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Effect of interactions on two-dimensional fermionic symmetry-protected topological phases with Z_2 symmetry

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We study the effect of interactions on two-dimensional fermionic symmetry-protected topological (SPT) phases using the recently proposed braiding statistics approach. We focus on a simple class of examples: superconductors with a Z_2 Ising symmetry. Although these systems are classified by $\mathbb{Z}$ in the noninteracting limit, our results suggest that the classification collapses to $\mathbb{Z}_8$ in the presence of interactions—consistent with previous work that analyzed the stability of the edge. Specifically, we show that there are at least eight different types of Ising superconductors that cannot be adiabatically connected to one another, even in the presence of strong interactions. In addition, we prove that each of the seven nontrivial superconductors have protected edge modes.

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Introduction. Recently it has become apparent that generalizations of topological insulators [1–7] known as "symmetry-protected topological (SPT) phases"[8–17] can be realized in large classes of interacting boson and fermion systems. Loosely speaking, SPT phases are characterized by two properties. First, they support robust gapless boundary modes which are protected by certain symmetries. Second, SPT phases can be adiabatically connected to a "trivial state" (i.e., an atomic insulator or product state) if the relevant symmetries are broken. While significant progress has been made in understanding SPT phases in one-dimensional (1D) systems [9,10,14–18], less is known about the higher dimensional case. Several approaches have been developed to understand these higher dimensional systems. One approach, which applies to bosonic SPT phases in general spatial dimension, is the cohomology classification scheme of Refs. [11–13]. Another approach, which applies to bosonic or fermionic two-dimensional (2D) SPT phases with chiral boson edge modes, is to study the edge theories of these systems using the K-matrix formalism [19–21].

In this Rapid Communication, we discuss a third approach which was introduced in Ref. [22] and applies to 2D SPT phases with unitary symmetry groups. The key idea behind this method is to study SPT phases by "gauging" their symmetries—i.e., coupling them to an appropriate gauge field, thereby transforming their global symmetries into gauge symmetries. One can then probe the structure of the original SPT phases by constructing the excitations of the gauged systems and computing their quasiparticle braiding statistics. This approach has several nice features. First, it provides a simple way to distinguish different SPT phases: if two gauged systems have different quasiparticle statistics, then it is clear that the corresponding "ungauged" systems cannot be adiabatically connected without breaking the symmetry. Second, it gives insight into the stability of the edge: As shown in Ref. [22], the quasiparticle braiding statistics of the gauged system can be used to prove the existence of protected edge modes.

While Ref. [22] focused on bosonic SPT phases, here we explore the fermionic case—a problem of particular interest because the classification of interacting fermionic SPT phases is not understood beyond 1D (although an interesting attempt was made in Ref. [23]). We focus on a simple class of examples: 2D superconductors with a Z_2 Ising symmetry. It was previously conjectured [24–26] that while these systems are classified by an integer invariant $\mathbb{Z}$ in the noninteracting limit, the classification collapses to $\mathbb{Z}_8$ when interactions are included. This claim was supported by an analysis of edge instabilities which established an *upper bound* of at most eight phases with protected gapless edge modes. Here we derive a *lower bound* for the classification, using both bulk and edge arguments. First, we show that there are at least eight different types of Ising superconductors that cannot be adiabatically connected to one another, even in the presence of strong interactions. Second, we prove that each of the seven nontrivial superconductors have protected edge modes.

Pseudospin notation. We begin with some notation. Consider a general fermion system with an on-site, unitary Z_2 symmetry S. Without loss of generality, we can assume that the Hamiltonian is built out of fermion operators that have a definite parity under S [27]. We will label the operators that are even under S with a pseudospin index $\uparrow$ and operators that are odd under S with an index $\downarrow$. In this notation, the system is composed of two species of fermions, $c_\uparrow$ and $c_\downarrow$, where

$$Sc_\uparrow S^{-1} = c_\uparrow; \quad Sc_\downarrow S^{-1} = -c_\downarrow. \tag{1}$$

In addition to the above Z_2 symmetry, locality dictates that the system must also conserve fermion parity P_f, defined by

$$P_f c_\uparrow P_f^{-1} = -c_\uparrow; \quad P_f c_\downarrow P_f^{-1} = -c_\downarrow. \tag{2}$$

Putting these two constraints together, we can see that the pseudospin-$\uparrow$ and pseudospin-$\downarrow$ fermions are *separately* conserved modulo 2.

The noninteracting limit. We next review the classification of *noninteracting* fermion SPT phases with Z_2 Ising symmetry. The key observation is that quadratic pseudospin mixing terms, e.g., $c_\uparrow^\dagger c_\downarrow$, are prohibited by the Z_2 symmetry. Therefore the $c_\uparrow$ and $c_\downarrow$ fermions are completely decoupled in the noninteracting limit. Applying the known integer classification of 2D topological superconductors [28,29], it follows that the different free fermion phases are classified by a pair of integers

$\nu=0$	1	$e_\uparrow$	$m_\uparrow$	$e_\downarrow$	$m_\downarrow$	$e_\uparrow e_\downarrow$	$m_\uparrow m_\downarrow$	$\varepsilon_\uparrow\varepsilon_\downarrow$	$e_\uparrow m_\downarrow$	$m_\uparrow e_\downarrow$	$\varepsilon_\uparrow$	$\varepsilon_\downarrow$	$e_\uparrow\varepsilon_\downarrow$	$\varepsilon_\uparrow e_\downarrow$	$m_\uparrow\varepsilon_\downarrow$	$\varepsilon_\uparrow m_\downarrow$
$\nu=8$	1	$e_\uparrow\varepsilon_\downarrow$	$m_\uparrow\varepsilon_\downarrow$	$\varepsilon_\uparrow e_\downarrow$	$\varepsilon_\uparrow m_\downarrow$	$m_\uparrow m_\downarrow$	$e_\uparrow e_\downarrow$	$\varepsilon_\uparrow\varepsilon_\downarrow$	$m_\uparrow e_\downarrow$	$e_\uparrow m_\downarrow$	$\varepsilon_\uparrow$	$\varepsilon_\downarrow$	$e_\uparrow$	$e_\downarrow$	$m_\uparrow$	$m_\downarrow$

$(\nu^\uparrow,\nu^\downarrow)$. Here, $(\nu^\uparrow,\nu^\downarrow) \in \mathbb{Z}^2$ corresponds to a phase where the pseudospin-$\uparrow$ and pseudospin-$\downarrow$ fermions form two decoupled topological superconductors with $\nu^\uparrow$ and $\nu^\downarrow$ chiral Majorana edge modes, respectively. (The sign of $\nu^\uparrow,\nu^\downarrow$ indicates the chirality of the edge mode—left or right moving). In this Rapid Communication, we only consider a *subset* of the above phases—namely, those satisfying $\nu^\uparrow = -\nu^\downarrow$. The reason for this restriction is that our definition for SPT phases requires that they be adiabatically connected to a trivial band insulator if the symmetry is broken, and only phases with $\nu^\uparrow = -\nu^\downarrow$ obey this condition. Hence, according to our definition, the noninteracting SPT phases are classified by a single integer $\nu = \nu^\uparrow = -\nu^\downarrow$.

The effect of interactions. We now investigate how this classification changes when we include interactions. In principle, the addition of interactions can have two effects on the classification of SPT phases: (1) interactions can increase the number of different phases by giving rise to new SPT phases that cannot be realized in noninteracting systems, and (2) interactions can decrease the number of different phases by allowing distinct noninteracting phases to be adiabatically connected to one another [18]. Here we focus on the latter effect. We ask, which of the noninteracting SPT phases remain distinct once we include interactions?

Our strategy for (partially) answering this question is as follows. First, we construct a lattice free fermion Hamiltonian H^ν for each SPT phase. We then couple the pseudospin-$\uparrow$ and pseudospin-$\downarrow$ fermions to two independent Z_2 gauge fields ($Z_2^\uparrow \times Z_2^\downarrow$) and we denote the resulting gauged Hamiltonian by H^ν_{gauge} (see the Supplemental Material for a precise definition of H^ν_{gauge} [30]). Finally, we study the braiding statistics of the Z_2 flux excitations of H^ν_{gauge}. We will show that these braiding statistics depend on ν modulo 8, and hence there must be at least eight distinct SPT phases, even in the presence of interactions.

To begin, it is useful to first think about a simpler system with only one pseudospin component and ν chiral edge modes. The quasiparticle braiding statistics of such a chiral superconductor were worked out by Kitaev in Ref. [31]. That calculation showed that the quasiparticle braiding statistics of the superconductor depends on the number of chiral edge modes ν, modulo 16. For example, if ν is even, the Z_2 gauge fluxes (i.e., superconducting vortices) are Abelian anyons with an exchange phase factor $e^{(\pi/8)i\nu}$. If ν is odd, the Z_2 fluxes are non-Abelian anyons with an exchange phase $(-)^{(\nu^2-1)/8}e^{(\pi/8)i\nu}$ when the two non-Abelian anyons are in the vacuum fusion channel.

Now, let us consider the two-component system H^ν_{gauge}. This system consists of a pseudospin-$\uparrow$ component with ν right-moving edge modes and a pseudospin-$\downarrow$ component with ν left-moving edge modes. Naively, one might guess that the braiding statistics of this system also depends on ν modulo 16, since it is made up of two independent chiral superconductors. However, this guess is incorrect: The braiding statistics of the "doubled" system only depends on ν modulo 8. To see this, we need to show that the braiding statistics for $\nu = 0,1,\ldots,7$ are all different, while the $\nu = 0$ case is equivalent to the $\nu = 8$ case. One way to establish the first statement is to compute the exchange phases of all the different types of $Z_2^\downarrow$ (or $Z_2^\uparrow$) flux excitations. Here, a $Z_2^\downarrow$ flux is defined to be any quasiparticle excitation that acquires a phase of -1 when braided around a pseudospin-$\downarrow$ fermion and acquires no phase when braided around a pseudospin-$\uparrow$ fermion. Using the results of Ref. [31], it is easy to see that for even ν there are four types of $Z_2^\downarrow$ fluxes with exchange statistics $\pm e^{(\pi/8)i\nu}$, while for odd ν there are two types of $Z_2^\downarrow$ fluxes with exchange statistics $\pm e^{-(\pi/8)i\nu}$. (In the latter case, we assume the fluxes are in the vacuum fusion channel). In particular, we see that the exchange statistics of the Z_2 fluxes are different for each of the eight possibilities $\nu = 0,1,\ldots,7$.

On the other hand, to see that $\nu = 0$ and $\nu = 8$ have the same braiding statistics, we need to construct an explicit isomorphism between the quasiparticles in the two systems. To this end, we consult Ref. [31] and note that for both $\nu = 0,8$ the gauge theory has four quasiparticles $1,e_\sigma,m_\sigma,\varepsilon_\sigma$ for each pseudospin direction, $\sigma = \uparrow,\downarrow$. Including all possible composites of pseudospin-$\uparrow$ and pseudospin-$\downarrow$ excitations, there are $4 \times 4 = 16$ quasiparticles all together. We can think of the ε_σ as the constituent fermions, while e_σ and m_σ are different types of Z_2^σ gauge fluxes which differ from one another by the addition of a fermion: $e_\sigma = m_\sigma\varepsilon_\sigma$. Using the results of Ref. [31], we can see that for both $\nu = 0,8$, the three particles $\varepsilon_\sigma,e_\sigma,m_\sigma$ acquire a phase of -1 when braided around each other. The only difference is that e_σ and m_σ are *bosons* for the case $\nu = 0$, while they are *fermions* for the case $\nu = 8$. With these properties in mind, one can easily see that the following map gives an isomorphism between the quasiparticles in the two systems:

Here, the table is organized so that the first ten quasiparticles are all bosons, while the other six are all fermions. We can see that the correspondence not only preserves braiding statistics and fusion rules, but also preserves the $Z_2^\uparrow \times Z_2^\downarrow$ gauge structure, mapping the $\downarrow$ fermions ($\varepsilon_\downarrow$) of one system onto the corresponding fermions in the other system, and likewise mapping the $Z_2^\downarrow$ fluxes ($e_\downarrow,m_\downarrow,\varepsilon_\uparrow e_\downarrow,\varepsilon_\uparrow m_\downarrow$) of one system onto the $Z_2^\downarrow$ fluxes of the other system (and similarly for $\uparrow$).

Two conclusions follow from the above analysis. First, we conclude that the Hamiltonians H^ν with $\nu = 0,1,\ldots,7$ cannot be adiabatically connected to one another in the presence of interactions, without breaking the Z_2 symmetry. Indeed, if there existed a gapped, Z_2 symmetric path connecting the different H^ν, then there would have to be a corresponding gapped path connecting the gauged systems H^ν_{gauge}, since we define our gauging procedure in such a way that the gauged and ungauged Hamiltonians, H_{gauge} and H, have *identical* energy spectra below the energy gap of the Z_2 flux excitations (see the definition of H_{gauge} in the Supplemental Material). But a gapped path connecting the gauged Hamiltonians is not possible since we have seen that they have different quasiparticle braiding statistics; hence a symmetry-preserving path connecting the ungauged Hamiltonians is also impossible. The second conclusion is that it is at least *plausible* that H^0 and H^8 can be adiabatically connected to one another in the presence of interactions, since the corresponding $Z_2^\uparrow \times Z_2^\downarrow$ gauge theories H^0_{gauge} and H^8_{gauge} share the same statistics and gauge structure.

The instability of $\nu = 8$ edge. In this section, we give additional evidence that the $\nu = 8$ system and $\nu = 0$ system belong to the same interacting phase: We show that the $\nu = 8$ edge can be gapped out by appropriate interactions, without breaking the Z_2 symmetry (explicitly or spontaneously). We note that a similar result was obtained previously in Refs. [24–26].

Our approach is based on bosonization. We note that the edge of the $\nu = 8$ free fermion system contains eight pseudospin-$\uparrow$ Majorana modes and eight pseudospin-$\downarrow$ Majorana modes moving in opposite directions. Pairing up the Majorana modes to form complex fermions, we can equivalently describe the edge using four pseudospin-$\uparrow$ and four pseudospin-$\downarrow$ complex fermions. We then bosonize these fermions, using four boson modes $\Phi_1,\ldots,\Phi_4$ for the pseudospin-$\uparrow$ fermions, and four boson modes $\Phi_5,\ldots,\Phi_8$ for the pseudospin-$\downarrow$ fermions. The edge is then described by the chiral boson Lagrangian

$$\mathcal{L}_{\text{edge}} = \frac{1}{4\pi}(K_{IJ}\partial_x\phi_I\partial_t\phi_J - V_{IJ}\partial_x\Phi_I\partial_x\Phi_J), \tag{3}$$

where $K = \text{diag}(1,1,1,1,-1,-1,-1,-1)$, and V_{IJ} is the velocity matrix. Here we use a normalization convention where the fermion creation operators are of the form $e^{i\Phi_k}$, $k = 1,\ldots,8$. In this language, the symmetry transformation is given by $S^{-1}\Phi S = \Phi + \pi K^{-1}\chi$, where $\chi^T = (0,0,0,0,1,1,1,1)$.

We now construct interaction terms that gap out the edge without breaking the Z_2 symmetry (either explicitly or spontaneously). We consider backscattering terms of the form $U(\Lambda) = U(x)\cos[\Lambda^T K \Phi - \alpha(x)]$. In order for $U(\Lambda)$ to be invariant under S, we require that

$$\Lambda^T\chi \equiv 0 \pmod 2. \tag{4}$$

In order to gap out the edge, we need to add four backscattering terms $\sum_i U(\Lambda_i)$: Each term can gap out a pair of counterpropagating edge modes. Such terms can gap out the edge as long as the $\{\Lambda_i\}$ vectors satisfy [32]

$$\Lambda_i^T K \Lambda_j = 0 \tag{5}$$

for all i,j. This "null-vector" condition guarantees that we can make a suitable change of variables mapping L_{edge} onto a system of four decoupled Luttinger liquids with four backscattering terms. It is then easy to see that the backscattering terms will gap out the corresponding Luttinger liquids (at least for large U [33]).

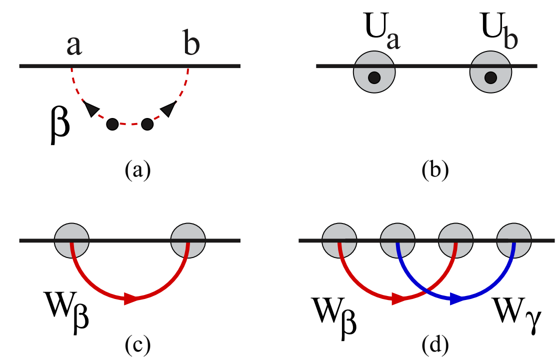

FIG. 1. (Color online) (a) We consider a thought experiment in which we create two $Z_2^\downarrow$ fluxes in the bulk and then move them along a path β to points a,b at the edge. (b) We argue that the two fluxes can be annihilated at the boundary by applying local operators U_a,U_b. (c) We define $\mathbb{W}_\beta$ to be an operator which describes a process in which the fluxes are created in the bulk, brought to the edge, and then annihilated. (d) To obtain a contradiction, we consider two paths β,γ that intersect one another, and we investigate the commutation algebra of the corresponding operators $\mathbb{W}_\beta,\mathbb{W}_\gamma$.

We now claim that the following $\{\Lambda_i\}$ will do the job:

$$\begin{aligned} \Lambda_1^T &= (1,-1,0,0,1,-1,0,0);\\ \Lambda_2^T &= (1,0,-1,0,1,0,-1,0);\\ \Lambda_3^T &= (1,0,0,-1,1,0,0,-1);\\ \Lambda_4^T &= (1,0,1,0,0,-1,0,-1). \end{aligned} \tag{6}$$

Indeed, it is easy to check that these $\{\Lambda_i\}$ obey the null-vector criterion (5), as well as the symmetry condition (4). To complete the argument, we need to check that the perturbation corresponding to $\{\Lambda_i\}$ does not spontaneously break the Z_2 symmetry. However, as explained in Ref. [19], we can rule out the possibility of spontaneous symmetry breaking if the $\binom{8}{4}$ 4×4 minors of the 8×4 matrix with columns $\Lambda_1,\ldots,\Lambda_4$ have no common factor. This property of $\Lambda_1,\ldots,\Lambda_4$ can be verified by direct calculation.

Protected edge states for $\nu \neq 0$ mod 8. On the other hand, we now show that the edge of H^ν is *protected* if $\nu \neq 0$ mod 8. To state our result more precisely, let us consider a disk geometry and a Hamiltonian of the form $H = H_{\text{bulk}} + H_{\text{edge}}$, where $H_{\text{bulk}} = H^\nu$, and H_{edge} is an arbitrary interacting Hamiltonian acting on fermions near the edge. In this setup, what we will show is that the ground state $|0\rangle$ cannot be both Z_2 symmetric and "short-range entangled" [34]. We believe that this result rules out the possibility of a Z_2 symmetric, gapped edge, and in this sense proves that the gapless edge excitations are protected.

As in Ref. [22], our argument is a proof by contradiction: We assume that $|0\rangle$ is short-range entangled and Z_2 symmetric and we show that these assumptions lead to a contradiction. The first step is to couple the pseudospin-$\uparrow$ and pseudospin-$\downarrow$ fermions to two independent Z_2 gauge fields. We then imagine creating a pair of $Z_2^\downarrow$ (or $Z_2^\uparrow$) fluxes in the bulk. After creating the $Z_2^\downarrow$ fluxes, we separate them and move them along some path β to points a,b at the boundary [Fig. 1(a)]. Formally, this process can be implemented by applying a unitary (stringlike) operator W_β to $|0\rangle$.

Next, we claim that the $Z_2^\downarrow$ fluxes can be annihilated at the boundary if we apply appropriate local operators. That is, there exist local operators U_a, U_b, acting near points a,b such that $U_a U_b W_\beta|0\rangle = |0\rangle$ [Fig. 1(b)]. Establishing this claim is the hardest step in the argument, and here we merely outline its proof [35]. The basic point is that when we bring the $Z_2^\downarrow$ fluxes to the boundary, we effectively create two $Z_2^\downarrow$ domain walls at a and b. Given that the ground state is $Z_2^\downarrow$ symmetric, these domain walls are *local* excitations: They only affect expectation values in the neighborhood of a and b. It then follows that these domain walls can be annihilated by local operators since local excitations of a short-range entangled state can always be annihilated locally.

In the third step, we consider a creation and annihilation process in which two $Z_2^\downarrow$ fluxes are created in the bulk, moved to the boundary, and then annihilated. Let $\mathbb{W}_\beta$ be a unitary operator describing this process [Fig. 1(c)]. (Formally, $\mathbb{W}_\beta = U_a U_b W_\beta$.) Now, consider a second path γ with the geometry shown in Fig. 1(d) and define $\mathbb{W}_\gamma$ in the same way. By construction, we have $\mathbb{W}_\beta|0\rangle = \mathbb{W}_\gamma|0\rangle = |0\rangle$. Hence

$$\mathbb{W}_\beta \mathbb{W}_\gamma|0\rangle = \mathbb{W}_\gamma \mathbb{W}_\beta|0\rangle = |0\rangle. \tag{7}$$

In the final step, we show that (7) leads to a contradiction if $\nu \neq 0 \pmod 8$. It is useful to consider separately the case where ν is even and ν is odd. First, suppose ν is even. In this case, the $Z_2^\downarrow$ fluxes are Abelian anyons and it follows from a general analysis of Abelian quasiparticle statistics (see, e.g., Refs. [22,36]) that

$$\mathbb{W}_\beta \mathbb{W}_\gamma|0\rangle = e^{2i\theta}\mathbb{W}_\gamma \mathbb{W}_\beta|0\rangle, \tag{8}$$

where $e^{i\theta}$ is the exchange phase of the $Z_2^\downarrow$ fluxes. According to the braiding statistics calculation outlined above, the four types of $Z_2^\downarrow$ fluxes have exchange statistics $\theta = \pm\frac{\pi\nu}{8}$. Hence if $\nu \neq 0 \pmod 8$, then $e^{2i\theta} \neq 1$ for any of the four types of fluxes, and Eqs. (7) and (8) are in contradiction.

Now suppose ν is odd. In this case, the $Z_2^\downarrow$ fluxes are non-Abelian anyons, so the above braiding statistics analysis is more complicated. However, we can avoid these complications using an alternative argument. We note that if ν is odd, then each $Z_2^\downarrow$ flux carries an *unpaired Majorana mode*. Thus, the state $W_\beta|0\rangle$ has unpaired Majorana modes localized near points a and b. But then it is clearly impossible for $U_a U_b W_\beta|0\rangle = |0\rangle$ since unpaired Majorana modes cannot be destroyed by any local operation. Once again, we encounter a contradiction, implying that our assumption is false and $|0\rangle$ cannot be both Z_2 symmetric and short-range entangled.

Conclusions and discussions. In this Rapid Communication we have studied SPT phases in interacting fermion systems using a braiding statistics approach. As a simple example, we considered superconductors with a Z_2 (Ising) symmetry. Although in the noninteracting case these Ising superconductors are classified by an integer invariant $\nu \in \mathbb{Z}$, we give evidence that the classification collapses to $\mathbb{Z}_8$ in the presence of interactions. We also give a general argument proving that the edge excitations are protected when $\nu \neq 0 \pmod 8$ and unprotected when $\nu = 0 \pmod 8$.

An interesting question is whether interactions allow for *additional* SPT phases with Z_2 symmetry that cannot be realized in free fermion systems. The method developed in this Rapid Communication may be useful in addressing this issue, as well as the more general problem of classifying fermionic SPT phases with unitary symmetries. Such a classification is especially desirable, as the group cohomology classification scheme [12,13] cannot be directly applied to fermionic systems.

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[34] Here, a fermionic state is said to be "short-range entangled" if it can be transformed into an atomic insulator by a local unitary transformation—a unitary operator generated from the time evolution of a local Hamiltonian over a finite time t.

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