Transverse magnetoresistance of nondegenerate semiconductors in strong magnetic fields

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The transverse magnetoresistance has been calculated for a nondegenerate semiconductor throughout the high-field region. The calculation has been performed taking into account the inelasticities in the electron-phonon scattering due to the finite energy of the phonons involved. The results are in agreement with previous calculations for the classical and quantum limits. However, our results are also valid in the intermediate region $\hbar\omega_c \sim kT$. Our results indicate that the classical limit is realized for $\hbar\omega_c/kT \leq 1/4$ while the quantum limit is obtained for $\hbar\omega_c/kT \geq 4$. We expect that these values are also valid for other scattering mechanisms.

I. INTRODUCTION

High-field galvanomagnetic phenomena are characterized by the condition that the product of the carrier cyclotron frequency and its mean time between collisions be much greater than unity, $\omega_c\bar{\tau} \gg 1$. Within this classification, two limiting cases are of interest. In the first, the classical limit, the magnetic field is strong enough to satisfy the high-field criterion, but not so large as to render the splitting of the Landau levels comparable to the average carrier energy: $\omega_c\bar{\tau} \gg 1$, $\hbar\omega_c/kT \ll 1$. In the other case, the quantum limit, the strong-field condition is obtained and, in addition, the splitting of the Landau levels is much greater than the average carrier energy: i.e., $\omega_c\tau \gg 1$ and $\hbar\omega_c/kT \gg 1$.

The theory of the transverse magnetoresistance of semiconductors in both the classical[1] and quantum[2-4] limits has been intensively studied in the past. In practice, however, the conditions necessary to obtain these limits are often difficult to realize. This is particularly true of the quantum limit, where impractically large magnetic fields are often required. It is then desirable to obtain expressions that apply throughout the strong-field regime (i.e., for all values of $\hbar\omega_c/kT$). Such a calculation not only depicts the behavior in the transition region, but, more importantly, serves to define exactly where the classical- and quantum-limit variations are obtained.

In this paper we calculate the transverse magnetoresistance of a nondegenerate semiconductor with isotropic parabolic energy bands throughout the strong-field regime. Only the case of scattering due to acoustic phonons via the deformation-potential mechanism is considered.

The scattering is treated in the Born approximation. Theoretical expressions for the strong-field transverse magnetoresistance in the elastic Born approximation diverge logarithmically.[2,3] This divergence is resolved by the more important of three cutoff mechanisms: inelasticities in the electron-phonon interaction,[2,3] collisional broadening of the energy levels,[2,3] and phonon drag.[5] In this study we assume that inelasticity is the dominant mechanism in resolving the divergence.

II. HIGH-FIELD TRANSVERSE MAGNETORESISTANCE

In the absence of collisions, the application of crossed electric and magnetic fields imparts a Hall velocity $\vec{V}_H = c(\vec{E}\times\vec{H})/H^2$ to each of the carriers. This corresponds to a nondissipative current $j_H = n_0 e V_H$ normal to the fields. Here n_0 is the carrier concentration. In a strong magnetic field, collisions are generally treated as a perturbation, this perturbation being calculated using the Born approximation.[2,3] Although one of us[6] has found reason to question the applicability of this approximation in strong magnetic fields, we employ it here.

In the Born approximation, the effect of collisions is to give rise to a dissipative current lying in the direction of the total electric field. For scattering due to acoustic phonons, this current is given by[7,8]

$$j_d = \frac{eL^2V_H\hbar}{kT}\sum_{ij}\frac{(k_y^i - k_y^j)^2}{2} f(\epsilon_i)[1 - f(\epsilon_j)]W_{ij}\,. \tag{2.1}$$

Here i and j represent the eigenstates of an electron in a magnetic field, $L = (\hbar/m^*\omega_c)^{1/2}$ is the classical radius of the lowest Landau level, k_y is the quantum number which determines the position along the electric field (i.e., $\langle i|x|i\rangle = -k_yL^2$), $f(\epsilon_i)$ is the thermal-equilibrium occupation probability for an electron in the state i, ϵ_i is the energy of an electron in the state i, and W_{ij} is the thermal-equilibrium transition rate between states i and j. In the Born approximation this rate is given by

$$W_{ij} = \frac{2\pi}{\hbar}\sum_q |C_q|^2\{|M_{ij}(\vec{q})|^2(N_q+1)\delta(\epsilon_i - \epsilon_j - \hbar\omega_q)$$

$$+ \left|M_{ij}(-\vec{q})\right|^2 N_q \delta(\epsilon_i - \epsilon_j + \hbar\omega_q)\} . \quad (2.2)$$

In the above, C_q is the electron-phonon coupling constant, $M_{ij}(\vec{q}) = \langle i | e^{i\vec{q}\cdot\vec{r}} | j \rangle$, N_q is the thermal-equilibrium number of phonons with wave vector $\vec{q}$, and $\hbar\omega_q$ is the phonon energy.

Equation (2.2) is the appropriate expression to be used when inelasticities in the electron-phonon interaction are the dominant mechanism removing the divergence in the transverse resistivity. This divergence is resolved by the appearance of the phonon energy in the energy-conserving δ function. If collisional broadening were to be the dominant mechanism, a collisional-corrected expression that recognizes that energy is only conserved within the limits imposed by the uncertainty principle would be employed,[3] while if phonon drag were to be the dominant mechanism, an additional term in Eq. (2.1) accounting for the current arising from disturbances in the phonon distribution would be required.[5,9]

The high-field magnetoresistance is generally measured in the configuration where no current flows normal to the applied field. In this, the so-called Hall configuration, the resistivity is given by $\rho_T = \vec{E}\cdot\vec{j}/j^2$. Noting that in a strong magnetic field $j_H \gg j_d$, this can be approximated by

$$\rho_T = E j_d/(n_0 e V_H)^2 , \quad (2.3)$$

where E is the total electric field.

We can now proceed with the calculation of the transverse resistivity. We confine our attention to the case of nondegenerate statistics so that

$$f(\epsilon_i) = 8\pi^{3/2} \frac{n_0}{q_T} L^2 \sinh\left(\frac{\hbar\omega_c}{2kT}\right) \times \exp\left[-\left(n+\frac{1}{2}\right)\frac{\hbar\omega_c}{kT} - \frac{k_z^2}{q_T^2}\right] . \quad (2.4)$$

Here

$$q_T = (2m^* kT/\hbar^2)^{1/2} \quad (2.5)$$

is the thermal de Broglie wave vector for the carriers, n is the quantum number indicating the Landau level (n can take on zero or any positive-integer value), and k_z is the quantum number designating the component of electron momentum parallel to the magnetic field (i.e., $\langle i|\hat{p}_z|i\rangle = \hbar k_z$). Furthermore, we employ the high-temperature approximation

$$\hbar s q_M \ll 4kT , \quad (2.6)$$

where s is the velocity of sound and q_M is the maximum wave vector for those phonons that significantly interact with the electrons. We find that for $\hbar\omega_c < kT$, $q_M \sim q_T$, while for $\hbar\omega_c > kT$, $q_M \sim 1/L$. An immediate consequence of this approximation is that $N_q \approx kT/\hbar\omega_q$. The high-temperature approximation is generally satisfied at temperatures where acoustic-phonon scattering dominates.

The matrix element $M_{ij}(\vec{q})$ can be evaluated in a straightforward manner. The result is that for $n_j \geq n_i$.

$$\begin{aligned} |M_{ij}(\vec{q})|^2 &= |\langle n_i, k_y^i, k_z^i | e^{i\vec{q}\cdot\vec{r}} | n_j, k_y^j, k_z^j\rangle|^2 \\ &= (2\pi)^2 (n_i!/n_j!) e^{-L^2 q_\perp^2/2} \left(\frac{L^2 q_\perp^2}{2}\right)^{n_j - n_i} \\ &\times \left[L_{n_i}^{n_j - n_i}\left(\frac{L^2 q_\perp^2}{2}\right)\right]^2 \\ &\times \delta(k_z^j - q_y - k_y^j)\delta(k_z^i - q_z - k_z^j) , \end{aligned} \quad (2.7)$$

while for $n_i > n_j$ the relationship $|M_{ij}(\vec{q})|^2 = |M_{ij}(-\vec{q})|^2$ can be employed. In the above $L_n^m(x)$ is the associated Laguerre polynomial,[10] and q_z, $q_\perp$, and q_y are the components of the phonon wave vector directed parallel to the magnetic field, normal to the magnetic field, and in the $\vec{H}\times\vec{E}$ direction, respectively.

Substituting Eqs. (2.2), (2.4), and (2.7) into (2.1), and with the aid of the generating function[11]

$$\left(\frac{x^2 z}{1-z}\right)^{-n/2} e^{-2xz/(1-z)} I_n\left(\frac{2xz^{1/2}}{1-z}\right) = \sum_{l=0}^{\infty} \frac{l!}{(l+n)!} [L_l^n(x)]^2 z^l , \quad (2.8)$$

where $I_n(y)$ is the Bessel function of imaginary argument, we can reduce the summations on n_i and n_j to a simple summation on $n = n_i - n_j$. The resulting expression for the dissipative current is

$$j_d = \frac{n_0 e V_H}{(2\pi)^2 \hbar\omega_c kT q_T \pi^{1/2}} \sum_{n=-\infty}^{\infty} \int d^3q \frac{q_y^2}{|q_z|} |C_q|^2 (N_q + 1) \times I_n\left[\frac{L^2 q_\perp^2}{2} \operatorname{csch}\left(\frac{\hbar\omega_c}{2kT}\right)\right] \exp\left[-\frac{L^2 q_\perp^2}{2}\coth\left(\frac{\hbar\omega_c}{2kT}\right) - \frac{q_z^2}{4q_T^2} - \frac{m^*(n\omega_c + \omega_q)^2}{2kTq_z^2}\right] . \quad (2.9)$$

In obtaining this result, the summations over k_y^i, k_y^j, k_z^i, and k_z^j were completed and the remaining summations converted to integrations. Also, the high-temperature approximation was not yet used at this stage.

For acoustic-phonon scattering via the deformation-potential-coupling mechanism, the electron-phonon coupling constant is given by[12]

$$|C_q|^2 = E_1^2 \hbar q / 2\rho_m s , \quad (2.10)$$

where E_1 is the deformation potential constant and ρ_m the mass density of the crystal. It is convenient to introduce the expression for the resistivity in the absence of a magnetic field due to deformation-potential coupling,[13]

$$\rho_0 = \frac{m^*}{n_0 e^2} \frac{E_1^2}{\rho_m s^2} \frac{3}{8\pi^{1/2}\hbar} q_T^3 . \quad (2.11)$$

Using Eqs. (2.3) and (2.9)–(2.11) we obtain in the high-temperature approximation the result

$$\frac{\rho_T}{\rho_0}=\frac{1}{3\pi^2 q_T^4}\sum_{n=-\infty}^{\infty}\int d^3q\,\frac{q_y^2}{|q_z|}\exp\left[-\frac{L^2q_\perp^2}{2}\coth\left(\frac{\hbar\omega_c}{2kT}\right)\right]\times I_n\left[\frac{L^2q_\perp^2}{2}\operatorname{csch}\left(\frac{\hbar\omega_c}{2kT}\right)\right]\times\exp\left[-\frac{q_z^2}{4q_T^2}-\frac{m^*(n\omega_c+\omega_q)^2}{2kTq_z^2}\right]. \quad (2.12)$$

We should point out that Gurevich and Firsov[14] have derived a similar expression using a Green's function approach. Their result, however, assumes that $\omega_c \gg \omega_q$, an assumption that is not valid throughout the strong-field regime.

The last exponential in Eq. (2.12) provides a lower cutoff on q_z to remove the divergence arising from the factor $|q_z|^{-1}$. The factor ω_q here arises from inelasticities. This exponential term is only important when its argument exceeds unity. For $n=0$, or $\omega_q \gg n\omega_c$, this implies $q_\perp^2/q_z^2 \gtrsim 2kT/(m^*s^2)$, which in the high-temperature approximation is much greater than unity. This allows us to replace the phonon energy $\hbar\omega_q$ appearing in this exponential term by $\hbar s q_\perp$. This approximation is also valid when $\omega_q \ll n\omega_c$ because here the ω_q term in the exponential is not important and could be ignored altogether.

Making this substitution and letting $\tau = q_z^2/4q_T^2$, $x = L^2q_\perp^2/2$, and $q_y = q_\perp \sin(\phi)$, Eq. (2.12) can be written

$$\frac{\rho_T}{\rho_0}=\frac{1}{6\pi}\left(\frac{\hbar\omega_c}{kT}\right)^2\sum_{n=-\infty}^{\infty}\int_0^\infty dx\,xe^{-x\coth(\hbar\omega_c/2kT)}\times I_n\left[x\operatorname{csch}\left(\frac{\hbar\omega_c}{2kT}\right)\right]\int_0^\infty\frac{d\tau}{\tau}\times\exp\left[-\tau-\left[\frac{n\hbar\omega_c+(2\hbar\omega_c m^*s^2x)^{1/2}}{2kT}\right]^2\frac{1}{4\tau}\right]. \quad (2.13)$$

The integral on τ can be identified as a modified Bessel function of the second kind. Using the integral representation[15]

$$K_\nu(z)=\frac{1}{2}\left(\frac{z}{2}\right)^\nu\int_0^\infty\frac{d\tau}{\tau^{\nu+1}}\,e^{-\tau-z^2/4\tau}, \quad (2.14)$$

we obtain

$$\frac{\rho_T}{\rho_0}=\frac{1}{3\pi}\left(\frac{\hbar\omega_c}{kT}\right)^2\sum_{n=-\infty}^{\infty}\int_0^\infty dx\,xe^{-x\coth(\hbar\omega_c/2kT)}\times I_n\left[x\operatorname{csch}\left(\frac{\hbar\omega_c}{2kT}\right)\right]K_0\left(\frac{|n\hbar\omega_c+(2\hbar\omega_c m^*s^2x)^{1/2}|}{2kT}\right). \quad (2.15)$$

Noting that the high-temperature approximation requires that $\hbar\omega_c m^*s^2x_M \ll 8(kT)^2$, where x_M is that maximum value of x that contributes significantly to the integration, we can use the approximations valid for $y \ll 1$,

$$K_0(|y|)\approx -\ln(y/2)-\gamma \quad (2.16)$$

and

$$K_0(|x+y|)\approx K_0(|z|)-|z|K_1(|z|)\ln|1+y/z|, \quad (2.17)$$

where $\gamma = 0.577$ is Euler's constant. This leads to the result

$$\frac{\rho_T}{\rho_0}=\frac{1}{3\pi}b^2\left\{-\int_0^\infty dx\,xe^{-x\coth(b/2)}I_0\left(x\operatorname{csch}\frac{b}{2}\right)\times\left[\gamma+\tfrac{1}{2}\ln\left(\frac{m^*s^2bx}{8kT}\right)\right]+2\sum_{n=1}^{\infty}K_0\left(\frac{nb}{2}\right)\int_0^\infty dx\,xe^{-x\coth(b/2)}I_n\left(x\operatorname{csch}\frac{b}{2}\right)-\sum_{n=1}^{\infty}\frac{nb}{2}K_1\left(\frac{nb}{2}\right)\int_0^\infty dx\,xe^{-x\coth(b/2)}\times I_n\left(x\operatorname{csch}\frac{b}{2}\right)\ln\left|1-\frac{2m^*s^2x}{n^2\hbar\omega_c}\right|\right\}, \quad (2.18)$$

where $b=\hbar\omega_c/kT$.

The first integral in Eq. (2.18) represents the contribution due to transitions within the same Landau level while the second represents the corresponding interlevel component. The last integral gives a small correction to this interlevel component due to inelasticities.

Let us consider the first term. Using the integral representation[16]

$$P_\mu^{-\nu}\left(\coth\frac{b}{2}\right)=\frac{1}{\Gamma(\mu+\nu+1)}\int_0^\infty dt\,t^\mu e^{-t\coth(b/2)}\times I_\nu\left(t\operatorname{csch}\frac{b}{2}\right), \quad (2.19)$$

where $P_n^m(z)$ is the associated Legendre function, and the relationship

$$t\ln t=\frac{d}{d\mu}t^\mu\bigg|_{\mu=1}, \quad (2.20)$$

this term can be expressed as

$$\frac{\rho_T}{\rho_0}\bigg|_{\text{intra}}=-\frac{b^2}{6\pi}\left\{\left[\psi(2)+\ln\frac{m^*s^2}{8kT}b+2\gamma\right]P_1\left(\coth\frac{b}{2}\right)+\frac{\partial P_\mu[\coth(b/2)]}{\partial\mu}\bigg|_{\mu=1}\right\}, \quad (2.21)$$

where $\psi(z)=(\partial/\partial z)\ln\Gamma(z)$ and $\Gamma(z)$ is the Γ function. Using the integral representation[17]

$$P_\nu(z)=\frac{1}{\pi}\int_0^\pi d\phi[z+\cos(\phi)(z^2-1)^{1/2}]^\nu, \quad (2.22)$$

it is easily shown that

$$\frac{\partial P_\mu[\coth(b/2)]}{\partial\mu}\bigg|_{\mu=1}=\left(\coth\frac{b}{2}\right)\ln\left[\frac{1+\coth(b/2)}{2}\right]$$

$$+\left(\coth\frac{b}{2}\right)-1\,, \qquad (2.23)$$

and with $\psi(2)=1-\gamma$, we obtain the desired result

$$\frac{\rho_T}{\rho_0}\bigg|_{\text{intra}}=\frac{b^2}{6\pi}\left(\left(\coth\frac{b}{2}\right)\left\{-(\gamma+1)+\ln\left[\frac{8kT}{m^*s^2}\,b\left(\frac{2}{1+\coth(b/2)}\right)\right]\right\}+1-\left(\coth\frac{b}{2}\right)\right). \qquad (2.24)$$

Now consider the second integral in Eq. (2.18), the one due to interlevel transitions. With the aid of Eq. (2.19) and the relationship[18]

$$P_1^{-n}(z)=\frac{1}{\Gamma(n+2)}\left(\frac{z+1}{z-1}\right)^{-n/2}(n+z)\,, \qquad (2.25)$$

this becomes

$$\frac{\rho_T}{\rho_0}\bigg|_{\text{intra}}=\frac{2b^2}{3\pi}\sum_{n=1}^{\infty}\left(n+\coth\frac{b}{2}\right)e^{-nb/2}K_0\left(\frac{nb}{2}\right). \qquad (2.26)$$

Finally, consider the last integral. The modified Bessel function $I_n(z)$ can be replaced by its asymptotic expansion for large argument when its argument is much greater than the square of its order.[19] Inspection of the rest of the integrand reveals that, for $nb \lesssim 1$, the major contribution to the integral arises when this condition is satisfied. Thus, we have for this correction term

$$\frac{\rho_T}{\rho_0}\bigg|_{\text{corr}}=-\frac{b^2}{3\pi^{3/2}}\frac{\cosh^2(b/2)}{\sinh(b/2)}\sum_{n=1}^{\infty}\frac{nb}{2}K_1\left(\frac{nb}{2}\right)\times\int_0^{\infty}dy\,y^{1/2}e^{-y}\ln\left|1-\frac{(2m^*s^2/kT)y}{n^2b\tanh(b/4)}\right|. \qquad (2.27)$$

The above expressions for transverse magnetoresistance were evaluated numerically. The computed resistivity is shown in Fig. 1 as a function of $b=\hbar\omega_c/kT$, with the quantity $[8kT/(m^*s^2)]^{1/2}$ as a parameter. The values chosen for this parameter range from the largest that would normally be encountered in practice to the smallest admitted by the high-temperature approximation.

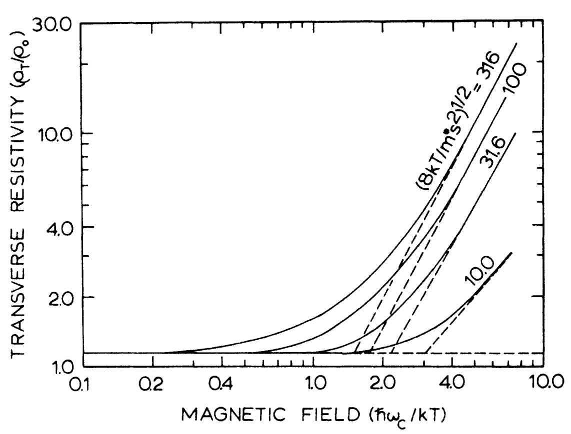

FIG. 1. The transverse magnetoresistance as a function of magnetic field for different values of the quantity $(8kT/m^*s^2)^{1/2}$. The variations in the classical and quantum limiting cases are depicted by the dashed curves.

In the classical region, the saturation magnetoresistance is seen to be $\rho_{T\,\text{sat}}=1.13\rho_0$, in agreement with earlier treatments.[1] This value was obtained for all but the most inelastic case studied. For this exceptional case [i.e., $(8kT/m^*s^2)^{1/2}=10$)] the saturation resistance was somewhat smaller. This difference is presumably due to the breakdown of the high-temperature approximation. As a result, we simply shifted the computed resistance to give $\rho_{T\,\text{sat}}=1.13\,\rho_0$.

From our expression for resistivity, it is readily shown that for $\hbar\omega_c/kT \lesssim \frac{1}{2}$ the derivation from saturation is given by[20]

$$\frac{\rho_T-\rho_{T\,\text{sat}}}{\rho_0}\le\frac{1}{3\pi}\frac{\hbar\omega_c}{kT}\ln\left(\frac{kT}{m^*s^2}\right). \qquad (2.28)$$

This deviation is seen to be small for all cases studied when $\hbar\omega_c/kT\le\frac{1}{4}$.

For still stronger fields, a transition to the quantum-limit behavior is seen, this limit being obtained for $\hbar\omega_c/kT\ge 4$.

In this development, we have assumed that the maximum values of the wave vector for phonons that interact with the electrons are of order q_T when $\hbar\omega_c/kT<1$, and of order $1/L$ when $\hbar\omega_c/kT>1$. To demonstrate the validity of this, it suffices to consider only the classical and quantum limiting cases.

In the quantum limit only the first term of Eq. (2.18) contributes to the resistivity. In the limit $\hbar\omega_c/kT\to\infty$ most of the contribution to the integral on x arises for $x\lesssim 1$. This corresponds to $q\lesssim 1/L$.

In the classical limit the second term of Eq. (2.18) provides the main contribution. Careful investigation of this term reveals that in the limit $\hbar\omega_c/kT\to 0$ the major contribution arises for $q\lesssim q_T$.

III. CONCLUSIONS

We have calculated the transverse magnetoresistance for nondegenerate semiconductors throughout the high-field region. This region includes the classical and quantum limits, and also the transition region between these limiting cases.

Our results are in agreement with previous calculations for the classical and quantum limits. We found that the classical limiting behavior is realized for fields $\hbar\omega_c/kT\le\frac{1}{4}$, while the quantum limit is obtained when $\hbar\omega_c/kT\ge 4$.

The calculation was performed assuming that scattering was due to acoustic-phonon scattering via the deformation-potential mechanism and further, that the dominant cutoff mechanism was the finite energy of the phonons. Although our results are limited to these restrictions, it seems reasonable to expect that the points at which the classical and quantum limits are realized should apply to other scattering and cutoff mechanisms.

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