The Toda lattice. II. Existence of integrals

H. Flaschka

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

(Received 22 August 1973)

Following recent computer studies which suggested that the equations of motion of Toda's exponential lattice should be completely integrable, Hénon discovered analytical expressions for the constants of the motion. In the present paper, the existence of integrals is proved by a different method. Our approach shows the Toda lattice to be a finite-dimensional analog of the Korteweg-de Vries partial differential equation. Certain integrals of the Toda equations are the counterparts of the conserved quantities of the Korteweg-de Vries equation, and the theory initiated here has been used elsewhere to obtain solutions of the infinite lattice by inverse-scattering methods.

The Toda lattice[1] is a system of unit masses, connected by nonlinear springs governed by an exponential restoring force. The equations of motion are derivable from the Hamiltonian

$$H=\sum_{n=1}^{n}\left(\tfrac{1}{2}P_n^2+e^{-(Q_n-Q_{n-1})}\right), \tag{1}$$

in which Q_n is the displacement of the nth mass from equilibrium, and P_n is the corresponding momentum. We assume periodic boundary conditions: $Q_{n+N}=Q_n$.

Recent computer experiments by Ford, Stoddard, and Turner[2] suggested that the Toda lattice is integrable. Subsequently, Hénon[3] confirmed this conjecture by exhibiting N integrals; the starting point of his investigations was the integrability of the hard-sphere gas, which is one limiting form of the Toda lattice. Here, we shall present a different, less computational proof of his result. Our method is based on the realization that the Toda lattice belongs to a class of evolution equations which can be studied, and in some cases solved, by utilization of a certain associated eigenvalue problem. All other known equations of this type are partial differential, the most famous of them being the Korteweg–de Vries equation.[4] It is interesting and suggestive that the latter is one continuum approximation to the Toda lattice.[5] In a subsequent paper we shall show that, in fact, the infinite Toda lattice can be solved by means of the inverse-scattering problem for a discrete Sturm-Liouville equation; the details are quite similar to those involved in the solution of the Korteweg–de Vries equation, as are the formulas for N-soliton solutions, the conservation laws, etc.

We begin by introducing new variables

$$a_n=\tfrac{1}{2}e^{-(Q_n-Q_{n-1})/2},$$

$$b_n=-\tfrac{1}{2}P_{n-1}.$$

It is easy to check that if Q_n, P_n satisfy the equations of motion derived from (1); then a_n, b_n satisfy

$$\dot a_n=a_n(b_{n+1}-b_n), \tag{2}$$

$$\dot b_n=2(a_n^2-a_{n-1}^2).$$

Now we define matrices L and B, functions of t, by[6]

$$L=\begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & 0 & & \\ 0 & a_2 & b_3 & a_3 & & \\ 0 & 0 & a_3 & b_4 & & \\ \vdots & & & & \ddots & \\ a_N & & & & & b_N \end{pmatrix},$$

$$B=\begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & -a_N \\ -a_1 & 0 & a_2 & 0 & \cdots & 0 \\ 0 & -a_2 & 0 & a_3 & \cdots & 0 \\ \vdots & & & & & \vdots \\ a_N & & & & & 0 \end{pmatrix}.$$

A simple computation shows that

$$\dot L=[B,L]\equiv BL-LB. \tag{3}$$

Now we appeal to a theory developed by Lax,[7] which has been used with success in the theory of certain partial differential equations.[8] Let $V=V(t)$ be the solution of $\dot V=BV$; since B is skew symmetric, V is orthogonal. A simple computation shows that $(d/dt)V^{-1}LV=0$; in fact, the derivative on the left reduces to $V^{-1}(\dot L-[B,L])V$, which vanishes by (3). Hence $V^{-1}LV=\text{const}$, so that $L(t)$ is always (orthogonally) similar to the same matrix. It follows[9] that the eigenvalues $\lambda_1,\ldots,\lambda_N$ of $L(t)$ are constant in time. In particular, in the characteristic polynomial

$$P(\lambda)=\lambda^N+I_1\lambda^{N-1}+\cdots+I_N-2^{-N+1}$$

of L (the extra power of 2 is a normalization consistent with Ref. 3), the quantities $2^j I_j$ are constant: they are the integrals of the Toda lattice constructed by Hénon.[3] Furthermore, it can be seen that the Hamiltonian (1) can be written as $H=\frac{1}{2}\sum_{n=1}(2\lambda_n)^2$, so that the $(2\lambda_n)$'s play the role of linear momentum variables for the Toda lattice. For an argument establishing the independence of these integrals, and for the reduction of the fixed-end lattice to the periodic lattice, we refer to Ref. 3.

Hénon[3] has also constructed a different set of integrals, which he denotes by J_n, which can be extended to the case of an infinite lattice, $N\to\infty$. He has observed[10] that these, too, have a simple interpretation in terms of the matrix L:

$$mJ_m=(-2)^m \operatorname{Tr} L^m \text{ for } m<N ,$$

$$NJ_N=(-2)^N \operatorname{Tr} L^N-2N(-1)^N .$$

It is interesting that the integrals of the Korteweg–de Vries equation can also be represented, in a certain sense, as traces of powers of a differential operator whose role in that theory is similar to the role of the matrix L in the present example.[11] We intend to comment further on this matter in the next paper.

I wish to thank Professor J. Ford and Professor M. Hénon for supplying preprints of their work, and for very helpful remarks.

---

[1]M. Toda, Prog. Theor. Phys. Suppl. 45, 174 (1970).
[2]J. Ford, S. D. Stoddard, and J. S. Turner, Prog. Theor. Phys. (to be published).
[3]M. Hénon, preceding paper, Phys. Rev. B 9, 1921 (1974).
[4]C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1967).
[5]M. Toda, M. Wadati, J. Phys. Soc. Jap. 34, 18 (1973).
[6]If $N=2$, slight modifications are necessary.
[7]P. D. Lax, Commun. Pure Appl. Math. 21, 467 (1968).
[8]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys.-JETP 34, 62 (1972)]; M. Wadati, J. Phys. Soc. Jap. 32, 1681 (1972).
[9]This consequence of a relation of the type (3) is the discovery of Lax (Ref. 7).
[10]M. Hénon (private communication).
[11]V. E. Zakharov and L. D. Faddeev, Funkts. Anal. i evo Prilozh. 5, 18 (1971).