Continuous transition between fractional quantum Hall and superfluid states

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We develop a theory of a direct, continuous quantum phase transition between a bosonic Laughlin fractional quantum Hall state and a superfluid, generalizing the Mott insulator to superfluid phase diagram of bosons to allow for the breaking of time-reversal symmetry. The direct transition can be protected by a spatial symmetry, and the critical theory is a pair of Dirac fermion fields coupled to an emergent Chern-Simons gauge field. The transition may be achieved in optical traps of ultracold atoms by starting with a $\nu = 1/2$ bosonic Laughlin state and tuning an appropriate periodic potential to change the topology of the composite fermion band structure.

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Introduction. One of the most celebrated examples of a continuous quantum phase transition is between a Mott insulator (MI) and a superfluid (SF) of bosons [1,2]. Over the last two decades, this transition has been successfully characterized, both theoretically and experimentally. In addition to the Mott insulator and the superfluid, it is expected that a fractional quantum Hall (FQH) state can be realized in strongly interacting bosonic systems, such as in optical traps of ultracold-atomic gases [3]. This raises a fundamental question of whether it is also possible to transition continuously between FQH states and Mott insulators or superfluids. While theories of continuous transitions between FQH states and Mott insulators have been developed [4–7], it has not been addressed whether the FQH state can directly and continuously transition to a superfluid as the kinetic energy of the bosons is increased relative to their interaction energy.

In this paper, we develop a theory of such a continuous transition, between a $\nu = 1/2$ bosonic Laughlin state and a superfluid, thereby providing a more general picture of the boson phase diagram (Fig. 1). Since the superfluid is described by an order parameter while the FQH state is a topological phase without a local order parameter, such a transition is conceptually quite exotic [8]. Realizing it in the laboratory would be an experimental example of a continuous quantum transition in a clean system (unlike QH plateau transitions) which lies outside the Ginzburg-Landau paradigm. Here, we will specialize to the case with fixed average particle number. We find that generically, in the absence of any additional symmetries besides particle number conservation, continuous transitions occur between the FQH state and Mott insulator or the Mott insulator and the superfluid. However in the presence of certain spatial symmetries, there may be a direct, continuous transition between the FQH state and the superfluid.

A simple way to understand the basic idea is through the composite fermion [9] framework. The $\nu = 1/2$ Laughlin state can be understood in terms of composite fermions attached to one flux quantum each, such that the mean-field state of composite fermions is a $\nu = 1$ integer quantum Hall (IQH) state. An externally applied periodic potential can change the band structure of the composite fermions such that they occupy bands with a total Chern number C. When $C = 1$, the state is still the $\nu = 1/2$ FQH state. However, when $C = 0$, the resulting state is a Mott insulator, and, as we explain below, when $C = -1$, the resulting state is a superfluid. Thus the

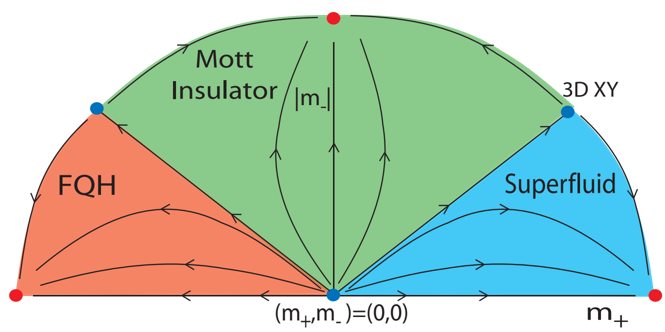

FIG. 1. (Color online) Proposed phase diagram and renormalization-group flows including the Mott insulator, superfluid, and $\nu = 1/2$ Laughlin FQH state, for fixed average particle number. We have defined $m_\pm \equiv m_1 \pm m_2$ [see Eq. (7)]; m_- is a symmetry-breaking field, so the direct transition between the FQH state and the SF can occur if the symmetry is preserved. The red points on the horizontal and vertical axes indicate the three stable phases, while the blue points at the origin and the diagonals indicate the unstable critical fixed points.

transitions between these states can be understood as Chern-number-changing transitions of the composite fermions. The critical theories for such transitions consist of gapless Dirac fermions coupled to a Chern-Simons (CS) gauge field.

Effective field theory constructions. In order to develop our theory, we need to provide a field-theoretic description that can naturally interpolate between the states of interest. To do this, we will use the parton/projective construction [10]. For the Laughlin FQH state, the Mott insulator, and the superfluid, the parton construction is essentially equivalent to the composite fermion construction, although the former is preferable because it can describe a wider class of FQH states [11] and can be formulated even in the absence of a background external magnetic field [12]. In this paper we will consider the situation where the bosons feel an external magnetic field, because it is more directly relevant to ultracold-atom proposals, though the theory can be generalized to cases without an external magnetic field. We write the boson operator $b(r)$ as

$$b(r) = f_1(r) f_2(r), \tag{1}$$

where f_1 and f_2 are charge $1/2$ fermions. This construction introduces an SU(2) gauge symmetry [13]. Since the f_i carry charge 1/2, they effectively see half as much magnetic field; thus for bosons at $\nu = 1/2$, the density of f_i is such that their effective filling fraction is $\nu_{f_i} = 1$. To describe the $\nu = 1/2$ Laughlin state, we assume a mean-field ansatz that breaks the SU(2) gauge symmetry to $U(1)$ and where f_i form $\nu_{f_i} = 1$ IQH states. Letting a denote the emergent $U(1)$ gauge field and A the background external gauge field, integrating out f_1 and relabeling $a \to a + \frac{1}{2}A$ gives

$$\mathcal{L} = f_2^\dagger i D_0 f_2 - \frac{1}{2m_{eff}} f_2^\dagger D^2 f_2 + \frac{1}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \delta\mathcal{L}, \quad (2)$$

where the covariant derivative is $D_\mu = \partial_\mu - i a_\mu - i A_\mu$, and $\delta\mathcal{L}$ includes additional interactions, external potentials, etc. This is the same theory obtained by the flux-attachment and flux-smearing mean-field approximation in the composite fermion theory, where f_2 is the composite fermion. At energies well below the gap of the f_1 state, a hole of f_1 can be created by inserting 2π flux; thus, for energies below the gap of the f_1 state, the boson b can be represented by the operator

$$b = \hat{M} f_2, \quad (3)$$

where $\hat{M}$ is an instanton operator that creates 2π flux of a. Integrating out f_2, which is assumed to form a $\nu_{f_2} = 1$ IQH state, and relabeling $a \to a - \frac{1}{2}A$ leads to the following effective action, to lowest order in the gauge fields and their derivatives:

$$\mathcal{L} = \frac{2}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (4)$$

This gives the correct Hall conductance and reproduces the correct topological degeneracies of the $\nu = 1/2$ Laughlin state [13].

Now suppose that $\delta\mathcal{L}$ is chosen in such a way that the lowest band for f_2 has a general Chern number, C. As we will discuss below, this may occur in the presence of an externally imposed periodic potential. Integrating out the fermions in Eq. (2) results in the following effective theory, to lowest order:

$$\mathcal{L} = \epsilon^{\mu\nu\lambda}\left[\frac{C+1}{4\pi} a_\mu \partial_\nu a_\lambda + \frac{C}{4\pi} A_\mu \partial_\nu A_\lambda + \frac{C}{2\pi} A_\mu \partial_\nu a_\lambda\right]. \quad (5)$$

When $C = 0$, (5) is simply $\mathcal{L} = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$, which describes a gapped state with a unique ground state on all closed manifolds. The gapped f_2 excitations are attached to a unit of flux, so they are bosonic excitations. After projecting onto the physical Hilbert space by following the analyses in Ref. [10], it can be verified that there are also no gapless protected edge states. Such a gapped state with solely bosonic excitations and unique ground-state degeneracies is a topologically trivial Bose-Mott insulator. This result can also be cast within the composite boson language [4], where the original boson is considered to be a composite boson ϕ attached to two units of flux. Performing the flux-smearing approximation gives composite bosons in no net magnetic field. The $\langle\phi\rangle \neq 0$ and $\langle\phi\rangle = 0$ states correspond to the FQH state and Mott insulator, respectively. This is just the bosonized description of the $C = 1$ and $C = 0$ composite fermion description of these states.

Since a is a dynamical gauge field, to describe a gapped state, the gauge fluctuations must be gapped and, to describe a fractionalized state, the gauge theory must be at a deconfined fixed point. Since CS gauge theories are gapped [14] and represent deconfined quantum field theories [15,16], the above construction can be used to represent FQH states. However, when $C = -1$, from (5) we see that there is no CS term for a. Restoring the Maxwell terms to (5), and performing a shift of variables $a \to a - \frac{1}{2}A$, the effective action is perturbatively, to lowest order, given by

$$\mathcal{L} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda + \frac{1}{g_1^2} f^2 + \frac{1}{g_2^2} F^2 + \frac{1}{g_3^2} f F, \quad (6)$$

where the Maxwell term is $f^2 \equiv f_{\mu\nu} f^{\mu\nu}$, and similarly for the last two terms, and we have assumed Lorentz invariance for simplicity. Since there is no CS term $\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$, we must reconsider whether the gauge fluctuations are gapped. Without the CS term, in 2+1 dimensional compact $U(1)$ gauge theory, instantons proliferate and condense at low energies, yielding a contribution $e^{-S_0}\hat{M} + \text{H.c.}$ to the effective action [17]. This induces a gap for a. However this term cannot be added to (6). From the mutual CS term $\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda$, we see that flux of a carries electric charge. $\hat{M}$, which instantly adds 2π flux, instantly causes a local depletion of the charge density; to satisfy charge conservation, it must create a current $j \sim \delta(t)$, which costs an infinite action. Thus instantons alone are suppressed at energies below the gap of the fermion states [18,19]. Since $\hat{M}$ creates a hole in the parton IQH states, the only possible instanton term that might be added to the effective action at low energies, below the fermion gap, is of the form $\hat{M} f_1^\dagger f_2^\dagger + \text{H.c.}$ The fermion operators fill in the hole created by the flux insertion, thus keeping the charge density uniform. Such a term is gauge-invariant if under a gauge transformation $f_i \to e^{i\gamma/2} f_i$, $A \to A - \partial\gamma$, $\hat{M} \to e^{i\gamma}\hat{M}$. Such a term does not gap out the gauge field, and leads to spontaneous symmetry breaking of the fermion number conservation [20]. Proliferation of these allowed instantons may be viewed as the mechanism within the gauge theory by which the fermion number conservation is spontaneously broken [20].

From the action (6), we see that magnetic fluctuations of a are charged under the external gauge field, which implies that they correspond to density fluctuations [18]. Thus a is dual to the superfluid Goldstone mode. In fact, (6) is dual to the standard superfluid action if instantons are ignored [18,21,22], as can be seen by introducing $\xi^\mu \equiv \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$ and a Lagrange multiplier φ to enforce the constraint $\partial_\mu \xi^\mu = 0$, and subsequently integrating out ξ_μ. This yields $\mathcal{L} \propto (\partial\varphi - A)^2$. Alternatively, integrating out a in (6) yields the standard superfluid response $\mathcal{L} \propto A_\mu(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})A_\nu$. We conclude that when f_2 fills bands with $C = -1$, the resulting state is a superfluid.[1] In the Appendix, we give a further discussion of how such a construction can describe a compressible state.

We note that within this effective field theory description, a deformation of the composite fermion band structure that causes the bands to overlap will result in a compressible

[1]While this appears surprising, we note that it is implicit in [6], where it was argued that the 3D XY critical point can be fermionized. However, where there is overlap, some of our results differ from those of Ref. [6]. Similarly, [18] uses an equivalent construction in a different context, for an XY Néel state.

non-Fermi-liquid state, with a composite fermion Fermi surface [23].

Since the theory here derives from a parton construction, there is a natural candidate series of many-body wave functions that can interpolate through the different phases, given by projecting the mean-field Slater determinant wave functions of f_1 and f_2 to the same coordinates.

Critical theory. The critical theories between the FQH state, MI, and SF therefore occur when the composite fermion f_2 bands touch and their net total Chern number changes. The transition between the SF and the $\nu = 1/2$ FQH state occurs when the total Chern number of f_2 changes from 1 to -1. This can happen either at a quadratic band touching or at two Dirac cones; the generic, stable case is two Dirac cones, because quadratic band touchings are marginally unstable to repulsive interactions [24]. To describe this, let $\psi(r)$ be a two-component fermion that describes the two f_2 bands that are involved in the transition, so that at low energies, $f_2(r) \sim c^T(r)\psi(r)$, where $c(r)$ is a two-component scalar function of r; i.e., at low energies $f_2(r)$ is a linear combination of the two bands described by ψ. Near the transition, at low energies $\psi(r) \sim \sum_{i=1}^{2} e^{iK_i r}\psi_i(r)$, where the Dirac points occur at momenta K_i and ψ_i are the two-component fermions obtained by linearizing about the Dirac points. The critical theory is

$$\mathcal{L} = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu \partial_\nu a_\lambda + \bar{\psi}_i \gamma^\mu D_\mu \psi_i + m_i \bar{\psi}_i \psi_i, \quad (7)$$

for $i = 1,2$, $\bar{\psi}_i = \psi_i^\dagger \sigma^z$, $\gamma_0 = \sigma_z$, $\gamma_x = \sigma_x$, $\gamma_y = \sigma_y$, where σ_i are the Pauli matrices. When both $m_i < 0$, we obtain the superfluid state, when $m_i > 0$, we obtain the FQH state, and if m_i have opposite signs, then we have the Mott insulator (see Fig. 1).

Critical points occur when some $m_i = 0$.[2] In the absence of any symmetries, the generic transition from FQH to SF therefore is through the Mott insulator. However, certain spatial symmetries may force $m_1 = m_2$ (see below), in which case there is a single tuning parameter that tunes between the superfluid and the FQH state.

Integrating out a Dirac fermion with mass m coupled to a gauge field a yields a CS term $\frac{\mathrm{sgn}(m)}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$. Thus, we consider the following Lagrangian [25]:

$$\mathcal{L}_{N_f,k} = \frac{N_f k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu \partial_\nu a_\lambda + \sum_{i=1}^{N_f}[\bar{\psi}_i \gamma^\mu D_\mu \psi_i + m\bar{\psi}_i \psi_i]. \quad (8)$$

The MI-SF transition is described by $\mathcal{L}_{1,1/2}$, the FQH-MI transition is described by $\mathcal{L}_{1,3/2}$, and the FQH-SF transition is described by $\mathcal{L}_{2,1/2}$ (see Fig. 1). This "fermionization" of the 3D *XY* transition was already conjectured in [6]. A crucial point is that the FQH-MI transition is different from the MI-SF transition because of the coefficient of the CS term, which affects the critical properties [5,6].

The critical exponents can be computed through a large N_f expansion, which has already been performed [6,26], motivated by the case $N_f = 1$. This is a relativistic transition, with dynamic critical exponent $z = 1$. The correlation length exponent ν is defined by $\xi \sim m^{-\nu}$, where ξ is the correlation length and m is the tuning parameter. ν can be determined by the dimension of the mass term. In the large-N_f limit, it was found to be [6,26]

$$\nu^{-1} = 1 + \frac{512\phi(1-2\phi)}{3\pi^2(1+\phi)^3}\frac{1}{N_f} + O\left(1/N_f^2\right), \quad (9)$$

where $\phi = (\frac{2\pi}{16k})^2$, although for $N_f = 1$ the leading $1/N_f$ correction was found [6] to be insufficient for accurately giving the 3D *XY* value of $\nu^{-1} \sim 1.5$. For the FQH-SF transition, $N_f = 2$, $k = 1/2$, we expect the large-N_f expansion to be more reliable.

At low energies the boson operator is $b \sim \hat{M}\psi$, so the scaling dimension Δ_b of b must be found by analyzing the dimension of the monopole operator combined with the fermion. If there are N_f Dirac points in the Brillouin zone, at momenta K_i, for $i = 1,\ldots,N_f$, then $\psi(r) \sim \sum_i e^{iK_i r}\psi_i(r)$. So far, the scaling dimension of an operator like $\hat{M}\psi_i$ is known only in the $N_f \to \infty$ limit. In that limit, the scaling dimension of b is [27] $\Delta_b = N_f(0.265...)$.

The order parameter exponent β for the superfluid is defined by $\langle b\rangle \sim m^\beta$. Following the arguments in Ref. [1], β can be seen to obey a generalized hyperscaling relation: $\beta = \nu\Delta_b$. Additionally, from general hyperscaling arguments, the superfluid susceptiblity scales like $\chi \sim m^{\nu(2\Delta_b - d - z)}$.

The scaling of the compressibility and conductivity follow from current-current correlation functions, which do not acquire any anomalous dimensions, and thus are similar to other two-dimensional transitions with $z = 1$ [28,29]:

$$\Pi_{\mu\nu}(k) = \left(\delta_{\mu\nu} + \frac{k_\mu k_\mu}{k^2}\right)\Pi_e(k) + \epsilon^{\mu\nu\lambda}k_\lambda \Pi_o(k). \quad (10)$$

Near the critical point, $\Pi_{\mu\nu} \sim \int d^d x dt \langle J_\mu(x,t)J_\nu(0,0)\rangle \sim \xi^{1-d}$. Therefore, $\Pi_e(k) \sim k$ and $\Pi_o(k) \sim O(k^0)$ at the critical point. From this we conclude that the compressibility vanishes at the critical point at zero temperature, while the conductivities are universal constants that can be computed in the large-N_f limit [6,30]. Therefore the dc longitudinal resistivity ρ_{xx} is zero on either side of the transition, but is a universal nonzero number of order h/e^2 at the transition, while the dc Hall resistivity ρ_{xy} is zero on the superfluid side, $h/2e^2$ on the FQH side, and a universal number of order h/e^2 at the critical point.

The temperature dependence of the polarization tensor at the critical point can be found by replacing k,ω by T. It follows that the compressibility at the transition scales like $\kappa \sim T$, while the conductivity is temperature independent at the transition. Finally, from general scaling considerations we can conclude that the specific heat scales like $C_v \sim T^2$.

Physical realizations. The transition described here is generic, and therefore can occur in principle in many different physical realizations involving strongly interacting systems of bosons. A particularly promising venue to realize bosonic FQH states is in optical traps of ultracold atoms [3], where strongly interacting bosons in a background effective magnetic field

[2] Note that in addition, chemical potential terms $\mu_i\psi_i^\dagger\psi_i$ are relevant operators that lead to a composite Fermi liquid. Nevertheless, spatial symmetries can impose $\mu_i = \mu$, and if particle number is held fixed, as in cold-atom settings, the composite Fermi liquid can be avoided and one can tune through these transitions with a single mass parameter.

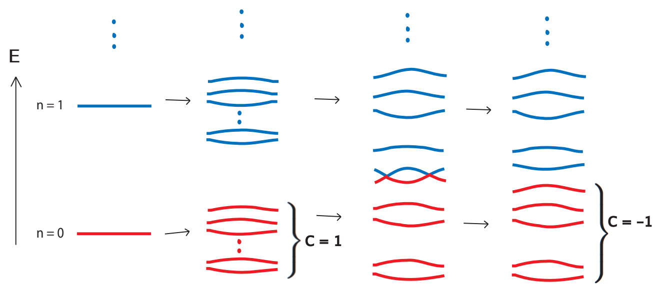

FIG. 2. (Color online) Evolution (schematic) of composite fermion bands as a periodic potential is turned on and tuned in an appropriate way. Red labels filled states and blue labels empty states. The flat bands on the far left indicate the Landau levels indexed by n.

can be realized. Now consider adding an external periodic potential $V_{pp}(r)$ with flux $2\pi p/q$ per plaquette. This induces a term $\delta H_{pp} = V_{pp}(r)b^\dagger(r)b(r)$ in the Hamiltonian of the bosons. Assuming that the composite fermion effective theory is the correct low-energy description,[3] the boson is represented by $b = \hat{M} f_2$, and therefore $b^\dagger b \propto f_2^\dagger f_2$, because $\hat{M}^\dagger \hat{M} \propto 1 + \alpha f^2 + \cdots$, where f^2 is the Maxwell term for a, α is a constant, and $\cdots$ indicate higher order derivatives of the gauge field. Therefore, to leading order, the composite fermion effective action obtains a contribution

$$\delta \mathcal{L}_{pp} \propto V_{pp}(r) f_2^\dagger f_2(r). \tag{11}$$

Such a periodic potential may be used to induce the Chern number of the composite fermions to change, as sketched in Fig. 2. For small V_{pp}, the Landau levels split into p subbands. As V_{pp} is increased, the top subband of the filled LL may eventually touch the bottom subband of the next empty LL, causing a change in the total Chern number of the filled bands. Spatial symmetries can force the Chern number to change by two units, causing a continuous FQH to superfluid transition. The necessary spatial symmetry depends on the nature of the V_{pp}. There can be many ways this can happen, and the most optimal one depends on the given experimental setup. One example is to turn on a honeycomb lattice with 2π flux per plaquette. In the limit of large V_{pp}, we can pass to the tight-binding limit with nearest and next-nearest neighboring hopping, with two low-lying bands with Chern number ± 1 for the two bands [31]. If the Chern number of the bottom band is 1, it is possible in principle to achieve this regime without closing the energy gap. As the second neighbor hopping is tuned through zero, there will be two band touchings, causing the Chern number to change directly from 1 to -1. It is the C_{3v} symmetry of the honeycomb lattice that protects the two Dirac cones in this case when the second neighbor hopping is zero [31].

Conclusions and discussion. We have presented a theory of a continuous transition between a Bose superfluid and a bosonic 1/2-Laughlin state. Remarkably, the theory predicts that the direct continuous transition is only possible in the presence of a spatial symmetry. When this symmetry is broken, the transition splits into two continuous transitions, with an intervening Mott insulator.

The superfluid we consider exists in a system with strong time reversal symmetry breaking (e.g., strong magnetic field). The reason that long-range phase coherence is possible in the presence of a strong magnetic field is the existence of a periodic potential. This gives a bandwidth to the single particle boson dispersion; when the bandwidth is strong compared to the interactions, the bosons condense into the bottom of the band. If the minimum of the band dispersion is not at zero momentum, the superfluid order parameter varies in space as set by the periodic potential, while simultaneously possessing long-range phase coherence. Since time-reversal symmetry is broken, one expects that the ground state of the superfluid will possess nonzero currents, as allowed by the symmetry. The critical point therefore also strongly breaks time-reversal symmetry.

Furthermore, we note that in order to properly also describe the Mott-insulating state, the theory presented here requires the bosons to be at integer filling of the lattice set by the periodic potential. If the bosons were instead at fractional filling, then the existence of the topologically trivial Mott insulator would require spontaneous breaking of the translation symmetry. Our theory does not include the more complicated physics associated with such spontaneous translation symmetry breaking in the phase diagram. Developing a proper critical theory that takes these effects into account is therefore a subject of future work.

Finally, we would like to make some cautionary remarks. The field theory presented here captures the asymptotic low energy properties of a putative superfluid-FQH transition. However, as is common in the theory of critical phenomena, the theory cannot reliably predict when such a continuous transition will occur in a microscopic model. In particular, when a periodic potential is applied to the bosons microscopically, the resulting periodic potential felt by the composite fermions in the effective long-wavelength theory is difficult to accurately predict. In other words, the exact $\delta \mathcal{L}_{pp}$ in Eq. (11) contains a structure factor, which is difficult to reliably compute in general, relating the microscopic periodic potential V_{pp} to that felt by the composite fermions. Furthermore, our work can of course not rule out the possibility of a different way for the superfluid-FQH transition to occur, which would be described by a different critical theory.

This work leaves a number of interesting questions open for future work. This includes more accurate estimates of the critical exponents of the transition, finding microscopic models where these continuous transitions are realized, including the case of fractional filling of the periodic potential, and studying the projected wave functions to show that they behave as predicted by the field theory. In particular, the idea that projecting a Chern number $+1$ free fermion band insulator with a Chern number -1 free fermion band insulator to obtain a bosonic wave function of a superfluid seems quite intriguing and deserving of further exploration.

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[3] This may not be the case if the gaps of the parton f_1 and f_2 bands are nearly equal.

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APPENDIX: COMPRESSIBILITY OF SLAVE PARTICLE/ COMPOSITE FERMION CONSTRUCTION OF SUPERFLUID STATE

In this section we will study in some more detail how the parton construction of the superfluid state manages to be compressible. As discussed in the main text, the parton construction of the superfluid state is as follows. We rewrite the boson operator as

$$b(r) = f_1(r) f_2(r), \tag{A1}$$

where f_i are fermions. Next, we consider a mean-field ansatz where f_1 forms a band insulator with Chern number 1, while f_2 forms a band insulator with Chern number -1, and suppose that these band insulators are created by the application of an external periodic potential. As discussed in the main text, such a construction yields a superfluid state for the bosons, because the emergent $U(1)$ gauge field a is gapless and can be associated with the dual of the superfluid Goldstone mode.

Since the fermions form band insulators due to an external periodic potential, by themselves they have a preferred density, which is set by the number of fermions per unit cell. Therefore it is not clear that the resulting state will be compressible, as changing the density would appear to cost a finite amount of energy. However, as we will explain below, such a construction does indeed yield a compressible state.

The zero-temperature compressibility κ of a quantum system is defined as

$$\kappa^{-1} = \frac{\partial \mu}{\partial n} = V \frac{\partial \mu}{\partial N} = V \frac{\partial^2 E}{\partial N^2} \sim N \frac{E(N+\delta) - 2E(N) + E(N-\delta)}{\delta^2}, \tag{A2}$$

where μ is the chemical potential, n is the density, V is the volume, and $E(N)$ is the ground-state energy for N particles, and the above derivatives are taken at constant volume. Thus we estimate the compressibility as

$$\kappa^{-1} \sim \frac{N \Delta_2(N,\delta)}{\delta^2}, \tag{A3}$$

where

$$\Delta_2(N,\delta) \sim E(N+\delta) - 2E(N) + E(N-\delta). \tag{A4}$$

The system is incompressible at zero temperature if, when we take $\delta \sim \sqrt{N}$ and $N \to \infty$, $\frac{N\Delta_2(N,\delta)}{\delta^2} \to \infty$ at fixed number density. In other words, the system is compressible at zero temperature if

$$\lim_{N\to\infty} \Delta_2(N,\sqrt{N}) < \infty. \tag{A5}$$

The choice $\delta \sim \sqrt{N}$ is for convenience; more generally, one must take the limit $\delta, N \to \infty$ with $\delta/N \to 0$.

In our slave-particle construction above, it was argued in the main text that the gauge field fluctuations of a are gapless. Therefore consider a system of fermions with a filled band with a nonzero Chern number, and subject it to a magnetic field that can vary with essentially zero energy cost. We now consider the energy $E_f(N,\phi)$, which is the ground-state energy of the fermionic sector of the parton theory, with N particles, and with additional ϕ flux quanta of a added to the system. Since the flux ϕ is a dynamical quantity, and the gauge field a is gapless, the ground-state energy $E(N+\delta) \approx E_f(N+\delta,\delta)$, where the optimal $\phi \sim \delta$ is approximately the additional number of particles added to the system.

Now, we would like to know the fate of

$$\kappa^{-1} \sim \frac{N \Delta_2^{\phi}(N,\delta)}{\delta^2}, \tag{A6}$$

where now

$$\Delta_2^{\phi}(N,\delta) \sim E_f(N+\delta,\delta) - 2E_f(N,0) + E_f(N-\delta,-\delta). \tag{A7}$$

When the fermions fill a Landau level, $\Delta_2^{\phi}(N,\sqrt{N}) < \infty$ as $N \to \infty$. This is because the ground-state energy of a filled lowest Landau level is $eBN/2m$, where we set $\hbar = c = 1$. From this, it follows that $\Delta_2^{\phi}(N,\delta) \sim \frac{\delta^2}{A}$, where A is the area of the system, so that $\kappa^{-1} \sim N/A$, which is bounded as $N \to \infty$ at fixed average number density N/A. Thus the Landau level problem gives a compressible state, if we allow the magnetic field to vary arbitrarily. This makes sense, since the density is only tied to the magnetic field, and once the magnetic field can vary arbitrarily, so can the density.

Now consider a Chern insulator, such as Haldane's honeycomb model with the lowest band filled [31]. We would like to know whether

$$\lim_{N\to\infty} \Delta_2^{\phi}(N,\sqrt{N}) < \infty. \tag{A8}$$

If so, we can then conclude that the parton Chern insulator construction of the superfluid will also be compressible if the gauge field a is gapless.

To establish that (A8) is true for such a situation, consider a continuum system with a constant magnetic field, i.e., a Landau level problem, and consider adding a small periodic potential. Let λ parametrize the strength of the periodic potential, and consider $\Delta_2^{\phi}(N,\delta,\lambda)$, where the last argument just parametrizes the value of λ in the Hamiltonian. Clearly for small $\lambda \ll eB/m$, we must have

$$\lim_{N\to\infty} \Delta_2^{\phi}(N,\sqrt{N},\lambda) < \infty. \tag{A9}$$

Furthermore, as long as we do not close the energy gap, continuously changing λ must always preserve the above inequality. This is because as long as we do not close the energy gap, the ground-state energy in the thermodynamic limit is analytic in λ, and so the above inequality must continue to be satisfied as λ is changed infinitesimally.

Now, we know that it is possible to, for instance, slowly turn on a honeycomb lattice potential with 2π flux per plaquette, such that even in the limit that the periodic potential is much stronger than eB/m, we do not close the energy gap. In this limit, we end up with two bands, and if the lower band has Chern number $+1$, then it is possible to adiabatically evolve from the continuum Landau level to this situation. For the Chern insulator with the lower band having $C = 1$ and 2π flux per plaquette, it follows that (A8) is satisfied, because we never had to close the energy gap as we increased the periodic potential. Flipping the sign of the second nearest neighbor hopping in such a model can flip the Chern number. We expect therefore that as long as $C = 1$ or $C = -1$ for the bottom band, (A8) will remain true.

We conclude that Chern insulators, in addition to filled Landau levels, will satisfy (A8) and are therefore compressible if the magnetic field is allowed to vary arbitrarily. Since the fluctuations of the emergent $U(1)$ gauge field are gapless in the parton construction of the superfluid, the magnetic field can indeed vary arbitrarily, so we see that the parton construction of the superfluid state is indeed compressible when the gauge fluctuations are taken into account.

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