

Statistics of spin fluctuations in quantum dots with Ising exchange

D. S. Lyubshin,^{1,2} A. U. Sharafutdinov,¹ and I. S. Burmistrov^{1,2}

¹*L.D. Landau Institute for Theoretical Physics RAS, Kosygina street 2, 119334 Moscow, Russia*

²*Moscow Institute of Physics and Technology, 141700 Moscow, Russia*

(Received 27 November 2013; revised manuscript received 20 March 2014; published 19 May 2014)

We explore the effect of single-particle level fluctuations on the Stoner instability in a quantum dot with a strong spin-orbit coupling in the framework of the universal Hamiltonian with the Ising exchange interaction. We reduce the problem to studying the statistics of extrema of a certain Gaussian process and demonstrate that, in spite of the randomness of the single-particle levels, the longitudinal spin susceptibility and all its moments diverge simultaneously at the point of the Stoner instability which is determined by the standard criterion involving the mean level spacing only.

DOI: [10.1103/PhysRevB.89.201304](https://doi.org/10.1103/PhysRevB.89.201304)

PACS number(s): 75.75.-c, 73.23.Hk, 73.63.Kv

Introduction. In the last two decades the physics of quantum dots (QDs) attracted a lot of interest from both experimentalists and theorists [1–5]. Under the assumption $E_{\text{Th}}/\delta \gg 1$, where E_{Th} and δ denote the Thouless energy and the mean single-particle level spacing, respectively, an effective zero-dimensional Hamiltonian has been derived [6]. This so-called universal Hamiltonian (UH) provides a convenient framework for the theoretical description of QDs. In the UH the electron-electron interaction that involves a set of matrix elements in the single-particle basis is reduced to just three parameters: the charging energy (E_c), the ferromagnetic exchange ($J > 0$), and the interaction in the Cooper channel.

At low temperatures $T \ll E_c$ the additional cost due to the charging energy restricts the probability of real electron tunneling through a QD, a phenomenon known as the Coulomb blockade [7]. This leads to suppression of the tunneling density of states (TDOS) in QDs for $T \ll E_c$ [8,9]. Although typically $E_c \gg \delta$, a small enough exchange interaction $J \lesssim \delta/2$ is important for a quantitative description of the experiments on transport through QDs at $T \lesssim \delta$ in a two-dimensional electron gas [10]. The exchange interaction of a large QD can be estimated by the Fermi-liquid interaction parameter (F_0^σ) of the corresponding bulk material: $J/\delta = -F_0^\sigma$. As well known for bulk materials, strong enough exchange interaction leads to a Stoner instability at $F_0^\sigma = -1$ and a corresponding quantum phase transition between a paramagnet and a ferromagnet. In QDs an intermediate case of the ground state (GS) having a finite value of spin can be realized for $\delta/2 \lesssim J < \delta$ in the case of the equidistant single-particle spectrum [6]. As J increases towards δ , the GS spin increases and at $J = \delta$ all electrons in a QD become spin polarized. This phenomenon of mesoscopic Stoner instability disappears in the thermodynamic limit $\delta \rightarrow 0$. Due to the entanglement of the charge and spin degrees of freedom in the UH, the mesoscopic Stoner instability affects the electron transport through a QD. For example, it leads to an additional nonmonotonicity of the energy dependence of the TDOS [11–13] and to the enhancement of the shot noise [14]. The Cooper channel interaction in the UH describes the superconducting correlations in QDs [15]. We shall assume that there is no attraction in the Cooper channel and, therefore, disregard it below [6]. We also neglect the corrections to the UH due to the fluctuations in the matrix elements of the interaction [16,17], which are small in the regime $\delta/E_{\text{Th}} \ll 1$ but can lead to interesting physics beyond the UH [5].

In the presence of a spin-orbit coupling the UH description of a QD breaks down. For a large spin-orbit length (weak spin-orbit coupling), $\lambda_{\text{SO}} \gg L$, where L is a typical size of the QD, fluctuations of the matrix elements of the interaction cannot be neglected [18,19]. However, for a QD fabricated in a two-dimensional electron gas the orbital degrees of freedom are coupled to in-plane components of the spin only. Thus in the regime $(\lambda_{\text{SO}}/L)^2 \gg (E_{\text{Th}}/\delta)(L/\lambda_{\text{SO}})^4 \gg 1$ the UH description is restored but with the Ising exchange interaction [18,20]. Contrary to case of the Heisenberg exchange, there is no mesoscopic Stoner instability within the UH with the Ising exchange ($J_z > 0$) for the equidistant single-particle spectrum [6]. As a consequence, the TDOS is almost independent of J_z , while the longitudinal spin susceptibility χ_{zz} is independent of T as in a clean Fermi liquid [11,21].

In the interacting electron systems a disorder-induced finite temperature transition between the paramagnetic and the ferromagnetic phases is possible in low dimensions $d \leq 2$ [22,23]. In $d = 3$ the Stoner instability can be shifted towards the smaller values of the exchange interaction due to disorder [24]. In the UH description the disorder is translated into randomness of the single-particle levels. The latter is crucial in the case of the Ising exchange since the average χ_{zz} acquires a T -dependent contribution of Curie type due to the level fluctuations [6]. The Curie-type contribution dominates at low enough T and for $\delta - J_z \ll \delta$. In this regime the level fluctuations become strong with respect to the distance $\delta - J_z$ to the average position of the Stoner instability at $J_z = \delta$ (albeit small when compared to the temperature). This implies that, although for $\delta - J_z \ll \delta$ the QD is in the paramagnetic phase on average, for a particular realization of the single-particle levels the QD can be fully spin polarized. Such events should affect the tail of the distribution function for χ_{zz} , but how exactly? Can it be possible that at $T = 0$ the level fluctuations shift the position of the Stoner instability from $J_z = \delta$ and lead to the existence of a finite temperature transition between the paramagnetic and the ferromagnetic phases?

In this Rapid Communication we address these questions within the UH with the Ising exchange interaction approach. We demonstrate that for $\delta - J_z \ll \delta$ the statistical properties of the spin susceptibility in the temperature range $\delta \ll T \ll \delta J_z/(\delta - J_z)$ are determined by the statistics of the extrema of a certain Gaussian process with drift that locally resembles

a fractional Brownian motion (FBM) with Hurst exponent $H = 1 - \epsilon$ with $\epsilon \rightarrow 0$ (recall that the FBM with Hurst exponent H is the Gaussian process $B_H(t)$ with zero mean and the two-point correlation function $\overline{[B_H(t) - B_H(t')]^2} = |t - t'|^{2H}$). We estimate the complementary cumulative distribution function (CCDF) for χ_{zz} and show that all moments of χ_{zz} are finite for $J_z < \delta$. Thus our results mean that at $T \gg \delta$, in spite of the presence of strong level fluctuations, the Stoner instability still occurs at $J_z = \delta$.

The partition function. We consider the following universal Hamiltonian with direct Coulomb and Ising exchange interactions [6]:

$$H = \sum_{\alpha,\sigma} \epsilon_\alpha a_{\alpha\sigma}^\dagger a_{\alpha\sigma} + E_c (\hat{N} - N_0)^2 - J_z \hat{S}_z^2. \quad (1)$$

For an isolated lateral QD with Rashba and Dresselhaus spin-orbit couplings the statistics of single-particle energies ϵ_α is described by the unitary Wigner-Dyson ensemble (class A) [20]. The operators of the total number of particles $\hat{N} = \sum_{\alpha,\sigma} a_{\alpha\sigma}^\dagger a_{\alpha\sigma}$ and the total spin $\hat{S} = (1/2) \sum_{\alpha\sigma\sigma'} a_{\alpha\sigma}^\dagger \sigma_{\sigma\sigma'} a_{\alpha\sigma'}$ are given as usual in terms of the single-particle creation ($a_{\alpha,\sigma}^\dagger$) and annihilation ($a_{\alpha,\sigma}$) operators and the Pauli matrices σ .

Since the operators for the number of spin-up and spin-down electrons commute with H , the grand partition function $Z = \text{Tr} e^{-\beta H + \beta \mu \hat{N}}$ can be written as [21]

$$Z = \sum_{n_\uparrow, n_\downarrow} Z_{n_\uparrow} Z_{n_\downarrow} e^{-\beta E_c (n - N_0)^2 + \beta J_z m^2 + \beta \mu n}. \quad (2)$$

The integers n_\uparrow and n_\downarrow represent the number of spin-up and spin-down electrons, respectively. The total number of electrons is $n = n_\uparrow + n_\downarrow$, and $m = (n_\uparrow - n_\downarrow)/2$ is the value of S_z . The factor Z_{n_\uparrow} (Z_{n_\downarrow}) is the canonical partition function for n_\uparrow (n_\downarrow) noninteracting spinless electrons. They take into account the contributions due to the single-particle energies and are given by the Darwin-Fowler integral: $Z_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} \prod_\gamma (1 + e^{i\theta - \beta \epsilon_\gamma})$. There is a convenient integral representation for Z which is exactly equivalent to Eq. (2):

$$Z = \sum_{k \in \mathbb{Z}} e^{-\beta E_c (k - N_0)^2} \int_{-\pi T}^{\pi T} \frac{d\phi_0}{2\pi T} e^{i\beta \phi_0 k} \tilde{Z}(\mu - i\phi_0), \quad (3)$$

$$\tilde{Z}(\mu) = \int_{-\infty}^{\infty} \frac{dh}{\sqrt{\pi} \beta J_z} e^{-h^2/\beta J_z} \prod_{\sigma} e^{-\beta \Omega_0(\mu + h\sigma/\beta)}.$$

Here $\Omega_0(\mu) = -T \ln \prod_\gamma (1 + e^{-\beta(\epsilon_\gamma - \mu)})$ stands for the thermodynamic potential of free spinless electrons; ϕ_0 and h are the zero-frequency Matsubara components of the electric potential and the magnetic field that can be used to decouple the direct Coulomb [8] and exchange interaction [21,25] terms, respectively.

At not very low temperatures $T \gg \delta$ we can perform integration over ϕ_0 in Eq. (3) in the saddle-point approximation [8,26]. In this case the grand canonical partition function factorizes: $Z = Z_C Z_S$, where

$$Z_C = \sqrt{\frac{\beta \Delta}{4\pi}} \sum_{k \in \mathbb{Z}} e^{-\beta E_c (k - N_0)^2 + \beta(\tilde{\mu} - \mu)k - 2\beta \Omega_0(\tilde{\mu})} \quad (4)$$

describes the effect of the charging energy. The exchange interaction is encoded in $Z_S = \exp[2\beta \Omega_0(\tilde{\mu})] \tilde{Z}(\tilde{\mu})$. Here $\tilde{\mu}$ is

the solution of the saddle-point equation $N_0 = -\partial \Omega_0(\tilde{\mu})/\partial \tilde{\mu}$, and $\Delta^{-1} = -\partial^2 \Omega_0(\tilde{\mu})/\partial \tilde{\mu}^2$ stands for the thermodynamic density of states for the chemical potential $\tilde{\mu}$. The thermodynamic potential $\Omega_0(\tilde{\mu})$ depends on a particular realization of the single-particle spectrum via the single-particle density of states $v_0(E) = \sum_\alpha \delta(E + \tilde{\mu} - \epsilon_\alpha)$. Provided $h^2 \ll \exp(\beta \tilde{\mu})$, we find

$$\beta \sum_{\sigma} [\Omega_0(\tilde{\mu}) - \Omega_0(\tilde{\mu} + h\sigma/\beta)] = \frac{h^2}{\beta \delta} - V(h), \quad (5)$$

$$V(h) = - \int_{-\infty}^{\infty} dE \delta v_0(E) \ln \left[1 + \frac{\sinh^2(h/2)}{\cosh^2(\beta E/2)} \right],$$

where $\delta v_0(E)$ stands for the deviation of the density of states $v_0(E)$ from its average value $1/\Delta \equiv 1/\overline{\Delta}$.

The longitudinal spin susceptibility is fully determined by the partition function, $\chi_{zz} = \partial \ln Z / \partial J_z$. Since Z_C is independent of J_z and therefore does not affect the spin susceptibility, we will discuss Z_S only. We note that the normalization is such that $Z_S = 1$ for $J_z = 0$. According to Eq. (2), Z increases with J_z ; it follows that $Z_S \geq 1$. It is useful to write down Z_S explicitly:

$$Z_S = \sqrt{\bar{J}_z/J_z} \int_{-\infty}^{\infty} \frac{dh}{\sqrt{\pi}} \exp[-h^2 - V(h\sqrt{\bar{J}_z})], \quad (6)$$

where $\bar{J}_z = \delta J_z / (\delta - J_z)$ is the renormalized exchange interaction. For the equidistant spectrum ($V = 0$), Eq. (6) yields $Z_S = \sqrt{\bar{J}_z/J_z}$ and $\chi_{zz} = 1/[2(\delta - J_z)]$.

Level fluctuations. Although the density of states $v_0(E)$ has non-Gaussian statistics, $V(h)$ is an even in h Gaussian random function for $\max\{|h|, T/\delta\} \gg 1$ [27]. It has zero mean and the following two-point correlation function (see the Supplemental Material [28]):

$$\overline{V(h_1)V(h_2)} = \sum_{\sigma=\pm} L(h_1 + \sigma h_2) - 2L(h_1) - 2L(h_2), \quad (7)$$

$$L(h) = \frac{2}{\pi^2 \beta} \int_0^{|h|} dt t \left[\text{Re} \psi \left(1 + \frac{it}{2\pi} \right) + \gamma \right].$$

Here $\psi(z)$ is the Euler digamma function, $\gamma = -\psi(1)$ is the Euler-Mascheroni constant, and $\beta = 2$ since the energy levels ϵ_α in Eq. (1) are described by the unitary Wigner-Dyson ensemble. The asymptotics of $L(h)$ are as follows [12,28]:

$$L(h) = \frac{h^2}{\pi^2 \beta} \begin{cases} \zeta(3)h^2/(8\pi^2), & |h| \ll 1, \\ \ln[|h|/(2\pi)] + \gamma - 1/2, & |h| \gg 1. \end{cases} \quad (8)$$

Average spin susceptibility. As well known, the level fluctuations are small at $T \gg \delta$. In this regime one can find from Eq. (8) that the variance of the level spacing is $\overline{(\Delta - \delta)^2}/\delta^2 = 3\zeta(3)\delta^2/(2\pi^4 \beta T^2) \ll 1$ [12]. Therefore it seems that in order to find the average spin susceptibility at $T \gg \delta$ it is enough to substitute $1/\Delta$ for $1/\delta$ in the expression for χ_{zz} obtained above for the equidistant spectrum. Performing the expansion to the second order in $\Delta - \delta$, we find the average spin susceptibility

$$\bar{\chi}_{zz} = \frac{1}{2(\delta - J_z)} \left[1 + \frac{3\zeta(3)}{2\pi^4 \beta} \frac{\delta \bar{J}_z^2}{J_z T^2} \right]. \quad (9)$$

This result indicates that the effect of fluctuations is small only at temperatures $T \gg \bar{J}_z$. For such temperatures the fluctuations of the level spacing are small in comparison with the distance to the average position of the Stoner instability, $(\Delta - \delta)^2 \ll (\delta - J_z)^2$. If $\delta - J_z \ll \delta$, the renormalized exchange interaction $\bar{J}_z \gg \delta$ and there is a wide interval of temperatures $\bar{J}_z \gg T \gg \delta$ where the effect of level fluctuations can be strong. We stress that the dependence of $\bar{\chi}_{zz}$ on T appears only due to level fluctuations. The result (9) can be also obtained from Eq. (6) by means of the second order perturbation theory in V using the asymptotic expression (8) at $|h| \ll 1$.

For temperatures $T \ll \bar{J}_z$ the integral in the right-hand side (rhs) of Eq. (6) is dominated by $|h| \sim \sqrt{\bar{J}_z/T} \gg 1$. Hence in this case to evaluate the integrals involved in the perturbation theory in V one has to use the asymptotic formula (8) at $|h| \gg 1$. Expanding Eq. (6) to the fourth order in V and performing the averaging of $\ln Z_S$ with the help of Eqs. (7) and (8) we obtain [28]

$$\bar{\chi}_{zz} = \frac{1}{2(\delta - J_z)} \left[1 + \frac{\bar{J}_z \ln 2}{\beta \pi^2 T} + a_2 \left(\frac{\bar{J}_z}{\beta \pi^2 T} \right)^2 \right], \quad (10)$$

where $a_2 \approx 0.29$. From Eq. (10) we see that the perturbation theory in V is justified only for $T \gg \bar{J}_z/(\pi^2 \beta)$. Therefore the result (10) is valid in the range $\bar{J}_z \gg T \gg \bar{J}_z/(\pi^2 \beta)$. In this regime the fluctuations of the spin susceptibility around its average value are small, $(\chi_{zz} - \bar{\chi}_{zz})^2/(\bar{\chi}_{zz})^2 \propto \bar{J}_z/(\pi^2 \beta T) \ll 1$.

Tail distribution for $\ln Z_S$. The perturbative result (10) suggests that the spin susceptibility can be strongly affected by level fluctuations at $\bar{J}_z/(\pi^2 \beta) \gg T \gg \delta$. Such regime is realized in the close vicinity of the average position of the Stoner instability $\delta - J_z \ll \delta/(\pi^2 \beta)$. However, if the effect of level fluctuations is strong, then it is useful to know not only the average spin susceptibility but also all its moments. With this in mind, we investigate the CCDF for $\ln Z_S$ at temperatures in the interval $\bar{J}_z/(\pi^2 \beta) \gg T \gg \delta$. In this range of temperatures, the integral in the rhs of Eq. (6) is dominated by the large values of $|h|$. Using asymptotic expression (8), one can check that for $|h_1|, |h_2| \gg 1$ the two-point correlation function (7) is homogeneous of degree two: $V(uh_1)V(uh_2) = u^2 V(h_1)V(h_2)$ [6]. We can therefore substitute $z v(h)$ for the random function $V(h\sqrt{\beta \bar{J}_z})$, where $z = \sqrt{\beta \bar{J}_z/(\pi^2 \beta)}$; the Gaussian random process $v(h)$ has zero mean, possesses the property $v(h) = v(-h)$, and its correlation function is given by

$$\overline{v(h_1)v(h_2)} = \frac{1}{2} \sum_{\sigma=\pm} (h_1 + \sigma h_2)^2 \ln(h_1 + \sigma h_2)^2 - h_1^2 \ln h_1^2 - h_2^2 \ln h_2^2. \quad (11)$$

Since $\overline{[v(h+u) - v(h)]^2} = -2u^2 \ln|u| + O(u^2) = O(u^{2H})$ for any $H = 1 - \epsilon < 1$, the trajectories of $v(h)$ are continuous and its increments are strongly positively correlated (see inset in Fig. 1). In fact the process $v(h)$ is in many aspects close to the ballistic case $\tilde{v}(h) = \xi|h|$, where ξ is a Gaussian random variable [recall that $\tilde{v}(h)$ is the unique process with $H = 1$]. We mention that the process $v(h)$ has arisen before in a seemingly unrelated context [29].

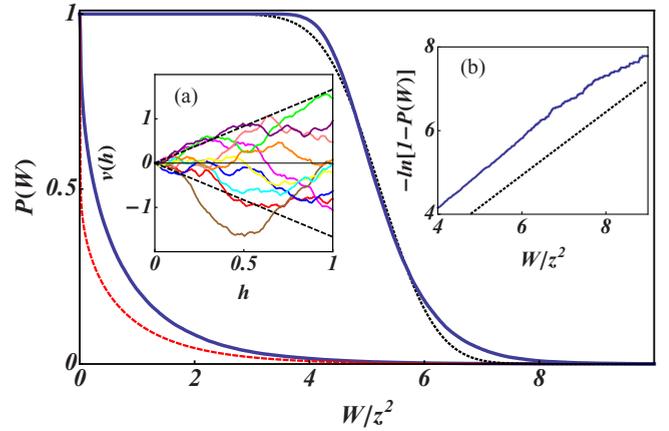


FIG. 1. (Color online) The dependence of $\mathcal{P}(W)$ on W/z^2 at $T = 3\delta$ obtained numerically for $J_z/\delta = 0.94$ ($z \approx 0.5$) (upper solid curve) and $J_z/\delta = 0.99994$ ($z \approx 16.8$) (lower solid curve). The black dotted curve is the CCDF for the normal distribution with mean and variance as one can find from the lowest order perturbation theory in V for $T = 3\delta$ and $J_z/\delta = 0.94$ [28]. The red dashed curve is the CCDF of the degenerate process $\tilde{v}(h)$ for $T = 3\delta$ and $J_z/\delta = 0.99994$ [28]. Inset (a): Several realizations of the process $v(h)$; dashed lines $\pm 2h\sqrt{\ln 2}$ are guides for the eye. Inset (b): Comparison of the tail of $\mathcal{P}(W)$ obtained numerically for $J_z/\delta = 0.99994$ ($z \approx 16.8$) and asymptotic result (15).

The average moments of $\ln Z_S$ can be conveniently written as $[\ln Z_S]^k = k \int_0^\infty dW W^{k-1} \mathcal{P}(W)$, where the function $\mathcal{P}(W)$ is the complementary cumulative distribution function, i.e., the probability for $\ln Z_S$ to exceed W : $\mathcal{P}(W) \equiv \text{Prob}\{\ln Z_S > W\}$. We note that $\mathcal{P}(0) = 1$, $\mathcal{P}(\infty) = 0$, and $\mathcal{P}(W)$ is monotonously decreasing. Although we cannot find a closed analytical expression for $\mathcal{P}(W)$, we bound it from above to prove that all moments of $\ln Z_S$ (and consequently all moments of χ_{zz}) are finite for $J_z < \delta$. We first split the Gaussian weight $\exp(-h^2)$ in the integral in the rhs of Eq. (6) and obtain ($0 < \gamma < 1$ is an arbitrary splitting parameter)

$$Z_S \leq \frac{2\sqrt{\bar{J}_z}}{\sqrt{\pi}\gamma J_z} \int_0^\infty dh e^{-(1-\gamma)h^2/\gamma} \max_{h \geq 0} \{e^{-h^2 - zv(h)/\sqrt{\gamma}}\}. \quad (12)$$

The inequality (12) allows us to reduce the problem of finding an upper bound for $\mathcal{P}(W)$ to studying the statistics of the maxima of the Gaussian process $Y(h) = -h^2 - (z/\sqrt{\gamma})v(h)$ which locally resembles a FBM with drift. Indeed, from Eq. (12) we find

$$\mathcal{P}(W) \leq \text{Prob} \left\{ \max_{h \geq 0} Y(h) > W + \frac{1}{2} \ln \frac{(1-\gamma)J_z}{\bar{J}_z} \right\}. \quad (13)$$

To give an upper bound for the probability $\text{Prob}\{\max_{h \geq 0} Y(h) > w\}$ we employ an auxiliary Gaussian process $X(h) = -h^2 + (2z\sqrt{\ln 2}/\sqrt{\gamma})B(h^2)$, where $B(h)$ is the standard Brownian motion (the Hurst exponent $H = 1/2$). The processes $Y(h)$ and $X(h)$ satisfy the conditions for Slepian's inequality [30], $\text{Prob}\{\max_{h \geq 0} Y(h) > w\} \leq \text{Prob}\{\max_{h \geq 0} X(h) > w\}$. Ruin probabilities for $X(h)$ are trivial to compute [31], and we find

the following upper bound for the CCDF [28]:

$$\mathcal{P}(W) \leq \exp \left\{ -\frac{\gamma}{2z^2 \ln 2} \left[W + \frac{1}{2} \ln \frac{(1-\gamma)J_z}{\bar{J}_z} \right] \right\}. \quad (14)$$

From Eq. (14) it follows that all moments of $\ln Z_S$ (and hence all moments of χ_{zz}) are finite for $J_z < \delta$ for temperatures in the range $\bar{J}_z/(\pi^2\beta) \gg T \gg \delta$. Therefore even in the presence of the strong level fluctuations the Stoner instability occurs at $J_z = \delta$. For $J_z < \delta$ and for temperatures $T \gg \delta$ the QD is in the paramagnetic state.

For $z \gg 1$ the saddle-point approximation in Eq. (6) becomes exact and the statistics of $\ln Z_S$ reduces to the statistics of maxima of $Y(h)$ directly. Since local behavior of $Y(h)$ may be compared to that of the FBM with Hurst exponent $H = 1 - \epsilon$, one can adapt the results of Ref. [32] for locally stationary processes and find that for $W \gg 2z^2 \ln 2$ with logarithmic accuracy [28]

$$\mathcal{P}(W) \propto \left(\frac{z^2}{W} \ln \frac{W}{z^2} \right)^{1/2} \exp \left(-\frac{W}{2z^2 \ln 2} \right). \quad (15)$$

This result, valid in the temperature range $\bar{J}_z/(\pi^2\beta) \gg T \gg \delta$, is consistent with the upper bound (14). To illustrate the result (15) we approximate the Gaussian process $v(h)$ by a degenerate one, $\tilde{v}(h) = \xi|h|$, where ξ is the Gaussian random variable with zero mean $\bar{\xi} = 0$ and variance $\bar{\xi}^2 = 4 \ln 2$. We estimate the partition function (6) as $Z_S \simeq \sqrt{\bar{J}_z/J_z} \exp(z^2 \bar{\xi}^2/4) [1 - \text{erf}(z\xi/2)]$. The large values of Z_S correspond to large negative values of ξ such that $\ln Z_S \approx z^2 \bar{\xi}^2/4$. Hence we find that for $z \gg 1$ the tail of the function $\mathcal{P}(W)$ is given by Eq. (15) without the logarithm in the preexponent [28]. As shown in Fig. 1 the overall behavior of $\mathcal{P}(W)$ for $z \gg 1$ is well enough approximated by the CCDF for the degenerate process $\tilde{v}(h)$. Also we mention that the behavior of $\mathcal{P}(W)$ for $z \gg 1$ is very different from its behavior at $z \lesssim 1$. For the latter, $\mathcal{P}(W)$ is given by the CCDF for the normal distribution (see Fig. 1).

Average moments of χ_{zz} . Equation (15) implies that the average moments of $\ln Z_S$ scale as $(\overline{\ln Z_S})^k \sim z^{2k}$ for $z \gg 1$. Hence for $\delta \ll T \ll \bar{J}_z/(\pi^2\beta)$ the k th moment of the spin susceptibility is given by

$$\overline{\chi_{zz}^k} \propto \left[\frac{\delta^2}{\pi^2\beta(\delta - J_z)^2 T} \right]^k, \quad k = 1, 2, \dots \quad (16)$$

The result (16) can be obtained from the saddle-point analysis of the integral in the rhs of Eq. (6), i.e., in essence, by Larkin-Imry-Ma-type arguments [33,34]. The scaling of the average spin susceptibility [Eq. (16) with $k = 1$] was indeed found in Ref. [6] using arguments of Larkin-Imry-Ma type.

Summary. To summarize, we have studied the Stoner instability in a QD within the UH with the Ising exchange interaction. We demonstrated that in the regime $\delta - J_z \ll \delta$ where the level fluctuations are dangerous all moments of the spin susceptibility $\chi_{zz}(T)$ are finite at temperatures $T \gg \delta$ for $J_z < \delta$. This means that (i) the Stoner instability is not shifted by the level fluctuations away from $J_z = \delta$, and (ii) randomness in the single-particle levels does not lead to transition at finite $T \gg \delta$ between the paramagnetic and ferromagnetic phases. Although we expect that these conclusions hold also for temperatures $T \lesssim \delta$ we cannot argue it within our approach; a separate analysis is needed.

Similar conclusions about the influence of randomness of the single-particle levels on the Stoner instability can be made for QDs with the Heisenberg exchange [35]. Our approach can be extended to the analysis of the effect of level fluctuations on the transverse spin susceptibility and the TDOS for the UH with the Ising exchange.

Our results, in principle, can be checked in QDs made of materials close to the Stoner instability such as Co impurities in a Pd or Pt host, Fe or Mn dissolved in various transition-metal alloys, Ni impurities in a Pd host, and Co in Fe grains, as well as nearly ferromagnetic rare-earth materials [36]. However, to test our most interesting results [Eqs. (15) and (16)] one needs to explore the regime $(\delta - J_z)/\delta \ll 1/(\pi^2\beta)$. At present the closest material to the Stoner instability we are aware of, $\text{YFe}_2\text{Zn}_{20}$, has the exchange interaction $J \approx 0.94\delta$ which is near the border of the regime with strong level fluctuations at low temperatures.

Acknowledgments. We acknowledge useful discussions with Y. Fyodorov, Y. Gefen, A. Iosevich, A. Shnirman, and M. Skvortsov. The research was funded in part by the Russian-Israel scientific research cooperation (RFBR Grant No. 11-02-92470 and IMOST 3-8364); the Council for Grant of the President of Russian Federation (Grant No. MK-4337.2013.2); Dynasty Foundation; RAS Programs ‘‘Quantum mesoscopics and disordered systems,’’ ‘‘Quantum physics of condensed matter,’’ and ‘‘Fundamentals of nanotechnology and nanomaterials’’; and by Russian Ministry of Education and Science.

-
- [1] Y. Alhassid, *Rev. Mod. Phys.* **72**, 895 (2000).
 [2] W. G. van der Wiel, S. De Franceschi, J. M. Elzerman, T. Fujisawa, S. Tarucha, and L. P. Kouwenhoven, *Rev. Mod. Phys.* **75**, 1 (2002).
 [3] I. Aleiner, P. Brouwer, and L. Glazman, *Phys. Rep.* **358**, 309 (2002).
 [4] R. Hanson, L. P. Kouwenhoven, J. R. Petta, S. Tarucha, and L. M. K. Vandersypen, *Rev. Mod. Phys.* **79**, 1217 (2007).
 [5] D. Ullmo, *Rep. Prog. Phys.* **71**, 026001 (2008).
 [6] I. L. Kurland, I. L. Aleiner, and B. L. Altshuler, *Phys. Rev. B* **62**, 14886 (2000).
 [7] R. I. Shekhter, *Sov. Phys. JETP* **36**, 747 (1973); I. O. Kulik and R. I. Shekhter, *ibid.* **41**, 308 (1975); E. Ben-Jacob and Y. Gefen, *Phys. Lett. A* **108**, 289 (1985).
 [8] A. Kamenev and Y. Gefen, *Phys. Rev. B* **54**, 5428 (1996).
 [9] N. Sedlmayr, I. V. Yurkevich, and I. V. Lerner, *Europhys. Lett.* **76**, 109 (2006).
 [10] J. A. Folk, C. M. Marcus, R. Berkovits, I. L. Kurland, I. L. Aleiner, and B. L. Altshuler, *Phys. Scr. T* **90**, 26 (2001); G. Usaj and H. U. Baranger, *Phys. Rev. B* **67**, 121308(R) (2003); Y. Alhassid and T. Rupp, *Phys. Rev. Lett.* **91**, 056801 (2003); Y. Alhassid, T. Rupp, A. Kaminski, and L. I. Glazman, *Phys. Rev. B* **69**, 115331 (2004).

- [11] M. N. Kiselev and Y. Gefen, *Phys. Rev. Lett.* **96**, 066805 (2006).
- [12] I. S. Burmistrov, Y. Gefen, and M. N. Kiselev, *JETP Lett.* **92**, 179 (2010).
- [13] I. S. Burmistrov, Y. Gefen, and M. N. Kiselev, *Phys. Rev. B* **85**, 155311 (2012).
- [14] B. Sothmann, J. König, and Y. Gefen, *Phys. Rev. Lett.* **108**, 166603 (2012).
- [15] M. Schechter, *Phys. Rev. B* **70**, 024521 (2004); Zu-Jian Ying, M. Cuoco, C. Noce, and Huan-Qiang Zhou, *ibid.* **74**, 012503 (2006); **74**, 214506 (2006); S. Schmidt, Y. Alhassid, and K. van Houcke, *Europhys. Lett.* **80**, 47004 (2007); S. Schmidt and Y. Alhassid, *Phys. Rev. Lett.* **101**, 207003 (2008); K. Van Houcke, Y. Alhassid, S. Schmidt, and S. M. A. Rombouts, [arXiv:1011.5421](https://arxiv.org/abs/1011.5421); Y. Alhassid, K. N. Nesterov, and S. Schmidt, *Phys. Scr. T* **151**, 014047 (2012); K. N. Nesterov and Y. Alhassid, *Phys. Rev. B* **87**, 014515 (2013).
- [16] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, *Phys. Rev. Lett.* **78**, 2803 (1997).
- [17] Ya. M. Blanter, A. D. Mirlin, and B. A. Muzykantskii, *Phys. Rev. Lett.* **78**, 2449 (1997).
- [18] Y. Alhassid and T. Rupp, [arXiv:cond-mat/0312691](https://arxiv.org/abs/cond-mat/0312691).
- [19] H. E. Türeci and Y. Alhassid, *Phys. Rev. B* **74**, 165333 (2006); G. Murthy, *ibid.* **77**, 073309 (2008); O. Zelyak and G. Murthy, *ibid.* **80**, 205310 (2009).
- [20] I. L. Aleiner and V. I. Fal'ko, *Phys. Rev. Lett.* **87**, 256801 (2001).
- [21] B. Nissan-Cohen, Y. Gefen, M. N. Kiselev, and I. V. Lerner, *Phys. Rev. B* **84**, 075307 (2011).
- [22] A. M. Finkel'stein, in *Soviet Scientific Reviews*, edited by I. M. Khalatnikov (Harwood Academic, London, 1990), Vol. 14.
- [23] A. V. Andreev and A. Kamenev, *Phys. Rev. Lett.* **81**, 3199 (1998).
- [24] X. Yang and Ch. Nayak, *Phys. Rev. B* **72**, 052405 (2005).
- [25] A. Saha, Y. Gefen, I. S. Burmistrov, A. Shnirman, and A. Altland, *Ann. Phys. (NY)* **327**, 2543 (2012).
- [26] K. B. Efetov and A. Tschersich, *Phys. Rev. B* **67**, 174205 (2003).
- [27] M. L. Mehta, *Random Matrices* (Boston, Academic, 1991).
- [28] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevB.89.201304> for details on (i) the analysis of the correlation function $V(h_1)V(h_2)$, (ii) the perturbative treatment of the average χ_{zz} , and (iii) the calculation of the probability $\text{Prob}\{\ln Z_S > W\}$.
- [29] T. Bojdecki, L. G. Gorostiza, and A. Talarczyk, *Potential Anal.* **28**, 71 (2008).
- [30] R. J. Adler, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes* (Hayward, California, 1990).
- [31] S. Asmussen and H. Albrecher, *Ruin Probabilities* (World Scientific, Singapore, 2010).
- [32] J. Hüslér and V. Piterbarg, *Stoch. Proc. Appl.* **83**, 257 (1999).
- [33] A. I. Larkin, *Sov. Phys. JETP* **31**, 784 (1970).
- [34] Y. Imry and Sh.-K. Ma, *Phys. Rev. Lett.* **35**, 1399 (1975).
- [35] A. U. Sharafutdinov, D. S. Lyubshin, and I. S. Burmistrov (unpublished).
- [36] L. D. Graham and D. S. Schreiber, *J. Appl. Phys.* **39**, 963 (1968); L. Shen, D. S. Schreiber, and A. J. Arko, *Phys. Rev.* **179**, 512 (1969); P. Gambardella, S. Rusponi, M. Veronese, S. S. Dhesi, C. Grazioli, A. Dallmeyer, I. Cabria, R. Zeller, P. H. Dederichs, K. Kern, C. Carbone, and H. Brune, *Science* **300**, 1130 (2003); A. M. Clogston, B. T. Matthias, M. Peter, H. J. Williams, E. Corenzwit, and R. C. Sherwood, *Phys. Rev.* **125**, 541 (1962); D. Shaltiel, J. H. Wrenick, H. J. Williams, and M. Peter, *ibid.* **135**, A1346 (1964); J. W. Loram and K. A. Mirza, *J. Phys. F: Met. Phys.* **15**, 2213 (1985); J. W. Loram, K. A. Mirza, and Z. Chen, *ibid.* **16**, 233 (1986); G. Mpourmpakis, G. E. Froudakis, A. N. Andriotis, and M. Menon, *Phys. Rev. B* **72**, 104417 (2005); S. Jia, S. L. Bud'ko, G. D. Samolyuk, and P. C. Canfield, *Nat. Phys.* **3**, 334 (2007).