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“Excitonic” matter in a superstrong magnetic field

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This paper is an investigation of the properties of excitonic matter in a superstrong magnetic field. The binding energy of an excitonic molecule is first calculated as a function of the magnetic field. The molecule is found to be bound relative to two excitons and so is expected to occur when the electron-hole concentration is small. When the electron-hole concentration is large, a new phase consisting of a two-dimensional van der Waals “solid” of electron-hole cylinders is proposed—analogous to situations at the surface of neutron stars. The binding energy of these cylinders is calculated for the case that the electron and hole have equal isotropic masses.

I. INTRODUCTION

This paper is stimulated by recent progress made in (i) matter on neutron stars, (ii) electron-hole drops. The electron-hole drop is the condensed phase obtained when a high-intensity laser is shone on a semiconductor crystal such as Ge. It is now believed that this condensed phase is a plasma of electrons and holes[1] (though there are still some who believe that this might be a molecular solid[2]) that form metallic droplets inside the semiconductor. The question that I am interested in here is what happens when a magnetic field is applied to these drops. When the magnetic field is small, nothing drastic is expected to happen (by that I mean that all that the B field does is to produce a small perturbation on the ground state); however, an interesting possibility arises when the magnetic energy becomes comparable to the excitonic energy. Then it is conceivable that major structural change can occur in the electron-hole fluid.

A similar problem arises in the investigation of matter on top of neutron stars where a superstrong magnetic field is present. In such cases Ruderman[3] has predicted that a new phase of matter may exist in which chains of atoms form a two-dimensional solid. These chains consist of nuclei arranged at periodic intervals with the electrons spread in cylinders around them. The kinetic energy of the electrons in these cylinders is much less than that in the atomic case due to the delocalization of the electrons in the z direction. On the other hand, as contrasted with the case of the atom, there is no sacrifice in the potential energy because here, as the electron goes away from one positive charge center and loses its potential energy relative to it, it approaches the next one on the chain and regains a comparable amount of potential energy.

The cylinders so formed can then attract each other and form a two-dimensional van der Waals “solid.” Such a phase is believed to exist for magnetic fields of the order of 10^{12} G; Ruderman also pointed out that because of the large dielectric constants and small effective masses in semiconductors the critical field in those cases could just be about 10^6–10^7 G. However, in his calculation Ruderman made heavy use of the fact that the proton mass is much heavier than the electron mass. In a typical semiconductor, the electron and hole mass are approximately equal, and the direct Coulomb potential that gives rise to Ruderman's binding goes to zero when the electron mass becomes equal to the hole mass. His calculation is therefore not applicable in the present situation. Another source of binding can come from exchange; from the exclusion principle the electrons tend to stay away from each other and thus this exchange Coulomb term can contribute a certain amount to the binding.

It is the purpose of this paper to investigate how much binding this exchange term can provide. The organization of the paper is as follows: the binding energy of an excitonic molecule is first calculated in a superstrong magnetic field by the vibrational method. These molecules should exist in the “excitonic matter” at low densities. The details of this calculation are presented in Secs. II–IV where the effect of the exchange is emphasized. The binding energy of the excitonic cylinders is next estimated. The various physical aspects of this new state and its possible detection are discussed in Secs. V–VII.

In this paper a model calculation will be performed. The model assumes one electron band and one hole band with equal masses. It is also assumed that all the spins are lined up because of the magnetic field. The latter assumption may not be exactly correct in the case of Ge, but its effects can be easily incorporated and will not affect our qualitative conclusions about the existence of the electron-hole-liquid cylinders. A second paper which applies this calculation to germanium is being prepared.

In doing the calculation, the field strength I have assumed is such that the magnetic energy is comparable to an exciton rydberg but smaller than the ordinary rydberg. With such a field strength one does not expect the basic band structure to change very much since the magnetic energy is of the order of an meV (the exciton rydberg in Ge is 3.6 meV) whereas the typical pseudopotential well depth is of the order of a Ry. One could also question whether there would be some sort of structural phase transition as the magnetic field is applied if the difference in energy between the two places is very small. In the absence of detailed calculation, it is of course impossible to answer the question definitely. But such a situation seems rather unlikely, and since nobody has ever observed it in the materials that we shall be interested in (Ge for example), we shall, therefore, assume that no such phase transitions occur. Also, no magnetic breakthrough can occur because the band gap of Ge is of the order of 0.56 eV, where as the excitonic rydberg is 3.6 meV.

II. VARIATIONAL WAVE FUNCTION

In this section, I shall propose the form of the variational wave function. The system of interest consists of two electrons and two holes with equal masses. Whereas in the atomic case the binding energy is only reduced by a factor of 4 when the hole mass becomes equal to the electron mass, from previous calculations[4,5] the molecular binding is reduced by a factor of 100 in the corresponding limit. Calculation of binding of "ordinary molecules" (i.e., hole mass much larger than electron mass) under strong magnetic fields has been done by Kadomtsev.[5] We expect intuitively that this binding energy will correspondingly be reduced dramatically when the electron mass is set equal to the hole mass. In the zero-magnetic-field case the spin wave function is antisymmetric so that the spatial wave function is symmetric, and one can say that there is some kind of "bond" between the two "atoms." The excitonic molecule remains bound as the electron-hole mass ratio goes from 0 to 1. In the high-B-field case, all the spins are lined up. The spin wave function is symmetric and the space wave function is then antisymmetric; the atoms are, so to speak, in an "antibonding" orbital. Whereas when the hole mass is very heavy, they can still be bound due to the large quadrupole moment of each atom, such moments vanish when the electron and hole mass become equal. Intuitively therefore, one would doubt very much whether the system would be bound relative to two excitons. Were it not for the exchange terms, such would indeed be the case, as we shall learn at the end of this calculation.

If the B field is fairly large, we expect the four particles to move around in circles in the lowest Landau level. In order to extremize the potential energy, they should move in orbits of the same radius and stay right on top of each other. The above reasoning therefore suggests that we should choose our trial function as Landau orbitals with the same center. Let us denote the electron coordinates by the subscripts 1 and 2 and the hole coordinates by the subscripts 3 and 4. The Landau wave functions can be represented in cylindrical coordinates by the set

$$\psi_{n,m,k_z} = e^{-\zeta/2}\zeta^{|m|/2}\omega(\zeta)\, e^{im\varphi}\, e^{ik_z z}\,, \tag{2.1}$$

where $z\hbar c\zeta = eB\rho^2$ and the energy E is given by

$$E_{n,m,k_z} = \frac{eB\hbar}{mc}\left(n + \tfrac{1}{2}|m| - \tfrac{1}{2}m + \tfrac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} \tag{2.2}$$

and

$$\omega(\zeta) = F(-n, |m| + 1, \zeta) \tag{2.2a}$$

is the appropriately normalized hypergeometric function.

The wave function ψ_{n,m,k_z} describes an electron state moving around the origin with a mean radius

$$r = (2m+1)^{1/2}\rho_0\,, \tag{2.2b}$$

where

$$\rho_0 = (\hbar c/eB)^{1/2}\,, \tag{2.3}$$

and a momentum in the z direction equal to k_z. Note that the energy (2.2) is independent of $|m|$. As the Coulomb interaction is turned on, it is found that this Coulomb energy becomes bigger as m increases and so the smallest possible m consistent with the exclusion principle should be chosen. From the introduction we know that the spatial wave function has to be antisymmetric. The above reasoning suggests the following form for the wave function:

$$\begin{aligned}\Phi = {} & [u_{m=1}(x_1)u_{m=0}(x_2) - u_{m=1}(x_2)u_{m=0}(x_1)] \\ & \times [u_{m=1}(x_3)u_{m=0}(x_4) - u_{m=1}(x_4)u_{m=0}(x_3)] \\ & \times v(x_1x_2x_3x_4)\,,\end{aligned} \tag{2.4}$$

where

$$u_m(x) = e^{-\rho^2/4\rho_0^2}\rho^m e^{im\varphi}\omega(\rho) \tag{2.5}$$

(the z dependence has to be different because the wave function has to be localized in the case of a bound state); v is symmetric with respect to the interchange of variables 1, 2 and 3, 4 and essentially takes into account the correlation effects that enter. As is mentioned, the maximum correlation in ρ is to have the orbits sitting on top of each other, and this has already been taken into account. Also, the maximum correlation in Φ would be to have the electrons (holes) sitting on opposite ends of the diameter of the circular orbit; the latter being fixed by the magnetic field. This has also been taken into account by the Hartree-Fock-type term multiplying v in (2.3).[6] v thus depends only on z. The choice of v is guided by the work of Hylleras and Ore.[7] Specifically, I choose

$$v(z)=\exp\left(-\alpha\sum_{i=1}^{4}|z_i|\right)e^{-\beta|z_1+z_2-z_3-z_4|}, \qquad (2.6)$$

where α and β are the variational parameters. Note the symmetry with respect to the variables 1, 2 and 3, 4. The first factor on the right-hand side of (2.6) localizes all the particles around the origin, while the second factor tries to keep the electrons and holes closer to each other than to members of their own species.

We thus finally arrive at the following wave function:

$$\begin{aligned}\psi=N\exp\left(-\alpha\sum_{i=1}^{4}|z_i|\right)e^{-\beta|z_1+z_2-z_3-z_4|}\\ \times\left(e^{i\varphi_1}\rho_1 e^{-\rho_1^2/4\rho_0^2}e^{-\rho_2^2/4\rho_0^2}-e^{i\varphi_2}\rho_2 e^{-\rho_2^2/4\rho_0^2}\right)\\ \times\left(e^{i\varphi_3}\rho_3 e^{-\rho_3^2/4\rho_0^2}e^{-\rho_4^2/4\rho_0^2}-e^{i\varphi_4}\rho_4 e^{-\rho_4^2/4\rho_0^2}e^{-\rho_3^2/4\rho_0^2}\right).\end{aligned} \qquad (2.7)$$

N here is the normalization constant. Note also that for the ease of calculation we have included only the leading term in $\omega(\rho)$ [see Eq. (2.1)] in (2.7).

III. ALGEBRAIC DETAILS

In this section the detail of the calculation of $\langle\psi|H|\psi\rangle$ will be presented. The result of this calculation will be presented in the next section. We have

$$H=H_0+V, \qquad (3.1)$$

$$H_0=\sum_{i=1}^{4}H_i=\sum_{i=1}^{4}\left[-\frac{\hbar^2}{2m}\left(\nabla-\frac{e_i}{c}A\right)^2\right] \qquad (3.2a)$$

$$\begin{aligned}=\sum_{i=1}^{4}-\frac{\hbar^2}{2m}\left[\frac{1}{\rho_i}\frac{\partial}{\partial\rho_i}\left(\rho_i\frac{\partial}{\partial\rho_i}\right)+\frac{\partial^2}{\partial z_i^2}+\frac{1}{\rho_i^2}\frac{\partial^2}{\partial\varphi_i^2}\right.\\ \left.-\frac{ie\hbar}{2mc}H\frac{\partial}{\partial\varphi_i}+\frac{e_i^2H^2}{8Mc^2}\rho_i^2\right],\end{aligned} \qquad (3.2b)$$

$$V=\sum_{i>j}\frac{e^2\epsilon_{ij}}{|r_i-r_j|}, \quad \epsilon_{ij}=\begin{cases}1, & ij=12 \text{ or } 34\\ -1, & \text{otherwise}.\end{cases} \qquad (3.3)$$

We split H_0 into two parts:

$$H_0=\sum_{i=1}^{4}H_i'+\sum_i\frac{\partial^2}{\partial z_i^2}. \qquad (3.4)$$

(We have changed to atomic units so that $\hbar^2/2m=1$.) Appendix A shows that, just as in the free-electron case, $\langle\psi|H'|\psi\rangle=2(\frac{1}{2}\hbar\omega_c)=\hbar\omega_c$, where ω_c is the cyclotron frequency defined by $\omega_c=eB/mc$. The calculations of $\langle\psi|\sum_i\partial^2/\partial z_i^2|\psi\rangle$ is straightforward but tedious. First, the normalization of the z part of the wave function, N_z, is obtained from the condition

$$N_z^2\int e^{-2\alpha\Sigma|z_i|}e^{-2\beta|z_1+z_2-z_3-z_4|}\prod_{i=1}^{4}dz_i=1. \qquad (3.5)$$

Because of the absolute values in the exponent, the regions of integration have to be split up for every possible sign of the exponent. After some tedious but straightforward arithmetic we obtain the result

$$N_z^2=\frac{1}{6(A_1+A_{1a}+B_1)+8(A_2+B_2)+2I_4}, \qquad (3.6)$$

$$\begin{aligned}A_1&=\frac{1}{2(\alpha-\beta)}\left(\frac{1}{8(\alpha-\beta)(\alpha+\beta)^2}-\frac{1}{32(\alpha-\beta)\alpha^2}-\frac{1}{32\alpha^2}\right),\\ A_{1a}&=\frac{1}{64\alpha^3}\frac{1}{(\alpha+\beta)},\\ \beta_1&=\frac{1}{32\alpha^2(\alpha+\beta)},\\ A_2&=\frac{1}{2(\alpha-\beta)}\left(\frac{1}{8(\alpha+\beta)^3}-\frac{1}{64\alpha^3}\right),\\ B_2&=\frac{1}{128\alpha^3(\alpha+\beta)},\\ I_4&=\frac{1}{16(\alpha+\beta)^4}.\end{aligned} \qquad (3.7)$$

The value of $\langle\psi|\sum_{i=1}^{4}\partial^2/\partial z_i^2|\psi\rangle$ is calculated in a similar manner as the normalization integral, and we obtain

$$\begin{aligned}\langle\psi|\sum_{i=1}^{4}\frac{\partial^2}{\partial z_i^2}|\psi\rangle=N_z^2\,2\alpha\beta(3A_1-3A_{1a}-3B_1\\ +2A_2-2B_2+I_4)+\alpha^2+\beta^2.\end{aligned} \qquad (3.8)$$

Lastly, we turn our attention to $\langle\psi|V|\psi\rangle$ which can be expressed as a sum of a direct term plus an exchange term, viz.,

$$\langle\psi|V|\psi\rangle=V_{\text{dir}}+V_{\text{ex}}, \qquad (3.9)$$

$$\begin{aligned}V_{\text{dir}}=-\int|u_{m=1}(1)|^2|u_{m=1}(z)|^2\frac{e^2}{r_{13}}|v|^2\\ -\int|u_{m=0}(1)|^2|u_{m=0}(3)|^2\frac{e^2}{r_{13}}|v|^2,\end{aligned} \qquad (3.10)$$

$$\begin{aligned}V_{\text{ex}}=-\quad u_1^*(3)u_2(3)\frac{e^2}{r_{34}}u_1(4)u_2^*(4)v^2\\ -\int u_1^*(2)u_1(1)\frac{e^2}{r_{12}}u_2(2)u_2^*(1)v^2.\end{aligned} \qquad (3.11)$$

Note that in the direct term, what is left is just the "self-energy" term between the electrons and the holes in the same states. The exchange terms, of course, are only effective between particles of the same species. The interaction potential can be expressed in a separable form in cylindrical coordinates as[8]

$$\frac{1}{|r_1 - r_2|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\varphi_1 - \varphi_3)} \cos k(z_1 - z_2) \times I_m\,(k\rho_<) K_m(k\rho_>) \,. \tag{3.12}$$

This expression is substituted into (3.10) and (3.11) and the resulting integration carried out. The arithmetic turns out to be rather complicated; one obtains

$$V_{\text{dir}} = V_{\text{dir}}^{(1)} + V_{\text{dir}}^{(2)} \,,$$

$$\begin{aligned} V_{\text{dir}}^{(i)} = -\frac{2e^2}{\pi} \Bigg\{ & \left(-\frac{\pi}{8\alpha^2(\alpha+\beta)} \ln 4\alpha\rho_{im} - \frac{\pi}{4\alpha(\alpha^2-\beta^2)} \ln 2\rho_{im}(\alpha+\beta) + \frac{\pi \ln 4\alpha\rho_{im}}{8\alpha^2(\alpha-\beta)} \right) + \frac{\pi}{8\alpha} \left(\frac{1}{2\alpha(\beta+3\alpha)} - \frac{\ln 4\alpha\rho_{im}}{(\beta+3\alpha)^2} \right. \\ & \left. - \frac{\ln 2(\alpha+\beta)\rho_{im}}{(\beta+3\alpha)^2} + \frac{1}{2\alpha(\alpha+\beta)} - \frac{\ln 4\alpha\rho_{im}}{(\alpha+\beta)^2} - \frac{\ln 2(\alpha+\beta)\rho_{im}}{(\alpha+\beta)^2} \right) + \frac{\pi}{4\alpha^2} \left[\frac{1}{4} \left(\frac{1}{\alpha-\beta} - \frac{1}{\alpha+\beta} \right) - \frac{1}{(\alpha+\beta)(\alpha^2-\beta^2)} \right] \\ & - \frac{\pi}{8(\alpha+\beta)^3} + \frac{\pi}{8} \left(\frac{1}{2\alpha(\alpha+\beta)^2} + \frac{1}{4\alpha^2(\alpha+\beta)} - \frac{1}{4\alpha^2(\alpha-\beta)} - \frac{1}{2\alpha(\alpha-\beta)^2} + \frac{1}{(\alpha-\beta)^2(\alpha+\beta)} \right) \Bigg\} \,, \end{aligned}$$

$$\rho_{2m} = \sqrt{3}\,\rho_0 \,, \quad \rho_{1m} = \rho_0 \,,$$

$$V_{\text{ex}} = -\frac{4e^2}{\pi} \left[-\frac{\pi}{8\alpha^2(\alpha+\beta)} + \frac{\pi}{8\alpha^2(\alpha-\beta)} + \frac{\pi}{4\alpha} \left(-\frac{1}{(\beta+3\alpha)^2} - \frac{1}{(\alpha+\beta)^2} \right) \right] .$$

We can now go ahead and try to minimize the total energy as a function of α and β. Since the expression for the energy is algebraically rather complicated, the minimization is not done explicitly but is rather done by means of the computer. The expression for the energy is evaluated for a range of values of α and β and the minimum is obtained by inspection. The result of this calculation is presented in the next section where its significance is also discussed.

IV. BINDING ENERGY OF THE MOLECULE

The results of the binding of the molecules are presented in Table I for both the case with exchange and that with the exchange term excluded. Note that the parameter that we use instead of B is the ratio of the cyclotron radius to the Bohr radius, viz.,

$$\frac{\rho}{a_0} = \frac{1}{a_0} \left(\frac{\hbar c}{eB} \right)^{1/2} = \frac{4.9 \times 10^4}{B^{1/2}} \quad .$$

One thing is immediately obvious from the table. The exchange term is very important! It caused the molecule to be about twice as bound as it would be without the exchange.

Ruderman[3] has shown that in the extreme-high-field limit the binding energy of an atom goes as

$$E_H^{(1)} = -\frac{\hbar^2}{m a_0^2} \left(\ln \frac{a_0}{\rho} \right)^2 . \tag{4.1}$$

Calling the molecular binding energy E_m, a plot of E_m vs ρ is shown in Fig. 1. The points fall roughly on a straight line, suggesting that the binding of the molecule depends on the magnetic field strength B roughly as

$$E_m = -\alpha (\ln B)^2 \tag{4.2}$$

The next question concerns the stability of the molecule against its decay into two excitons. The binding energy of the hydrogen atom in a strong B field has been calculated by Cohen *et al.*[9,10] using a variational method. However, the binding energy of the exciton is not just half that of the atom. Indeed, in the high-B-field limit, one expects the binding energy of the exciton to be

$$E_{\text{ex}}^{(1)} = \frac{\hbar^2}{\mu a_0'^2} \ln^2\!\left(\frac{a_0'}{\rho} \right) , \tag{4.3}$$

TABLE I. Binding energy of the molecule as a function of the B field ($\rho_m/a_0 = 4.9 \times 10^4/B^{1/2}$ where B is in G) with and without exchange. Note that the binding energy is expressed in terms of the ordinary rydberg, not the exciton rydberg.

ρ_m/a_0	E(Ry)	$(-E)^{1/2}$
	Without exchange	
0.1	−3.6	1.89
0.05	−6.119	2.5
0.01	−18.5	4.3
0.001	−59.3	7.7
	With exchange	
0.1	−6.60	2.6
0.05	−10.9	3.3
0.01	−28.9	5.4
0.001	−80.9	9.0

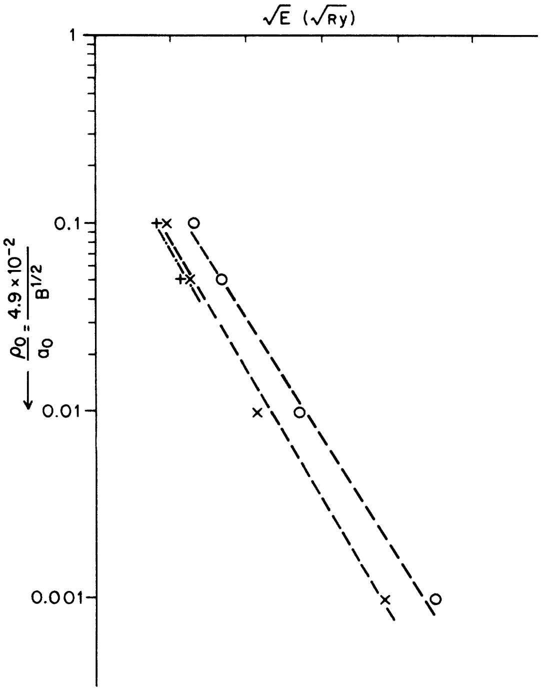

FIG. 1. Binding energy of the excitonic molecule with exchange (O), without exchange (×), and that of two excitons (+). The latter is estimated according to Sec. V, using the values of Cohen *et al.* (Ref. 9). The straight lines are arbitrarily drawn between the points to illustrate that they roughly behave linearly (but not quite—as is seen in the deviation).

where a_0', the exciton Bohr radius, is twice that of the atom. μ here is the effective mass and is equal to $\frac{1}{2}m$.

Thus in this approximation

$$E_{\mathrm{ex}}^{(1)} = \frac{1}{2}\,\frac{\hbar^2}{ma_0^2}\left(\ln 2 + \ln\frac{a_0}{\rho}\right)^2 \tag{4.4}$$

[and not just as $(\hbar^2/2ma_0^2)\ln^2(a_0/\rho)$].

For a field strength $B = 2.4\times10^{11}$ G (corresponding to $\rho_0 = 0.1$). Cohen *et al.*[9] obtained a value 8.25 Ry for their binding energy of hydrogen whereas formula (4.1) gives a value of -10.12 Ry. This suggests that formula (4.3) is not exactly applicable to the region of field strength in which we have calculated the binding energy of the molecule, but it suggests the following estimate. We take Cohen, Lodenquai, and Ruderman's binding energy in Ry, divide it by 2 Ry and subtract ln2 from its square root, and then square it. Since we are interested in two excitons, the above result has to be multiplied by 2. We find that the total binding of two excitons for the case $\rho_0 = 0.1$ (corresponding to $B = 2.4\times10^{11}$ G) to be -3.6 Ry. This value is smaller than our binding energy which is -6.61 Ry. Cohen *et al.*[9,10] have only done the calculation for a limited value of the B field. Our estimate using his results is shown in Fig. 1. As is seen, the binding energy of the molecule without exchange is so close to that of two excitons that it is not possible to say that the molecules (without exchange) are bound. On the other hand, it is fairly obvious that the molecules are bound relative to two excitons when the effect of exchange is included.

V. ELECTRON-HOLE CYLINDER–THE TRIAL WAVE FUNCTION

In Secs. II–IV we have calculated the binding energy of an excitonic molecule in a magnetic field, and we find that even though the exchange does increase the binding, it does not change the logarithmic dependence of the binding energy on the magnetic field. In this section we shall investigate what happens in the high-density limit and what role the exchange plays in this case. As contrasted with the low density case, it will be found that the binding energy has a power-law dependence on the strength of the magnetic field, viz., $E \propto B^{2/3}$.

Suppose we ignore the Coulomb interaction. Then the ground state of such a system is just the Hartree-Fock wave function of the Landau wave functions described in Eq. (2.1). The levels are filled in the usual manner; i.e., one first takes $k_z = 0$ and starts filling up all the m states until one reaches a $m_{\max}$ such that $\rho_{\max} = (2m_{\max}+1)^{1/2}\rho_0$ equals the sample size; then one goes to a higher k_z, and so on. When the Coulomb interaction is included, it turns out that the exchange interaction is essentially proportional to $1/m$, so that it might be more profitable to sacrifice some kinetic energy and gain more potential energy by going to a smaller $m_{\max}$ and a larger k_F. In this way one will have cylindrical electron-hole drops, the radii of which can be tuned by a B field.

To be more precise then, the wave function we are proposing is

$$\psi(\varphi, \rho, z) = \det(\psi_m^k(\rho, \psi, z))$$

$$= \det\left[\prod_{m=1}^{M_{\max}}\prod_{k=0}^{k_F}\left(\frac{1}{m!(2\rho_0^2)^{m+1}\pi}\right)^{1/2} \times e^{im\varphi_i}\rho_i^m e^{-\rho_i^2/4\rho_0^2} e^{ikz_i}\frac{1}{\sqrt{L}}\right].$$

Our task is to find $\langle\psi|H|\psi\rangle$ and determine what $m_{\max}$ and k_F should be for a magnetic field of strength B. We shall come back and discuss the suitability of this wave function after the calculation.

VI. ALGEBRAIC DETAIL

We shall follow the calculation in Sec. III and shall use the same notation with the generalization that particle indices i now runs from 1 to n instead of from 1 to 2. We have a cylinder of N electrons and N holes spread out in $m_{\max}$ shells of increasing radius, each shell has N_l electrons (holes) so that the total number of electrons (holes) N is equal to $N_l \times m_{\max}$. From Sec. III and Appendix A, we have

$$\langle \psi | H' | \psi \rangle = 2N \times \tfrac{1}{2}\hbar\omega_c = N\hbar\omega_c \ . \tag{6.1}$$

The evaluation of $\langle \psi | \sum \partial^2/\partial z_i^2 | \psi \rangle$ is much easier. One just proceeds as in the free-electron theory and finally obtains the energy per electron (hole) (see Appendix B for details):

$$E'_{\text{kin}} = \frac{4\pi^2}{3} \frac{\hbar^2}{ml^2} \left(\frac{\rho_0}{r}\right)^2 . \tag{6.2}$$

The total kinetic energy is that due to the electrons and the holes and is therefore $2E'_{\text{kin}}$, i.e.,

$$E_k = \frac{8\pi^2}{3} \frac{\hbar^2}{ml^2} \left(\frac{\rho_0}{r}\right)^2 , \tag{6.3}$$

where

$$l = \frac{L}{N_l} \approx \frac{r^2}{2\rho_0^2} \frac{L}{N} , \tag{6.4}$$

$$r = (2M_{\max} + 1)^{1/2} \rho_0 \ .$$

We next turn our attention to the potential energy terms

$$\langle \psi | V | \psi \rangle = V_{\text{dir}} + V_{\text{ex}} \ . \tag{6.5}$$

As indicated, they can again be decomposed as a sum of a direct and an exchange term. Here

$$V_{\text{dir}} = -e^2 \sum_{km} \int |\psi_m^k(1)|^2 \frac{1}{r_{12}} |\psi_m^k(2)|^2 dr_1\, dr_2 \ , \tag{6.6}$$

$$V_{\text{ex}} = -2e^2 \sum_{\substack{mk \\ m_1 k_1}} \int \psi_{km}(1) \psi^*_{km}(2) \frac{1}{r_{12}} \psi^*_{m_1 k_1}(1) \psi_{m_1 k_1}(2) \ .$$

The factor of 2 that appears in V_{ex} is due to the fact that there are contributions from both the electron-electron and the hole-hole terms. We shall first concentrate on the exchange term because, as it turns out, it is the only term of importance here.

Using Eq. (3.12) we have

$$\begin{aligned} V_{\text{ex}} = -2 \sum_{\substack{m_1 k_1 \\ m'_2 k'_2}} N^2_{m_1 k_1} N^2_{m'_2 k'_2} \frac{2}{\pi} \sum_m \int & e^{i(\varphi_1 - \varphi_2)(m_1 - m'_2 + m)} \\ & \times (\rho_1 \rho_2)^{m_1 + m_2 + 1} e^{-(\rho_1^2 + \rho_2^2)/2\rho_0^2} e^{i(z_1 - z_2)(k_1 - k'_2)} \\ & \times \cos k(z_1 - z_2) I_m(k\rho_<) K_m(k\rho_>) \\ & \times d\varphi_1\, d\varphi_2\, dz_1\, dz_2\, d\rho_1\, d\rho_2\, dk \ . \end{aligned} \tag{6.8}$$

Here

$$N_{mk} = \left(\frac{1}{\pi L m! (2\rho_0^2)^{m+1}}\right)^{1/2} \tag{6.9}$$

is the normalization constant. Using the fact that

$$\int_{-\infty}^{\infty} du\, e^{ixu} \cos ub = \pi[\delta(x+b) + \delta(x-b)] , \tag{6.10}$$

we can do both the z and the φ integration, arriving at the following expression for V_{ex} :

$$\begin{aligned} V_{\text{ex}} = -L16\pi^2 \sum_{\substack{m_1 k_1 \\ m_2 k_2}} & N^2_{m_1 k_1} N^2_{m_2 k_2} \\ & \times \int (\rho_1 \rho_2)^{m_1 + m_2 + 1} e^{-(\rho_1^2 + \rho_2^2)/2\rho_0^2} d\rho_1\, d\rho_2 \\ & \times I_{m_2 - m}(|k_1 - k_2| \rho_<) K_{m_2 - m_1}(|k_1 - k_2| \rho_>) \ . \end{aligned} \tag{6.11}$$

L here is the total length of the cylinder.

Let us write

$$V_{\text{ex}} = V_A + V_B \ , \tag{6.12}$$

where V_A is the contribution from those terms for which $m_1 = m_2$ and V_B is from those terms with $m_1 \neq m_2$.

The small expansion for $I_m(x) K_m(x)$ will be used, in particular $I_0(x) K_0(x) \rightarrow -\ln(x)$ as $x \rightarrow 0$. With this simplification, the k integral can be easily done, and we have

$$\begin{aligned} V_A = \frac{4e^2}{l} N_L \sum N_m^4 \int & (\rho_1 \rho_2)^{2m+1} \\ & \times e^{-(\rho_1^2 + \rho_2^2)/4\rho_0^2} d\rho_1\, d\rho_2 \left(\ln \frac{2\pi\rho_>}{l} - 2\right) . \end{aligned} \tag{6.13}$$

Here

$$l = \frac{L}{N_l} , \quad N_m = \left(\frac{2}{(2\rho_0^2)^{m+1} m!}\right)^{1/2} . \tag{6.14}$$

After some tedious algebra we obtain an approximate formula for V_A [11]:

$$V_A = 2N \left[\frac{2e^2}{l} \ln \frac{\sqrt{2}\,\pi r}{l} - \frac{e^2}{l}\left(6 - \frac{(2.72)^2}{\sqrt{2}}\right)\right] . \tag{6.15}$$

From Appendix B, we have

$$V_B = -\frac{e^2}{l} 2N\alpha \ , \tag{6.16}$$

where α is of the order of unity. Thus

$$V_{\text{ex}} = 2N \left(\frac{2e^2}{l} \ln \frac{\sqrt{2}\,\pi r}{l} - \frac{e^2}{l} (0.4 + \alpha)\right) . \tag{6.17}$$

V_{dir}, as it turns out, is also much smaller. In fact,

$$\begin{aligned} V_{\text{dir}} = \sum_{mk} 8\pi L N^4_{mk} \int & (\rho_1 \rho_2)^{2m+1} e^{-(\rho_1^2 + \rho_2^2)/2\rho_0^2} d\rho_1\, d\rho_2 \\ & \times \int_0^{\infty} dk \int_{-1/2}^{1/2} \cos k(z_1 - z_2)\, dz_1\, dz_2 \\ & \times I_0(k\rho_<) K_0(k\rho_>) \ . \end{aligned} \tag{6.18}$$

Since there is only one sum over k (those electron-hole terms for which $k_1 \neq k_2$ are zero because the corresponding contribution from the electron-electron terms exactly cancels it out), this term should be $1/N_l$ times smaller than the exchange terms.

Collecting (6.5), (6.11), (6.15), (6.16), and (6.18), we finally have

$$V = 2N\left(\frac{2e^2}{l}\ln\frac{\sqrt{2}\,\pi r}{l} - \frac{e^2}{l}\times(0.4+\alpha)\right) . \tag{6.19}$$

VII. BINDING OF THE ELECTRON-HOLE CYLINDER

From the previous section, the total energy per unit electron-hole pair can be written as (apart from $N\hbar\omega_c$)

$$\frac{E}{N} = \frac{8\pi^2}{3}\,\frac{\hbar^2}{ml^2}\left(\frac{\rho_0}{r}\right)^2 - \frac{e^2}{l}\left(4\ln\frac{l}{\sqrt{2}\,\pi r} + (0.8+\alpha)\right) . \tag{7.1}$$

Minimizing with respect to both l and r, we obtain (taking α to be zero)

$$r = 1.1\, a_0(\rho_0/a_0)^{2/3} , \tag{7.2}$$

$$l = 0.58\, a_0(\rho_0/a_0)^{2/3} , \tag{7.3}$$

$$E = -13.4(a_0/\rho_0)^{2/3} e^2/a_0 . \tag{7.4}$$

One might wonder how good Eq. (7.4) is since correlation has not been included. Recently, Horing, Danz, and Glasser[12] have evaluated the correlation energy of a free-electron gas in a magnetic field. Realizing our cylinder is just a free-electron gas with a Fermi energy

$$\zeta = \frac{\hbar^2}{2m}\,\frac{1}{l^2} , \tag{7.5}$$

we can directly apply their formula. They obtain

$$E_{\text{corr}} = \frac{1}{16\pi^2}\left(\frac{\hbar\omega_c}{\zeta}\right)\ln\gamma , \tag{7.6}$$

$$\gamma = \hbar^2\omega p^2/4\zeta\hbar\omega_c . \tag{7.7}$$

$\ln\gamma$ is typically of the order unity, e.g., with $\rho_0 = 10\, a_0$, $\ln\gamma \approx 5$. Substituting Eq. (7.5) into Eq. (7.6), we obtain

$$E_{\text{corr}} \cong \frac{1}{60}\left(\frac{a_0}{\rho_0}\right)^{2/3} \text{Ry} , \tag{7.8}$$

which is much smaller than Eq. (7.4), thus justifying the neglect of the correlation energy.

It is conceivable that the above-mentioned cylinders can attract each other by van der Waals forces and form some sort of a two-dimensional structure. However, one has to have an idea of the surface energy involved to make a prediction as to whether there should be one or more than one cylinder in an electron-hole drop. Intuitively, one would expect the surface energy to be much smaller than the volume energy that we just calculated so that at low pumping power level there should just be a few cylinders present in the drop. These cylinders should be easily detected from light scattering. Recently, Pokrovskii and Svistunova[13] did light scattering on the electron-hole drops and from the angular dependence of this Raleigh scattering was able to obtain an estimate of the dimensions of the drops. It is expected that this light scattering will be strongly anisotropic in the present strong-magnetic-field case.

Without detailed calculation taking the full band structure of a crystal into account, it is not possible to make statements as to how large a field is required for the above to be seen. A rough estimate could be made from the fact that the magnetic field must be such that the exciton binding energy be comparable with the binding energy [Eq. (7.14)], i.e.,

$$13.4(a_0/\rho_0)^{2/3}\times 2 = 1 . \tag{7.9}$$

Since we shall be mainly interested in Ge, we take a_0 to be the excitonic radius in Ge. The ρ_0 that appears in Eq. (7.9) should in this case be $[(\hbar c/eB)\epsilon_0^{1/2}]^{1/2}$, where ϵ_0 is the static dielectric constant and is equal to 15.4 in germanium. (Since typical excitonic energy is much less than the band gap, the use of a static dielectric constant is justified.) With this, we obtain a necessary B field of 4 G. This estimate is of course too rough to be true, but it does indicate that the B field required by condition (7.9) is not very large. Another criterion that has to be satisfied is that only the lowest Landau orbital is occupied. That results in a B field limit of about 10^5 G.

VIII. CONCLUSION

In this paper we have calculated the binding energy of an excitonic molecule with equal electron and hole masses and showed that a large amount of the binding relative to the state of two excitons comes from exchange. We have also showed how it is possible for the electron-hole drops to form a new phase consisting of cylinders as the magnetic field becomes large and have estimated its binding energy in the model for which the electron and hole masses are equal. In these pseudo-one-dimensional cylinders it is conceivable that there can be some Peierls-type[14] instabilities that convert the cylinder into an insulator if the temperature is low enough. This will not affect our estimate of the binding energy very much since such effects will have been included in the correlation-energy calculation already and they are shown to be small. Lastly, it is also possible that there can be some spin-density-wave instabilities, but this is also expected not to change our binding energy very much.

Although the stability of the electron-hole liquid relative to the excitonic liquid depends a lot on the band structure in the zero-magnetic-field case, this is not so in the high-magnetic-field situation. Indeed, as is shown in this paper, the binding energy of the electron-hole liquid goes as $B^{1/3}$ [Eq. (2.3) and (7.4)] whereas that of the exciton goes as $\ln B_0$ so that when the magnetic field is high enough the electron-hole cylinder is always more stable than the excitonic phase. This is true not only in Ge.

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APPENDIX A

We shall confine our interest to the ρ and ϕ coordinates and write that part of the wave function as

$$\psi = \psi_1(1)\psi_2(2) - \psi_1(2)\psi_2(1) \ . \tag{A1}$$

Here

$$\psi_1 = N_1 e^{i\varphi}\rho\, e^{-\rho^2/4\rho_0^2}, \quad \psi_2 = N_2 e^{-\rho^2/4\rho_0^2} \ .$$

Then

$$\langle\psi|H'|\psi\rangle = \langle\psi_1|H'|\psi_1\rangle + \langle\psi_2|H'|\psi_2\rangle \ . \tag{A2}$$

The cross term vanishes because H_i' commutes with L_z and ψ_1, ψ_2 are different eigenfunctions of L_z (and hence orthogonal to each other).

H_i' could be further decomposed as $H'_{\varphi_i} + H'_{\rho_i}$ where

$$H'_{\varphi_i} = -\frac{\hbar^2}{2m}\frac{1}{\rho_i^2}\frac{\partial^2}{\partial\varphi_i^2} + \frac{ie\hbar H}{2mc}\frac{\partial}{\partial\varphi_i}, \tag{A3}$$

$$H'_{\rho_i} = -\frac{\hbar^2}{2m}\frac{1}{\rho_i}\frac{\partial}{\partial\rho_i}\left(\rho_i\frac{\partial}{\partial\rho_i}\right) + \frac{e^2H^2}{8mc^2}\rho_i^2 \ . \tag{A4}$$

Then

$$\langle\psi_1|H_{\varphi_1}|\psi_1\rangle = -\frac{\hbar^2}{2m}\frac{1}{2}\frac{1}{\rho_0^2} , \tag{A5}$$

$$\langle\psi_2|H_{\varphi_2}|\psi_2\rangle = 0 , \tag{A6}$$

$$\langle\psi_1|H_{\rho_1}|\psi_1\rangle = \frac{\hbar^2}{2m}\frac{1}{\rho_0^2}\frac{3}{2} , \tag{A7}$$

$$\langle\psi_2|H_{\rho_2}|\psi_2\rangle = \frac{\hbar^2}{2m}\frac{1}{\rho_C^2} , \tag{A8}$$

so that

$$\langle\psi_1|H_1'|\psi_1\rangle = \frac{\hbar^2}{2m}\frac{1}{\rho_0^2} = \tfrac{1}{2}\hbar\omega c = \langle\psi_2|H_2'|\psi_2\rangle \tag{A9}$$

in agreement with what one would expect using the full solution in (2.1). In fact, in general we have

$$\begin{aligned}\langle\psi_m|H'_{\varphi_m}|\psi_m\rangle &= -(\tfrac{1}{4}\hbar\omega_c)m = \frac{-\hbar^2}{2M}\frac{1}{2\rho_0^2}m ,\\ \langle\psi_m|H'_{\rho_m}|\psi_m\rangle &= \frac{\hbar^2}{2M}\frac{1}{2\rho_0^2}(m+1),\end{aligned} \tag{A10}$$

so that

$$\langle\psi_m|H_m'|\psi_m\rangle = \tfrac{1}{2}\hbar\omega_c \ .$$

This fact is used in Sec. VI where one is interested in electron-hole cylinders.

APPENDIX B

In this appendix, we shall calculate the kinetic energy of the one-dimensional electron gas described in Secs. V and VI. The density of states being $L/2\pi$ (we use periodic boundary condition), the total energy is

$$\frac{L}{2\pi}\frac{\hbar^2}{2m}\frac{k_F^3}{3} \ . \tag{B1}$$

The Fermi momentum is obtained from the requirement that

$$\frac{L}{2\pi}\int^{k_F} dk = N_l , \tag{B2}$$

where N_l is not the total number of electrons because we have not yet included the degeneracy of the states. In fact, N_l is related to the total electron density N by

$$N_{lm} \approx N_l r^2/2\rho_0^2 = N \ . \tag{B3}$$

Here r is the maximum radius of the cylinder and m is the number of shells in the cylinder. In writing down (B3) we have assumed $m \gg 1$ so that $r = (2m+1)^{1/2}\rho_0 \approx (2m)^{1/2}\rho_0$. Putting (B3) and (B2) into (B1), we finally have

$$\frac{E_{\text{kin}}}{N} = \frac{4\pi^2}{3}\frac{\hbar^2}{ml^2}\left(\frac{\rho_0}{r}\right)^2 . \tag{B4}$$

APPENDIX C

In this appendix we shall argue that $V_B/m_{\max} \to \infty$.

Using the fact that

$$I_{m_1-m_2}(x)K_{m_1-m_2}(x) \xrightarrow[x\to 0]{} \frac{1}{|m_1-m_2|} , \tag{C1}$$

we obtain, after some algebra ,

$$V_B = -2N_l\frac{e^2}{l}\sum_{m_1\neq m_2}\frac{\{[\frac{1}{2}(m_1+m_2)]!\}^2}{m_1!\,m_2!}\frac{1}{|m_1-m_2|} .$$

Using Sterling's approximation for $\Gamma(m)$, we obtain

$$V_B \cong -\left(\sum\frac{[\frac{1}{2}(m_1+m_2)]^{m_1+m_2+1}}{m_1^{m_1+1/2}m_2^{m_2+1/2}}\frac{1}{|m_1-m_2|}\right)2N_l\frac{e^2}{l} \ . \tag{C2}$$

This can be written as

$$V_B \simeq - \sum_{m_1 \neq m_2} \left(\frac{\frac{1}{2}(m_1 + m_2)}{m_1} \right)^{m_1+1/2} \left(\frac{\frac{1}{2}(m_1 + m_2)}{m_2} \right)^{m_2+1/2} \times \frac{1}{|m_1 - m_2|} \, 2N_l \frac{e^2}{l} \qquad \text{(C3)}$$

It is obvious from (C3) that the dominant contribution comes from those terms for which m_1 is close to m_2. We thus write

$$V_B = -\alpha \, 2N_l \, m_{\max} \frac{e^2}{l} \quad . \qquad \text{(C4)}$$

To evaluate α one has to go to the computer. Since the present calculation is a rather simplified model, it is not worthwhile to carry out the exact evaluation of α.

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$$c = \begin{cases} 0 & \varphi_1 = \varphi_2 \\ 2 \quad \text{when} & \varphi_1 - \varphi_2 = \frac{1}{2}\pi \\ 4 & \varphi_1 - \varphi_2 = \pi \end{cases}$$

which is what we want.

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$$(\rho_0^2)^{2m+2} \left[2m - 1) + 2 \left(\frac{1}{m} + \frac{1}{m-1} \right) m(m-1)(2m-2)! + 2^2 \left(\frac{1}{m} + \frac{1}{m-1} + \frac{1}{m-2} \right) m(m-1)(m-2)! + \cdots \right]$$

by

$$\tfrac{1}{4}(2\rho_0^2)^{m+1} - \tfrac{1}{2}(2\rho_0^2)^{m+1} m! \, \tfrac{1}{2}\sqrt{\pi} \; m \, e^{-(m-1)} (m-1)^{m-1/2} \quad .$$

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