Saturation and coherence properties of three-magnon nonlinear processes*

Cid B. de Araujo and S. M. Rezende[†]

Instituto de Física, Universidade Federal de Pernambuco, 50000, Recife, Brasil

(Received 5 October 1973)

A theoretical account of the saturation and coherence properties of the states generated in three-magnon nonlinear processes is given. The results show that above the nonlinear threshold the parametric magnons become coherent and their saturation is achieved through the back reaction on the uniform mode.

Since the work of Suhl[1] and other authors[2–4] it has been known that it is possible to excite magnons with wave vector $k \neq 0$ in microwave experiments with strongly magnetic materials. The excitation is due to the oscillation of the coupling parameter between two or more magnon modes, and for this reason the processes are called parametric. As in other nonlinear processes, the excitation is very intense when the driving field exceeds a threshold value. In the "perpendicular-pumping" case the coupling between the magnon modes is made through the uniform precession mode, which is excited by the external field.[1,2] In the "parallel-pumping" case there is a direct coupling between magnon pairs and the driving field.[3,4] The previous theoretical discussions of these processes have been concerned with macroscopic aspects of the excitations, such as the threshold fields and the susceptibilities.[1–6] There has been little attempt to describe the microscopic aspects of the parametric magnons above the threshold. On the other hand, the only discussion of the statistical properties of the states generated under parametric excitation[7] was made under several simplifications rendering incomplete results. In the present paper we give a theoretical account of the saturation and coherence properties of parametric magnons in the three-magnon process known as subsidiary absorption.[1,2] The study is developed taking into account the interaction of the k modes with the heat bath and their back reaction on the uniform precession. The results obtained here differ from those of Savchenko and Tarasenko,[8] who have used a perturbation approach, which is not very convenient for nonlinear processes.

We use the Holstein-Primakoff formalism to describe the excitations of a Heisenberg ferromagnet. Assume that a sample is uniformly magnetized in the z direction. The Hamiltonian taken includes Zeeman, exchange, and anharmonic interactions, the interaction between the system and the heat bath, and the classical excitation by an external rf magnetic field perpendicular to the static field. Following Balucani *et al.*,[6] we write

$$H = H_S + H_R + H_{RS} + H_1(t), \tag{1}$$

where

$$H_S = \omega_0 a_0^\dagger a_0 + \sum_k [\omega_k a_k^\dagger a_k + (f_k a_0^\dagger a_k a_{-k} + \text{H.c.})], \tag{2}$$

$$H_R = \sum_\alpha \omega_\alpha R_\alpha^\dagger R_\alpha , \tag{3}$$

$$H_{RS} = \sum_{k,\alpha} (g_{k\alpha}^* R_\alpha^\dagger a_k + g_{k\alpha} a_k^\dagger R_\alpha), \tag{4}$$

$$H_1(t) = 2ch[a_0 e^{i\omega t} + a_0^\dagger e^{-i\omega t}], \tag{5}$$

where ω_0, $\omega_k = \omega_{-k}$ are the frequencies of the magnon modes of interest; a_0, a_k, and a_{-k} are the corresponding annihilation operators; f_k is the dipolar three-magnon coupling coefficient[5]; $g_{k\alpha}$ is the coupling constant between the magnon system and the heat bath, which is described by the frequencies ω_α and operators R_α; h is the amplitude of the microwave driving field; and c is an appropriate conversion factor $c = \frac{1}{4} g \mu_B (2NS)^{1/2}$.

The equations of motion of the magnon operators obtained from the Heisenberg equation are given by

$$\frac{\partial a_k}{\partial t} = -i \frac{\delta H}{\delta a_k^\dagger} - \eta_k a_k + F_k(t), \tag{6}$$

where

$$\eta_k = \pi f(\omega_k) |g_{kk}|^2 , \tag{7}$$

$$F_k(t) = -i \sum_\alpha g_{k\alpha} R_\alpha(t_0) e^{-i\omega_\alpha t} , \tag{8}$$

where η_k is the relaxation rate of the mode k, $f(\omega_k)$ is the density of the heat-bath modes, and $F_k(t)$ represents a Langevin random force with correlators of Markoffian-systems type.[6] It is very convenient to work in the representation of coherent magnon states,[7] which are defined as the eigenstates of the annihilation magnon operator $a_k |v_k\rangle = v_k |v_k\rangle$. From Eq. (6) we obtain a nonlinear system of equations for the three interacting modes with wave vectors 0, $\vec{k}$, and $-\vec{k}$. Elimination of all variables except v_k and v_0 leads to

$$\frac{dv_k(t)}{dt} - a[b - |v_k(t)|^2] v_k(t) = S_k(t), \tag{9}$$

where

$$a = 32 |f_k|^4 (2ch)^2 \eta_0^{-3} (\eta_k^2 + 4 |f_k|^2 |v_0|^2)^{-1}, \tag{10}$$

$$b=2|f_k|^2\eta_0^{-1}a^{-1}\left(\frac{(2ch)^2}{\eta_k\eta_0}-\frac{\eta_k\eta_0}{4|f_k|^2}\right), \tag{11}$$

$$S_k(t)=\tfrac{1}{2}F_k(t)\,e^{i\omega_k t}+(f_k v_0/2\eta_k)F^*_{-k}(t)e^{-i\omega_k t}, \tag{12}$$

$$|v_0|^2=\frac{(2ch)^2}{\eta_0^2}\left(1-\frac{16|f_k|^2\eta_k|v_k|^2}{\eta_0(\eta_k^2+4|f_k|^2|v_0|^2)}\right). \tag{13}$$

Equations (9)–(13) contain all the information carried by Eq. (6), except for the assumption that the uniform mode interacts with a single mode pair and that $\omega=\omega_0=2\omega_k$. Equation (9) is a typical nonlinear Langevin equation which appears in Brownian-motion studies and laser theory.[9] It shows that the magnon-pair modes, whose amplitude is v_k, are driven (thermally) by the heat-bath modes and by the uniform precession. The term ab can be positive or negative and therefore can produce damping or amplification of the modes. The term $a|v_k|^2$ represents a nonlinear damping effect which can saturate the rate of creation of magnon-pair modes. The onset of the instability process can be obtained with $b=0$, which gives

$$\langle n_0\rangle_{\rm crit}=|v_0|^2_{\rm crit}=\left(\frac{2c}{\eta_0}h_{\rm crit}\right)^2=\left(\frac{\eta_k}{2|f_k|^2}\right)^2 . \tag{14}$$

This value agrees with the results for the threshold of the subsidiary absorption process.[1,5] For $b>0$ the magnon pairs are driven unstable. In the steady state one can readily show that the expectation value of the occupation number of k modes in this case is given by

$$\langle n_k\rangle=|v_k|^2=\frac{\eta_0(\eta_k^2+4|f_k|^2|v_0|^2_{\rm sat})}{16|f_k|^2\eta_k}\times\left(1-\frac{(\eta_k\eta_0)^2}{4|f_k|^2(2ch)^2}\right), \tag{15}$$

where $|v_0|^2_{\rm sat}$ is the number of $k=0$ modes at the saturation. Equation (15) was obtained under the assumption that $\dot v_0=0$, which is obviously true in the steady state. In this approximation one loses information about the saturation value of $|v_0|^2$. However, clearly $|v_0|$ does not vary much above the threshold and one can assume that $|v_0|^2_{\rm sat}=|v_0|^2_{\rm crit}$, which is given in Eq. (14). This assumption is consistent with the previous one in which the uniform mode was supposed to interact only with a single mode pair. In fact the interaction with other magnon modes would simply introduce a correction in $|v_0|^2_{\rm sat}$, which would have a small effect on the occupation number of each k mode. In the present context, $\langle n_k\rangle$ does not go to infinity as in the previous treatments of this process. The saturation in the k magnon population, which comes from term $a|v_k|^2$ in Eq. (9), is simply a result of the conversion of the magnon pairs back into uniform precession modes. The neglect of this back-reaction term leads to the infinity of the previous studies of this process. Note that when we consider $S_k(t)$ in Eq. (9) there is a stable solution $v_k\neq 0$ for $b<0$, which leads to a threshold condition different than Eq. (14). This correction in the critical field has been pointed out before[10] but it is very small in the materials of greater interest [e.g., yttrium iron garnet (YIG)].

In order to study the coherence properties of the magnon states created in the nonlinear three-magnon process we use the density-matrix formalism. In the coherent-state representation[9] the density matrix can be expanded as

$$\rho=\int P(v_k)\,|v_k\rangle\langle v_k|\,d({\rm Re}v_k)\,d({\rm Im}v_k), \tag{16}$$

where v_k is the complex eigenvalue of the coherent state $|v_k\rangle$ and $P(v_k)$ is the proper expansion coefficient and might be thought as a probability density. Substituting Eq. (16) into the equation of motion for the density matrix $i\hbar\dot\rho=[H,\rho]$, with $v_k=re^{i\phi}$ and the same approximations used previously, we obtain

$$\frac{\partial P}{\partial y}+\frac{1}{x}\frac{\partial}{\partial x}[(D-x^2)x^2P]=\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right)+\frac{1}{x^2}\frac{\partial^2 P}{\partial\phi^2}, \tag{17}$$

where

$$x=(a/\Gamma_k)^{1/4}r,\quad y=(a\Gamma_k)^{1/2}t,$$

$$D=(a/\Gamma_k)^{1/2}b,\quad \Gamma_k=\tfrac{1}{2}\eta_k(\bar n_k+\bar n_{-k}+1),$$

where $\bar n_k$ and $\bar n_{-k}$ are the thermal number of magnons with $\vec k$ and $-\vec k$ wave vectors. This is a Fokker-Planck[9] equation which is stochastically equivalent to Eq. (9).

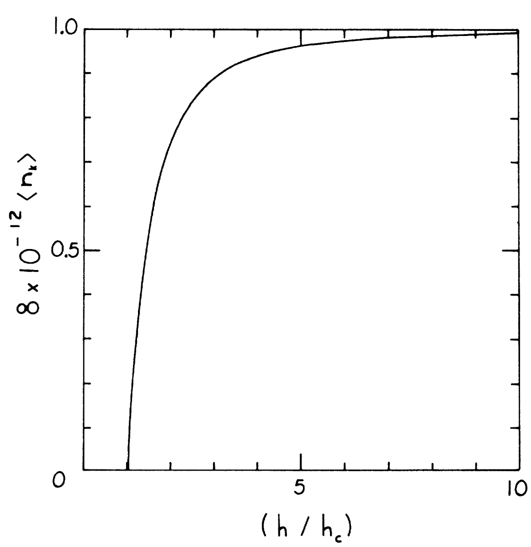

FIG. 1. Number of parametric spin waves excited by three-magnon nonlinear processes vs the ratio between the microwave field amplitude and the threshold field in YIG. The relevant parameters are $|f_k|\sim 1$ sec^{-1} and $\eta_0\cong\eta_k\simeq 10^6$ sec^{-1}.

Here we are interested only in the stationary solution of Eq. (17), that is, the one which is independent of y and ϕ.[9] Straightforward integration gives

$$P(x) = N \exp(\tfrac{1}{2}Dx^2 - \tfrac{1}{4}x^4), \qquad (18)$$

where

$$N = \left(2\pi \int_0^\infty x \exp(\tfrac{1}{2}Dx^2 - \tfrac{1}{4}x^4)\,dx\right)^{-1}.$$

Note that as a and Γ_k are positive definite, D has the sign of b. Hence for $b<0$, i.e., below the threshold field, $D<0$ and the function $P(x)$ behaves as a Gaussian distribution. This distribution is similar to that of systems in thermal equilibrium.[9] For $D \gg 1$ the amplitude of the stationary state consists of two components, a coherent one convoluted with a much smaller fluctuation with Gaussian distribution.[9] For YIG this condition is satisfied for fields just above the threshold ($\sim 1.005 h_{crit}$). This behavior is similar to that of photon states in lasers.[9]

The authors would like to acknowledge helpful discussions with Professor N. Zagury and with all members of the Solid State Group of Universidade Federal de Pernambuco.

---

*Work partially supported by Conselho Nacional de Pesquisas (Brazilian Government).

†Under a grant of FORGE Foundation (U.S.A.).

[1]H. Suhl, J. Phys. Chem. Solids 1, 209 (1957).

[2]N. Bloembergen and S. Wang, Phys. Rev. 93, 72 (1954).

[3]E. Schlömann, J. J. Green, and U. Milano, J. Appl. Phys. 31, 386S (1960).

[4]F. R. Morgenthaler, J. Appl. Phys. 31, 95S (1960).

[5]H. Le Gall, Phys. Status Solidi 28, 495 (1968).

[6]U. Balucani, F. Barochi, and V. Tognetti, Phys. Rev. A 5, 442 (1972).

[7]N. Zagury and S. M. Rezende, Phys. Rev. B 4, 1201 (1971).

[8]M. A. Savchenko and V. V. Tarasenko, Zh. Eksp. Teor. Fiz. 51, 482 (1966) [Sov. Phys.-JETP 24, 325 (1967)].

[9]See, for example, H. M. Nussenzweig, *Topics in Quantum Optics* (Gordon and Breach, New York, to be published) or J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

[10]U. Balucani, F. Barocchi, and V. Tognetti, Nuovo Cimento B 12, 83 (1972).