PHYSICAL REVIEW B VOLUME 9, NUMBER 4 15 FEBRUARY 1974

# Cyclotron phase resonance in a thin slab: The variational method

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(Received 20 August 1973)

The transmission of electromagnetic energy through a thin plasma slab in the anomalous-skin-effect regime is calculated using a new variational principle. The principle is shown to be closely related to the iterative scheme used in the multiple-reflection method. Using this new technique, we show that, for slabs even as thin as 1/10 of an electron mean free path, the higher-order multiple reflections are negligible and the first-order multiple-reflection result provides an excellent approximation to the true transmission. In a succeeding paper, this variational technique will be extended to include Fermi-liquid correlation effects.

## I. INTRODUCTION

This is the second of a series of three papers devoted to the theoretical study of the phenomenon of cyclotron phase resonance in a thin slab of metal.[1]

The first paper[2] was concerned with the effect of the Fermi-liquid-theory parameter $A_1$ on the fields deep in the interior of a thick slab. The present paper considers anew the problem of transmission through a thin slab, one whose thickness is a fraction of an electron mean free path, when there are no Fermi-liquid effects. There are two reasons for doing so. One is to investigate certain aspects of the thin-slab problem which have received inadequate attention in the past. The second is to introduce the variational technique which will make it possible, in the third paper of the series, to calculate the effect of Fermi-liquid-theory correlations on the phenomenon of cyclotron phase resonance.

The impetus for our study is provided by a recent letter in which Phillips, Baraff, and Dunifer (PBD)[3] reported measurements of microwave transmission through thin (approximately one-tenth of an electron mean free path) slabs of sodium and potassium at a microwave frequency of 116 GHz. The most striking feature of the experiment was a strong sharp peak in the intensity of the transmitted signal when the ambient magnetic field (which was directed normal to the face of the slab) was swept through that value for which the cyclotron frequency of the carriers was equal to the microwave frequency.

Although there were, and still are, strong reasons for believing that many of the characteristic features of the PBD measurement were dominated by electron-correlation effects, there is seemingly a possibility that the existence of the strong sharp peak was more a consequence of the thinness of the slab than of the correlations. This possibility appears viable because fully self-consistent evaluations of the transmission of microwave field through a metallic slab under cyclotron-phase-resonance conditions[4] have been carried out only to first order in the multiple-internal-reflection series.[5] (In the multiple-reflection scheme, the zeroth-order field is the microwave field which would be found in the infinite medium. The first-order term arises from the action of the emergent surface which reflects fields back towards the incident surface. The second-order term arises from the action of the incident surface on the reflected field. This sends an additional field forward towards the emergent surface, and so on.) Thus, there remains the possibility that higher-order internal reflections might, when the cyclotron frequency and microwave frequency are equal, add coherently, much as do the fields in a Fabry-Perot interferometer on resonance, to produce the strong sharp peak which is observed. This paper is concerned with the evaluation of the contribution of the higher-order multiple-reflection terms.

Although it would be perfectly possible to carry through the iterative procedures (described in Refs. 4 and 5) to generate the higher-order reflection terms sequentially, we have found another procedure for evaluating these terms which is much simpler to carry out. It is based on the use of a variational principle to evaluate the transmission. The principle, which we shall derive and use here, is apparently a new one. It can easily be generalized to include the effects of electron correlations, something which cannot readily be included in the multiple-reflection-series formalism.

In Sec. II of this paper, we shall derive the variational principle to be used for the free-electron situation wherein correlations are ignored.

In Sec. III of this paper, we demonstrate that when the transmission is evaluated using (as trial fields) the fields which would exist in the semi-infinite slab, i.e., the *zeroth*-order fields, the result is exactly the same as that obtained using the *first*-order term of the multiple-reflection series. By adding suitable terms to the zeroth-order fields (terms which are to be varied so as to render the calculated transmission stationary), we can, at a single stroke, represent the effect of *all* the higher terms in the multiple-reflection series.

In Sec. IV of the paper, we carry out this evaluation, again for the free-electron case. The results of the calculation, which are presented in Sec. V, indicate that even for slabs as thin as those used in the PBD experiments, higher-order multiple reflections have slight effect on the transmission. Therefore the first-order calculation provides a reasonable description of the fields in the slab. This first-order calculation does *not* yield the strong sharp peak which is observed and we can conclude, with some confidence, that this feature is *not* a result of coherent internal reflections. In the third and concluding paper of the series, we shall extend this variational principle to include correlation effects and use it to evaluate the transmission, this time when correlations are present.

## II. DERIVATION OF VARIATIONAL PRINCIPLE

Consider Maxwell's wave equation relating the transverse circularly polarized microwave electric field $e(z)$ to the transverse circularly polarized current $j(z)$. Supressing the $e^{-i\omega t}$ time dependence, we have

$$\left(\frac{d^2}{dz^2}+k_0^2\right)e(z)=-i\omega\mu_0 j(z),\quad 0<z<Ll$$

where $k_0=\omega/c$ and where the slab, which is normal to the $z$ axis, has a thickness of $L$ times the electron mean free path $l$. It is convenient to take $l$ as the unit of length so that $z=xl$ defines $x$ as the dimensionless distance coordinate, to let the electric field $e(z)$ be denoted as $\psi_0(x)$, and to introduce a function $\psi_1(x)$ which is proportional to the current. Then the wave equation becomes

$$\left(\frac{d^2}{dx^2}+k_0^2l^2\right)\psi_0(x)=+ib\psi_1(x)\ , \tag{2.1}$$

$$b\equiv(\omega_p V_F/\omega c)^2(\omega\tau)^3\ . \tag{2.2}$$

Although the constant $b$ could be chosen arbitrarily, the choice here [where $\omega_p=(4\pi ne^2/m)^{1/2}$ is the plasma frequency, $V_F$ is the Fermi speed, and $\tau=l/V_F$ is the mean free time of the conduction electrons] has the consequence that no dimensional constants appear in the equation which describes the current as being driven by the electric field via a nonlocal conductivity. The constant $b$ is a huge number, $\approx 10^{10}$ in the situation of interest here, while $b^{-1/3}$, a very small number, is essentially the ratio of the anomalous skin depth to the mean free path.

The boundary conditions we apply to (2.1) are those appropriate to having a wave of amplitude $A$ incident on the slab at $x=0$:

$$\left(1+\frac{1}{ik_0l}\frac{d}{dx}\right)\psi_0(x)=2A,\quad x=0$$

$$\left(1-\frac{1}{ik_0l}\frac{d}{dx}\right)\psi_0(x)=0,\quad x=L\ .$$

We integrate (2.1) subject to these boundary conditions to obtain

$$\psi_0(x)=A\,e^{ik_0lx}+\frac{b}{2k_0l}\int_0^L e^{ik_0l|x-y|}\psi_1(y)\,dy \tag{2.3}$$

The transmission amplitude $f$ is defined as the ratio between the microwave field $\psi_0(L)$ at the emergent face of the slab and $A\ e^{ik_0lL}$, the field which would have been found at that same plane had the slab been absent. Using (2.3), we have

$$f\equiv[\psi_0(L)/Ae^{ik_0lL}]=1+(b/2Ak_0l)\times\int_0^L e^{-ik_0ly}\psi_1(y)\,dy\ . \tag{2.4}$$

Solving (2.4) for $A$ in terms of $f$, we use the value of $A$ so determined in (2.3) to write[6]

$$\psi_0(x)=\left(\frac{b}{2k_0l}\right)\int_0^L[(f-1)^{-1}e^{ik_0l(x-y)}+e^{ik_0l|x-y|}]\psi_1(y)\,dy$$

and, rearranging slightly, we have

$$\mu\int_0^L e^{ik_0l(x-y)}\psi_1(y)\,dy=\psi_0(x)+\frac{ib}{k_0l}\int_0^L\theta(y-x)\times\sin k_0l(x-y)\psi_1(y)\,dy\ , \tag{2.5}$$

where

$$\theta(y-x)=1,\quad y>x \tag{2.6a}$$

$$=0,\quad y<x \tag{2.6b}$$

and

$$\mu\equiv-(b/2k_0l)[f/(1-f)]\ . \tag{2.7}$$

For the free-electron gas with diffuse scattering of electrons at the slab faces (at $x=0$ and $x=L$) and a uniform magnetic field directed along the direction normal to the slab, the nonlocal conductivity which relates the circularly polarized component of current to the circularly polarized component of field can be calculated by solving the Boltzmann transport equation. That solution, expressed in the units we are using here, is[7,8]

$$\psi_1(x)+\int_0^L K_{11}(x-y)\psi_0(y)\,dy=0\ , \qquad (2.8)$$

where

$$K_{11}=\tfrac{3}{4}\int_0^\infty(1/t-1/t^3)e^{-at|x-y|}\,dt\ , \qquad (2.9)$$

$$a\equiv 1-i(\omega-\omega_c)\tau\ , \qquad (2.10)$$

and where $\omega_c=eH_0/mc$ is the cyclotron frequency of the carriers.

We regard (2.5) and (2.8) together as a pair of coupled homogeneous integral equations for the two fields $\psi_0$ and $\psi_1$. These two equations can be written as

$$\mu\sum_{n=0}^{1}\int_0^L \mathfrak{M}_{mn}(x-y)\psi_n(y)\,dy$$

$$-\sum_{n=0}^{1}\int_0^L \mathfrak{L}_{mn}(x-y)\psi_n(y)\,dy=0\ , \qquad (2.11a)$$

or symbolically, as

$$\mu\mathfrak{M}\psi-\mathfrak{L}\psi=0\ , \qquad (2.11b)$$

where only

$$\mathfrak{M}_{01}(x-y)\equiv e^{ik_0 l(x-y)} \qquad (2.12)$$

differs from zero and where

$$\mathfrak{L}_{mn}(x-y)\equiv\delta_{mn}\delta(x-y)+\begin{pmatrix}0 & \dfrac{ib}{k_0 l}\theta(y-x)\sin k_0 l(x-y)\\ K_{11}(x-y) & 0\end{pmatrix} \qquad (2.13)$$

Equation set (2.11) defines an eigenvalue problem for $\mu$. To extract the eigenvalue, multiply the $m$th equation of the set by $\Phi_m(x)$ [where $\Phi_m$ is as yet arbitrary], integrate over $x$ and sum over $m$ to get

$$\sum_{m=0}^{1}\sum_{n=0}^{1}\int_0^L\int_0^L \Phi_m(x)[\mu\mathfrak{M}_{mn}(x-y)$$

$$-\mathfrak{L}_{mn}(x-y)]\psi_n(y)\,dx\,dy=0\ , \qquad (2.14a)$$

or symbolically,

$$\mu(\Phi\mathfrak{M}\psi)-(\Phi\mathfrak{L}\psi)=0\ , \qquad (2.14)$$

from which we have

$$\mu=(\Phi\mathfrak{L}\psi)/(\Phi\mathfrak{M}\psi)\quad . \qquad (2.15)$$

The value of $\mu$ calculated using (2.15) will be exact, even for arbitrary $\Phi_m$, provided that the fields $\psi_n$ satisfy (2.11). Note that these equations (2.11) are, in fact, the variational consequences of demanding that $\mu$, as given in (2.15), be independent of the choice of $\Phi_m$, i.e., that

$$\delta\mu/\delta\Phi_m(x)=0\ . \qquad (2.16a)$$

Suppose now that instead of being arbitrary, the fields $\Phi_m(x)$ were to satisfy the variational consequences of demanding that $\mu$, as given by (2.15), be independent of the choice of $\psi_n$,

$$\delta\mu/\delta\psi_n(y)=0\ , \qquad (2.16b)$$

namely

$$\sum_{m=0}^{1}\int_0^L dx\,\Phi_m(x)[\mu\mathfrak{M}_{mn}(x-y)$$

$$-\mathfrak{L}_{mn}(x-y)]=0\ . \qquad (2.17)$$

Then $\mu$, calculated by (2.15), will again be exact even if the fields $\psi_n$ are completely arbitrary.

In practice, we shall have neither the exact fields $\psi$ satisfying (2.11) nor the exact fields $\Phi$ satisfying (2.17), but hopefully we can propose trial fields $\psi^T$ and $\Phi^T$ which differ from the exact fields by quantities which are first-order small. Using these trial fields in (2.15) to evaluate

$$\mu^T\equiv(\Phi^T\mathfrak{L}\psi^T)/(\Phi^T\mathfrak{M}\psi^T) \qquad (2.18)$$

gives a result which differs from the exact $\mu$ by a second-order small quantity, which is why this method is useful. The program of course is to postulate functional forms for $\Phi^T$ and $\psi^T$, letting these forms contain certain parameters which are varied so as to satisfy (2.16). The resulting fields are used in (2.18) to calculate $\mu$. With $\mu$ evaluated, the transmission amplitude $f$ follows from (2.7).

## III. RELATION TO MULTIPLE-REFLECTION SERIES

Note that $\mu$, as given by (2.15), depends on four fields, $\psi_0$, the electric field in the slab, $\psi_1$, (proportional to) the electric current in the slab, and $\Phi_0$ and $\Phi_1$, fields which have no obvious physical role but which are formally the mathematical adjoints. It turns out, however, that by comparing (2.11a) and (2.17), and making use of the specific forms (2.12) and (2.13), one can show that

$$\Phi_0(x)=\psi_1(L-x)\ , \qquad (3.1a)$$

$$\Phi_1(x)=\psi_0(L-x)\ . \qquad (3.1b)$$

Furthermore, one can show that it is valid to make use of (3.1) before carrying out the variation, so that as a practical matter, $\mu$ depends only on the two $\psi$ fields. In this section, we are going to let $\psi_0$ and $\psi_1$ be the fields which would exist in the semi-infinite slab, and we are going to show that the transmitted field calculated using (2.15) is the same as that calculated by using the multiple-reflection scheme to first order. That is, *the variational integral itself automatically produces the iteration which is at the heart of the multiple-reflection technique.*

We denote the semi-infinite medium fields by a superscript zero. Taking the $L\to\infty$ limit of (2.5) and (2.8), and noting that the transmission amplitude $f$ (and therefore the eigenvalue $\mu$) must decrease exponentially with increasing slab thickness, we have

$$\psi_1^0(x)+\int_0^\infty K_{11}(x-y)\psi_0^0(y)\,dy=0\ , \tag{3.2a}$$

$$\psi_0^0(x)+\frac{ib}{k_0 l}\int_0^\infty \theta(y-x)\sin k_0 l(x-y)\times\psi_1^0(y)\,dy=0 \tag{3.2b}$$

as the equations governing the semi-infinite medium field.

The numerator of (2.15) consists of four terms, namely,

$$\int_0^L\Phi_0(x)\psi_0(x)\,dx+(ib/k_0l)\int_0^L\int_0^L\Phi_0(x)\theta(y-x)\sin k_0l(x-y)\psi_1(y)\,dx\,dy+\int_0^L\int_0^L\Phi_1(x)K_{11}(x-y)\psi_0(y)\,dx\,dy+\int_0^L\Phi_1(x)\psi_1(x)\,dx\ . \tag{3.3}$$

We use (3.1) to express this in terms of the $\psi$ fields only, and we choose, for the $\psi$ fields, the semi-infinite medium fields which satisfy (3.2). As a result, the last two terms in (3.3) may be combined as

$$\int_0^L\psi_0^0(L-x)\,dx\left[\psi_1^0(x)+\int_0^L K_{11}(x-y)\psi_0^0(y)\,dy\right]=-\int_0^L\psi_0^0(L-x)\,dx\times\int_L^\infty K_{11}(x-y)\psi_0^0(y)\,dy\ .$$

For later convenience, we substitute $L-x=x'$, $y=L+y'$, so that, after dropping primes, this term is

$$-\int_0^L\psi_0^0(x)\,dx\int_0^\infty K_{11}(x+y)\psi_0^0(y+L)\,dy\ . \tag{3.4}$$

Next, consider the second term in (3.3). An immediate consequence of (3.2) is that

$$\left(\frac{d^2}{dx^2}+k_0^2l^2\right)\psi_0^0(x)=ib\,\psi_1^0(x) \tag{3.5}$$

so that the second term in (3.3) can also be written

$$\frac{1}{k_0l}\int_0^L dx\,\psi_1^0(L-x)\int_0^L\theta(y-x)\sin k_0l(x-y)\left(\frac{d^2}{dy^2}+k_0^2l^2\right)\psi_0^0(y)\,dy\ .$$

Let the $\theta$ function act to limit the $y$ integration to the range $x<y<L$. Then, integrating by parts, we have

$$\int_x^L\sin k_0l(x-y)\left(\frac{d^2}{dy^2}+k_0^2l^2\right)\psi_0^0(y)=\psi_0^0(L)k_0l\cos k_0l(L-x)-\left(\frac{d\psi_0^0}{dy}\right)_L\sin k_0l(L-x)-k_0l\psi_0^0(x)\ .$$

Hence, combining the first and second term in (3.3) gives

$$\int_0^L\psi_1^0(L-x)\,dx\left[\psi_0^0(L)\cos k_0l(L-x)-\frac{1}{k_0l}\left(\frac{d\psi_0^0}{dy}\right)_L\sin k_0l(L-x)\right]\ . \tag{3.6}$$

To evaluate the integrals here, we again use (3.5), and again integrate by parts,

$$\int_0^L e^{\pm ik_0lx}\psi_1^0(x)\,dx=\frac{1}{ib}\int_0^L e^{\pm ik_0lx}\left(\frac{d^2}{dx^2}+k_0^2l^2\right)\psi_0^0(x)\,dx=\frac{1}{ib}\left[e^{\pm ik_0lL}\left(\frac{d\psi_0^0}{dx}\right)_L-\psi_0^0(L)(\pm ik_0l)e^{\pm ik_0lL}-\left(\frac{d\psi_0^0}{dx}\right)_0+\psi_0^0(0)(\pm ik_0l)\right]\ . \tag{3.7}$$

The evaluation of (2.15) is independent of how we normalize the fields. Let us choose $\psi_0^0(0)=1$ and let $Z$ be the dimensionless surface admittance, so that

$$\psi_0^0(x=0)\equiv 1\ , \tag{3.8a}$$

$$\left(\frac{d\psi_0^0}{dx}\right)_{x=0}\equiv ik_0lZ\ . \tag{3.8b}$$

To within an accuracy of $\psi_0^0(L)/Z\psi_0^0(0)$ (which will be of the order $b^{-5/6}$) we can ignore the terms in (3.7) arising from the upper limit, and write

$$\int_0^L e^{\pm ik_0lx}\psi_1^0(x)\,dx=-\left(\frac{k_0l}{b}\right)(Z\mp1)\ . \tag{3.9}$$

We have evaluated (3.6) as

$$-\frac{k_0 l}{b}\left[Z\psi_0^0(L)+\frac{1}{ik_0 l}\left(\frac{d\psi_0^0}{dy}\right)_L\right] ,$$

so the four terms in the numerator of (2.15) are

$$-\frac{k_0 l}{b}\left(Z+\frac{1}{ik_0 l}\frac{d}{dx}\right)\psi_0^0(L)-\int_0^L\psi_0^0(x)\,dx\int_0^\infty K_{11}(x+y)\times\psi_0^0(y+L)\,dy . \quad (3.10a)$$

The single term in the denominator of (2.15) may also be evaluated using (3.9) as

$$\int_0^L\int_0^L dx\,\Phi_0(x)\,e^{ik_0l(x-y)}\psi_1(y)\,dy = e^{ik_0lL}(k_0l/b)^2(Z+1)^2 \quad (3.10b)$$

and thus $\mu$, evaluated by using the semi-infinite medium fields, is

$$\mu^0\equiv\frac{(\Phi^0,\mathcal{L},\psi^0)}{(\Phi^0,\mathfrak{M},\psi^0)} =\left\{-e^{-ik_0lL}\left[\left(\frac{k_0l}{b}\right)\left(Z+\frac{1}{ik_0l}\frac{d}{dx}\right)\psi_0^0(L)+\int_0^L\psi_0^0(x)\,dx\int_0^\infty K_{11}(x+y)\psi_0^0(y+L)\,dy\right]\right\}\bigg/\left(\frac{k_0l}{b}\right)^2(Z+1)^2 . \quad (3.11)$$

For slabs appreciably thicker than the anomalous skin depth, the transmission amplitude $f$ will be very small, say of order $\psi_0^0(L)/Z\psi_0^0(0)$, so that to the same accuracy as we retained in (3.9), we can write (2.7) as

$$f=-2k_0l\mu/b .$$

Furthermore, it follows from the definition of $f$ that the field $e(L)$ at the emergent face of a slab on which is incident a field of unit amplitude is $e(L)=fe^{ik_0lL}$. Using (3.11) we then have

$$e(L)=\frac{2}{(1+Z)^2}\left[\left(Z+\frac{1}{ik_0l}\frac{d}{dx}\right)\psi_0^0(L)+\frac{1}{ik_0l}\int_0^L\psi_0^0(x)\,dx\int_0^\infty ibK_{11}(x+y)\psi_0^0(y+L)\,dy\right] . \quad (3.12)$$

This expression is identical to that for the field given as Eq. (3.19) of Ref. 5 (apart from differences in notation and an obvious typographical error on the upper limit of the integral), an expression obtained by evaluating the multiple-reflection series up to and including the term which arises from the first reflection at the emergent face.

Let us stress that this first reflection enhances the transmission radically in the neighborhood of the emergent face. The question at issue is how much effect the *higher* reflections—i.e., those beyond the first—will have.

## IV. VARIATIONAL CALCULATION

The true fields in the finite slab, including all the contributions of all the internal reflections, differ from the semi-infinite medium fields by functions which we shall parametrize in as simple and reasonable way as possible—namely, by exponentials. We therefore choose trial fields of the form

$$\psi_0(x)=\psi_0^0(x)+Ae^{-px} , \quad (4.1a)$$

$$\psi_1(x)=\psi_1^0(x)+Be^{-qx} . \quad (4.1b)$$

We surmise that by choosing the four parameters $A$, $B$, $p$, and $q$ correctly, these trial fields will provide a reasonable representation of the actual fields in the slab and therefore, that the value of $\mu$ calculated using (4.1) will be more accurate than the value of $\mu^0$ calculated using the zeroth-order fields only.

The four parameters are to be determined variationally: We use (3.1) for the adjoint fields, (4.1) for the trial fields, and evaluate the resulting $\mu$ using (2.15). The result is of the form

$$\mu=(F_0+2F_1A+2F_2B+F_3A^2+2F_4AB+F_5B^2)/(G_0+2G_1B+G_2B^2) , \quad (4.2)$$

where $F_1$ and $F_3$ depend on $p$; $F_2$, $F_5$, $G_1$, and $G_2$ depend on $q$; and $F_4$ depends on both $p$ and $q$. The techniques for evaluating these coefficients are the same as those used in Sec. III, and we merely state results here: $F_0$ and $G_0$ are given by (3.10a) and (3.10b), respectively,

$$G_1=-(k_0l/b)(Z+1)e^{ik_0lL}\int_0^L e^{-(q+ik_0l)x}dx , \quad (4.3a)$$

$$G_2=e^{ik_0lL}\left(\int_0^L e^{-(q+ik_0l)x}dx\right)^2 , \quad (4.3b)$$

$$F_1=-\int_0^L e^{-px}dx\int_0^\infty K_{11}(x+y)\,\psi_0^0(y+L)\,dy , \quad (4.3c)$$

$$F_2=\int_0^L e^{-qx}dx\left[\psi_0^0(L)\cos k_0lx-\frac{1}{k_0l}\left(\frac{d\psi_0^0}{dx}\right)_L\sin k_0lx\right] , \quad (4.3d)$$

$$F_3=\int_0^L\int_0^L e^{-p(L-x)}K_{11}(x-y)\,e^{-py}\,dx\,dy , \quad (4.3e)$$

$$F_4=\int_0^L e^{q(L-x)}e^{-px}dx , \quad (4.3f)$$

$$F_5 = \frac{ib}{k_0 l} \int_0^L \int_0^L e^{-qx(L-x)} \theta(y-x) \times \sin k_0 l(x-y) e^{-qy}\, dx\, dy \quad . \tag{4.3g}$$

The parameters $A$, $B$, $p$, and $q$ are to be chosen by the variational requirement that $\mu$ be stationary From (4.2), we obtain two of the variational equations $\partial\mu/\partial A = 0$ and $\partial\mu/\partial B = 0$ in the form

$$F_1 + F_3 A + F_4 B = 0 \quad ,$$
$$(F_2 - \mu G_1) + F_4 A + (F_5 - \mu G_2) B = 0 \quad ,$$

which gives

$$A = -[F_1(F_5 - \mu G_2) - F_4(F_2 - \mu G_1)]/[F_3(F_5 - \mu G_2) - F_4^2] \quad , \tag{4.4a}$$
$$B = -[F_3(F_2 - \mu G_1) - F_1 F_4]/[F_3(F_5 - \mu G_2) - F_4^2] \quad . \tag{4.4b}$$

Consider the size of the various terms: In the appendix, we study $\psi_0^0(x)$ and discover that when $x$ is appreciably larger than the skin depth, $\psi_0^0$ drops rapidly from its value of 1 (at $x=0$) to $b^{-1/2}$. We also show that the admittance $Z$ is of the order $b^{1/3}$. The quantity $\mu$ will be of order $\mu^0 = F_0/G_0$, the value given in Sec. III. Using these estimates of size, and the assumptions that $p$ and $q$ will be of order unity, we obtain the following order-of-magnitude estimates of the size of the terms in (4.2): $F_0 \sim Z^{-3}$; $F_1$, $F_2 \sim Z^{-3/2}$; $F_3$, $F_4 \sim 1$; $F_5 \sim Z^3$; $G_0 \sim Z^{-4}$; $G_1 \sim Z^{-2}$; $G_2 \sim 1$; $\mu \sim Z$. Use of (4.4) then gives us $A \sim Z^{3/2}$, $B \sim Z^{-4}$. We insert these size estimates back into (4.2) and learn that each of the terms involving $B$ could have been deleted from (4.2) without changing the size of $\mu$ more than one part in $Z^2$.

Let us stress this point: *The variational principle here is rejecting (i.e., maintaining exceedingly small values for) additions to the current* $\psi_1^0$. This same conclusion will also be reached with more sophisticated choices of trial function. *Because of this, we shall be able to take the current in the finite slab as a given quantity, namely, the same as the current in the semi-infinite medium, which is exceedingly small everywhere except in the anomalous skin depth at the incident face* and concentrate our efforts only on varying $\psi_0$.

If we now drop $B$ from the problem, assuming that

$$\psi_0(x) = \psi_0^0(x) + A e^{-px} \quad ,$$
$$\psi_1(x) = \psi_1^0(x) \quad ,$$

we obtain

$$\mu = (F_0 + 2F_1 A + F_3 A^2)/G_0 \quad ,$$

and, from the variational condition $\partial\mu/\partial A = 0$, we obtain

$$A = -F_1/F_3$$

so that

$$\mu = (F_0 - F_1^2/F_3) G_0^{-1} \quad ,$$

a form which depends on $p$. It is easy to show that because we have satisfied the condition $\partial\mu/\partial A = 0$, the remaining equation $\partial\mu/\partial p = 0$ is equivalent to demanding that (4.5) be stationary with respect to $p$. Since $F_0$ and $G_0$ are independent of $p$, the equation determining $p$ is

$$d(F_1^2/F_3)/(dp) = 0 \tag{4.6}$$

and this we handle numerically. The variational parameter $p$ depends on the physical parameters which describe the slab and the electron gas within it. It turns out that an exceedingly good description of the solution to (4.6) is provided by the equation

$$-(p + a)L = \alpha + \beta \ln(aL + \gamma) \quad , \tag{4.7}$$

where $a$ is defined by (2.10) and where $\alpha$, $\beta$, and $\gamma$ are real positive constants which depend only on the two parameters $b$ and $\omega\tau$. If we use (4.7) to compute the piece added to the field, we find a field which grows in the direction of the emergent face and whose direction of twist in the magnetic field is opposite to that of the field $\psi_0^0(x)$.

Clearly, the variational principal is asking that the added part of $\psi_0$ should represent those terms which, from the multiple-reflection point of view, are launched at the emergent face of the slab.

## V. RESULTS AND DISCUSSION

In order to carry out the program of evaluating the transmission amplitude, it is necessary to have an explicit representation for the field $\psi_0^0(x)$. Such a representation has been given elsewhere.[9] However, that form, while useful for certain analytic manipulations, is not the best form for carrying out the numerical manipulations needed here.

In the two appendixes of this paper, we give an alternative form for $\psi_0^0(x)$ which is more suited to our use. The alternative form, like the original, is obtained by evaluating the Wiener-Hopf solution to lowest order in the parameter $(a/b^{1/3})$. This number is essentially the ratio of the anomalous skin depth to either the Gantmakher-Kaner wavelength or the mean free path, whichever is shorter. In the PBD experiments, this parameter ranges from about $5\times10^{-4}$ at $\omega_c/\omega = 1$ to about $5\times10^{-2}$ at $\omega_c/\omega = 0.65$ or 1.35. Thus, the expansion of $\psi_0^0$ we use here is virtually exact at cyclotron phase resonance.

In Figs. 1 and 2, we have plotted the transmission amplitude as a function of magnetic field, expressed in terms of $\omega_c/\omega$, the ratio of cyclotron frequency to the microwave frequency. The parameters chosen are comparable to those encountered experimentally in the PBD letter. In Fig. 1, the field is calculated to zeroth order in the multiple-reflection series, i.e., as though the emergent surface played no role in the physics beyond defining the plane on which the field is observed. In Fig. 2, the dashed curve is the field calculated to first order in the multiple-reflection series using the calculation described in Sec. III. Note that the effect of the emergent surface is to enhance the transmission over the zeroth-order result (Fig. 1), and that the enhancement is greatest at cyclotron phase resonance, at $\omega_c/\omega = 1.0$. It is clear that the emergent surface radically alters the field within the slab. The solid curve is the field calculated variationally, using the method described in Sec. IV, which includes the effects of the higher-order multiple reflections. Only a slight difference is apparent between the variational calculation (solid curve) and the first-order calculation (dashed curve). This means, of course, that the second- and higher-order terms in the multiple-reflection series are small. It explains why the variational principle forced the piece $Ae^{-px}$ to represent a field launched from the emergent face: Because the multiple-reflection series converges

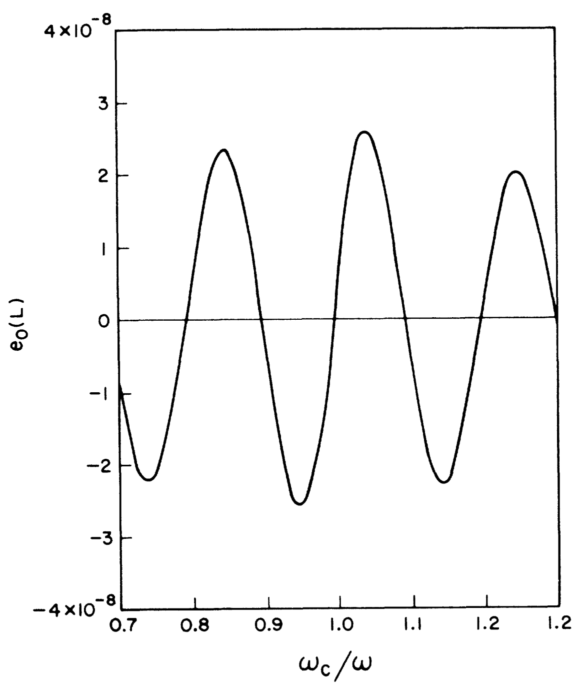


FIG. 1. Imaginary (i.e., out-of-phase) component of the electric field at a depth $L = 0.1$ electron mean free paths from the incident surface of a conducting slab irradiated by a microwave field of unit amplitude. The parameters describing the material of the slab are $\omega\tau = 300$, $b = 9.45 \times 10^9$.

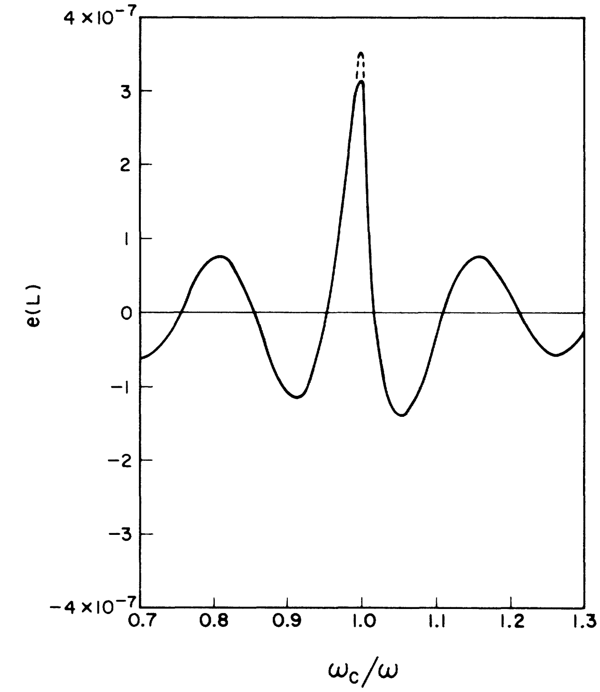


FIG. 2. Real (i.e., in-phase) component of the electric field at the emergent surface of a slab whose thickness is $L = 0.1$ electron mean free paths but which is otherwise identical to that used in Fig. 1. (Note the difference in scale between Figs. 1 and 2.) The dashed curve is the first-order multiple-reflection result. The solid curve includes the effect of the higher-order reflections.

very rapidly, there is only a negligible second-order field launched from the incident face; the zeroth-order term launched at the incident face and the first-order term launched from the emergent face provides an adequate description of the field in the slab.

As the thickness $L$ of the slab is made greater, the very slight difference between the first-order calculation (dashed curve) and the variational one (solid curve) decreases still further.

It is especially interesting to notice that the effect of higher-order terms included in the variational calculation is actually to *decrease* slightly, rather than to enhance, the transmitted amplitude near $\omega_c/\omega = 1$. This certainly means that there is no possibility of multiple reflection being responsible for the sharp peak in the experimentally measured transmission. We have repeated these variational calculations with more flexible trial functions for $\psi_0$ (that is, with more parameters to vary and more labor required) but the results for the calculated transmission differ imperceptibly from those we have reported here.

It seems fairly certain that the calculations we have reported accurately represent the transmission amplitude for the model of the slab in which the electrons are treated as a degenerate gas of free fermions which suffer diffuse reflection at the faces of the slab. In Fig. 3 we have reproduced the data from the PBD letter. It is clear that there are many features of the data which are not correctly described by this model.

In the third and final paper of this series, we shall include the effects of Fermi-liquid correlations on the transmission and we shall show that some, but not all, of the features of the transmission can be ascribed to the correlation effects.


### ACKNOWLEDGMENTS

I am indebted to J. F. Carolan, J. M. Rowell and W. M. Walsh, Jr. for useful discussions during the course of this work.


### APPENDIX A: GENERAL COMMENT ON SOME INTEGRALS INVOLVING $\psi_0^0$

In order to be able to carry out the calculations described in the body of the paper, it is necessary to have a numerical or analytic representation of $\psi_0^0(x)$, the semi-infinite medium electric field. In Appendix B, we shall, using the Wiener-Hopf method, give such a representation in a form which is especially useful for the numerical calculations we have to perform. The essential result is that, at distances $x$ which are large compared to an anomalous skin depth, we can write

$$\psi_0^0(x)=\int_0^\infty \psi(u)e^{-(a+u)x}\,du \quad , \tag{A1}$$

where $a=1-i(\omega-\omega_c)\tau$ and where a specific form for $\psi(u)$ will be developed in Appendix B. Because the form (A1) is *not* valid for $x$ comparable to the skin depth, this representation is not immediately useful for the term appearing in (3.4). It turns out, however, that the Wiener-Hopf method provides us, as a by-product of the calculation of $\psi_0^0(x)$, with $J(k)$, the Fourier transform of a function $j(x)$,

$$J(k)\equiv\int_{-\infty}^{\infty} j(x)e^{-ikx}\,dx \quad , \tag{A2}$$

which is defined as

$$j(x)\equiv\int_0^\infty K_{11}(x+y)\psi_0^0(y)\,dy\ ,\quad x>0 \tag{A3a}$$

$$\equiv 0\ ,\qquad x<0\quad . \tag{A3b}$$

A specific representation of $J(k)$, valid for $k$ smaller than the reciprocal of the anomalous skin depth, is also given in Appendix B.

Now consider (3.4). Using (A3), we have

$$\begin{aligned}\int_0^L\psi_0^0(x)\,dx\int_0^\infty K_{11}(x+y)\psi_0^0(y+L)\,dy\\ =\int_0^\infty\psi_0^0(x)\,dx\int_0^\infty K_{11}(x+y)\psi_0^0(y+L)\,dy-\int_L^\infty\psi_0^0(x)\,dx\int_0^\infty K_{11}(x+y)\psi_0^0(y+L)\,dy\\ =\int_{-\infty}^{+\infty} j(y)\psi_0^0(y+L)\,dy-\int_0^\infty\int_0^\infty dx\,dy\,\psi_0^0(x+L)K_{11}(x+y+L)\psi_0^0(y+L)\quad .\end{aligned} \tag{A4}$$

The thickness $L$ of the slab is always much greater than an anomalous skin depth, and so we can use (A1) to evaluate the integrals on the right-hand side of (A4). The first of these is

$$\int_{-\infty}^{\infty} j(y)\,dy\int_0^\infty du\,\psi(u)e^{-(a+u)(y+L)}=e^{-aL}\int_0^\infty J[k=(u+a)/i]\,\psi(u)e^{-uL}\,du \tag{A5}$$

The second integral in (A4) can be evaluated using (A1) and (2.9),

$$\int_0^\infty\int_0^\infty dx\,dy\,\psi_0^0(x+L)K_{11}(x+y+L)\psi_0^0(y+L)=\tfrac{3}{4}\int_0^\infty dt\,(1/t-1/t^3)e^{-atL}\Phi(t)^2 \quad , \tag{A6}$$

where

$$\Phi(t)\equiv e^{-aL}\int_0^\infty\psi(u)e^{-uL}\,du/(u+a+at)\quad . \tag{A7}$$

The integral $F_1$, defined in (4.3c), is also readily evaluated using (A1) and (2.9) as

$$F_1(p)=-\tfrac{3}{4}\int_1^\infty(1/t-1/t^3)\times\left[\frac{(1-e^{-(at+p)L})}{at+p}\right]\Phi(t)\,dt \tag{A8}$$

### APPENDIX B: WIENER-HOPF SOLUTION FOR $\psi_0^0$ AND $J(k)$

Our starting point is the integrodifferential equation for the field $\psi_0^0(x)$. Combining (3.2a) and (3.5) into a single equation, we have

$$\left(\frac{d^2}{dx^2}+k_0^2l^2\right)\psi_0^0(x)+ib\int_0^\infty K_{11}(|x-y|)\psi_0^0(y)\,dy=0\ , \tag{B1a}$$

which is to be solved to the boundary conditions

$$\psi_0^0(x=0)=1\ , \tag{B1b}$$

$$\psi_0^0(x\to\infty)=0\ . \tag{B1c}$$

This defines $\psi_0^0$ only for $x=0$, and, following the usual Wiener-Hopf prescription,[10] we take

$$\psi_0^0(z)\equiv 0,\quad x<0 \tag{B2}$$

$$h(x)\equiv 0,\quad x>0 \tag{B3a}$$

$$\equiv -ib\int_0^\infty K_{11}(|x-y|)\psi_0^0(y)\,dy\ , \tag{B3b}$$

so that

$$\left(\frac{d^2}{dx^2}+k_0^2l^2\right)\psi_0^0(x)+ib\int_{-\infty}^{\infty}K_{11}(|x-y|)$$

$$\times\psi_0^0(y)\,dy+h(x)=0 \qquad \text{(B4)}$$

is an equation which coincides with (B1) for $x>0$ and, unlike (B1), is both valid for all $x$ and Fourier transformable. Its transform is

$$[-k^2+k_0^2l^2+ibK(k)]E(k)+H(k)$$

$$=ike(0)+e'(0) \quad , \qquad \text{(B5)}$$

where

$$E(k)\equiv\int_{-\infty}^{\infty}dx\,\psi_0^0(x)e^{-ikx} \quad , \qquad \text{(B6)}$$

and where $H$ and $K$ are the transforms of $h$ and $K_{11}$, respectively. $e(0)$ and $e'(0)$ are the initial value and derivative of $\psi_0^0(x)$ as $x\to 0+$.

At this point, one uses the Wiener-Hopf factorization. That is, denoting the coefficient of $E(k)$ as $-Q(k)$, one constructs two functions $f^+(k)$ and $f^-(k)$ with the following four properties:

$$f^+(k)/f^-(k)=Q(k)\equiv k^2-k_0^2l^2-ibK(k) \quad ; \qquad \text{(B7a)}$$

$f^-(k)$ is analytic for $k$ in the lower half plane , (B7b)

$f^+(k)$ is analytic and free from zeros in the upper half plane ; (B7c)

they both exhibit algebraic, rather than exponential growth as $k\to\infty$ . (B7d)

Equation (B5) now becomes

$$f^-(k)E(k)=f^+(k)[H(k)-ike(0)-e'(0)] \quad ,$$

and, using the standard analyticity arguments, each side of the above equation is equal to some polynomial which, in this case, can be shown to be a constant $a_0$. Hence, we have

$$E(k)=a_0/f^-(k) \quad , \qquad \text{(B8a)}$$

$$H(k)=a_0/f^+(k)+ike(0)+e'(0) \qquad \text{(B8b)}$$

Usually, one is concerned only with $E(k)$ and not with $H(k)$, the Fourier transform of a function introduced to make the Wiener-Hopf method work. In our case, however, comparing (A3) and (B3) reveals that $j(x)=ih(-x)/b$ so that the function $J(k)$ needed for the integrals in Appendix A is given by

$$J(k)=iH(-k)/b \quad . \qquad \text{(B9)}$$

Using the large-$k$ expansion of (B6),

$$E(k\to\infty)=e(0)/(ik)+e'(0)/(ik)^2+\cdots \quad ,$$

our knowledge that at large $k$, the expansion of $f^-(k)$ will take the form

$$1/f^-(k)=1/k+C/k^2+\cdots \qquad \text{(B10)}$$

gives us

$$a_0=e(0)/i=-i \quad , \qquad \text{(B11a)}$$

$$e'(0)=-a_0C=iC \quad , \qquad \text{(B11b)}$$

where $C$ is a constant we can compute as soon as we have $f^-(k)$.

Specific forms for $f^+(k)$ and $f^-(k)$ follow from a straightforward application of the Wiener-Hopf technique. $Q(k)$ is an even function of $k$ which is analytic except for branch points at

$$k=\pm ia=\pm[(\omega-\omega_c)\tau+i] \quad .$$

We choose the branch cuts to run from the branch points outward to infinity, parallel to the imaginary axis. We shall displace the branch cuts slightly in the neighborhood of

$$k=iK_0\equiv i\left(\tfrac{3}{4}\pi b\right)^{1/3} \qquad \text{(B12)}$$

in such a way that the special points $k=\pm iK_0$ are always to the right-hand side of the cut as one faces away from the origin of the $k$ plane (Fig. 4). This choice of cuts has two effects, one of which is to produce the form (A1), and the other of which is to assure that the equation $Q(k)=0$ has only one pair of roots, namely, at $k=\pm k_1$. To the accuracy which we shall be working,

$$k_1=K_0e^{i\pi/6} \quad . \qquad \text{(B13)}$$

The Wiener-Hopf factorization is then accomplished by having

$$f^-(k)=(k-k_1)e^{-S^-(k)} \quad , \qquad \text{(B14a)}$$

$$f^+(k)=(k+k_1)^{-1}e^{-S^+(k)} \quad , \qquad \text{(B14b)}$$

where

$$S^{\pm}(k)\equiv\frac{1}{2\pi i}\int_{-\infty\mp i}^{\infty\mp i}(z-k)^{-1}dz\,\ln[Q(z)/(z^2-k_1^2)] \ . \qquad \text{(B15)}$$

Note also that

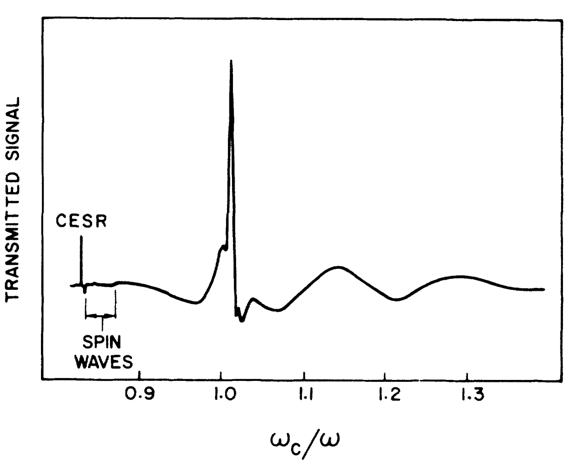


FIG. 3. Transmission data from the PBD letter.

$$f^+(k) = -1/f^-(-k) \quad . \tag{B16}$$

We consider $S^-(k)$: Sweep the contour upward to surround the branch cut in the upper half-plane. If we let $k_c$ denote a value of $z$ along the cut, and we let $Q^\pm(k_c)$ denote boundary values of $Q(z)$ on the outgoing (+) and the incoming (−) side of the cut, then the integral in (B15) becomes

$$S^-(k) = \frac{1}{2\pi i}\int_{ia}^{i\infty} (k_c - k)^{-1}\, dk_c \ln G(k_c) \ , \tag{B17}$$

$$G(k_c) \equiv Q^+(k_c)/Q^-(k_c) \ . \tag{B18}$$

This form arises because the integrand does not contribute to the tiny semicircle around the branch points and because $Q(z)$ is the only part of the integrand which is different on the incoming and outgoing sides of the cut.

We can now take the Fourier inverse of (B6), using (B8a), (B11a), and (B14a) as

$$\psi_0^0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(k) e^{ikx}\, dk$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dk}{k - k_1}\, e^{S^-(k) + ikx} \ . \tag{B19}$$

We also evaluate this integral by sweeping the contour upwards, letting it surround the pole of this integrand and the branch cut. The pole contributes a residue with spatial dependence $e^{ik_1 x}$, a quantity which is essentially $e^{-x/\delta}$, where $\delta$ is the anomalous skin depth. We shall always work with $x$ much larger than the skin depth, so the residue contribution will be exponentially negligible.

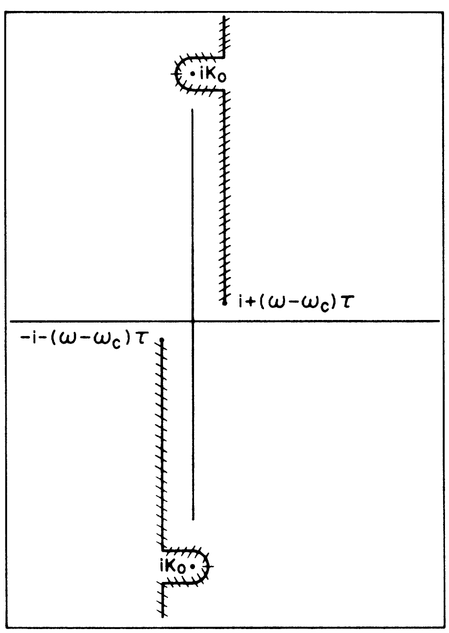


FIG. 4. Location of the branch cuts.

The branch cut contribution consists only of the integrals along the outgoing and incoming sides of the cut, again because the integral around the semicircle surrounding the branch point vanishes. Thus,

$$\psi_0^0(x) = \frac{1}{2\pi i}\int_{ia}^{i\infty} \frac{dk_c}{k_c - k_1}\, e^{ik_c x}\left(e^{S^-(k_c^+)} - e^{S^-(k_c^-)}\right) \ . \tag{B20}$$

Points on the outgoing (+) and incoming (−) sides of the cut can be designated as

$$k_c^\pm = k_c(1 \mp i\eta) \ , \quad \eta \to 0+$$

so that

$$S^-(k_c^\pm) = \frac{1}{2\pi i}\int_{ia}^{i\infty} \frac{dk_c' \ln G(k_c')}{k_c' - (k_c \mp i\eta)}$$
$$= \frac{1}{2\pi i}\, P\!\int_{ia}^{i\infty} \frac{dk_c' \ln G(k_c')}{k_c' - k_c} \mp \tfrac{1}{2}\ln G(k_c) \tag{B21a}$$
$$\equiv S^0(k_c) \mp \tfrac{1}{2}\ln G(k_c) \ . \tag{B21b}$$

We define

$$I(k_c) \equiv \frac{1}{2\pi i}\ln G(k_c) \tag{B22}$$

and choose the branch cuts such that

$$k_c = ia + iu, \quad 0 < u < \infty$$
$$dk_c = i\,du \ . \tag{B23}$$

Then (B20) takes the form (A1), with

$$\psi(u) = \frac{i}{\pi(k_1 - k_c)}\, e^{S^0(k_c)} \sin[\pi I(k_c)] \ . \tag{B24}$$

Now, to return to (B9): We use (B11a), and (B16) to obtain

$$J(k) = (i/b)[i(k - k_1)e^{-S^-(k)} - ike(0) + e'(0)] \ . \tag{B25}$$

All that remains is to get specific evaluations for $S(k)$, and $e'(0)$. Using the definitions of $K(k)$ and of $K_{11}$,

$$K(k_c^\pm) = \frac{3i}{4k_c}\left[\left(1 + \frac{a^2}{k_c^2}\right)\left(\ln\frac{k_c + ia}{k_c - ia} \mp i\pi\right) + \frac{2ia}{k_c}\right]$$
$$\equiv A(k_c) \mp iB(k_c) \ . \tag{B26}$$

Then

$$I(k_c) = \frac{1}{2\pi i}\ln\frac{1 - iC(k_c)}{1 + iC(k_c)} \tag{B27a}$$
$$= -\frac{1}{\pi}\tan^{-1}C(k_c) \tag{B27b}$$

where

$$C(k_c) \equiv \frac{B(k_c)}{A(k_c) - i(k_c^2 - k_0^2 l^2)/b} \ . \tag{B28}$$

We can perform some of the integrals we need analytically if we use approximate forms for $I(k_c)$. The approximate forms below are, with the exception of (B29a) below, valid to an accuracy $|a|/K_0$. The approximation (B29a) is a purely numerical one, and it is based on the power series expansion:

range 1:

$$k_c < |aK_0|^{1/2},$$

$$C(k_c) \approx \frac{B(k_c)}{A(k_c)} = \left\{ \pi(1 + a^2/k_c^2) \Big/ \left[ \left(1 + \frac{a^2}{k_c^2}\right) \ln\left(\frac{k_c + ia}{k_c - ia}\right) + \frac{2ia}{k_c} \right] \right\}$$

$$I(k_c) = -\tfrac{1}{2} + A_1(ia/k_c) + A_3(ia/k_c)^3 + A_5(ia/k_c)^5 + A_9(ia/k_c)^9, \quad \text{(B29a)}$$

where

$$A_1 = 4/\pi^2 = 0.405\,285,$$

$$A_3 = 8/3\pi^2 - 64/3\pi^4 = 0.051\,182,$$

$$A_5 = 12/5\pi^2 - 128/3\pi^4 + 1024/5\pi^6 = 0.018\,181,$$

$$A_9 = \tfrac{1}{2} - A_1 - A_3 - A_5 = 0.025\,352.$$

The maximum numerical error in $I(k_c)$ is $1.6 \times 10^{-3}$ at $k_c/ia = 1.1$. That error drops to $8 \times 10^{-4}$ at $k_c/ia = 1.01$ and $1.2$, and is still less for other values of $k_c/ia$ between 1.0 and 1.01 and between $k_c/ia = 1.2$ and infinity:

range 2:

$$|aK_0|^{1/2} < k_c < |a|^{1/6} K_0^{5/6},$$

$$I(k_c) = -\tfrac{1}{2} + A_1(ia/k_c); \quad \text{(B29b)}$$

range 3:

$$|a|^{1/6} K_0^{5/6} < k_c < K_0,$$

$$I(k_c) = -\tfrac{1}{2} + A_1 \frac{ia}{k_c} + \frac{1}{\pi}\left[\left(\frac{k_c}{K_0}\right)^3 - \frac{1}{3}\left(\frac{k_c}{K_0}\right)^9 + \frac{1}{5}\left(\frac{k_c}{K_0}\right)^{15} \cdots \right]; \quad \text{(B29c)}$$

range 4:

$$k_c > K_0,$$

$$I(k_c) = -\frac{1}{\pi}\left[\left(\frac{K_0}{k_c}\right)^3 - \frac{1}{3}\left(\frac{K_0}{k_c}\right)^9 + \frac{1}{5}\left(\frac{K_0}{k_c}\right)^{15} - \frac{1}{7}\left(\frac{K_0}{k_c}\right)^{21} + \cdots \right]. \quad \text{(B29d)}$$

These approximations give us

$$S^-(k) = \int_0^{|aK_0|^{1/2}} i\,dt\,(k_c - k)^{-1}\left[A_3\left(\frac{ia}{k_c}\right)^3 + A_5\left(\frac{ia}{k_c}\right)^5 + A_9\left(\frac{ia}{k_c}\right)^9\right] + \int_0^{K_0} i\,dt\,(k_c - k)^{-1}\left(-\tfrac{1}{2} + A_1\frac{ia}{k_c}\right) + \frac{1}{\pi}\sum_{n\ \mathrm{odd}>0}\frac{(-1)^{(n-1)/2}}{n}\left[\int_0^{K_0} i\,dt\,(k_c - k)^{-1}\left(\frac{k_c}{K_0}\right)^{3n} - \int_{K_0}^{\infty} i\,dt\,(k_c - k)^{-1}\left(\frac{K_0}{k_c}\right)^{3n}\right]. \quad \text{(B30)}$$

For $k$ smaller than $|aK_0|^{1/2}$ (and we shall never need $k$ larger than this), we can, to within the $|a/K_0|$ accuracy, expand $(k_c - k)^{-1}$ in the integrand of the infinite series term as $1/k_c + k/k_c^2$, carry out the indicated integrations, and sum the resultant series to obtain, for the entire infinite series term, the value $-\frac{1}{12}i\pi - k/(2K_0\sqrt{3})$. The other terms may be evaluated with no particular difficulty.

The only other moderately subtle point involves the evaluation of $Z$, the dimensionless surface impedance. From (B14a), (B17), and (B22), we have, as $k \to \infty$,

$$1/f^-(k) = (k - k_1)^{-1} e^{S^-(k)} = (1/k)\left[1 + k_1/k - (1/k)\int_{ia}^{i\infty} dk_c\, I(k_c)\right] \quad \text{(B31)}$$

and the integral here is

$$\int_{ia}^{i\infty} dk_c\, I(k_c) = \int_0^{|aK_0|^{1/2}} i\,dt\left[A_3\left(\frac{ia}{k_c}\right)^3 + A_5\left(\frac{ia}{k_c}\right)^5 + A_9\left(\frac{ia}{k_c}\right)^9\right] + \int_0^{K_0} i\,dt\left(-\tfrac{1}{2} + A_1\frac{ia}{k_c}\right) + \frac{1}{\pi}\sum_{n\ \mathrm{odd}>0}\frac{(-1)^{(n-1)/2}}{n}\left[\int^{iK_0} dk_c\left(\frac{k_c}{K_0}\right)^{3n} - \int_{iK_0}^{\infty} dk_c\left(\frac{K_0}{k_c}\right)^{3n}\right].$$

The infinite series may be integrated term by term and summed to give $K_0/(2\sqrt{3})$. Then the coefficient of $1/k^2$ in (B31) (or the constant $C$ in (B10) is

$$C = K_0(e^{i\pi/6} + \tfrac{1}{2}i - 1/2\sqrt{3}) - A_1 ia \ln(K_0/a) = 2K_0 e^{i\pi/3}/\sqrt{3} - (4ia/\pi^2)\ln(K_0/a).$$

This means that the dimensionless surface impedance, using (3.8) and (B11b), is

$$Z=\frac{1}{ik_0 l}\left(\frac{d\psi_0}{dx}\right)_0=\frac{2}{\sqrt{3}}\left(\frac{K_0}{k_0 l}\right)e^{i\pi/3}-\frac{4}{\pi^2}\left(\frac{ia}{k_0 l}\right)\ln\left(\frac{K_0}{a}\right) . \tag{B32}$$

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