Magnetic penetration depth and vortex structure in anharmonic superconducting junctions with an interfacial pair breaking

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The penetration depth l_j in superconducting junctions is identified within the Ginzburg-Landau theory as a function of the interfacial pair breaking, of the magnetic field, and of the Josephson coupling strength. When the interfacial pair breaking goes up, l_j increases and an applicability of the local Josephson electrodynamics to junctions with a strong Josephson coupling is extended. In the junctions with strongly anharmonic current-phase relations, the magnetic field dependence of l_j is shown to lead to a significant difference between the weak-field penetration depth and the characteristic size of the Josephson vortex. For such junctions a nonmonotonic dependence of l_j and of the lower critical field on the Josephson coupling constant is found, and the specific features of spatial profiles of the supercurrent and the magnetic field in the Josephson vortex are established.

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I. INTRODUCTION

The magnetic self-effects of the Josephson current in wide superconductor-thin interlayer-superconductor junctions result in the screening inside the junction interlayer of the magnetic field applied at the junction edge, and in the formation of Josephson vortices. Spatial variations of the magnetic field and of the current density along the junction interface are coupled with variations of the phase difference and should be determined jointly.

The corresponding results for standard tunnel junctions with a sufficiently weak Josephson coupling have been obtained within the Ginzburg-Landau (GL) theory since the early days of studying the Josephson effect [1–4]. The results imply that the Josephson penetration depth λ_J significantly exceeds the London penetration depth λ_L, as is commonly observed. The Meissner screening being significantly stronger than the Josephson one is consistent with the definition of "weak superconductivity," although it is not generally an integral feature of weak links.

The conventional definition of weak links and, in particular of tunnel junctions, requires the critical current j_c to be significantly less than the depairing current j_{dp} deep within the superconducting leads. On the other hand, the condition $\lambda_J \gg \lambda_L$ is equivalent to $j_c^{1/2} \ll (\frac{j_{dp}}{\kappa})^{1/2}$, where κ is the GL parameter. For junctions involving strongly type-II superconductors, the relation presented can prove to be more restrictive for the critical current than the weak-link requirement $j_c \ll j_{dp}$. Hence, the standard results only apply to tunnel junctions satisfying the condition $j_c^{1/2} \ll (\frac{j_{dp}}{\kappa})^{1/2}$. In the opposite case $j_c^{1/2} \gtrsim (\frac{j_{dp}}{\kappa})^{1/2}$, $j_c \ll j_{dp}$, which ensures the relation $\lambda_J \lesssim \lambda_L$ for weak links, the electrodynamics of tunnel junctions acquires a nonlocal character [5,6]. In a strongly nonlocal regime $\lambda_J \ll \lambda_L$, the characteristic scale of an isolated Josephson vortex along the junction plane is $\frac{\lambda_J^2}{\lambda_L}$, which is substantially less than the Josephson penetration depth λ_J [5]. Recently the nonlocality has been experimentally identified in planar junctions with thin superconducting electrodes [7], where the conditions for observing the nonlocal effects [8–13] are modified, and monitored more easily as compared with the junctions with thick leads.

A distinctive feature of superconducting junctions considered in this paper is the presence of an interfacial pair breaking. An intense interfacial pair breaking can take place, for example, in junctions involving unconventional superconductors and/or magnetic or normal metal interlayers. Since in a small transition region weak links are quite sensitive to local conditions, an interface-induced local weakening of the superconducting condensate density can have a profound influence on the whole of the Josephson effect. As the result, the interplay of the Josephson coupling strength and interfacial pair activity controls the behavior of the supercurrent.

A weak Josephson coupling leads to the sinusoidal (harmonic) current-phase relation, whereas a strongly anharmonic supercurrent emerges at the large values of the coupling constant. In planar junctions with a strong Josephson coupling and vanishing interfacial pair activity the critical current j_c becomes comparable with the depairing current j_{dp}, and the junctions do not represent weak links [14]. Conversely, the critical current of the junctions with an intense interfacial pair breaking is strongly suppressed, as compared to the case of no pair breaking, and can only be substantially less than j_{dp}, irrespective of the Josephson coupling strength [15]. Thus the interfacial pair breaking maintains the planar junctions with a pronounced Josephson coupling as weak links $j_c \ll j_{dp}$ with strongly anharmonic current-phase relations.

This paper addresses effects of the interfacial pair breaking and of the Josephson coupling strength on the magnetic penetration depth l_j and the Josephson vortex structure in wide planar junctions involving strongly type-II superconductors. For a fixed Josephson coupling, the quantity l_j is shown to go up with the interfacial pair breaking. This substantially extends an applicability domain of the condition $l_j \gg \lambda_L$ and, hence, of the local Josephson electrodynamics to the junctions with a strong Josephson coupling in the presence of an intense interfacial pair breaking.

The magnetic field dependence of the penetration depth is studied below both for harmonic and anharmonic superconducting junctions. In the junctions with the harmonic supercurrent described by local Josephson electrodynamics, the Josephson penetration depth λ_J is the only characteristic scale of the problem. Along with the critical current, it depends substantially on the strength of the interfacial pair breaking.

Under a weak applied field, λ_J exactly coincides with the penetration depth, while the latter is shown to depend on the magnetic flux Φ through the junction and to approach the value $l_{jv} = l_j(\frac{\Phi_0}{2}) = \frac{\pi}{2}\lambda_J$ at half of the flux quantum.

While in harmonic junctions a characteristic size of the Josephson vortex l_{jv} (a half of its effective width) is of the same order as λ_J, in junctions with strongly anharmonic current-phase relations the magnetic field dependence $l_j(\Phi)$ is demonstrated to become pronounced and to result in a significant difference between l_{jv} and a weak-field penetration depth l_{j0}. As a specific feature of the strongly anharmonic current-phase relation, a nonmonotonic dependence of the Josephson vortex size l_{jv} and of the lower critical field on the Josephson coupling strength, for a fixed and intense interfacial pair breaking, is identified within the local Josephson electrodynamics. Finally, the spatial structure of an isolated Josephson vortex in the junctions with an intense interfacial pair breaking is studied. In particular, narrow peaks in the current-phase relation of strongly anharmonic junctions are shown to transform into narrow peaks in a spatial profile of the supercurrent density in the vortex.

The paper is organized as follows. The magnetic field dependence of the penetration depth in harmonic junctions is described in Sec. II. In Sec. III the penetration depth in anharmonic junctions is obtained as a function of the magnetic field, of the Josephson coupling constant, and of the strength of the interfacial pair breaking. Section IV addresses spatial profiles of the phase difference, of the magnetic field, and of the supercurrent density in an isolated Josephson vortex in anharmonic junctions. The lower critical field in such junctions is found in Sec. V. Section VI contains discussions and Sec. VII concludes the paper.

II. l_j IN HARMONIC JUNCTIONS

Let the static magnetic field $\boldsymbol{H} = H\boldsymbol{e}_z$ be applied along the z axis to a symmetric planar junction involving thick leads made of strongly type-II superconductors (see Fig. 1). A homogeneous plane rectangular interlayer at $x = 0$ is supposed to be of zero length within the GL approach. The spatially constant widths L_y, L_z of the junction are considered to significantly exceed the penetration depths: $L_y, L_z \gg l_j, \lambda_L$. Under such conditions the magnetic field is independent of the z coordinate inside the interlayer and in the superconductors.

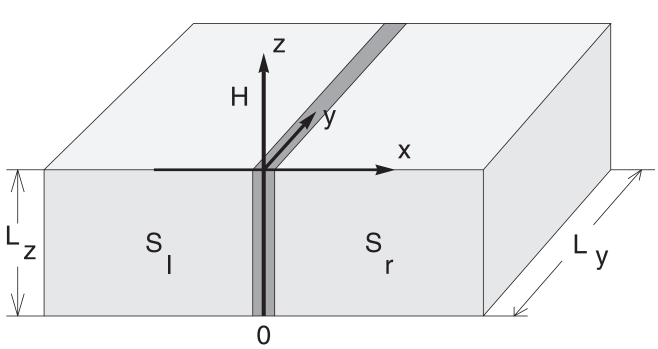

FIG. 1. Schematic diagram of the junction.

The applied field is assumed to be substantially less than the critical fields of the leads, and to produce a negligibly small influence on the Josephson current as a function of the phase difference $j(\chi)$. At the same time the self-field effects, generated by the current flowing through wide junctions, interconnect the magnetic field $H[\chi(y)]$, the supercurrent density $j[\chi(y)]$, and the spatially dependent phase difference $\chi(y)$, and can have a profound influence on their spatial distributions.

Within the local Josephson electrodynamics, which presupposes the condition $l_j \gg \lambda_L$, the spatially dependent static phase difference $\chi(y)$ in the junctions with a harmonic current-phase relation $j(\chi) = j_c \sin\chi$ satisfies a well-known one-dimensional sine-Gordon equation [1–4]

$$\frac{d^2\chi(y)}{dy^2} = \frac{1}{\lambda_J^2}\sin\chi(y). \tag{1}$$

Here λ_J is the Josephson penetration depth $\lambda_J = (c\Phi_0/16\pi^2\lambda_L j_c)^{1/2}$ and $\Phi_0 = \pi\hbar c/|e|$ is the superconductor flux quantum.

The self-consistent results of the GL theory for the Josephson current $j(\chi)$ in planar junctions [15–17] is being used below. The order parameters in the two superconducting leads is written as $f_{1(2)}(x)e^{i\chi_{1(2)}(x)}$, where the moduli $f_{1(2)}(x)$ are normalized to their values in the bulk in the absence of the supercurrent. In symmetric junctions $f = f(|x|)$, i.e., $f_2(x) = f_1(-x)$, and the boundary conditions for f are

$$\left(\frac{df}{d\overline{x}}\right)_{\pm} = \pm\left(g_\delta + 2g_\ell\sin^2\frac{\chi}{2}\right)f_0, \tag{2}$$

where f_0 is an interface value of $f(x)$, $\overline{x} = x/\xi(T)$, $\xi(T)$ is the temperature dependent superconductor coherence length, and χ is the phase difference $\chi = \chi_- - \chi_+$.

The coefficient g_ℓ in (2) is the effective dimensionless Josephson coupling constant, and g_δ is the effective dimensionless interface parameter. The parameters g_δ and g_ℓ are the main characteristics of the interface in the GL theory. They are assumed to be positive and, therefore, resulting in an interfacial pair breaking in accordance with (2). In the absence of the current, i.e., at $\chi = 0$, the suppression of the order parameter at the interface is described solely by g_δ. When the supercurrent flows, the Josephson coupling contributes to the phase dependent suppression of the order parameter at the interface [17].

In macroscopic samples of strongly type-II superconductors, the influence of the interfacial pair breaking on the Meissner effect is small, in the measure of $\kappa^{-1} \ll 1$, and will be disregarded below. Thus the local penetration depth of the Meissner effect is considered to be spatially constant, irrespective of the boundary conditions for the order parameter, and equal to λ_L which is related to the bulk condensate density. Contrary to its negligible influence on the Meissner effect, the interfacial pair breaking can have a considerable impact both on the critical current and, in the presence of a pronounced Josephson coupling, on the current-phase relation. For this reason the standard expression and estimates for $j(\chi)$, which do not take into account effects of the interfacial pair breaking and of the Josephson coupling strength, can fail.

The harmonic current-phase relation $j(\chi) = j_c \sin\chi$ takes place under the condition $g_\ell \ll \max(1, g_\delta)$, which incorporates not only tunnel junctions, defined as $g_\ell \ll 1$, but also the junctions with a strong Josephson coupling $1 \lesssim g_\ell \ll g_\delta$ in the presence of an intense interfacial pair breaking [16,17]. The junctions satisfying the generalized condition $g_\ell \ll \max(1, g_\delta)$ will be called harmonic junctions. With the corresponding expression for the critical current of harmonic junctions (see (S10) in [17]) and with those for λ_L and ξ, the quantity λ_J in (1) can be written as

$$\lambda_J = \left(\frac{c\Phi_0}{16\pi^2\lambda_L j_c}\right)^{1/2} = \left(\frac{\lambda_L\xi}{g_\ell}\right)^{1/2} \frac{1}{\sqrt{2 + g_\delta^2} - g_\delta}. \quad (3)$$

Hence, the characteristic length scale λ_J substantially depends on the strength of the interfacial pair breaking g_δ.

In standard tunnel junctions with $g_\delta \ll 1$ one gets $\lambda_J \gg \lambda_L$, since the parameter g_ℓ is proportional to the junction transparency and in this case extremely small. The characteristic length (3) decreases $\propto g_\ell^{-1/2}$ with increasing the effective Josephson coupling constant and becomes comparable with λ_L at the characteristic value $g_\ell^{1/2} \sim \kappa^{-1/2} \ll 1$, which can be still small in the strongly type-II superconductors. At the same time, λ_J increases with the interfacial pair breaking. In junctions with an intense pair breaking $g_\delta \gg 1$ the limiting relation $\lambda_J \approx (\frac{\lambda_L\xi}{g_\ell})^{1/2} g_\delta$ follows from (3). One sees that λ_J considerably exceeds λ_L under the condition $g_\ell^{1/2} \ll g_\delta\kappa^{-1/2}$, which allows the strong coupling constant $g_\ell \gtrsim 1$, provided $g_\delta \gg \kappa^{1/2}$. Thus, in the presence of an intense interfacial pair breaking the local electrodynamics can be applied to describing the harmonic junctions with a large Josephson coupling.

If a strongly nonlocal regime $\lambda_J \ll \lambda_L$ takes place, one can combine the results of Ref. [5] with Eq. (3), where the effects of the interfacial pair breaking are taken into account. This leads to the following characteristic scale of an isolated Josephson vortex:

$$\frac{\lambda_J^2}{\lambda_L} = \frac{\xi}{g_\ell\left(\sqrt{g_\delta^2 + 2} - g_\delta\right)^2}. \quad (4)$$

Further on the condition $\lambda_J \gg \lambda_L$ will be assumed, which ensures an applicability of the local theory. For the quantitative analysis, let us consider the junction of Ferrell and Prange [1,4], i.e., a wide junction occupying the half-space $y > 0$, $L_y \to \infty$ under the magnetic field applied at the junction edge $y = 0$. The magnetic field is assumed to be fully screened far inside the junction plane ($y \to \infty$), where the supercurrent density also vanishes. In describing the screening effects, the magnetic flux Φ through the junction will be considered not exceeding half of the flux quantum $|\Phi| \leqslant \frac{\Phi_0}{2}$. The magnetic field H_0 at the junction edge at $\Phi = \frac{\Phi_0}{2}$ is known to be the highest field, for which a solution with no vortex precursors is possible, and, therefore, the magnetic field as well as the current density decay monotonically with increasing the distance y from the interlayer edge. The screening of such an external field is only metastable, since it exceeds the lower critical field [2,18]. At the same time, the spatial distributions of the quantities $H(y)$, $j(y)$, and $\chi(y)$, controlled by the screening effect at $\Phi = \frac{\Phi_0}{2}$, coincide with their spatial profiles in the half of an isolated Josephson vortex involving single flux quantum Φ_0. Hence, when Φ is equal to half of a flux quantum, the penetration depth $l_j(\frac{\Phi_0}{2})$ represents a characteristic size l_{jv} of the vortex, a half of its effective width along the y axis. One also notes that the magnetic field in the center of the vortex, produced by the vortex Josephson current, coincides with the magnetic field H_0 at the junction edge at $\Phi = \frac{\Phi_0}{2}$.

As a weak applied field $\Phi \ll \Phi_0$ induces only a small supercurrent in the junction ($|\sin\chi| \ll 1$), one can consider small phase differences and linearize the sine function in Eq. (1). This results in a simple exponentially decaying solution of (1): $\chi = \chi_0 \exp(-y/\lambda_J)$, $H(y) = -[\Phi_0\chi_0/(4\pi\lambda_L\lambda_J)]\exp(-y/\lambda_J)$. The latter expression signifies that the quantity (3) coincides with the weak-field penetration depth exactly: $l_{j0} = \lambda_J$ [4,18]. With the increasing magnetic flux through the junction, the linearized description fails and one should use the solution of Eq. (1) found in Ref. [1]. For the magnetic field at $x = 0$ inside the junction $y > 0$, with a maximum at the junction edge $y = 0$, one has $H(y) = \mp\Phi_0/\{2\pi\lambda_L\lambda_J\cosh[(y + y_0)/\lambda_J]\}$, $y_0 \geqslant 0$, and the phase difference is $\chi(y) = \pm 2\arcsin \operatorname{sech}[(y + y_0)/\lambda_J]$.

Since the spatial profile $H(y)$ of the magnetic field in the junction interlayer ($x = 0$) can substantially differ from the exponential one, the equality

$$\int_0^{+\infty} H(y)dy = l_j H(0) \quad (5)$$

will be put to use for a quantitative description of the junction penetration depth l_j. Equation (5) is in agreement with the standard definition of magnetic penetration depths in various other circumstances [18,19]. Here $H(0)$ is the magnetic field at the junction edge $y = 0$, and (5) defines a characteristic size of an adjacent region, where the magnetic field as well as the dc supercurrent are confined within the junction.

Substituting the solution for $H(y)$ in (5) and taking the integral, one gets l_j as a function of y_0. Since y_0 and Φ are implicitly related to each other in accordance with the condition $\Phi = 2\lambda_L l_j H(0)$, one obtains eventually the dependence of the Josephson penetration depth on the magnetic flux through the junction

$$l_j^{-1}(\Phi) = \lambda_J^{-1}\frac{\sin(\pi\Phi/\Phi_0)}{\pi\Phi/\Phi_0}, \quad |\Phi| \leqslant \frac{1}{2}\Phi_0, \quad (6)$$

and the relation

$$H(0) = \frac{\Phi_0}{2\pi\lambda_J\lambda_L}\sin\left(\frac{\pi\Phi}{\Phi_0}\right). \quad (7)$$

Thus, $l_j(\Phi)$ goes up with the increase of the magnetic flux within the given limits. While $l_j(\Phi) \approx l_{j0} = \lambda_J$ for $\pi|\Phi| \ll \Phi_0$, one gets $l_j(\frac{\Phi_0}{2}) \equiv l_{jv} = \frac{\pi}{2}\lambda_J$ when half of the flux quantum pierces the junction. Here both the weak-field penetration depth l_{j0} and the characteristic size of the Josephson vortex l_{jv} are associated with one and the same length scale λ_J. The difference between them, though quantitatively noticeable, is not significant.

III. l_{j} IN JUNCTIONS WITH ANHARMONIC CURRENT-PHASE RELATIONS

In anharmonic junctions the equation for a spatially dependent phase difference takes the form [cf. (1)]

$$\frac{d^2\chi[(y)]}{dy^2} - \frac{16\pi^2\lambda_{\mathrm{L}}}{c\Phi_0} j[\chi(y)] = 0, \tag{8}$$

and the defining relation for the penetration depth $l_{\mathrm{j}}(\Phi)$ as a function of the magnetic flux is [17]

$$l_{\mathrm{j}}^{-1}(\Phi) = \left[\frac{8\lambda_{\mathrm{L}}\Phi_0}{c\Phi^2}\int_0^{\frac{2\pi\Phi}{\Phi_0}} j(\chi)d\chi\right]^{1/2}. \tag{9}$$

Here $j(\chi)$, the Josephson current density in the absence of the magnetic field, is assumed to be an odd function of χ.

Equation (9) describes the junction penetration depth and its magnetic flux dependence, assuming the current-phase relation of the junction to be known. Therefore, making use of the results of the GL theory for the anharmonic phase dependence $j(\chi)$, allows one to obtain from (9) the quantity $l_{\mathrm{j}}(\Phi)$. Substituting $j = j_{\mathrm{c}}\sin\chi$ in (9), one easily reproduces Eq. (6) for harmonic junctions.

The results for harmonic junctions remain applicable to the anharmonic case under sufficiently weak applied magnetic fields, when a spatially dependent current density is small enough throughout the junction plane allowing the linearization of the current-phase relation: $j \approx j_0'\chi,\ \ j_0' = (\frac{dj(\chi)}{d\chi})_{\chi=0}$. Then the integration of the current density in (9) results in the penetration depth $l_{\mathrm{j}0} = (\frac{c\Phi_0}{16\pi^2\lambda_{\mathrm{L}} j_0'})^{1/2}$. Though for anharmonic junctions j_0', in general, is not the critical current, it is so for the harmonic ones. With this proviso, the weak-field penetration depth $l_{\mathrm{j}0}$ coincides with λ_{J}.

An analytical expression for l_{j} can be obtained for arbitrary values of the Josephson coupling g_ℓ in the regime of a pronounced interfacial pair breaking $g_\delta^2 \gg 1$, which guarantees that the planar junctions are the weak links. Using the corresponding current-phase relation of the GL theory, one gets from (9) [17]

$$l_{\mathrm{j}}(\Phi, g_\ell, g_\delta) = \frac{2(\pi|\Phi|/\Phi_0)\sqrt{(g_\delta + g_\ell)\lambda_{\mathrm{L}}\xi}}{\ln^{1/2}\left[1 + \frac{4g_\ell(g_\delta+g_\ell)}{g_\delta^2}\sin^2\frac{\pi\Phi}{\Phi_0}\right]}. \tag{10}$$

Under the condition $4g_\ell(g_\delta + g_\ell)\sin^2\frac{\pi\Phi}{\Phi_0} \ll g_\delta^2$, which is satisfied in the weak-field and/or in the tunneling limits, Eq. (10) reduces to (6) with λ_{J} defined in (3) and taken at $g_\delta^2 \gg 1$.

The junction penetration depth (10), as a function of the magnetic flux, monotonically increases from $l_{\mathrm{j}0} = g_\delta(\lambda_{\mathrm{L}}\xi/g_\ell)^{1/2}$ in the weak-field limit to

$$l_{\mathrm{jv}}(g_\ell, g_\delta) = \frac{\pi\sqrt{(g_\delta + g_\ell)\lambda_{\mathrm{L}}\xi}}{\ln^{1/2}\left[1 + \frac{4g_\ell(g_\delta+g_\ell)}{g_\delta^2}\right]}, \tag{11}$$

when half of the flux quantum pierces the junction.

In strongly anharmonic junctions ($g_\ell \gg g_\delta$) the relation $l_{\mathrm{jv}} \gg l_{\mathrm{j}0}$ takes place, which signifies a pronounced magnetic field dependence $l_{\mathrm{j}}(\Phi)$. The corresponding quantitative condition follows from (11):

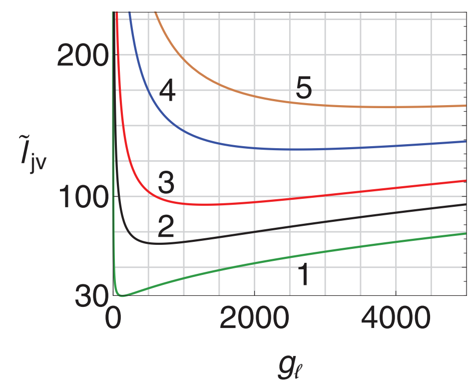

FIG. 2. (Color online) The dimensionless characteristic size of the Josephson vortex $\tilde{l}_{\mathrm{jv}}$ as a function of g_ℓ taken for various $g_\delta \gg 1$: (1) $g_\delta = 100$, (2) $g_\delta = 500$, (3) $g_\delta = 1000$, (4) $g_\delta = 2000$, and (5) $g_\delta = 3000$.

$$l_{\mathrm{jv}} \approx \frac{g_\ell}{g_\delta}\frac{\pi}{\sqrt{2}\ln^{1/2}\frac{2g_\ell}{g_\delta}} l_{\mathrm{j}0} \gg l_{\mathrm{j}0}. \tag{12}$$

A significant difference between the characteristic size of the Josephson vortex $l_{\mathrm{j}}(\frac{\Phi_0}{2}) = l_{\mathrm{jv}}$ and the weak-field penetration depth $l_{\mathrm{j}0}$ is in striking contrast with the harmonic junctions, where $l_{\mathrm{jv}} = \frac{\pi}{2}l_{\mathrm{j}0} = \frac{\pi}{2}\lambda_{\mathrm{J}}$.

A substantial increase of l_{jv} as compared to $l_{\mathrm{j}0}$ is associated with the behavior of the quantity $\int_0^\pi j(\chi)d\chi$, which enters the right-hand side of (9) at $\Phi = \frac{\Phi_0}{2}$. For the harmonic current $\int_0^\pi j(\chi)d\chi = 2j_{\mathrm{c}}$ that leads to $l_{\mathrm{jv}} = \frac{\pi}{2}l_{\mathrm{j}0}$. In the strongly anharmonic regime, when $g_\ell \gg g_\delta$ and $g_\delta^2 \gg 1$, the Josephson current $j(\chi)$ is small outside a pronounced narrow peak of the width $\Gamma \sim \frac{g_\delta}{g_\ell} \ll 1$ in a vicinity of $\chi = \chi_c$ (see (S11) and (S12) in [17] and also Fig. 2 in Ref. [15]). The critical current j_{c} is determined by the height of the peak: $j_{\mathrm{c}} = j(\chi_c)$. Therefore, a qualitative estimate is $\int_0^\pi j(\chi)d\chi \sim \Gamma j_{\mathrm{c}} \ll j_{\mathrm{c}}$. In accordance with Eq. (9), this results in an increase of l_{jv}. Such an unconventional behavior of $j(\chi)$ originates from the phase-dependent proximity effect near the interface, which takes place when $g_\ell \gg g_\delta$ and $g_\delta^2 \gg 1$ [17].

To ensure the applicability of the result (12), the conditions $g_\ell \gg g_\delta,\ \ g_\delta^2 \gg 1$, allowing strongly anharmonic effects to manifest themselves in the junctions, have to be restricted further as the consequence of applying the local electrodynamics. This leads to the relation $l_{\mathrm{j}}(\Phi) \gg \lambda_{\mathrm{L}}$, which is sensitive to the magnetic flux. In weak fields one gets $l_{\mathrm{j}0} \gg \lambda_{\mathrm{L}}$ and ultimately $\kappa^{1/2}g_\ell^{1/2} \ll g_\delta$. Joining the conditions results in strong inequalities $\kappa^{1/2}g_\ell^{1/2} \ll g_\delta \ll g_\ell,\ g_\delta^2 \gg 1$. If these are satisfied, the relation $l_{\mathrm{j}}(\Phi) \gg \lambda_{\mathrm{L}}$ and the results (10)–(12) would take place in the whole region $|\Phi| \leqslant \frac{1}{2}\Phi_0$. The conditions $\kappa^{1/2}g_\ell^{1/2} \ll g_\delta \ll g_\ell$ are quite restrictive and uncommon as they can be satisfied only at huge values of g_ℓ. On account of a monotonic increase of $l_{\mathrm{j}}(\Phi)$ with Φ, substantially weaker conditions $l_{\mathrm{jv}} \gg \lambda_{\mathrm{L}}$ emerge at $|\Phi| = \frac{1}{2}\Phi_0$. Then the local approach is justified in describing the Josephson vortex and, in particular, its size (11), and can fail at smaller values of $|\Phi|$. This results in $g_\ell^{1/2} \gg \kappa^{1/2}$, up to a logarithmic factor.

As seen from (11), the dimensionless characteristic size of the Josephson vortex $\tilde{l}_{\rm jv}(g_\ell,g_\delta)=l_{\rm jv}/\sqrt{\lambda_{\rm L}\xi}$ is expressed, within the GL theory, solely via the parameters g_ℓ and g_δ. The quantity $\tilde{l}_{\rm jv}$ is shown in Fig. 2 for various strengths of the interfacial pair breaking, as a function of g_ℓ. The numerical results have been obtained by carrying out the evaluation of $j(\chi)$ with the exact self-consistent formulas of the GL theory [15,17], and further by calculating the junction penetration depth (9) at $\Phi=\frac{1}{2}\Phi_0$. All the curves in Fig. 2 are perfectly approximated by Eq. (11). As a function of the Josephson coupling strength g_ℓ, the quantity $l_{\rm jv}$ shows a nonmonotonic behavior. In the harmonic regime $g_\ell \ll g_\delta$ the penetration depth decreases with g_ℓ as $l_{\rm j}\propto g_\ell^{-1/2}$ [see (3)]. When the parameter g_ℓ increases further and the anharmonic features of the current-phase relation become pronounced, the integral of the supercurrent over the phase difference in Eq. (9) diminishes, as discussed above. As a consequence, the junction penetration depth (9) [and, in particular, (10) and (11)] gradually goes up with increasing g_ℓ in the region $g_\ell \gg g_\delta$, $g_\delta^2\gg 1$. As follows from (11), a minimum of $l_{\rm jv}(g_\ell,g_\delta)$ as a function of g_ℓ at fixed g_δ takes place at $g_\ell^2\sim g_\delta^2\gg 1$. Specifically, the minima of $\tilde{l}_{\rm jv}(g_\ell,g_\delta)$, which correspond to curves 1 and 2 of Fig. 2, are $l_{\rm jv,min}(g_\delta=100)\approx 29.7675$ at $g_\ell\approx 129.562$, and $l_{\rm jv,min}(g_\delta=500)\approx 66.5611$ at $g_\ell\approx 647.781$.

After rewriting the condition $l_{\rm jv}(g_\ell,g_\delta)\gg\lambda_{\rm L}$ in the form $\tilde{l}_{\rm jv}(g_\ell,g_\delta)\gg\sqrt{\kappa}$, one can see that the applicability domain of the results shown in the figures and obtained within the local theory, depends on the GL parameter $\kappa\gg 1$. For example, all the curves in Fig. 2 satisfy the condition $\tilde{l}_{\rm jv}(g_\ell,g_\delta)\geqslant 30\gg 3$ and, therefore, they are applicable to the case $\kappa=10$. However, for $\kappa=100$ a substantial part of curve 1 does not satisfy the condition $\tilde{l}_{\rm jv}(g_\ell,g_\delta)\gg 10$ in the given region of g_ℓ. It is in contrast to the other curves, which remain wholly justified. Similar remarks would apply in fact to all subsequent figures of the paper (Figs. 3–8).

With increasing the interface parameter g_δ, the critical current $j_{\rm c}$ and the integral $\int_0^\pi j(\chi)d\chi$ decrease irrespective of the relation between g_δ and g_ℓ. For this reason and in accordance with (9) and (10), the junction penetration depth monotonically increases with increasing g_δ at fixed Φ and g_ℓ. The quantity $\tilde{l}_{\rm jv}$ as a function of g_δ, taken for various g_ℓ, is depicted in Fig. 3. In the region $g_\delta^2\gg 1$ the exact results are in agreement with those following from (11). As a consequence of the nonmonotonic dependence on g_ℓ, the curves in Fig. 3, which correspond to different g_ℓ, can cross each other.

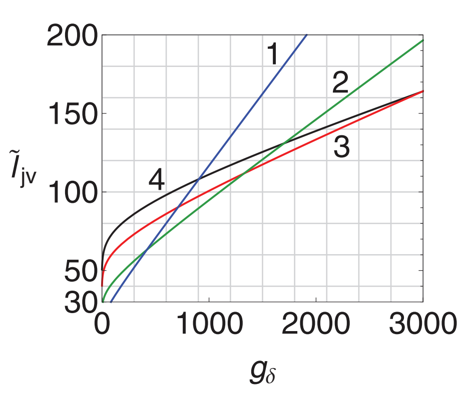

FIG. 3. (Color online) The dimensionless characteristic size of the Josephson vortex $\tilde{l}_{\rm jv}$ as the function of g_δ taken for various g_ℓ: (1) $g_\ell=300$, (2) $g_\ell=1000$, (3) $g_\ell=3000$, and (4) $g_\ell=5000$.

IV. THE SPATIAL STRUCTURE OF AN ISOLATED JOSEPHSON VORTEX

While in harmonic junctions the Josephson screening of the magnetic field is characterized by the only length scale $\lambda_{\rm J}$, in the strongly anharmonic junctions the spatial distributions of $\chi(y)$, $H(y)$, and $j(y)$ along the junction plane contain two characteristic lengths, at a fixed value of the applied magnetic field. Since in the strongly anharmonic regime the current density $j(\chi)$ has a pronounced narrow peak as a function of χ, the Josephson current experiences abrupt spatial changes in a small region of y, where varying in space phase difference passes through the vicinity of $\chi(y_c)=\chi_c$ with a change of y. The quantities $H(y)$ and $\frac{d\chi(y)}{dy}$ change comparatively quickly in that small space region, so that a smaller characteristic length is determined by the particular form of the anharmonic current-phase relation. As a result, the spatial profile of $j[\chi(y)]$ contains narrow peaks, while $\chi(y)$ and $H(y)$ acquire more angular shape as compared to that in the harmonic junctions. The greater characteristic length is the junction penetration depth $l_{\rm j}(\Phi)$, which can evolve considerably with the varying magnetic field, as shown in the preceding section. Due to a small value of the supercurrent outside the peak region, in strongly anharmonic junctions $\frac{d\chi(y)}{dy}$ is almost constant and $\chi(y)$ is nearly a linear function of y over the scale $\sim l_{\rm j}(\Phi)$.

Consider here an isolated Josephson vortex deep inside the junction plane, which is known to contain a single flux quantum. The magnetic field is symmetric and the current density is antisymmetric with respect to the vortex center, while $\chi(y)$ changes monotonically overall by 2π. In the presence of a strong interfacial pair breaking $g_\delta^2\gg 1$, the equation describing the spatial dependence of the phase difference within the local Josephson electrodynamics can be written as [17]

$$\frac{d\chi}{d\tilde{y}}=\pi\,\frac{\ln^{1/2}\left[1+\frac{4g_\ell(g_\delta+g_\ell)}{g_\delta^2}\sin^2\frac{\chi(\tilde{y})}{2}\right]}{\ln^{1/2}\left[1+\frac{4g_\ell(g_\delta+g_\ell)}{g_\delta^2}\right]}. \qquad (13)$$

Here the dimensionless coordinate $\tilde{y}=y/l_{\rm jv}$ is introduced and $H(y)>0$ assumed.

The spatial profiles of the phase difference $\chi(\tilde{y})$, of the magnetic field $H(\tilde{y})$ and of the supercurrent density $j(\tilde{y})$ in an isolated Josephson vortex, are depicted in Figs. 4–6. The vortex center is taken here at $y=0$, and the asymptotic values of the phase difference are $\chi_{-\infty}=0$, $\chi_\infty=2\pi$. All the distributions obtained confirm that the overall large scale of the spatial variations is $l_{\rm jv}$. In this respect, even a comparatively small difference between $\lambda_{\rm J}$ and $l_{\rm jv}=\frac{\pi}{2}\lambda_{\rm J}$ in harmonic junctions can be discerned. In each of these figures curve 1 describes the behavior of the corresponding quantity in harmonic junctions with a strong interfacial pair breaking. It is similar to the analogous profiles in standard tunnel junctions. A particularly interesting case of strongly anharmonic junctions is described by curve 3 in the figures. Curve 3 in Fig. 4 demonstrates a sharp

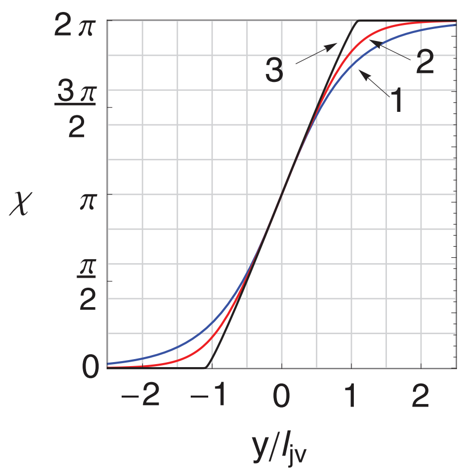

FIG. 4. (Color online) The spatial profile of the phase difference in the Josephson vortex in the junctions with $g_\delta = 10^2$ and various g_ℓ: (1) small Josephson couplings $g_\ell \ll 1$, (2) $g_\ell = 10^2$, and (3) $g_\ell = 10^4$.

crossover of the gradual behavior of the phase difference and its asymptotic value. It is in contrast to curve 1, which smoothly varies over the only scale $l_{\rm jv}$. Curve 2 shows an intermediate behavior. Similarly, in contrast to curves 1 and 2, curve 3 in Fig. 5, which describes the profile of the magnetic field in strongly anharmonic junctions, shows no noticeable tails of the field at distances $|y| > l_{\rm jv}$.

Curve 3 in Fig. 6 demonstrates that the supercurrent flows in strongly anharmonic junctions mostly in a small narrow part of the Josephson vortex. As was noted above, these are the narrow peaks in the current-phase relation of strongly anharmonic junctions, which transform into spatial peaks of the supercurrent density due to a spatial dependence of the phase difference. An effect of similar origin, but with a transformation into the magnetic flux dependence, has been recently predicted in strongly anharmonic junctions, whose widths are much less than the junction penetration depth [20]. The narrow central Fraunhofer peak of the total critical current

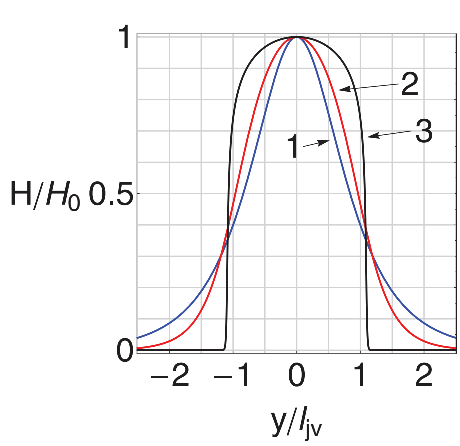

FIG. 5. (Color online) The spatial profile $H(\tilde{y})$ in the Josephson vortex, normalized to the field value H_0 in its center, in junctions with $g_\delta = 10^2$ and various g_ℓ: (1) small Josephson couplings $g_\ell \ll 1$, (2) $g_\ell = 10^2$, and (3) $g_\ell = 10^4$.

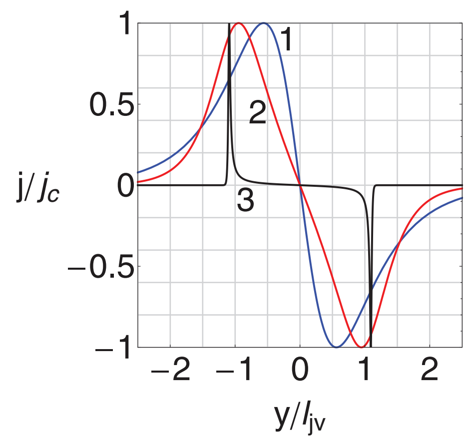

FIG. 6. (Color online) The spatial profile $j(\tilde{y})$, normalized to its critical value j_c, in the Josephson vortex in junctions with $g_\delta = 10^2$ and various g_ℓ: (1) small Josephson couplings $g_\ell \ll 1$, (2) $g_\ell = 10^2$, and (3) $g_\ell = 10^4$.

was found to possess the following half-width at the half of the peak $(\Delta\Phi/\Phi_0) \approx 1.35 g_\delta/g_\ell \ll 1$, under the conditions $g_\ell \gg g_\delta$ and $g_\delta^2 \gg 1$.

V. THE LOWER CRITICAL FIELD

The lower critical field of the junction is known to satisfy the relation $H_{\rm jc1} = 4\pi\Omega_l/\Phi_0$, where Ω_l is the thermodynamic potential of the Josephson vortex per unit length. For the junctions with an intense interfacial pair breaking $g_\delta^2 \gg 1$ one gets [17]

$$H_{\rm jc1} = \frac{\Phi_0 g_\delta \int_{-\infty}^{\infty} \ln\left[1 + \frac{4g_\ell(g_\delta+g_\ell)}{g_\delta^2}\sin^2\frac{\chi(\tilde{y})}{2}\right] d\tilde{y}}{8\pi\lambda_L\lambda_{\rm J}\sqrt{g_\ell(g_\delta+g_\ell)\ln\left[1+\frac{4g_\ell(g_\delta+g_\ell)}{g_\delta^2}\right]}}, \quad (14)$$

where $\chi(\tilde{y})$ is the solution of Eq. (13) and $\lambda_{\rm J} \approx (\lambda_L\xi/g_\ell)^{1/2} g_\delta$ [see (3)]. For harmonic junctions, when $g_\ell \ll g_\delta$, the logarithmic functions in (14) can be expanded and the integral calculated with the solution $\chi(\tilde{y}) = -2\arcsin\mathrm{sech}(\pi\tilde{y}/2)$. This results in $H_{\rm jc1} = \Phi_0/(\pi^2\lambda_L\lambda_{\rm J})$, in agreement with the conventional expression [2,18].

To single out the dependence of $H_{\rm jc1}$ on the effective interface parameters g_ℓ and g_δ, it is convenient to introduce the dimensionless lower critical field of the junctions $\tilde{H}_{\rm jc1} = H_{\rm jc1}/H^*$, taken in units of $H^* = \Phi_0/(\lambda_L^{3/2}\xi^{1/2})$. Figure 7 displays $\tilde{H}_{\rm jc1}$ as a function of the strength of the Josephson coupling, for various values of the strong interfacial pair breaking. It is a nonmonotonic function of g_ℓ. In tunnel junctions the field increases with g_ℓ and decreases with g_δ as $H_{\rm jc1} = 2\Phi_0/(\pi^2\lambda_J 2\lambda_L) \propto g_\ell^{1/2} g_\delta^{-1}$. A decrease with g_ℓ under the conditions $g_\ell \gg g_\delta \gg 1$ takes place for the same reason, for which the penetration depth increases (see Fig. 2 above and its discussion).

The relation $H_{\rm jc1} \ll H_{\rm c1}$ is to a large extent close to the strong inequality $l_{\rm jv} \gg \lambda_L$, which determines the applicability of the local theory to describing the structure of the Josephson vortex. Here $H_{\rm c1} = \Phi_0(\ln\kappa + 0.08)/(4\pi\lambda_L^2)$ is the lower critical field of the massive strongly type-II superconductor

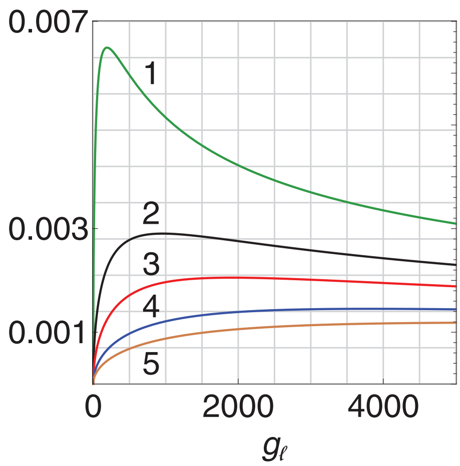

FIG. 7. (Color online) The dimensionless lower critical field $\tilde{H}_{\rm jc1}$ as a function of g_ℓ, taken for various $g_\delta \gg 1$: (1) $g_\delta = 100$, (2) $g_\delta = 500$, (3) $g_\delta = 1000$, (4) $g_\delta = 2000$, and (5) $g_\delta = 3000$.

without weak links. After reducing the relation $H_{\rm jc1} \ll H_{\rm c1}$ to the form $\tilde{H}_{\rm jc1} \ll (\ln\kappa + 0.08)/(4\pi\sqrt{\kappa})$, one sees that the applicability domain of the condition $H_{\rm jc1} \ll H_{\rm c1}$, in terms of Fig. 7, depends on the GL parameter $\kappa \gg 1$. For example, for $\kappa = 10$ one obtains a strong inequality $\tilde{H}_{\rm jc1} \ll 0.06$, which applies to all the curves in Fig. 7. For $\kappa = 100$ one gets the relation $\tilde{H}_{\rm jc1} \ll 0.037$. It is fully applicable to curves 2–5 and only partially to curve 1. This is very similar to what was said above regarding Fig. 2 plotted for the dimensionless quantity $\tilde{l}_{\rm jv}$, taken for the same set of parameters.

In tunnel junctions, the field H_0 in the center of the Josephson vortex is related to $H_{\rm jc1}$ as $H_{\rm jc1} = \frac{2}{\pi}H_0$ [2,18]. In strongly anharmonic junctions with an intense interfacial pair breaking, the ratio $H_{\rm jc1}/H_0$ depends on g_ℓ and g_δ. As shown in Fig. 8, the ratio varies between $\frac{2}{\pi}$ and unity. It monotonically increases with g_ℓ, while the pair breaking tends to suppress it towards its standard value.

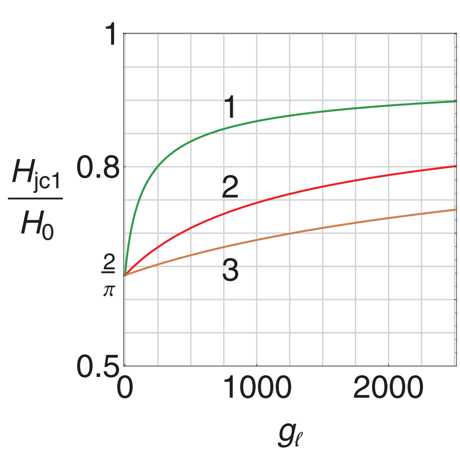

FIG. 8. (Color online) The ratio $H_{\rm jc1}/H_0$ as a function of g_ℓ, for various $g_\delta \gg 1$: (1) $g_\delta = 100$, (2) $g_\delta = 1000$, and (3) $g_\delta = 3000$.

VI. DISCUSSION

A pronounced unconventional behavior of the magnetic properties of the planar junctions emerges, when the parameters g_ℓ and g_δ are large and satisfy the conditions $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$. A phase dependent suppression of the order parameter at the junction interface, taking place under such conditions due to the proximity effect (see (S13) in [17]), is of key importance here. Qualitatively, the Josephson current (S11) increases and the junction penetration depth (10) and (11) decreases with increasing g_ℓ, when $g_\ell \ll g_\delta$, $g_\delta^2 \gg 1$ and the suppression does not substantially depend on g_ℓ. However, if $g_\ell \gg g_\delta$, then a phase dependent local decrease of the condensate density at the interface takes place with g_ℓ and results in the anharmonic Josephson current, which substantially decreases in the wide region of the phase difference. Though the critical current does not diminish in this regime (see (S12) in [17]), the integral $\int_0^\pi j(\chi)d\chi$ decreases with g_ℓ. This induces an increase of the Josephson vortex size l_{jv}, demonstrating an important role the anharmonic effects play in the problem in question.

A possibility of achieving large values of g_ℓ and g_δ in experiments has not been established as yet. However, a number of microscopic models persuasively indicate that the strong inequalities $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$, resulting in a pronounced anharmonic current-phase relation, can be satisfied under certain conditions. More restrictive relations emerge due to the application of local Josephson electrodynamics at large g_ℓ. In weak fields this results in the uncommon conditions $\kappa^{1/2} g_\ell^{1/2} \ll g_\delta \ll g_\ell$, which are only satisfied at huge values $g_\ell \gtrsim 10^4$, and could be challenging in an experimental realization. For the Josephson vortices the conditions are substantially weaker: $g_\ell^{1/2} \gg \kappa^{1/2}$, $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$.

There are no fundamental upper bounds to large values of the parameters g_ℓ and g_δ. Microscopic model results for g_ℓ can be obtained based on the corresponding studies of the Josephson current near T_c [14,16,21,22], or of the boundary conditions for the superconductor order parameter at the interface in the GL theory [23–27]. As follows from the microscopic results, in dirty junctions with small and moderate transparencies, g_ℓ can vary from vanishingly small values in the tunneling limit to those well exceeding 10^2 and leading to a substantially anharmonic behavior of the Josephson current. The parameter g_ℓ goes up, when the interface transparency increases. In highly transparent planar junctions g_ℓ can generally take huge values. As $g_\ell \propto \xi(T)$, an additional increase of g_ℓ occurs near T_c.

Large g_δ corresponds to a strong suppression of the order parameter at the junction interface. A strong interfacial pair breaking can be induced by proximity to superconductor-normal metal interfaces and to magnetically active boundaries in various superconductors, including isotropic s-wave ones [21,23,28–31]. In unconventional superconductors a significant pair breaking can be present also near superconductor-insulator and superconductor-vacuum interfaces [32–38]. Under certain conditions, the order parameter can be fully suppressed on the boundary, in particular, for symmetry reasons in unconventional superconductors. This signifies that g_δ can, in general, take huge values.

A specific microscopic example studied in detail theoretically [14,21], in which the parameters g_ℓ and g_δ of the GL theory can satisfy the conditions $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$, is the dirty SNS junction where the normal conductivity of the leads is significantly less than the conductivity of a thin normal metal interlayer.

VII. CONCLUSIONS

The problem of the magnetic field screening and of the Josephson vortex structure in superconducting planar junctions with anharmonic current-phase relations has been solved in this paper within the GL theory. Since a strongly anharmonic behavior only appears due to a pronounced Josephson coupling, an intense interfacial pair breaking needs to be present for the planar junctions to be weak links. Another reason for a pronounced interfacial pair breaking to play a crucial role for the theory developed, is that an intense pair breaking significantly increases the penetration depth and thereby substantially extends an applicability domain of the local Josephson electrodynamics, which in this case applies to the junctions with the strong Josephson coupling.

The magnetic penetration depth l_{j} in the junctions is identified theoretically as a function of the magnetic flux, of the Josephson coupling strength, and the interfacial pair breaking. Due to a nonexponential spatial profile of the screened magnetic field in the junction plane, a quantitative definition of l_{j} is put to use, similar to the standard definitions of magnetic penetration depths in various other circumstances. In harmonic junctions a characteristic size of the Josephson vortex along the junction plane l_{jv} and the weak-field penetration depth l_{j0} are shown to be related as $l_{\mathrm{jv}} = \frac{\pi}{2} l_{\mathrm{j0}}$. A pronounced magnetic field dependence of l_{j}, which induces a significant increase of l_{jv} as compared to l_{j0}, is predicted in strongly anharmonic junctions. A nonmonotonic dependence of l_{jv} on the Josephson coupling strength is obtained in such junctions, as a consequence of the phase-dependent proximity effect, and demonstrated to result in an applicability of the local approach to sufficiently large g_ℓ at a fixed g_δ.

A narrow peak in an anharmonic current-phase relation was found to induce a peak in the spatial profile of the supercurrent density, which is narrow compared to the size of the Josephson vortex. A nonmonotonic dependence on the Josephson coupling as well as a monotonic one on a strength of the interfacial pair breaking, was obtained for the lower critical field of the junctions.

An inclusion of the nonlocal effects into the theory developed above is desirable. Possible restrictions on the results obtained could be also associated with imperfections of the planar geometry of the junctions. Their study would require a significant extension of the theoretical approach used, and lies outside the scope of the paper.

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