eq. (1.1), the argument being as follows: The net result of a fundamental m-particle tunneling process is the creation of two quasiparticles and the transfer of m electrons from one side of the junction to the other.[13] At the same time a proper number of ground-state pairs are created and annihilated. In the m-electron transfer we gain an energy meV. The threshold for such a process is that this energy can create the two quasiparticles, i.e., $meV = 2\Delta$. If the m electrons in the process emit n "photons" to the microwave field in the barrier of the junction, the threshold condition becomes $meV = 2\Delta + n\hbar\omega$, which directly yields Eq. (1.1). Since the condition corresponds to a threshold the shape of the structure will be steplike, as observed, e.g., at the onset of quasiparticle tunneling in ordinary thin-film tunnel junctions.

In order to find the microwave-power dependence of the subharmonic gap structure due to multi-particle tunneling we shall study the process in more detail. From ordinary higher-order perturbation theory we have for the transition probability for m electrons in one process[14]

$$w = \frac{2\pi}{\hbar}\sum_F \left| \sum_{i_1 i_2 \cdots i_{m-1}} \frac{(I|i_1)(i_1|i_2)\cdots(i_k|i_{k+1})\cdots(i_{m-1}|F)}{(E_I - E_{i_1})(E_I - E_{i_2})\cdots(E_I - E_{i_{m-1}})} \right|^2 \delta(E_I - E_F) , \tag{2.1}$$

where $|I\rangle$ is the initial state with the energy E_I, $|F\rangle$ is the final state with the energy E_F, and $|i_k\rangle$ is an intermediate state with the energy E_{i_k}. $(i_k|i_{k+1})$ is the matrix element $\langle i_k|H_T|i_{k+1}\rangle$ of the tunneling Hamiltonian[15] H_T between state i_k and state i_{k+1}. The applied voltage V enters through the energy differences in the denominator of Eq. (2.1). The expression Eq. (2.1) will yield structure at voltages given by $eV = 2\Delta/m$, $2\Delta/(m-2)$, ..., 2Δ or $2\Delta/2$ and integer multiples of these. We will here only be interested in the structure at $2\Delta/m$.[13]

The result of an applied microwave field will be to change the matrix elements which enters Eq. (2.1). Before the time integration leading to Eq. (2.1) the matrix element was

$$(i_k|i_{k+1})\exp[(i/\hbar)(E_{i_k} - E_{i_{k+1}})t] . \tag{2.2}$$

The energy difference $E_{i_k} - E_{i_{k+1}}$ must be changed to

$$E_{i_k} - E_{i_{k+1}} + eV^{rf}\cos(\omega t) \tag{2.3}$$

when microwaves are applied, since one electron has been transferred. The phase difference in Eq. (2.2) which is $i/\hbar$ times the time integrated energy difference then becomes [Eq. (1.3)]

$$\frac{i}{\hbar}(E_{i_k} - E_{i_{k+1}})t + i\alpha\sin(\omega t) . \tag{2.4}$$

Therefore instead of the expression Eq. (2.2) we should insert

$$(i_k|i_{k+1})\exp[(i/\hbar)(E_{i_k} - E_{i_{k+1}})t + i\alpha\sin(\omega t)] = (i_k|i_{k+1})\sum_{n_{k+1}} J_{n_{k+1}}(\alpha)\exp[(i/\hbar)(E_{i_k} - E_{i_{k+1}} + n_{k+1}\hbar\omega)t] \tag{2.5}$$

in Eq. (2.1). The final result with applied microwaves is that Eq. (2.1) should be replaced by

$$w = \frac{2\pi}{\hbar}\sum_{F,n} \left| \sum_{\substack{n_1 n_2 \cdots n_m \\ \Sigma n_k = n}} \sum_{i_1 i_2 \cdots i_{m-1}} \frac{J_{n_1}(\alpha)J_{n_2}(\alpha)\cdots J_{n_m}(\alpha)(I|i_1)\cdots(i_{m-1}|F)}{(E_I - E_{I_1} + n_1\hbar\omega)[E_I - E_{i_2} + (n_1+n_2)\hbar\omega]\cdots} \right|^2 \delta(E_I - E_F + n\hbar\omega) . \tag{2.6}$$

The main contribution to the sturcture at $eV = 2\Delta/m$ from Eq. (2.1) comes from terms where the smallest energy factor in the denominator is close to $2\Delta/m$. We therefore can neglect the n_k dependence of the energy denominators in Eq. (2.6) as long as

$$\hbar\omega \ll 2\Delta/m . \tag{2.7}$$

Then the summation over n_k in Eq. (2.6) can be done immediately using

$$\sum_{\substack{n_1 n_2 \cdots n_m \\ \Sigma n_k = n}} J_{n_1}(\alpha)J_{n_2}(\alpha)\cdots J_{n_m}(\alpha) = J_n(m\alpha), \tag{2.8}$$

We get

$$w=\frac{2\pi}{\hbar}\sum_{n,F}J_n^2(m\alpha)\left|\sum_{i_1 i_2\cdots i_{m-1}}\frac{(I\,|\,i_1)(i_1|\,i_2)\cdots(i_{m-1}|\,F)}{[E_I-E_{i_1}+(n/m)\hbar\omega][E_I-E_{i_2}+2(n/m)\hbar\omega]\cdots}\right|^2\delta(E_I-E_F+n\hbar\omega)\,. \tag{2.9}$$

The expression [Eq. (2.9)] shows first that the microwave-induced satellites occur at the observed location Eq. (1.1) since the energy difference $E_I - E_I$ in the δ function contains meV caused by transferred electrons. Further, it shows the observed microwave power dependence of Eq. (1.2). The relative magnitude of the different subharmonic gap structures has been calculated for $m=1$ and $m=2$[9,10] within the WKB approximation of Harrison[16] for a one-dimensional-geometry tunnel junction. In Appendix A we have calculated the general step magnitude within the same approximation by the formalism of Ref. 17 and find that the current step at $eV=2\Delta/m$ is

$$\left[2\left(\frac{m}{2}\right)^m\frac{1}{m!}\right]^2\left(\frac{\exp[-2(d/\hbar)(2m_e\phi_W)^{1/2}]}{16}\right)^{m-1}\times\frac{2(d/\hbar)(2m_e\phi_W)^{1/2}+1/m}{2(d/\hbar)(2m_e\phi_W)^{1/2}+1} \tag{2.10}$$

times the current step at $eV=2\Delta$. Here d is the width of the insulating barrier, ϕ_W is the work function of the barrier, and m_e is the free-electron mass. From Eq. (2.10) it follows that multiparticle tunneling will yield a steadily decreasing series.

From the fact that two quasiparticles are created in a fundamental process it follows that for two different superconductors with gap parameters Δ_1 and Δ_2 the results Eqs. (1.1) and (1.2) still apply with (Fig. 1)

$$2\Delta\to\Delta_1+\Delta_2\quad\text{for odd } m \tag{2.11}$$

and (Fig. 2)

$$2\Delta\to 2\Delta_1\ \text{ and/or } 2\Delta_2\ \text{ for even } m\,. \tag{2.12}$$

The relative magnitude of the $2\Delta_1/m$ structure and the $2\Delta_2/m$ structure for even m becomes rather complicated, but for $m=2$ it is simply[9,10]

$$\Delta_1\tanh(\tfrac{1}{2}\beta\Delta_1)/\Delta_2\tanh(\tfrac{1}{2}\beta\Delta_2)\ , \tag{2.13}$$

where β is $1/k_BT$, k_B is Boltzmann's constant, and T the temperature.

III. SELF-COUPLING

In the Sec. II we have ignored the rf-voltage induced by the ac Josephson current I_J. For a Josephson junction this current is given by[18]

$$I_J=I_0\sin\phi\ , \tag{3.1}$$

where I_0 is the maximum dc Josephson current and ϕ is the phase difference between the two superconductors. ϕ follows from the general relation[18]

$$\frac{d\phi}{dt}=\frac{2eV}{\hbar}\ , \tag{3.2}$$

where V is the voltage across the junction.

The self-induced rf voltage contains the same frequencies as the Josephson ac current. In order to determine this voltage we note that the driving current through the junction is equal to the Josephson current I_J plus the displacement current $I_{\rm dpl}$ given by

$$I_{\rm dpl}=C\frac{dV}{dt}=\frac{\hbar C}{2e}\frac{d^2\phi}{dt^2}\ , \tag{3.3}$$

where Eq. (3.2) has been used. C is the capacitance between the two superconductors. This means that the driving current $I_{\rm dr}$ becomes[3,12]

$$I_{\rm dr}=\frac{\hbar C}{2e}(\ddot\phi+\omega_J^2\sin\phi)\ , \tag{3.4}$$

where ω_J is the Josephson plasma frequency determined by

$$\omega_J^2=2eI_0/\hbar C\ . \tag{3.5}$$

Without any applied current (microwaves) Eq. (3.4) becomes a simple pendulum equation which can be solved.[19] The solution with a dc voltage V_0 across the junction can be written[3,12]

$$\phi=\omega_0t+\sum_{l=1}^{\infty}2\alpha_l\sin l\omega_0t\ , \tag{3.6}$$

where ω_0 is the Josephson frequency [Eq. (3.2)]

$$\omega_0=2eV_0/\hbar \tag{3.7}$$

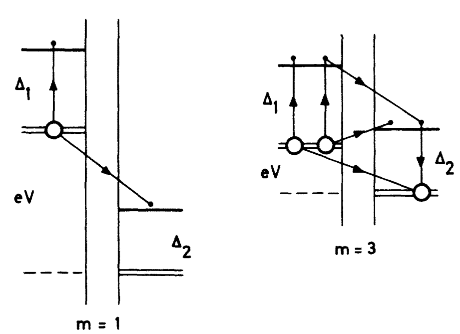

FIG. 1. Diagrams showing processes contributing to the odd-m structure at $(\Delta_1+\Delta_2)/m$ for $m=1$ and 3. The open circles are ground-state pairs and the dots are quasiparticles.

and α_l are expansion coefficients determined by a coupling constant Γ given by[12]

$$\Gamma = (\omega_J/\omega_0)^2 = \hbar I_0/2eCV_0^2 \; . \tag{3.8}$$

The coefficients α_l are related to the corresponding voltage V_l^{rf} through Eq. (1.3), i.e., $\alpha_l = eV_l^{rf}/\hbar l\omega_0$.

For small coupling $\Gamma \ll 1$ (large dc voltage)[20]

$$\alpha_l = (2/l)(\tfrac{1}{4}\Gamma)^l \; , \tag{3.9}$$

giving the solution, retaining only $l=1$ in Eq. (3.6),

$$\phi = \omega_0 t + 2\alpha_1 \sin\omega_0 t \; . \tag{3.10}$$

For large coupling $\Gamma \gg 1$ (small dc voltage) the α_l's saturate at the values[20]

$$\alpha_l = 1/l \; . \tag{3.11}$$

In both cases the solution has the *leading terms* of Eq. (3.10) but for large coupling there is a not quite negligible content of higher harmonics of the Josephson frequency ω_0.

With applied microwaves we will have a driving current in Eq. (3.4) with the microwave frequency ω. In Appendix B we have determined the magnitude of this current due to the microwave voltage $-V^{rf}\cos\omega t$ and get

$$I_{dr} = CV^{rf}\omega\sin\omega t = \frac{\hbar C}{2e}\,2\alpha\omega^2\sin\omega t \; , \tag{3.12}$$

where α is determined by Eq. (1.3). The equation from which the self-coupled voltage is to be determined is therefore

$$\ddot{\phi} + \omega_J^2 \sin\phi = 2\alpha\omega^2\sin\omega t \; . \tag{3.13}$$

In Appendix B an expression for the phase difference is derived when microwaves are applied with small self-coupling parameter Γ and small rf-microwave voltage amplitude V^{rf}. In the same way as in the case with zero rf-microwave voltage we suggest that the derived expression [Eq. (B9)] may be the fundamental self-coupling term even for large Γ

$$\phi = \omega_0 t - 2\alpha\sin\omega t + 2\alpha_1\sin(\omega_0 t - 2\alpha\sin\omega t) \; . \tag{3.14}$$

We now try to find the effect of the phase Eq. (3.14) on the current through a junction.

A. Odd *m*

The current through a tunnel junction may be found from Eq. (11) of Werthamer,[11] where the tunnel current is derived from second-order perturbation theory. Introducing the Fourier transform of the quasiparticle current amplitude $j_1(\omega)$ and the pair current amplitude $j_2(\omega)$ of Werthamer[11]

$$J_{1,2}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} j_{1,2}(\omega) \; . \tag{3.15}$$

Equation (11) of Ref. 11 for the tunnel current may be written

$$J(t) = \mathrm{Im}\Big[e^{-i\phi(t)/2}\Big(\int_{-\infty}^{\infty} dt'\, e^{i\phi(t+t')/2} J_1(t') + \int_{-\infty}^{\infty} dt'\, e^{-i\phi(t-t')/2} e^{i\beta} J_2(t')\Big)\Big] \; , \tag{3.16}$$

where β is a phase factor (α of Ref. 11). Equation (3.16) should replace Eq. (3.1), at least for tunnel junctions, but as long as the dc voltage is less than and not too close to the gap voltage $2\Delta/e$, Eq. (3.1) is a good approximation to Eq. (3.16) with $I_0 = \mathrm{Re}\, j_2(0)$.

Introducing the phase of Eq. (3.14) into Eq. (3.16) and taking the dc part we obtain for the dc tunnel current (neglecting the Josephson steps)

$$I_{dc} = \sum_{M=0}^{\infty}\sum_n J_n^2((2M+1)\alpha)\{[J_M^2(\alpha_1) - J_{M+1}^2(\alpha_1)]\,\mathrm{Im}\, j_1((M+\tfrac{1}{2})\omega_0 - n\omega) + J_M(\alpha_1)J_{-M-1}(\alpha_1)\sin\beta\, 2\,\mathrm{Re}\, j_2((M+\tfrac{1}{2})\omega_0 - n\omega)\} \; . \tag{3.17}$$

The first term in Eq. (3.17) is the quasiparticle current and the second is the pair current. Characteristic structures occur when the argument of $j_1(\omega)$ and $j_2(\omega)$ becomes $2\Delta/\hbar$. The structures from j_1 are steps reflecting the quasiparticle step and the structures from j_2 are peaks reflecting the Riedel singularity. This means that we find structures at voltages given by Eq. (1.1) for $m = 2M+1$, i.e., odd m. We also find the microwave-power dependence of Eq. (1.2).

The relative magnitudes of the different subharmonic gap structure are given by the Bessel functions of the argument α_1. Since α_1 depends on the dc voltage in a complicated way it is difficult to say anything definite about this. From the case without microwaves it follows that α_1 increases with decreasing dc voltage and saturates at the value 1. This indicates a steadily decreasing series. The alternating sign of the last term in Eq. (3.17) need not be observed in an experiment since β could adjust in such a way that the sign remains unchanged.

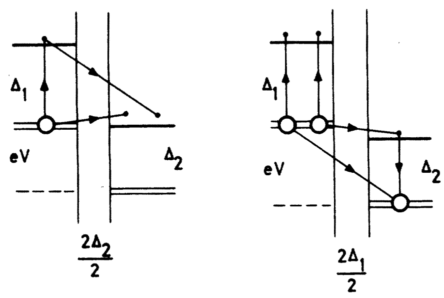

FIG. 2. Diagrams showing processes contributing to the even-m structure at $2\Delta_1/m$ and $2\Delta_2/m$ for $m=2$.

B. Even m

The even-m structures cannot be found from the tunneling treatment. One can get these structures by invoking pair breaking in one or the other of the two superconductors due to the electromagnetic field in the junction. This pair breaking is caused by the rf magnetic field penetrating the superconductors.

The energies of the electromagnetic field at different frequencies may be found in the following way: By inserting Eq. (3.14) into Eq. (3.1) we get for the Josephson current through the junction

$$I_J = I_0 \sum_{N,n} J_n(2\alpha_1) J_n(2(N+1)\alpha) \times \sin[(N+1)\omega_0 t - n\omega t] \quad . \quad (3.18)$$

The different frequency parts of the self-coupled voltage are proportional to the corresponding frequency parts of the current divided by that frequency as may be seen from Eq. (3.13). One could just as well take the $(N+1)$th term of the full solution to Eq. (3.13) which is not shown in Eq. (3.14). Therefore the energy of the electromagnetic field with the frequency $(N+1)\omega_0 - n\omega$ is proportional to

$$J_N^2(2\alpha_1) J_N^2(2(N+1)\alpha) \; . \quad (3.19)$$

Pair breaking can take place when $\hbar$ times this frequency is equal to the energy gap 2Δ. This yields structures at voltages given by Eq. (1.1) for $m = 2(N+1)$. The microwave-power dependence of Eq. (1.2) follows directly from Eq. (3.19) if we assume that the influence on the tunnel current is proportional to the energy. The onset of the pair breaking would influence both the quasiparticle part and the pair part of the current and one could expect both steps and peaks in the structure as for the odd-m structure. From Eq. (3.19) one also expects the structures to form a steadily decreasing series, but from the description given here one might not expect the even and odd structures taken together to give a steadily decreasing series.

For different superconductors with gap parameters Δ_1 and Δ_2 it is obvious that 2Δ should be replaced in accordance with Eqs. (2.11) and (2.12). The relative magnitude of the $2\Delta_1/2$ structure and the $2\Delta_2/2$ structure for the self-coupling case is *not* in accordance with Eq. (2.13) for multiparticle tunneling. One would expect in the self-coupling case the structure belonging to the largest penetration depth, i.e., the smallest gap parameter, to be the largest. Another fact pointing in the same direction is that the onset of the different dissipative processes would hurt the structures at higher voltages worst.

C. Other processes

The arguments leading to the even-m series (Fig. 3) are in a way more general than the arguments leading to the odd-m series (Fig. 4). For the odd-m series, the structures reflect structure in the dc characteristic at $2\Delta/e$. For the even-m series the structures are the result of the onset of energy absorption from the ac-Josephson-current-induced electromagnetic field in the junction. Therefore in order to get the even-m series one need not have a dc characteristic with pronounced structure. A similar general *odd-m* structure can be obtained if, as absorption processes, we look at pair breaking, where one quasiparticle is created at each side of the junction (Fig. 5). Such processes would yield structures at the locations of Eq. (1.1) and with approximately the power dependence of Eq. (1.2). But in order to give a considerable contribution the transmission probability for the junction (barrier) should be of the order one meaning that there should be very close coupling between the two superconductors. In that case also processes where the two quasiparticles are created at the side of the junction opposite to the side of the pair annihilation (Fig. 6) could contribute leading to an even-m series. The two last-mentioned processes supplement the two first in the self-coupling description.

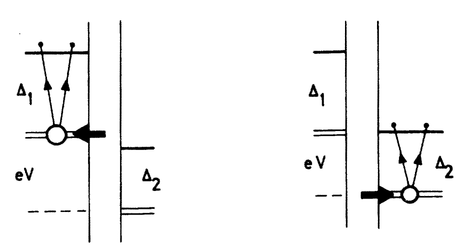

FIG. 3. Diagrams showing the pair breaking due to the Josephson rf voltage yielding the even-m structure. The heavy arrows symbolize Josephson "photons."

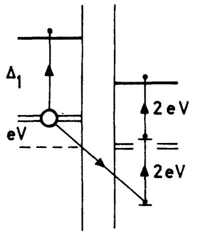

FIG. 4. Diagram showing the process leading to odd-m structure. The diagram shows a tunneling process assisted by two Josephson "photons."

IV. DISCUSSION

Both multiparticle tunneling and self-coupling are in accordance with the observed voltage location Eq. (1.1) of the microwave-induced satellites of the subharmonic gap structure and the microwave-power dependence Eq. (1.2). They also both are in accordance with the distinction between odd- and even-m structures for different superconductors Eq. (2.11) and (2.12).

In order to distinguish the two explanations we have to consider other, unfortunately less specific, properties of the theories.

A. Line shapes

Multiparticle tunneling yields steplike structures for the subharmonic gap structures, whereas the self-coupling yields both steplike and peak structures. When the coupling Γ and therefore α_1 increases the step structure of Eq. (3.17) will appear first, since J_M^2 initially increases faster than $J_M J_{M+1}$. Experiments show structures which sometimes are steplike, sometimes peak structures, and sometimes a mixture of steplike and peak structures.[1-5,8] This means that multiparticle tunneling alone could not explain all the observed structures.

B. Relative magnitude of the different m structures

The magnitude of the mth subharmonic gap step relative to the gap-structure ($m=1$) step from multiparticle tunneling follows from Eq. (2.10). The leading factor here is

$$[\tfrac{1}{16}\exp(-(2m_e\phi_W)^{1/2}2d/\hbar)]^{m-1} \quad , \tag{4.1}$$

the remaining factor lying between 0.4 and 4 as long as $m \leqq 6$. Looking at, for instance, $m=3$ the maximum value for the factor is $\frac{1}{256} \cong 0.004$. The corresponding (maximum) factor for the self-coupling ($m=3$) case $[J_1^2(1) - J_2^2(1)]$ from Eq. (3.17) becomes 0.18. Experimentally this factor can exceed 0.004 by a factor of one order of magnitude.[8] This again means that multiparticle tunneling alone cannot explain all the subharmonic gap structure. The conclusion in this case is more uncertain because the expression Eq. (2.10) is derived under the assumption that the tunneling matrix elements are small.

C. Matrix-element dependence

The dependence of the mth subharmonic gap structure due to multiparticle tunneling would be proportional to $|M|^{2m}$, where M is some average matrix element for the tunneling process.

For the self-coupling odd-m case Eq. (3.17), j_1 and j_2 are proportional to $|M|^2$. If we go to the limit of saturation for $\alpha_1(\alpha_1 = 1)$[12] the odd series will go as $|M|^2$. The experiment of Giaever and Zeller[3] on tunnel junctions with light-sensitive barriers shows that all the subharmonic gap structures are proportional to the square of the matrix element $|M|^2$. It also shows that initially the higher subharmonic gap structures grow more rapidly than $|M|^2$. This supports the self-coupling case for odd m with an eventual saturation of the self-induced Josephson rf voltage.

The self-coupling for even m is also supported by their experiment. Here the explanation is different. If the rf-voltage saturates, the electromagnetic energy in the junction also saturates. This means that the even-m structure would be proportional to the tunnel current, i.e., also proportional to $|M|^2$.

D. Different superconductors

For two different superconductors with gap parameter Δ_1 and Δ_2 the relative magnitude of the structures at Δ_1/e and Δ_2/e due to multiparticle tunneling should be Δ_1/Δ_2 at zero temperature [Eq. (2.13)]. For self-coupling the relative magnitude of the same structures would be reversed. All experiments on different superconductors[1-4] seem to support the self-coupling explanation in this respect, namely, the structure at the smaller of the two voltages Δ_1/e and Δ_2/e is the largest.

V. CONCLUSION

The explanations of the subharmonic gap structures in superconducting junctions considered here, multiparticle tunneling, and self-coupling, yield the same results for the most pronounced properties of Eqs. (1.1), (1.2), (2.11), and (2.12). The less distinct properties where the two theories disagree seem to support self-coupling as the main cause of the observed subharmonic gap

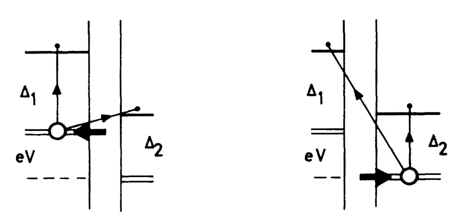

FIG. 5. Diagrams showing processes contributing to the odd-m structure due to pair breaking.

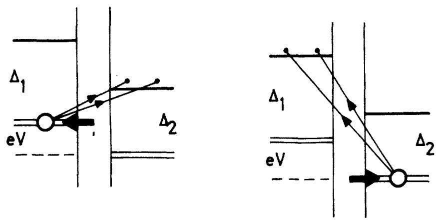

FIG. 6. Diagrams showing processes contribution to the even-m structure due to pair breaking, where the quasiparticles are created at a side opposite to the side of the pair annihilation.

structures, but not excluding multiparticle tunneling as being in part responsible for some of the observed structures.

There is one drawback concerning the self-coupling, namely the two conceptually different explanations for the odd and the even series. This difference, however, may only be apparent, as indicated by the possibility also of an odd series from pair breaking, in which one quasiparticle is created at each side of the junction. This question can only be settled by including the pair breaking in the tunneling treatment.

ACKNOWLEDGMENTS

The authors are grateful to Sidney Shapiro and John Wilkins for initiating and stimulating our collaboration. Further, we would like to acknowledge P. E. Gregers-Hansen, L. Hultén, H. Højgaard Jensen, B. Kofoed, N. F. Pedersen, K. Saermark, and O. Hoffmann Soerensen for continued interest and helpful discussions. One of us (L. E. H.) is grateful to NORDITA for financial support.

APPENDIX A

We suppose that the multiparticle tunneling can be properly described by a tunneling Hamiltonian[15]

$$H_T = \sum_{\vec k \vec q \alpha} T_{\vec k\vec q} c^\dagger_{\vec k\alpha} c_{\vec q\alpha} + T^*_{\vec k\vec q} c^\dagger_{\vec q\alpha} c_{\vec k\alpha} , \tag{A1}$$

where the tunneling matrix elements are approximated by[10]

$$|T_{\vec k\vec q}|^2 \sim \delta_{\vec k_\parallel, \vec q_\parallel} \exp[-(2d/\hbar)(2m_e\phi_W + \hbar^2 k_\parallel^2)^{1/2}] . \tag{A2}$$

$\vec k$ and $\vec q$ are the wave vectors of the electrons in the left (l) and right (r) superconductor, respectively, and $\vec k_\parallel$ means the component of $\vec k$ parallel to the barrier. ϕ_W is a constant barrier potential and d is the barrier thickness. α is a spin index and $c^\dagger$ is a creation operator for electrons.

By the aid of the formalism outlined in Ref. 17, it is possible to derive the following expression for the total tunneling current density:

$$i(t) = -\frac{es}{4\pi d^2\hbar} \mathrm{Re}\Bigg[\sum_{m=1}^{\infty}\left(\frac{-e^{-s}}{16}\right)^m \frac{1}{m}\left(1+\frac{1}{ms}\right) \times \frac{1}{-i\beta} \sum_{E_0, E_1, \ldots, E_{2m-1}} \mathrm{Tr}\left(\underline{M}_0^r(t)\underline{M}_1^l\underline{M}_2^r \cdots \underline{M}_{2m-1}^l\right) e^{i(E_{2m-1}-E_0)(t/\hbar)}\Bigg] , \tag{A3}$$

where

$$s = \frac{2d}{\hbar}(2m_e\phi_W)^{1/2} ,$$

$$\beta = 1/k_B T \quad (T \text{ is the temperature}),$$

$$\underline{M}_0^r(t) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{eV^{rl}}{\hbar\omega}\right)\begin{pmatrix} h_n^+(t)n^r(E_0) & h_n^+(t)p^r(E_0) \\ h_n^-(t)p^r(E_0) & h_n^-(t)n^r(E_0)\end{pmatrix}$$

and

$$\underline{M}_p^{r(l)} = \sum_{n=-\infty}^{\infty} J_n\left(\frac{eV^{rl}}{\hbar\omega}\right)\begin{pmatrix} \delta_{p,n}^{(\mp)} n^{r(l)}(E_0) & \delta_{p,n}^{(\pm)} p^{r(l)}(E_p) \\ -\delta_{p,n}^{(\pm)} p^{r(l)}(E_p) & -\delta_{p,n}^{(\pm)} n^{r(l)}(E_p)\end{pmatrix}$$

$$(p = 1, 2, 3, \ldots) .$$

The integer m tells us that it is a m-particle process and E_p takes the discrete imaginary values $(\pi/-i\beta)\times$(odd number). Finally the functions h, n, p, and δ are defined by

$$h_n^\pm(t) = \exp\{\pm i[(n\omega - eV/\hbar)t - \tfrac{1}{2}\phi]\} , \tag{A4}$$

$$n^{r(l)}(E) = E/(E^2 - \Delta^2_{r(l)})^{1/2} \tag{A5}$$

$$p^{r(l)}(E) = \Delta_{r(l)}/(E^2 - \Delta^2_{r(l)})^{1/2} \tag{A6}$$

and

$$\delta^{\pm}_{p,n} = e^{\mp i\phi/2}\delta_{[E_{p-1}-E_p],[\pm(eV-n\hbar\omega)]}, \tag{A7}$$

where ϕ is the dc phase difference between the superconductors.

When the $\underline{M}$ matrices are multiplied together and the trace is taken of the final 2×2 matrix, products of the density of state functions n and p will appear. The dc-current step at $2\Delta/m$ (identical superconductors) from the mth-order term is independent of ϕ and arises always from terms with two n factors and $2m-2$ p factors because of the Kronecker δ functions $\delta_{p,n}$. The approximation $\hbar\omega \ll \Delta$ is used and we get

$$i^{dc} = -\frac{es}{2\pi d^2\hbar}\sum_{m=1}^{\infty}\left(\frac{-e^{-s}}{16}\right)^m\left(1+\frac{1}{ms}\right)\sum_{m=-\infty}^{\infty}J_n^2\left(\frac{meV^{rf}}{\hbar\omega}\right)\frac{1}{-i\beta}$$

$$\times\sum_E n(E_0+n\hbar\omega)p(E_1)p(E_2)\cdots p(E_{m-1})n(E_m)p(E_{m+1})\cdots p(E_{2m-1}), \tag{A8}$$

where

$$E_p = E + \sum_{j=p+1}^{2m-1}[eV\,\mathrm{sgn}(j-m-0.5) + i\times(\text{pos. infinitesimal})].$$

p and j are integers and eV and $\hbar\omega$ are of the form $(\pi/-i\beta)\times$(even number). Finally, the analytic continuation from discrete imaginary to continuous real-valued energies are performed in the same way as in Ref. 17. Then we can calculate the mth-order current steps at the voltages $eV=(2\Delta+n\hbar\omega)/m$:

$$i^{step}_{m,n} = \frac{es\Delta}{4\pi d^2\hbar}\tanh\frac{\beta\Delta}{2}\left(\frac{e^{-s}}{16}\right)^m\left(1+\frac{1}{ms}\right)\left[\frac{2}{m!}\left(\frac{m}{2}\right)^m\right]^2 J_n^2\left(\frac{meV^{rf}}{\hbar\omega}\right). \tag{A9}$$

APPENDIX B

We first consider the case in Sec. III when $\omega_J=0$. From Eq. (3.4) we get

$$\frac{\hbar}{2e}\dot{\phi}(t) = \frac{1}{C}\int^t I_{dr}(t')\,dt' + C_1. \tag{B1}$$

When the microwaves are applied, the entities $I_{dr}(t)$ and C_1 have to take the values

$$I_{dr}(t) = CV^{rf}\omega\sin\omega t \tag{B2}$$

and

$$C_1 = V^{dc} \tag{B3}$$

in order to get the effective voltage

$$V(t) = \frac{\hbar}{2e}\dot{\phi}(t) = V^{dc} - V^{rf}\cos\omega t \tag{B4}$$

across the junction.

Next, we let ω_J be nonzero but much less than $\omega_0 = 2eV^{dc}/\hbar$. Then the first-order solution in ω_J becomes

$$\phi(t) = \omega_0 t - 2\alpha\sin\omega t - \omega_J^2\int^t dt'$$

$$\times\int^{t'} dt''\sin(\omega_0 t'' - 2\alpha\sin\omega t'') + C_2, \tag{B5}$$

where the indefinite integral becomes

$$\omega_J^2\sum_n J_n(2\alpha)\frac{\sin(\omega_0-n\omega)t}{(\omega_0-n\omega)^2} \tag{B6}$$

and the integration constant is determined so $\phi(0)=0$, i.e.,

$$C_2 = 0. \tag{B7}$$

If $J_n(2\alpha)\ll1$ when $|n\omega|\sim\omega_0$ then the denominator of Eq. (B6) can be approximated by ω_0^2 and the following inequality is fulfilled

$$V^{rf} \ll V^{dc}. \tag{B8}$$

The final expression for $\phi(t)$ becomes

$$\phi(t) = \omega_0 t - 2\alpha\sin\omega t + \Gamma\sin(\omega_0 t - 2\alpha\sin\omega t) \tag{B9}$$

where

$$\Gamma = (\omega_J/\omega_0)^2 \ll 1 \tag{B10}$$

is the coupling constant.

*Present address: NORDITA, Copenhagen and Physics Laboratory I, Technical University of Denmark, DK-2800 Lyngby, Denmark.

†Present address: Department of Astronomy, The University of Texas at Austin, Austin, Texas 78712.

[1]B. N. Taylor and E. Burstein, Phys. Rev. Lett. 10, 14 (1963); C. J. Adkins, Philos. Mag. 8, 1051 (1963); Rev. Mod. Phys. 36, 211 (1964); I. K. Yanson, V. M. Svistunov, and I. M. Dmitrenko, Zh. Eksp. Teor. Fiz. 47, 2091 (1964) [Sov. Phys.-JETP 20, 1404 (1965)]; S. M. Marcus, Phys. Lett. 19, 623 (1966); Phys. Lett. 20, 236 (1966); A. A. Bright and J. R. Merrill, Phys. Rev. 184, 446 (1969); G. I. Rochlin, Phys. Rev. 153, 513 (1967).

[2]J. M. Rowell and W. L. Feldman, Phys. Rev. 172, 393 (1968).

[3]I. Giaever and H. R. Zeller, Phys. Rev. B 1, 4278 (1970).

[4]S. Bermon and R. M. Mesak, Solid State Commun. 9, 2143 (1971); H. J. Levenstein and J. E. Kunzler, Phys. Lett. 20, 581 (1966); L. J. Barnes, Phys. Rev. 184, 434 (1969); M. Puma and B. S. Deaver, Jr., Appl. Phys. Lett. 19, 539 (1971).

[5]A. Longacre, Jr. and S. Shapiro, Bull, Am. Phys. Soc. 16, 399 (1971).

[6]Recently, subharmonic gap structure and microwave-induced satellites have been observed in microbridges [P. E. Gregers-Hansen, E. Hendricks, M. T. Levinsen, and G. R. Pickett, Phys. Rev. Lett. 31, 524 (1973)].

[7]P. K. Tien and J. P. Gordon, Phys. Rev. 129, 647 (1963). M. R. Samuelsen, B. Kofoed, and O. Hoffman Soerensen, Phys. Status Solidi A 11, K79 (1972).

[8]O. Hoffman Soerensen, B. Kofoed, N. F. Pedersen, and Sidney Shapiro, Phys. Rev. B 9, 3746 (1974).

[9]J. R. Schrieffer and J. W. Wilkins, Phys. Rev. Lett. 10, 17 (1963).

[10]J. W. Wilkins, in *Tunneling Phenomena in Solids*, edited by E. Burstein and S. Lundquist (Plenum, New York, 1969).

[11]N. R. Werthamer, Phys. Rev. 147, 255 (1966).

[12]S. Strässler and H. R. Zeller, Phys. Rev. B 3, 266 (1971).

[13]A process where more than two quasiparticles are created will only give higher-order contributions to lower-order processes (renormalization). The same will be the case for a process where M electrons go from one side to the other and at the same time $m-M$ go from the other side to the first.

[14]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, England 1962); J. McConnell, *Quantum Particle Dynamics* (North-Holland, Amsterdam, 1960).

[15]M. H. Cohen, L. M. Falicov, and J. C. Phillips, Phys. Rev. Lett. 8, 316 (1962); R. E. Prange, Phys. Rev. 131, 1083 (1963).

[16]W. A. Harrison, Phys. Rev. 123, 85 (1961).

[17]L.-E. Hasselberg, J. Phys. F 3, 1438 (1973); J. Phys. F 3, 1453 (1973).

[18]B. D. Josephson, Adv. Phys. 14, 419 (1965).

[19]H. C. Corben and P. Stehle, *Classical Mechanics* (Wiley, New York, 1950).

[20]A factor of 4 missing in Fig. 2 and in Eqs. (3), (5), and (6) of Ref. 12.