Integrals of the Toda lattice

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The exponential lattice introduced by Toda is shown to be an integrable dynamical system. An explicit set of n integrals is given for a lattice of n particles with periodic boundary conditions. The case of fixed-end boundary conditions is also covered as a particular case. An alternative set of integrals is obtained, which can be extended to the case of an infinite lattice.

Toda[1] introduced and studied a one-dimensional lattice in which the force between neighbor particles is an exponentially decreasing function of their distance. With an appropriate unit of length, the equations of motion are

$$\dot{x}_i = u_i, \quad \dot{u}_i = C(e^{-(x_i - x_{i-1})} - e^{-(x_{i+1} - x_i)}), \tag{1}$$

where x_i is the displacement of the ith particle from its equilibrium position and u_i is the corresponding velocity. C is a constant.

Extensive numerical investigations by Ford, Stoddard, and Turner[2] strongly suggested that the Toda lattice is an integrable dynamical system. We prove here that this is indeed the case and we give explicit expressions for the integrals.

Let us define

$$X_i = C\, e^{-(x_{i+1} - x_i)}. \tag{2}$$

The equations of motion become

$$\dot{X}_i = (u_i - u_{i+1}) X_i, \quad \dot{u}_i = X_{i-1} - X_i. \tag{3}$$

We consider first the case of a periodic lattice: $x_{i+n} = x_i$, with n given. The system is defined by one full period, for instance by particles 1 to n. Then the following expressions are n-independent integrals of the motion:

$$I_m = \sum u_{i_1} \cdots u_{i_k} (-X_{j1}) \ldots (-X_{jl}) \quad (m = 1, \ldots, n), \tag{4}$$

where the summation is extended to all terms which satisfy the following conditions: (i) the indices $i_1, \ldots, i_k, j_1, j_1 + 1, \ldots, j_l, j_l + 1$, which appear in the term (either explicitly, or implicitly through a factor X_j) are all different (modulo n); (ii) the number of these indices is m, i.e., $k + 2l = m$. Two terms differing only in the order of the factors are not considered different, and therefore only one of them appears in the sum. For example, for $n = 3$ the integrals are

$$\begin{aligned} I_1 &= u_1 + u_2 + u_3, \\ I_2 &= u_1 u_2 + u_2 u_3 + u_3 u_1 - X_1 - X_2 - X_3, \\ I_3 &= u_1 u_2 u_3 - u_1 X_2 - u_2 X_3 - u_3 X_1. \end{aligned} \tag{5}$$

For the proof it will be convenient to write symbolically $u_i = [i]$, $(-X_j) = [j, j+1]$. The relations (3) become

$$\begin{aligned} \frac{d}{dt}[i, i+1] &= ([i] - [i+1])[i, i+1], \\ \frac{d}{dt}[i] &= -[i-1, i] + [i, i+1]. \end{aligned} \tag{6}$$

Therefore $\dot{I}_m$ is a sum of derived terms of the same form as the original terms, except that one index may eventually appear twice. We consider all possible cases:

(a) A derived term has no doubled index. This can only result from the derivation of a factor $[i]$ in an original term of (4), in which one of the neighboring indices $i-1$ and $i+1$ is not used. For instance, if $i+1$ is not used, then the derived term containing $[i, i+1]$ has no doubled index. But then there exists another original term where $[i]$ is replaced by $[i+1]$, the rest being unchanged, and this term produces a derived term containing $-[i, i+1]$, which destroys the previous one. Thus all derived terms of this kind disappear.

(b) A derived term has a doubled index i, common to a factor $[i]$ and a factor $[i, i+1]$. This can result from an original term containing either $[i, i+1]$ or $[i]$ $[i+1]$ (through the derivation of $[i+1]$). The sign is + in the first case, − in the second case; for any original term containing $[i, i+1]$ there is another term where this is replaced by $[i]$ $[i+1]$, and vice versa, as shown by the formation rules; therefore derived terms of this kind also disappear.

(c) The case of a doubled index i common to a factor $[i-1, i]$ and a factor $[i]$ is treated in the same way.

(d) A derived term has a doubled index i, common to a factor $[i-1, i]$ and a factor $[i, i+1]$. This results from an original term containing either $[i-1]$ $[i, i+1]$ or $[i-1, i]$ $[i+1]$, and again the derived terms cancel each other in pairs. This completes the proof of $\dot{I}_m = 0$.

To show that the n integrals I_m are independent, we remark that if a general relation existed between them, it would exist in particular for $C = 0$. But in that case I_m reduces to the symmetric function of order m of the velocities

$$I_m = \sum u_{i_1} \cdots u_{i_m} , \tag{7}$$

with $i_1, \ldots, i_m$ all different, and the n symmetric functions of n variables are independent.

I_1 is simply the total momentum of the n particles (assuming that each particle has unit mass), while I_2 is related to the total energy H by

$$I_2 = \tfrac{1}{2} I_1^2 - H. \tag{8}$$

No simple physical meaning has been found for the other integrals $I_3, \ldots, I_n$. Flaschka[3] has found an elegant derivation of the constancy of the I_m, based on Lax's[4] formalism.

We consider now the case of fixed-end boundary conditions, which has been investigated numerically by Saito *et al.*[5] x_0 and x_{n+1} are set permanently equal to zero, and the motion of particles 1 to n is studied. It turns out that this case can be reduced to the previous one.[6] Consider a periodic lattice of period $2n+2$; define it by the particles $-n$ to $+(n+1)$; and take initial conditions such that

$$x_{-i} = -x_i , \quad u_{-i} = -u_i . \tag{9}$$

If these conditions are satisfied at $t=0$, they will be satisfied at all times because the initial symmetry will be preserved. Therefore, in particular, $x_0 = 0$, $x_{n+1} = 0$ at all times, and particles 1 to n behave as a fixed end system. We have the integrals $I_1, \cdots, I_{2n+2}$ of the periodic system, which can be expressed in terms of the positions and velocities of particles 1 to n. For m odd, I_m vanishes because symmetrical terms destroy each other. I_{2n+2} reduces to a constant: Consider one of its terms which contains factors u_i, and let i_0 be the smallest value of $|i|$ among these factors. Since all indices must be used in a term of I_{2n+2}, the only possible configurations between $-i_0$ and $+i_0$ are

$$[-i_0, -i_0+1][-i_0+2, -i_0+3] \cdots [i_0-2, i_0-1][i_0], \tag{10}$$

and the symmetrical one. But then there exists in I_{2n+2} another term with the sequence from $-i_0$ to $+i_0$ inverted, the rest being unchanged, and the two terms destroy each other. The only terms left in I_{2n+2} are those which do not contain factors u_i. There are two of them; one is

$$[-n, -n+1] \; [-n+2, -n+3] \cdots [n, n+1] \tag{11}$$

and the other has the same form with all indices shifted one place to the right or left. Therefore $I_{2n+2} = 2(-C)^{n+1} = \text{const}$. We are left with the n integrals $I_2, I_4, \ldots, I_{2n}$, i.e., the required number for the system of n particles with fixed ends. To show that these are nontrivial independent integrals, consider a term in I_{2m} which contains only factors u_i. Assume that some factor u_i is present without the symmetrical factor u_{-i}. Then there exists in I_{2m} an opposite term obtained by replacing u_i by u_{-i}, and these two terms cancel. The only terms of the form considered which are left are products of pairs $u_i u_{-i}$. Therefore in the particular case $C=0$, we have

$$I_{2m} = \sum (-u_{i_1}^2) \cdots (-u_{i_m}^2) , \tag{12}$$

with $i_1, \cdots, i_m$ all different: the n integrals I_{2m} are the symmetric functions of the n quantities $(-u_i^2)$, and therefore they are independent.

Finally we consider the case of an infinite lattice. The integrals (4) of the periodic case cannot be readily extended to the limit $n \to \infty$ because they involve a multiple summation on the indices. But it is possible to write another set of n independent integrals for the periodic case, namely:

$$J_m = \sum_{i=1}^{n} \sum A(\alpha_0, \ldots, \alpha_p, \beta_0, \ldots \beta_{p-1}) u_i^{\alpha_0} u_{i+1}^{\alpha_1} \cdots u_{i+p}^{\alpha_p} X_i^{\beta_0} X_{i+1}^{\beta_1} \cdots X_{i+p-1}^{\beta_{p-1}} \quad (m = 1, \ldots, n) , \tag{13}$$

where the second sum is extended, for a given i, to all terms which satisfy

$$p \geq 0, \quad \alpha_j \geq 0, \quad \beta_j \geq 1, \quad \sum_{j=0}^{p} \alpha_j + 2 \sum_{j=0}^{p-1} \beta_j = m , \tag{14}$$

and the numerical coefficients A are given by

$$A(\alpha_0, \ldots, \alpha_p, \beta_0, \ldots \beta_{p-1}) = \prod_{j=0}^{p} \frac{(\alpha_j + \beta_{j-1} + \beta_j - 1)!}{\alpha_j !} \prod_{j=0}^{p-1} \frac{1}{\beta_j ! (\beta_j - 1)!} . \tag{15}$$

In the first product it is understood that $\beta_{-1} = \beta_p = 0$. Note also that if $p=0$, there is no X factor in (13). The explicit expressions for $m = 1-5$ are

$$J_1 = \sum_{i=1}^{n} u_i ,$$
$$J_2 = \sum_{i=1}^{n} [\tfrac{1}{2} u_i^2 + X_i] ,$$

$$J_3=\sum_{i=1}^{n}\left[\tfrac{1}{3}u_i^3+(u_i+u_{i+1})X_i\right],$$
$$J_4=\sum_{i=1}^{n}\left[\tfrac{1}{4}u_1^4+(u_i^2+u_iu_{i+1}+u_{i+1}^2)X_i+\tfrac{1}{2}X_i^2+X_iX_{i+1}\right],\tag{16}$$
$$J_5=\sum_{i=1}^{n}\left[\tfrac{1}{5}u_i^5+(u_i^3+u_i^2u_{i+1}+u_iu_{i+1}^2+u_{i+1}^3)X_i+(u_i+u_{i+1})X_i^2+(u_i+2u_{i+1}+u_{i+2})X_iX_{i+1}\right].$$

A straightforward computation shows that $\dot{J}_m=0$. The J_m's are independent, since for $C=0$ they reduce to

$$J_m=\frac{1}{m}\sum_{i=1}^{n}u_i^m .\tag{17}$$

There are, of course, relations between the two sets of integrals I and J; the first of these relations are

$$J_1=I_1,\quad J_2=-I_2+\tfrac{1}{2}I_1^2,\; J_3=I_3-I_1I_2+\tfrac{1}{3}I_1^3,$$
$$J_4=-I_4+I_1I_3+\tfrac{1}{2}I_2^2-I_1^2I_2+\tfrac{1}{4}I_1^4 .\tag{18}$$

Note that J_2 is the total energy H, as shown by (8). Using again the particular case $C=0$, we find that these relations are the standard relations between the sums of equal powers (17) and the symmetric functions (7) of the variables u_i.

The integrals J_m involve a single summation over i, for a given value of m. Therefore they can be extended to the limit $n\to\infty$, provided that the sums converge. In order to secure the convergence, it will usually be necessary to subtract appropriate constants from the brackets of (16) before going to the limit $n\to\infty$. Consider, for instance, the soliton described by Toda,[1] given in our notations by

$$u_i=\beta(\tanh\theta_i-\tanh\theta_{i-1}),$$
$$X_i=1+\beta^2(1-\tanh^2\theta_i),\tag{19}$$

with $\theta_i=\alpha i-\beta t$, $\beta=\sinh\alpha$. For $i\to\pm\infty$, there holds $u_i\to 0$, $X_i\to 1$, and the terms in brackets in (16) tend to definite limits C_m. It can be shown that $C_m=0$ for m odd, $C_m=(m-1)!/(\frac{1}{2}m)!^2$ for m even. We define new integrals K_m, for n finite, by

$$K_m=J_m-nC_m .\tag{20}$$

For the above soliton solution (19), the integrals K_m converge for $n\to\infty$. The first values are

$$K_1=2\sinh\alpha,\; K_2=\sinh 2\alpha,$$
$$K_3=\tfrac{2}{3}\sinh 3\alpha+2\sinh\alpha,\tag{21}$$
$$K_4=\tfrac{1}{2}\sinh 4\alpha+2\sinh 2\alpha.$$

The integrals J_m will also be of interest when going to the continuum limit, where they reduce to "constants of local conservation type," as defined by Miura, Gardner, and Kruskal.[7]

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