Symmetry enforced non-Abelian topological order at the surface of a topological insulator

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The surfaces of three-dimensional topological insulators (3D TIs) are generally described as Dirac metals, with a single Dirac cone. It was previously believed that a gapped surface implied breaking of either time-reversal $\mathcal{T}$ or U(1) charge conservation symmetry. Here, we discuss a possibility in the presence of interactions, a surface phase that preserves all symmetries but is nevertheless gapped and insulating. Then, the surface must develop topological order of a kind that can not be realized in a two-dimensional (2D) system with the same symmetries. We discuss candidate surface states, non-Abelian quantum Hall states which, when realized in 2D, have $\sigma_{xy} = \frac{1}{2}$ and hence break $\mathcal{T}$ symmetry. However, by constructing an exactly soluble 3D lattice model, we show they can be realized as $\mathcal{T}$-symmetric surface states. The corresponding 3D phases are confined, and have $\theta = \pi$ magnetoelectric response. Two candidate states have the same 12-particle topological order, the (Read-Moore) Pfaffian state with the neutral sector reversed, which we term T-Pfaffian topological order, but differ in their $\mathcal{T}$ transformation. Although we are unable to connect either of these states directly to the superconducting TI surface, we argue that one of them describes the 3D TI surface, while the other differs from it by a bosonic topological phase. We also discuss the 24-particle Pfaffian-antisemion topological order (which can be connected to the superconducting TI surface) and demonstrate that it can be realized as a $\mathcal{T}$-symmetric surface state.

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I. INTRODUCTION

Three-dimensional (3D) topological insulators host unusual surface states that can be described in a number of different ways [1–3]. In models of free electrons that respect time-reversal and charge conservation symmetries which are necessary to describe this phase, the surface is metallic. The surface electronic structure is comprised of an odd number of two-dimensional (2D) Dirac cones, which is impossible to realize in a purely 2D system with time-reversal invariance. It is crucial that electrons transform as Kramers pairs, i.e., time reversal $\mathcal{T}$ acting twice on electrons gives $\mathcal{T}^2 = -1$.

Other surface terminations of the topological bulk are also interesting. If $\mathcal{T}$ is broken only at the surface, the metallic edge can be gapped to yield an insulating surface. The topological bulk properties are revealed in the properties of domain walls between opposite $\mathcal{T}$ breaking regions on the surface. The domain walls are necessarily metallic and host an odd number of chiral Dirac fermions $c_- = 2n + 1$ [4]. Thus, each domain is associated with Hall conductance $\sigma_{xy} = n + \frac{1}{2}$ (and $\kappa_{xy}/T = n + \frac{1}{2}$ in the natural units for thermal Hall conductance), where n is an integer. Such a half-integer Hall conductance is forbidden in a 2D system in the absence of electron fractionalization and is directly related to the magnetoelectric effect [5,6]. The magnetoelectric polarizability of topological insulators is $\theta = \pi$ in contrast to trivial, time-reversal-symmetric insulators which have $\theta = 0$. If, on the other hand, we break charge conservation on the surface, by pairing and condensing Cooper pairs, the resulting surface superconductor also has exotic properties (its vortices host Majorana zero modes) [7], which only occur in 2D systems if $\mathcal{T}$ symmetry is broken.

It was believed that these were the only possible surface states of the 3D topological insulator, i.e., they must either be gapless (e.g., metallic) or break one of the defining symmetries. Recently, inspired by the study of bosonic topological phases [8–13], and fermionic topological superconductors [14], a different surface termination has been recognized that preserves all symmetries and develops an energy gap at the surface [12–16]. In this situation the surface develops topological order, i.e., there are anyonic excitations bound to the surface. While the topological order itself may be realized in 2D, the transformation properties of anyons under action of the symmetry is unlike in any 2D system.

A close analog of this problem was recently discussed in the context of fermionic topological superconductors protected by $\mathcal{T}$ symmetry, where a non-Abelian surface topological order was identified [14]. The discussion in this paper will closely follow this earlier work. Of course, such surface states will only be realized in systems with strong electron correlations, and represent a qualitatively new property of interacting topological insulators. This is analogous to the argument that in a 2D quantum spin system with $S = \frac{1}{2}$ per unit cell, a gapped, symmetric state must be topologically ordered [17]. Indeed, the absence of ordering is taken as a sign of topological order. Similarly, if on the surface of a 3D topological insulator, no sign of superconductivity or $\mathcal{T}$ breaking is present but an energy gap opens, this will be indicative of topological order. In fact, as discussed in the context of fermionic topological superconductors [14], and as we will see in the following, the topological order here is required to be *non-Abelian*. This mechanism may provide a route to realizing non-Abelian topological order, which would provide further impetus in the search for strongly correlated topological insulators such as SmB_6 [18–20].

Let us discuss some of the key physical requirements that the topologically ordered surface of a 3D topological insulator (TI) should satisfy. (a) "Z_2 ness": Given the Z_2 classification of free-fermion TIs, a pair of topologically ordered surfaces should "unwind," and should be connected to a trivial confined

TABLE I. Topological spin and electrical charge assignments of T-Pfaffian state. Excitations are labeled by X_k where $X \in \{1, \sigma, \psi\}$ are the rows and $k \in \{0, 1, \ldots, 7\}$ are columns. Charge assignments are at the bottom of the table. The particle ψ_4 is the electron, a charge e fermion with trivial mutual statistics with all other particles.

	0	1	2	3	4	5	6	7
I	1		i		1		i	
σ		1		−1		−1		1
ψ	−1		−i		−1		−i	
Charge	0	e/4	e/2	3e/4	e	5e/4	3e/2	7e/4

phase. (b) Magnetoelectric response $\theta = \pi$: In the absence of topological order, surface domains of the $\mathcal{T}$-breaking insulating surface can be assigned a Hall conductivity $\sigma_{xy} = \frac{1}{2}$ and thermal Hall conductivity $\kappa_{xy}/T = \frac{1}{2}$ (modulo integers). These are equivalent to the statement that the magnetoelectric response is $\theta = \pi$. The implication for the topologically ordered states is that in their 2D versions, which break $\mathcal{T}$, should have $\sigma_{xy} = \frac{1}{2}$ and $\kappa_{xy}/T = \frac{1}{2}$ modulo integers. (c) On breaking charge conservation symmetry, it should be possible to remove the topological order while preserving $\mathcal{T}$. The resulting superconductor should host $hc/2e$ vortices with Majorana zero modes in their core.

At first, a promising choice appears to be the Read-Moore Pfaffian state [21]. One picture of this phase is to consider a topological superconductor of spin-polarized electrons in a $p_x + ip_y$ pairing state, while the Cooper pairs form a $\nu = \frac{1}{8}$ bosonic Laughlin state. The latter has anyon excitations with fractional charge $q_k = 2e\frac{k}{8}$ and a charged chiral edge state. The bound state of the superconductor quasiparticle and the charge e excitation is identified as the electron and only those excitations that braid trivially with the electron are retained. This phase has 12 quasiparticles (including the electron) and $\sigma_{xy} = \frac{(2e)^2}{8h} = \frac{1}{2}\frac{e^2}{h}$. While this satisfies one of our criteria, it is readily seen from the topological spins of the associated topological order that this can not be realized in a $\mathcal{T}$-invariant way even on a surface.

However, a simple modification produces a more promising candidate which we call the T-Pfaffian ($\mathcal{T}$-preserving Pfaffian) [22]. Consider the time-reversed superconductor ($p_x - ip_y$) combined in the identical way with the Abelian topological order of Cooper pairs. This theory also has $\sigma_{xy} = \frac{1}{2}$ and $\kappa_{xy}/T = \frac{1}{2}$ when realized in 2D as required. Moreover, the topological spins of the quasiparticles (Table I) now appear compatible with time-reversal symmetry. We construct an exactly soluble 3D model that explicitly demonstrates that this state can be realized on the surface of a 3D bulk system while retaining $\mathcal{T}$ and charge U(1) symmetries. Since the surface topological order is forbidden in a strictly 2D system, we have realized *a* 3D topological insulator. Moreover, this phase has magnetoelectric response $\theta = \pi$. The remaining question is as follows: Is this the same phase as the free-fermion topological insulator?

A necessary condition to make this identification is that in the absence of charge conservation symmetry (i.e., induced by proximity coupling the surface to a superconductor), one should recover the TI surface superconductor, without topological order, but with Majorana zero modes in the vortex cores. The T-Pfaffian state, however, allows no simple way to exit the topological phase even when charge conservation is absent, while retaining $\mathcal{T}$. On the other hand, we find that there are two versions of the T-Pfaffian state (T-Pfaffian$_\eta$, with $\eta = \pm 1$) which differ in the way the non-Abelian particles transform under time-reversal symmetry. The non-Abelian particles with bosonic (fermionic) topological spin are assigned $\mathcal{T}^2 = \eta$ ($\mathcal{T}^2 = -\eta$). We can then demonstrate the following fact: (i) The two states corresponding to $\eta = +1$ and -1 differ by the surface topological order of a bosonic topological superconductor (BTSc) with a $\mathcal{Z}_2$ classification and (ii) they can differ from the 3D free-fermion TI surface at most, by the same BTSc surface topological order. This implies that one of them *must be* the free-fermion TI surface state, while the other represents a mixture with a BTSc although, unfortunately, we can not at present specify which of the $\eta = \pm 1$ is the 3D TI surface.

We also discuss a second topological order, the Pfaffian-antisemion state obtained recently [23,24] by a series of elegant physical arguments. This state is a tensor product of the Read-Moore Pfaffian state as discussed above, with a neutral antisemion theory $\{1, \bar{s}\}$ and has 24 quasiparticles. Its statistics is compatible with $\mathcal{T}$ symmetry, and furthermore passes the necessary requirements for being identified with the 3D TI surface state, including realizing the TI surface superconductor on breaking charge conservation [23,24]. Here, we prove that it is indeed realizable as the surface state of a 3D bulk system with the requisite symmetries, by constructing an exactly soluble 3D lattice model. In both this and the T-Pfaffian case, we find time-reversal symmetry is respected *only* if the electrons transform projectively, i.e., as $\mathcal{T}^2 = -1$.

A central tool will be the Walker-Wang construction [14,16,25,26] of an exactly soluble lattice Hamiltonian, that realizes the desired surface topological order, while maintaining a topologically trivial bulk [27]. The model realizes a ground-state wave function which is a superposition of 3D loops, one for each quasiparticle of the surface topological phase. The amplitude for any configuration is obtained as follows. The topological order is represented in terms of the R and F symbols that are associated with certain basic loop moves [28]. First, one projects the loop configuration onto a 2D plane, and relates it to a reference configuration using the elementary moves. For each move, the amplitude acquires a factor that is related to the R and F symbols. Based on the resulting state, one can readily show that an anyon that has nontrivial mutual statistics with some other excitation is necessarily confined to the surface. Furthermore, the surface excitations realize the required topological order. A minor caveat here is that for simplicity we work with bosonic Walker-Wang models, without elementary fermions in the Hilbert space [14]. Since the surface topological order is nonmodular, i.e., it contains a fermion excitation that has trivial mutual statistics with everything else, this excitation is deconfined in the bulk. Hence, the 3D state that is realized is a Z_2 gauged version of the topological insulators, i.e., it involves bulk fermions that carry gauge charge and loops carrying π gauge flux. This state may thought of as being obtained from a bosonic model, from a parton construction $b = f_\uparrow f_\downarrow$, where the Z_2 gauge charged fermions $f_{\uparrow,\downarrow}$ are placed in a 3D topological phase [29]. This can be readily rectified by introducing elementary fermions c_σ and condensing the product $c_\sigma^\dagger f_\sigma$ which confines the gauge flux and removes the bulk topological order.

Before moving to the technical details, we raise the following question that may have occurred to some readers. How is the $\theta = \pi$ magnetoelectric response of 3D topological insulator reconciled with the topologically ordered insulating surfaces with $\mathcal{T}$ symmetry? Note that $\theta = \pi$ implies that a weak applied magnetic field produces a surface charge density of $e/2$ (modulo integer multiples of e) per flux quantum. Of course, such a charge density is meaningful only if the surface is also insulating. Also, charge is only determined modulo e since integer multiples of e can be screened by surface electrons. The fractional part, however, can not be screened by electrons and is a bulk property. Specifying whether this charge density is $\pm e/2$ leads to the usual argument for breaking of time-reversal symmetry at the surface. The key observation here is that the candidate states both contain charge $e/2$ excitations which can screen the induced charge, and hence breaking of $\mathcal{T}$ symmetry is not required.

II. T-PFAFFIAN TOPOLOGICAL ORDER

First let us introduce the topological order in the T-Pfaffian state, including the anyon types and their fusion and braiding rules. The T-Pfaffian state is a twisted version of the Pfaffian state (such that the state could potentially be time-reversal invariant). Similar to Pfaffian, it is the combination of the non-Abelian Ising theory with the Abelian $U(1)_8$ theory. The Ising theory describes a gauged $p + ip$ superconductor with Z_2 fluxes and contains anyons labeled by I, σ, ψ. I labels the trivial vacuum, ψ is the fermion in the superconductor, and σ is the Z_2 flux. They fuse according to

$$\sigma \times \sigma = I + \psi,$$
$$\sigma \times \psi = \sigma,$$
$$\psi \times \psi = I,$$

σ is hence non-Abelian and has quantum dimension $\sqrt{2}$ while ψ has quantum dimension 1. The topological spins for the anyons are

$$\theta_I = 1, \quad \theta_\sigma = e^{-i\frac{\pi}{8}}, \quad \theta_\psi = -1. \tag{1}$$

The braiding of the fermion ψ around the Z_2 flux results in a phase factor of -1. The Ising sector is neutral and does not carry charge.

The $U(1)_8$ theory describes the $\nu = \frac{1}{2}$ quantum Hall state where charge $2e$ electron pairs form an effective $\nu = \frac{1}{8}$ bosonic Laughlin state. The Chern-Simons effective theory for this state is

$$\mathcal{L} = \frac{8}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}. \tag{2}$$

There are eight Abelian anyons in the theory labeled by $k = 0,1,2,\ldots,7$ which add in the usual way when they fuse

$$k_1 \times k_2 = (k_1 + k_2) \text{ mod } 8. \tag{3}$$

The topological spins for the anyons are given by

$$\theta_k = e^{i\frac{\pi}{8}k^2}. \tag{4}$$

A full braiding of particle k around particle k' results in a phase factor of

$$\theta_{kk'} = e^{i\frac{\pi}{4}kk'}. \tag{5}$$

The $U(1)_8$ sector carries fractional charge with the k particle carrying $\frac{ke}{4}$mod$2e$ charge.

The T-Pfaffian theory is then obtained by combining I and ψ of the Ising theory with even k of the $U(1)_8$ theory and σ with odd k. That is, the T-Pfaffian theory has anyons

$$\begin{array}{cccc} I_0 & I_2 & I_4 & I_6 \\ \psi_0 & \psi_2 & \psi_4 & \psi_6. \\ \sigma_1 & \sigma_3 & \sigma_5 & \sigma_7 \end{array}$$

The charge assignment of the T-Pfaffian theory is carried over from the $U(1)_8$ theory. The fusion and braiding statistics of the combined anyons are the product of the fusion and braiding of the Ising part and the U(1) part. In particular, the topological spin for the combined anyons are the quantum dimensions of the I_k and ψ_k (even k) anyons are 1 and that of the σ_k (odd k) anyons are $\sqrt{2}$. The particle and antiparticle pairs in the T-Pfaffian theory are

$$I_k \sim I_{(8-k)\text{mod } 8}, \quad \psi_k \sim \psi_{(8-k)\text{mod } 8}, \quad \sigma_k \sim \sigma_{(8-k)\text{mod } 8}. \tag{6}$$

Obviously, I_0 is the vacuum in the theory. Moreover, it is easy to check that a full braiding of ψ_4 around any other particle in the theory leads to a phase factor of 1. That is, the ψ_4 particle is a local excitation of the system and can not be seen by a braiding operation far away. In fact, ψ_4 has topological spin of -1, carries charge e, and is therefore the electron in the system.

Note that while the T-Pfaffian state has the same anyon types as the Pfaffian state, the statistics of the two are different. In particular, the statistics of the Ising part is taken to be the complex conjugate of that in Pfaffian. Therefore, for example, the topological spin of the σ_1 anyon is 1 in the T-Pfaffian theory while it is $e^{i\frac{\pi}{4}}$ in the Pfaffian theory.

III. TIME-REVERSAL SYMMETRY ON T-PFAFFIAN

Is it possible to realize the T-Pfaffian topological order in a time-reversal-invariant system? For pure 2D system, the answer is no. This is easy to see if we look at the edge of the system. The Ising part of the theory has chiral central charge $c_- = -\frac{1}{2}$ and is neutral, hence does not contribute to charge Hall conductance σ_{xy}. On the other hand, the $U(1)_8$ part of the theory has chiral central charge $c_- = 1$ and $\sigma_{xy} = \frac{1}{2}$. Therefore, the T-Pfaffian theory has total chiral central charge $c_- = \frac{1}{2}$ and charge Hall conductance $\sigma_{xy} = \frac{1}{2}$. Obviously, this is not possible in a pure 2D system with time-reversal symmetry.

However, such argument fails if the T-Pfaffian theory is realized on the surface of a 3D gapped system because the 2D surface of a 3D system does not have an edge of its own. Therefore, the chiral edge of the T-Pfaffian state no longer presents an obstruction to realization in a time-reversal-invariant system. On the surface of a 3D system, we can only probe how time-reversal symmetry acts on the excitations in the bulk of the 2D system, namely, the anyons. First, the action of time-reversal symmetry takes complex conjugation of all the braiding and fusion processes among the anyons.

Moreover, time-reversal symmetry can map one anyon type to another. Therefore, if the T-Pfaffian theory can be realized on the surface of a 3D time-reversal-invariant system, then the time-reversal symmetry must exchange the anyon types in such a way that the fusion and braiding amplitudes are invariant under both complex conjugation and this exchange of anyon types.

Such an exchange of anyon type does seem to exist if we consider the topological spins of the anyons, which describe the self-rotating processes of the anyons. From Table I, we can see that the topological spins remain invariant if the time-reversal symmetry performs the following exchange:

$$I_2 \leftrightarrow \psi_2, \quad I_6 \leftrightarrow \psi_6 \tag{7}$$

together with taking complex conjugation. Compare to the Pfaffian state where such an exchange of anyon type, hence time-reversal symmetry, can not exist. In particular, the topological spins for the four σ anyons in the Pfaffian theory are

$$\theta_{\sigma_1} = e^{i\frac{\pi}{4}}, \quad \theta_{\sigma_3} = e^{i\frac{5\pi}{4}}, \quad \theta_{\sigma_5} = e^{i\frac{5\pi}{4}}, \quad \theta_{\sigma_7} = e^{i\frac{\pi}{4}}, \tag{8}$$

which do not form time-reversal-invariant pairs.

Moreover, we can check that the exchanges I_2 with ψ_2 and I_6 with ψ_6 is consistent with the fusion rules of the T-Pfaffian theory. For example, the fusion process of

$$I_2 \times \psi_4 = \psi_6 \tag{9}$$

is mapped to

$$\psi_2 \times \psi_4 = I_6 \tag{10}$$

under this exchange, which is again a valid fusion process in the T-Pfaffian theory.

Therefore, we have found an exchange of anyon types which, together with complex conjugation, keeps the fusion rules and the topological spins of the T-Pfaffian theory invariant.

IV. LOCAL TIME-REVERSAL-SYMMETRY ACTION

The exchange of anyon types, however, does not completely specify the action of time-reversal symmetry on the T-Pfaffian state. In particular, for anyon types which do not change under time reversal, one can ask whether time reversal acts as $T^2 = 1$ or -1 on the anyon locally. This is a legitimate question to ask because away from the anyonic excitations, the state remains invariant under time reversal. If the anyon type does not change under time reversal, then time reversal acts effectively locally around the anyon. As shown in [30], the local action of time reversal can only square to 1 or -1. If $T^2 = -1$ on a particular anyon, then the anyon has an extra spin label and leads to a local twofold degeneracy under time-reversal symmetry when it is separated from all other anyons. On the other hand, if the anyon changes type under time reversal, then the effective action of time reversal is nonlocal on the state and it is not well defined to talk about T^2 locally for the anyon.

Note that for non-Abelian anyons (like the σ's), time-reversal action can be nonlocal even though it does not change the anyon type because it can change the fusion channel of two anyons. For example, under time-reversal symmetry, the fusion channels of two σ_1's ($\sigma_1 \times \sigma_1 = I_2$ and $\sigma_1 \times \sigma_1 = \psi_2$) map into each other. Therefore, it might seem not well defined to talk about the T^2 value for the σ's. However, we can always find a time-reversal-invariant fusion channel involving the σ's, for example, by fusing σ_k with σ_{8-k} into vacuum, and ask if there are local Kramer degeneracies associated with each anyon in this fusion channel. Therefore, we can assign T^2 values to both Abelian and non-Abelian anyons as long as the anyon type does not change under time reversal.

The local time-reversal-symmetry action represented by this $T^2 = \pm 1$ information is important because with different local action, the topological state can correspond to totally different bulk phase. For example, the statistics of Z_2 gauge theory with anyons $\{I,e,m,\epsilon\}$ is time-reversal invariant with no exchange of anyons. If the two bosonic particles e and m both transform as $T^2 = 1$, the state can be realized in 2D time-reversal-invariant system. However, if they both transform as $T^2 = -1$, the state can only be realized on the surface of 3D bosonic topological superconductors [12,15]. For the T-Pfaffian state, we would be interested in the T^2 transformation for the charged boson I_4, the chargeless fermion ψ_0, the electron ψ_4, and all the non-Abelian σ_k particles. In particular, the T^2 transformation of ψ_4 would tell us whether we are dealing with a $T^2 = 1$ or -1 topological insulator.

In this section, we state the general rules for determining the T^2 information for anyons. We give the motivation for setting these rules and apply them to the T-Pfaffian state. In Appendix B, we will provide an algebraic proof for these rules in terms of the exactly solvable Walker-Wang model realizing the particular topological state on the surface.

The local action of time reversal on the anyons can be determined from the following two rules that apply to both Abelian and non-Abelian anyons:

(i) Rule 1: If anyons i and j fuse into k and none of i, j, k change type under time reversal, then $T_k^2 = T_i^2 \times T_j^2$.

(ii) Rule 2: If i maps into $\bar{i}$ (different from i) under time reversal and the braiding of i around $\bar{i}$ in the fusion channel k resulting in a phase factor of 1 (or -1), then $T_k^2 = 1$ (or -1).

The first rule comes from considering a region with two anyons i and j. If i and j are separated far enough (larger than correlation length), then they each have well defined T^2. The total local time-reversal-symmetry action on the whole region is composed of that on i and j separately. However, for an observer very far away from this region, the total anyonic charge in the region is k. Therefore, $T_k^2 = T_i^2 \times T_j^2$. Note that one consequence of this rule is that a particle and its antiparticle have the same T^2.

The second rule comes from considering a region with i and $\bar{i}$. Applying time reversal, i changes into $\bar{i}$ and vice versa. This is equivalent to rotating the $(i,\bar{i})$ pair by 180°. Applying time reversal twice, we have rotated the pair by 360° which is equivalent to a full braiding of i around $\bar{i}$ and results in a phase factor of 1 (or -1). However, for an observer far away from this region, the total anyonic charge in the region is k and the 1 (or -1) phase factor comes from T_k^2. Applying these two rules, we can find out possible T^2 action on each anyon locally. There may be more than one set of T^2 values for all the anyons satisfying these rules.

Let us now apply these rules to T-Pfaffian and determine the local time-reversal-symmetry action. First of all, the electron ψ_4 is the fusion product of I_2 and ψ_2 which map into each

TABLE II. Time-reversal-symmetry action on the T-Pfaffian. The semion-antisemion (I_2, ψ_2) [as well as (I_6, ψ_6)] are exchanged by time-reversal symmetry. For the remaining quasiparticles, Kramers degeneracy ($\mathcal{T}^2 = \pm 1$) can be assigned as shown above. The electron ψ_4 is a Kramers doublet as is I_4. The non-Abelian anyons can have two possible $\mathcal{T}^2$ assignments given by $\eta = \pm 1$.

	0	1	2	3	4	5	6	7
I	1		×		−1		×	
σ		η		$-\eta$		$-\eta$		η
ψ	1		×		−1		×	

other under time reversal. Because the braiding of I_2 around ψ_2 gives a -1, ψ_4 transform as $T^2 = -1$. Therefore, we are indeed dealing with a $T^2 = -1$ topological insulator. Next, because σ_1 and σ_3 can fuse both to ψ_4 and I_4 and they do not change type under time reversal, using the first rule we find that I_4 transforms in the same way as ψ_4. That is, $T^2 = -1$ for the charged boson. From this, we can easily tell that ψ_0, the chargeless fermion, transforms as $T^2 = 1$. Finally, $\{\sigma_1, \sigma_7\}$ have the same $T^2 = \eta = \pm 1$, while $\{\sigma_3, \sigma_5\}$ have $T^2 = -\eta$. This information is summarized in Table II of T^2 values:

Two T-Pfaffians. Hence, there are two possible ways that time-reversal symmetry can act on the T-Pfaffian state, labeled with $\eta = \pm 1$. These two states can be mapped into each other by combining with the following Z_2 gauge theory. Consider a Z_2 gauge theory where the gauge charge e and the gauge flux m both transform as $T^2 = -1$ and e carries U(1) charge -1 (in units of the electron charge) while m is neutral. This particular Z_2 gauge theory can not be realized in a purely 2D system as discussed in Ref. [12], where it was termed the eTmT state, and realizes the surface topological order of a bosonic symmetry protected topological (SPT) phase protected by time-reversal symmetry (charge attached to the e particle is an unimportant difference). Bring such a Z_2 gauge theory on top of the $\eta = 1$ T-Pfaffian state and condense the boson pair eI_4. The combination of e and I_4 is charge neutral and transforms as $T^2 = 1$, therefore, the condensate preserves both symmetries. After the condensation, the gauge flux m gets bounded to the σ particles in the T-Pfaffian state in order to commute with the condensate while all the Abelian particles in the T-Pfaffian state remain. Therefore, the particle content in the resulting theory is the same as the original T-Pfaffian state, but the time-reversal representation of the σ particles change from $\eta = 1$ to -1. The U(1) charge carried by the particles remains the same. Thus, these two states differ by a particular bosonic topological superconductor.

V. COMPARISON WITH TOPOLOGICAL INSULATOR SURFACE STATE

From the discussion in the previous sections we see that a possible definition of time-reversal and charge conservation symmetry action does exist for the anyonic excitations in the T-Pfaffian state. Therefore, the T-Pfaffian state could potentially be realized on the surface of 3D systems with $\mathcal{T}$ and charge conservation symmetry, although not in purely 2D due to the chiral edge modes in T-Pfaffian. In the next section, we show that such a 3D realization indeed exists by presenting an exactly solvable model using the Walker-Wang construction [25]. Due to the nontrivial symmetry action in T-Pfaffian, the 3D bulk of the system must have some nontrivial symmetry protected topological order. That is, the model is a 3D topological insulator. But, is it *the* topological insulator realized in free-fermion systems or some other previously unknown topological insulator which is only possible in strongly interacting systems?

To answer this question definitely, we would need to find some topological invariants for different topological insulators and compute them for this system. However, we do not know how to do this. In the following, we will check certain properties of this model and see if it is consistent with what we know about the free-fermion topological insulator. We find the following: (i) two copies of T-Pfaffian are trivial which is consistent with the Z_2 classification of the free-fermion topological insulator; (ii) by breaking time reversal but not charge conservation symmetry, the topological order can be removed. Now, surface domain walls between regions with opposite $\mathcal{T}$ breaking carry gapless 1D modes with $c_- = 1$ and $\sigma_{xy} = 1$, which is known to happen in the free-fermion topological insulator. This also implies $\theta = \pi$; (iii) for one of the T-Pfaffian states ($\eta = 1$ or -1), the topological order can be removed by breaking charge conservation but not time-reversal symmetry, a property expected for free-fermion topological insulator surface states. While we do not explicitly construct the route to removing topological order, we demonstrate this to be a logical consequence.

A. Two copies of T-Pfaffian are trivial

The free-fermion topological insulator (TI) has a Z_2 classification. That is, if we take two copies of the free-fermion TI and allow interactions between them, the surface state can be made trivial without breaking either time-reversal or charge conservation symmetry. Therefore, if the T-Pfaffian state is realized on the surface of the free-fermion TI, we should be able to take two copies of it and remove the topological order without breaking time-reversal or charge conservation symmetry. This is indeed the case as we show in the following.

Suppose that we have two T-Pfaffian states whose anyons are labeled as $\{I_k, \sigma_k, \psi_k\}$ and $\{\tilde{I}_k, \tilde{\sigma}_k, \tilde{\psi}_k\}$. We can condense the following set of composite bosonic particles without breaking time reversal or charge conservation:

$$I_2\tilde{\psi}_6,\ \psi_2\tilde{I}_6,\ I_6\tilde{\psi}_2,\ \psi_6\tilde{I}_2,\ I_4\tilde{I}_4,\ \psi_0\tilde{\psi}_0,\ \psi_4\tilde{\psi}_4. \tag{11}$$

Note first that each composite particle listed above has bosonic self- and mutual statistics, therefore, they can be condensed together. Also, each composite particle has charge 0mod 8, hence condensing them does not break charge conservation. Moreover, the composite particles either map to themselves under time reversal or appear in time-reversal pairs. Finally, they all transform as $T^2 = 1$ under time reversal. Therefore, the condensate does not break time reversal either.

After condensing these particles, the non-Abelian σ_k and $\tilde{\sigma}_k$ particles are all confined. Some of the composite $\sigma\tilde{\sigma}$-type particles remain, which up to the condensed particles include

$$\sigma_1\tilde{\sigma}_3,\ \sigma_1\tilde{\sigma}_7. \tag{12}$$

The Abelian particles that remain include (up to the condensed particles)

$$I_4,\ \psi_0,\ \psi_4. \tag{13}$$

In the resulting theory, the $\sigma\tilde{\sigma}$ particle splits into two Abelian particles and the theory is equivalent to the product of a free-fermion part $\{I,\psi_4\}$ and a simple Z_2 gauge theory part

$$I,\ e,\ m,\ \epsilon. \tag{14}$$

e and m come from $\sigma_1\tilde{\sigma}_7$. They are bosons, carry charge 0, and are invariant and transform as $T^2 = 1$ under time reversal. ϵ comes from ψ_0. It is a fermion, has charge 0, and maps to itself and transforms as $T^2 = 1$ under time reversal. Therefore, the total theory is trivial under time reversal and charge conservation symmetry and can be realized in 2D.

B. Breaking time-reversal symmetry and confinement

To remove the topological order in the T-Pfaffian surface state by breaking time reversal but not charge conservation symmetry, we can bring a 2D fractional quantum Hall state with the T-Pfaffian topological order and couple it to the T-Pfaffian surface state. The 2D state has $c_- = \frac{1}{2}$ and $\sigma_{xy} = \frac{1}{2}$, therefore is not time-reversal invariant. But, it does have charge conservation symmetry.

We label the anyons in the surface T-Pfaffian state as $\{I_k,\sigma_k,\psi_k\}$ and those in the 2D T-Pfaffian state as $\{I'_k,\sigma'_k,\psi'_k\}$. Condense the following composite particles:

$$I_2\psi'_6,\ \psi_2 I'_6,\ I_6\psi'_2,\ \psi_6 I'_2,\ I_4 I'_4,\ \psi_0\psi'_0,\ \psi_4\psi'_4. \tag{15}$$

Note that this condensation is very similar to the one we applied in the previous section to two copies of T-Pfaffian surface states. However, here the operation breaks time reversal from the beginning because the 2D T-Pfaffian state breaks time-reversal symmetry. After this condensation, the surface state is reduced to the product of a charge neutral Z_2 gauge theory with anyons $\{I,e,m,\epsilon\}$ together with a charged fermion. We can further remove the topological order by condensing the e particle in the Z_2 gauge theory.

To break the time-reversal symmetry in the opposite way and remove the topological order, we bring a time-reversed copy of the 2D T-Pfaffian state with anyons $\{\bar{I}'_k,\bar{\sigma}'_k,\bar{\psi}'_k\}$ and couple it in the time-reversed way to the surface T-Pfaffian state. The statistics in the time-reversed copy of the 2D T-Pfaffian is the complex conjugate of that in the original 2D T-Pfaffian state. Therefore, in this new combination, we would condense

$$\psi_2\bar{\psi}'_6,\ I_2\bar{I}'_6,\ \psi_6\bar{\psi}'_2,\ I_6\bar{I}'_2,\ I_4\bar{I}'_4,\ \psi_0\bar{\psi}'_0,\ \psi_4\bar{\psi}'_4. \tag{16}$$

The resulting theory is again composed of a neutral Z_2 gauge theory with $\{I,\bar{e},\bar{m},\bar{\epsilon}\}$ and a charged fermion. By condensing $\bar{e}$, we remove the topological order completely.

Between the 2D T-Pfaffian state and its time-reversal copy, there is a $c_- = 1$ and $\sigma_{xy} = 1$ edge. Condensation in the system does not affect c_- and σ_{xy}. Therefore, we can break time-reversal symmetry in opposite ways on the surface T-Pfaffian state, remove any topological order, and be left with a $c_- = 1$ and $\sigma_{xy} = 1$ chiral edge between the two regions. This is what is known to happen on the free-fermion TI surface starting from the gapless Dirac-cone surface state.

C. Breaking charge conservation symmetry

When the surface of the free-fermion topological insulator is gapless, the surface Dirac cone can be gapped out (in a topologically trivial way) by inducing superconductivity on the surface without breaking time-reversal symmetry. If the T-Pfaffian can be realized as the topologically ordered surface state of the free-fermion TI, we would like to see that the topological order can be removed by condensing charge without breaking time-reversal symmetry. In the T-Pfaffian state, it is not obvious how this can be achieved. For example, one might want to condense the charged boson I_4 and simplify the topological order. However, I_4 transforms under time reversal as $T^2 = -1$. Therefore, condensing I_4 necessarily breaks time-reversal symmetry. The other Abelian particles are not bosons and can not be directly condensed. We show that such a removal of topological order can be achieved for one of the T-Pfaffian states ($\eta = 1$ or -1) by combining with a 2D topological order which has time reversal but not charge conservation symmetry and then condensing composite bosonic particles. Therefore, one of the T-Pfaffian states is consistent with being the surface state of the free-fermion TI. Our argument proceeds in the following steps:

(1) The "modularized" T-Pfaffian is a time-reversal-symmetric bosonic topological state. As a fermionic topological order, the T-Pfaffian state has Z_2 fermion parity symmetry. We can gauge the Z_2 symmetry and obtain a modularized bosonic topological theory where the local fermion has a mutual -1 statistics with the Z_2 fluxes. Such a gauging process is not unique and we find that one of the possible modularized theories has time-reversal-symmetric fusion and braiding statistics. The gauging process and the resulting theory are described in detail in Appendix A. This step works for both versions of the T-Pfaffian ($\eta = \pm 1$).

(2) One of the "modularized" T-Pfaffians can be realized in 2D with time-reversal symmetry. As a bosonic topological order with time-reversal-symmetric fusion and braiding statistics, the modularized T-Pfaffian states can be either realized in 2D time-reversal-symmetric systems or on the surface of 3D bosonic topological superconductors. By simply looking at the theories, it is hard to tell which is the case. However, useful information can be obtained from our knowledge of bosonic topological superconductors. We know that bosonic topological superconductors have a $Z_2 \times Z_2$ classification [8,9,12]. These are composed of (i) the nontrivial state in the first Z_2 has three fermion surface topological order (which is chiral when realized in 2D) while (ii) the nontrivial state for the second Z_2 has both the electric and magnetic charges transforming as $\mathcal{T}^2 = -1$ and is a nonchiral surface topological order [12,15,16], labeled eTmT in Refs. [12,15]. Because the modularized T-Pfaffian theories are nonchiral, they must belong to either the trivial or the nontrivial case of the second Z_2 (i.e., eTmT surface topological order). Moreover, the two modularized T-Pfaffians differ by exactly eTmT. To see this, take the eTmT surface topological order, and bring it on top of the $\eta = 1$ modularized T-Pfaffian. Condense the composite particle eI_4. Because both e and I_4 are both Kramer doublets, such a condensation does not violate time-reversal symmetry. The resulting theory after the condensation is exactly the $\eta = -1$ modularized T-Pfaffian state. Therefore, one of the modularized theories can be realized in 2D with time-reversal symmetry, while the other is the surface state of a

nontrivial bosonic topological superconductor (eTmT surface topological order). One last piece of missing information is which one is which. We do not know the answer to this question right now.

(3) Combining the "modularized" T-Pfaffian with T-Pfaffian, the topological order can be removed without breaking time-reversal symmetry. Now, bring the modularized T-Pfaffian state (complex conjugated) on top of the corresponding T-Pfaffian state. Condense fermion-fermion pair to confine the Z_2 gauge field and we obtain a doubled T-Pfaffian theory. The topological order in this doubled theory can be completely removed by condensing pairs of corresponding particles, e.g., $I_2 I_2'$, which does not violate time-reversal symmetry. Therefore, for one of the T-Pfaffians, the surface topological order can be removed through combination with a 2D $\mathcal{T}$-invariant topological state and condensing bosons in a time-reversal-invariant way.

Therefore, we can conclude from the previous analysis that one of the T-Pfaffian states, on breaking charge conservation symmetry, is equivalent to the superconducting surface of the free-fermion topological insulator, while the other differs from it by a bosonic topological superconductor (with surface topological order eTmT). Thus, one of the T-Pfaffian states passes all the physical requirements expected of TI surface topological order Z_2"ness," thermal and electrical Hall (equivalent to $\theta = \pi$) conductivity on $\mathcal{T}$-breaking surface domain walls, and surface superconductor with $\mathcal{T}$ symmetry and free of topological order. However, at this moment, we can not tell whether this is the $\eta = 1$ state or the $\eta = -1$ T-Pfaffian state.

D. Connecting surface topological order to the free-fermion TI

We mention here a simple argument that allows us to connect one of the T-Pfaffian states with the free-fermion TI, based on the classification result of [31]. We note, however, that the argument above did not utilize this result. In Ref. [31], U(1) $\rtimes Z_2^T$ fermionic topological insulators in 3D were proposed to have a Z_2^3 classification, one Z_2 corresponds to the free-fermion topological insulator while the remaining Z_2^2 refers to neutral bosonic SPTs with time-reversal symmetry. The ambiguity here is whether the T-Pfaffian(s) describe a mixture of the free-fermionic topological insulator with 3D bosonic SPT phases. We can prove that there is a T-Pfaffian state which is to be identified with the free-fermion TI, with no bosonic SPT mixture assuming the classification result above.

First, let us discuss possible bosonic SPT phases with U(1) $\rtimes Z_2^T$ symmetry which have a Z_2^3 classification. The root states in terms of surface topological order are (i) three-fermion state, (ii) the toric code with e and m transforming as $T^2 = -1$ (eTmT state), and (iii) the toric code where both e and m are charge $\frac{1}{2}$ of the Cooper pair. However, in the presence of electrons, the classification is reduced to Z_2^2 for these bosonic SPTs. One can combine, say, the electron with (iii) to obtain a mixture of (i) and (ii). Therefore, one can take the two neutral states (i) and (ii) as the relevant topological orders [31].

The remaining question is as follows: Is the T-Pfaffian a mixture of fermionic TI and one or both of the bosonic SPTs (i), (ii)? Note, if it is a mixture, then breaking charge conservation symmetry is not sufficient to eliminate the topological order. The T-Pfaffian can not contain the state (i) since T-Pfaffian has $\kappa_{xy} = \frac{1}{2}\kappa_0$ and $\sigma_{xy} = \frac{1}{2}\sigma_0$ in one realization while the mixture with state (i) would have an additional thermal Hall conductivity of $\pm 4\kappa_0$, while retaining the same charge Hall conductivity. Here, $\sigma_0 = e^2/h$ and $\kappa_0 = L_0\sigma_0 T$ where $L_0 = \frac{\pi^2}{3}(\frac{k_B}{e})^2$ is the Lorentz number and T is the temperature.

Therefore, the T-Pfaffian can at best differ from the free-fermion TI by the eTmT state (ii). However, the two versions of the T-Pfaffian differ from one another by exactly this state, which has a Z_2 classification. Hence, one of the two T-Pfaffian states represents the surface of the free-fermion TI.

VI. WALKER-WANG CONSTRUCTION AND MORE ON TIME-REVERSAL SYMMETRY

In this section, we construct the 3D model with time-reversal symmetry which realizes the T-Pfaffian theory on its surface and have time reversal acting in the way expected. In Sec. VI A, we introduce the basic idea of the Walker-Wang construction and explain in Sec. VI B how it allows us to determine the local time-reversal-symmetry transformation ($T^2 = \pm 1$) for the anyons. We try to first present the basic picture and the general idea underlying the Walker-Wang construction in this section without going into too much details, which is saved for Sec. VI C where we give the exactly solvable Hamiltonian and address some related subtleties.

A. Walker-Wang: General idea

The Walker-Wang construction provides a way to write an exactly solvable 3D model which realizes a particular topological order on the 2D surface of the system [25]. Given all the fusion and braiding data of a 2D anyon theory, the Walker-Wang prescription gives the local Hilbert space, terms in the Hamiltonian and ground-state wave function of a 3D model such that the 2D anyon theory emerges on the surface of the system. While it is not surprising that 2D anyon theories can be realized on the surface of 3D systems, the Walker-Wang construction is useful in the following ways: (1) it provides exactly solvable 3D models to realize "chiral" 2D topological orders, for which a 2D exactly solvable model is not known to exist; (2) the 3D Walker-Wang model can have extra symmetry than is possible on the topological order in a purely 2D system. That is, the surface of the Walker-Wang model can realize symmetry enriched topological orders that are not possible in 2D, which is a result of the nontrivial symmetry protected topological order in the 3D bulk of the system. In our previous works, we have explored this property of the Walker-Wang model in the case of bosonic and fermionic topological superconductors, demonstrating the existence of time-reversal-invariant topological orders which are impossible in purely 2D systems but can be realized on the 3D surface. Here, we use a similar strategy to study fermionic topological insulators and show that the T-Pfaffian state can be realized on the surface of a 3D system with time reversal and charge conservation symmetry while it is not possible in 2D with the same symmetry.

In this section, we are not going to explain all the details related to the exactly solvable Hamiltonian, but only focus on the basic idea of the Walker-Wang construction and show how it allows us to determine that $T^2 = -1$ for the electrons. (T^2 for other anyons can also be determined.)

The basic idea underlying the Walker-Wang construction is very intuitive. The model is constructed such that the ground-state wave function is a superposition of 3D loops (more precisely "string nets," in the sense of [32]) labeled by the anyon types, which describes the $(2+1)$D space-time trajectory of the anyons. The amplitude for a given configuration of these loops C in the $(3+1)$D wave function $\Psi_{3D}(C)$ is determined by the expectation value of the corresponding Wilson loop operators in the $(2+1)$D TQFT (topological quantum field theory) which describes how the anyon world lines twist and intertwine with each other; i.e., we have

$$\Psi_{3D}(C) = \langle W(C)\rangle_{2+1\text{TQFT}}. \tag{17}$$

This is similar in spirit to, e.g., quantum Hall wave functions, which are related to the space-time correlations of their edge states. Here, since we demand a topologically ordered boundary state, the expectation values are taken in the boundary TQFT.

The wave function for the T-Pfaffian Walker-Wang model hence contains 12 different string types corresponding to the 12 different anyons in the theory which can braid and fuse according to the fusion rules of T-Pfaffian. The strings have directions. If the direction of a string related to anyon i is reversed, it becomes a string related to the antiparticle i^*. Since the twisting and intertwining of the anyon world lines may depend on the angle of view, in order to calculate the amplitude of the string-net configurations, we need to pick a particular projection of the 3D loops onto a 2D surface. The projection we will use is also shown in Fig. 5.

Having fixed a projection, the amplitude of each configuration can be obtained using the braiding and fusion rules given by the R and F matrices of the T-Pfaffian theory, which is the product of the R and F matrices of the Ising part and the $U(1)_8$ part. The R matrix for the Ising part reads as

$$\begin{aligned} &R_*^{I,*} = R_*^{*,I} = 1, \quad R_I^{\sigma,\sigma} = e^{i\frac{\pi}{8}}, \quad R_\psi^{\sigma,\sigma} = e^{-i\frac{3\pi}{8}},\\ &R_\sigma^{\sigma,\psi} = R_\sigma^{\psi,\sigma} = i, \quad R_I^{\psi,\psi} = -1. \end{aligned} \tag{18}$$

The R matrix in the U(1) part is

$$R_{(k_1+k_2)\,\text{mod}\,8}^{k_1,k_2} = e^{i\frac{2\pi}{16}k_1k_2}. \tag{19}$$

The F matrix for the Ising part is

$$\begin{aligned} &\left[F_\sigma^{\sigma,\sigma,\sigma}\right]_{\alpha,\beta} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\\ &\left[F_\psi^{\sigma,\psi,\sigma}\right]_{\sigma,\sigma} = \left[F_\sigma^{\psi,\sigma,\psi}\right]_{\sigma,\sigma} = -1, \end{aligned} \tag{20}$$

where $\alpha,\beta = I,\psi$, all other terms being 1.

The F matrix for the U(1) part is

$$\begin{aligned} &\left[F_{(k_1+k_2+k_3)\,\text{mod}\,8)}^{k_1,k_2,k_3}\right]_{(k_1+k_2)\,\text{mod}\,8,(k_2+k_3)\,\text{mod}\,8}\\ &\quad = e^{i\frac{\pi}{8}k_1[k_2+k_3-(k_2+k_3)\,\text{mod}\,8]}. \end{aligned}$$

It can only take value ± 1.

Using the braiding and fusion moves as illustrated in Fig. 1, we can deform any string-net configurations configuration to a set of isolated loops. The change in the amplitude of the configurations is given by the F and R matrices. The isolated loops can be removed with the change in amplitude by

$$\begin{aligned} &\Delta_{I_k} = \Delta_{\psi_k} = 1, \quad k = 0,2,4,6\\ &\Delta_{\sigma_k} = -\sqrt{2}, \quad k = 1,3,5,7. \end{aligned} \tag{21}$$

Using this set of rules, the amplitude of any string-net configuration can be determined (relative to the all I_0 configuration).

Such a bulk wave function encodes the statistics on the surface, as we show in the following. Anyonic excitations can be created by adding open strings to the surface. The wave function becomes a superposition of all string-net configurations in which the corresponding strings end at the positions of the excitations. Then, we can check the statistics of the excitations by tracking these open strings. Suppose we exchange two string ends of the same type α [as shown in Fig. 2(a)] by crossing two red string segments on the surface. The two α anyons fuse to a β anyon. [Figure 2(a) shows one possible string-net configuration.] This twist in the string-net configuration (relative to the string-net configuration before exchange) can be removed to bring the strings back to their original form, but this results in a factor of $R_\beta^{\alpha,\alpha}$. Therefore, exchanging end of strings of the same type adds a $R_\beta^{\alpha,\alpha}$ factor to the total wave function, which is equivalent to saying that the ends of the strings are anyons with self-statistics given by $R_\beta^{\alpha,\alpha}$. Similarly, one can check, with linked loops on the surface as shown in Fig. 2, that string ends of different types have mutual statistics given by the corresponding R matrix element.

Open strings in the bulk also create excitations in the ground state. However, if the corresponding anyon has nontrivial braiding with any other anyon, the excitation energy grows

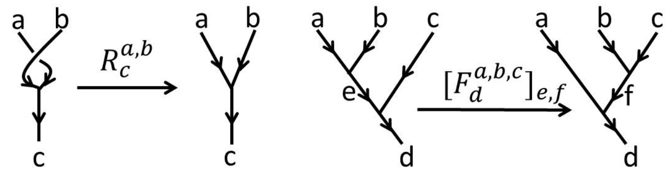

FIG. 1. Braiding and fusion moves on the string-net configurations.

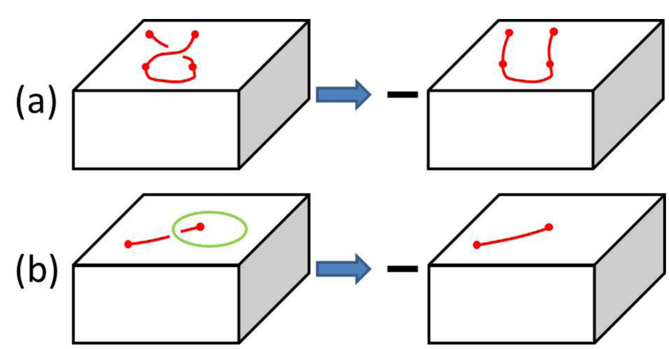

FIG. 2. (Color online) The anyonic excitations on the surface are created by open strings. The end of strings of type α are anyonic excitations of type α with the expected statistics. This can be seen from the braiding statistics of the strings generating (a) the exchange and (b) the braiding of the end of strings.

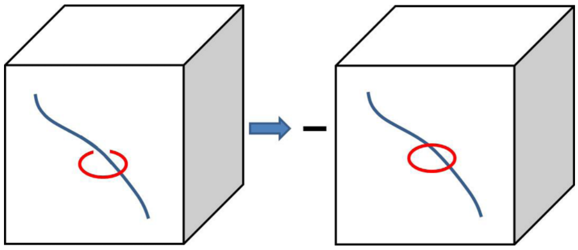

FIG. 3. (Color online) If an anyon α has nontrivial braiding with some other anyon β, then the open string of type α in the bulk can change the quantum fluctuation phase factors of β loops along its length, which costs finite energy. Therefore, the end of strings of type α are confined in the bulk. Otherwise, the end of string of type α is a deconfined point excitation, which must be either bosonic or fermionic in a 3D system.

linearly with the string length, leading to confinement of the particles at the ends of the strings. To see the confinement, consider an open string of type α (colored blue) in the bulk which is circled by a small ring of a different string type β (colored red), as shown in Fig. 3. Using the braiding rule between the loops, we find that the linking between the ring and the open string can be removed together with a phase factor

$$S_{\alpha,\beta} = \frac{1}{D}\sum_{\gamma} d_\gamma R_\gamma^{\bar{\beta}\alpha} R_\gamma^{\alpha\bar{\beta}}, \tag{22}$$

where γ is the fusion product of α and β, d_γ is its quantum dimension, and $D = \sqrt{\sum_\alpha d_\alpha}$. If $S_{\alpha,\beta} \neq 1$, the open string changes the quantum fluctuation phase factors of small loops along its length, which costs finite energy. Therefore, the string's end points can not be separated very far, and the corresponding anyonic excitations in the bulk are confined. If $S_{\alpha,\beta} = 1$ for all β, then the end of string of type α can be a deconfined point excitation in the 3D bulk of the system, which has to be either a boson or a fermion.

In the T-Pfaffian state, all anyons have nontrivial braiding with some other anyon except the electron I_4. The electron is a local excitation in the 2D topological order and hence has trivial braiding with any other anyon. Therefore, in the T-Pfaffian Walker-Wang model, the only deconfined excitation in the 3D bulk is the electron. However, open strings lying on the surface, where no loops can encircle them, give rise to deconfined excitations [26]. Therefore, the 3D Walker-Wang model written in terms of the fusion and braiding rules of the T-Pfaffian theory has a deconfined electron in the bulk and deconfined quasiparticles corresponding to all the anyons in the T-Pfaffian theory on the surface. We will postpone describing all details of the exactly solvable model to Sec. VI C. First, let me see how the Walker-Wang construction tells us more about time-reversal-symmetry action in the T-Pfaffian state.

B. Time-reversal symmetry of the Walker-Wang model

The Walker-Wang model provides us with not only an exactly solvable model to realize the T-Pfaffian surface state, but also a more concrete setup to study the time-reversal symmetry in the system. Because the ground-state wave function is determined by the F and R matrices of the T-Pfaffian state, in order for the wave function to be time-reversal symmetric, the time-reversal-symmetry action must leave the F and R matrices invariant. However, a quick check shows that the F and R matrices are not invariant under the exchange of $I_2 \leftrightarrow \psi_2$, $I_6 \leftrightarrow \psi_6$, and complex conjugation. For example [33],

$$\left(R_{\psi_4}^{\psi_2 I_2}\right)^* = -R_{\psi_4}^{I_2\psi_2}. \tag{23}$$

In order to fix this, we need to introduce extra phase factors α_k^{ij} in the time-reversal-symmetry action to the vertices where three strings meet. With proper choice of α_k^{ij}, the F and R matrices can be invariant as

$$\left[F_l^{ijk}\right]_{m,n}^* = \left[F_{\bar{l}}^{\bar{i}\bar{j}\bar{k}}\right]_{\bar{m},\bar{n}} \frac{\alpha_m^{ij}\alpha_l^{mk}}{\alpha_n^{jk}\alpha_l^{in}},$$

$$\left(R_k^{ij}\right)^* = R_{\bar{k}}^{\bar{i}\bar{j}} \frac{\alpha_k^{ij}}{\alpha_k^{ji}},$$

where $\bar{i}$ is the time-reversal partner of i. That is, we need to introduce some extra vertex degrees of freedom at the branching point of strings, which gets a phase factor under time-reversal-symmetry action. With a proper choice of gauge for F and R (explained in detail in Appendix B), a possible set of α's is $\alpha = i$ for the vertices

$$\begin{gathered}(\sigma_1,\sigma_5,I_2),\ (\sigma_3,\sigma_7,I_6),\ (I_2,I_2,I_4),\\ (I_2,\psi_2,\psi_4),\ (I_6,I_6,I_4),\ (I_6,\psi_6,\psi_4)\end{gathered} \tag{24}$$

and $\alpha = -i$ at their time-reversal partners

$$\begin{gathered}(\sigma_1,\sigma_5,\psi_2),\ (\sigma_3,\sigma_7,\psi_6),\ (\psi_2,\psi_2,I_4),\\ (\psi_2,I_2,\psi_4),\ (\psi_6,\psi_6,I_4),\ (\psi_6,I_6,\psi_4),\end{gathered} \tag{25}$$

and $\alpha = 1$ for all other allowed vertices. Note that here we are labeling the strings at each vertex such that the corresponding anyons fuse to the vacuum, i.e., the directions of the strings are all pointing towards the vertex. The strings at each vertex are ordered in a clockwise way. [With this choice of α, Eq. (24) is satisfied, not for the F and R given above, but with some other gauge choice of F and R. This is explained in detail in Appendix B.]

With this choice of α, we find that $T^2 = -1$ on all the vertices listed above in Eqs. (24) and (25), while $T^2 = 1$ on all other allowed vertices. But where are these $T^2 = -1$ vertex degrees of freedom coming from? Actually, they are related to the $T^2 = \pm 1$ transformation law for each anyon type under time reversal, as we explain below.

A simple way to understand the vertex $T^2 = -1$ degrees of freedom is to "attach Haldane chains" to the strings labeled by σ_3, σ_5, I_4, and ψ_4. Imagine adding pairs of spins $\frac{1}{2}$ to each string segment. Along the strings of types

$$\sigma_3,\ \sigma_5,\ I_4,\ \psi_4, \tag{26}$$

the spin $\frac{1}{2}$'s are put into a "Haldane chain" state where spins on neighboring string segments are connected into singlet pairs, as shown in Fig. 4. Along strings of all other types, the two spins on the same segment are connected into singlet pairs. A sample configuration is shown in Fig. 4.

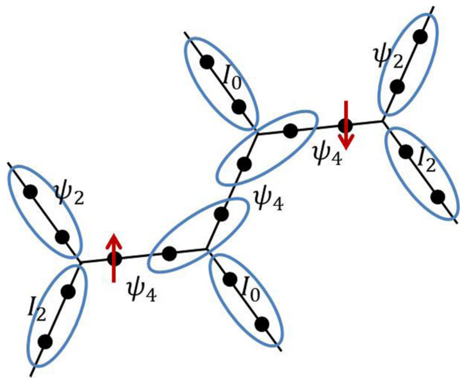

FIG. 4. (Color online) Sample spin configuration in the decorated Walker-Wang model. Black dots represent spin $\frac{1}{2}$'s, and blue ellipses represent spin singlets. The leftmost vertex $\{I_2,\psi_2,\psi_4\}$ prefers a down spin while the rightmost vertex $\{\psi_2,I_2,\psi_4\}$ prefers an up spin.

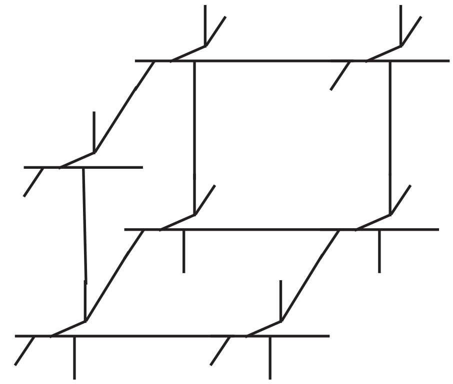

FIG. 5. A trivalent 3D lattice obtained by splitting the vertices in a cubic lattice (taken from [25]).

Because of the two different ways of pairing the spins, there are unpaired spins at vertices such as $\{I_2,\psi_2,\psi_4\}$ and $\{\psi_2,I_2,\psi_4\}$ as shown in Fig. 4. Note that because we have fixed a projection on the system, vertices $\{I_2,\psi_2,\psi_4\}$ and $\{\psi_2,I_2,\psi_4\}$ are different with different chiralities. We can choose to have the spin to point up at vertex $\{I_2,\psi_2,\psi_4\}$ and point down at vertex $\{\psi_2,I_2,\psi_4\}$. In this way, it becomes natural that time reversal adds an extra $\alpha = i$ at vertex $\{I_2,\psi_2,\psi_4\}$ and an extra $\alpha = -i$ at vertex $\{\psi_2,I_2,\psi_4\}$. In general, an unpaired spin always appears at vertices listed in Eqs. (24) and (25). We choose to have the spin to point up at vertices in Eq. (24) and to point down at vertices in Eq. (25). Therefore, all these vertices transform under time reversal as $T^2 = -1$. There are other possible ways to attach Haldane chains to realize the desired time-reversal symmetry. We summarize them in Appendix B, but the different choices do not affect our discussion in the following.

Such a spin configuration not only fixes the time-reversal symmetry action on the ground state, but also determines the transformation law of the deconfined anyonic excitations on the surface of the system. For example, deconfined excitations of anyon type I_4 on the surface are created by adding open strings of type I_4 to the surface. At the point of excitation, the wave function contains vertices such as $\{I_0,I_0,I_4\}$. Because I_4 carries a Haldane chain with it (while I_0 does not), at the vertex where the I_4 string ends, there is an extra spin-$\frac{1}{2}$ degree of freedom which transforms projectively under time reversal. Therefore, the anyonic excitation of type I_4 on the surface carries a projective representation of time reversal. If the symmetry if not broken, the excitation gives rise to at least a twofold degeneracy locally.

C. Walker-Wang construction: Details

In this section, we discuss the details about the exactly solvable model with T-Pfaffian surface state using the Walker-Wang construction. Readers not interested in the exact form of the Hamiltonian can skip this section. Our construction here follows closely the strategy outlined in [25,26] and we refer the reader there for further details.

The T-Pfaffian Walker-Wang model is defined on a 3D trivalent lattice. A trivalent 3D lattice can be obtained by splitting the vertices in a cubic lattice as shown in Fig. 5. We have taken a particular projection of the 3D lattice onto the 2D plane. Each link in the lattice carries a 12-dimensional spin degree of freedom, corresponding to the 12 anyon types in T-Pfaffian. The Hamiltonian contains vertex terms A_v involving all three links ending at a particular vertex and plaquette terms B_p involving links around a particular plaquette

$$H = \sum_v A_v + \sum_p B_p. \tag{27}$$

The vertex term enforces the fusion rules at each vertex by giving a higher energy to all disallowed vertices where the three strings do not fuse to the vacuum.

The plaquette term is a sum over terms labeled by anyon type s, $B_p = \sum_s d_s B_p^s$, weighted by the quantum dimension of s. Each B_p^s can change the labels of links in a plaquette $(abcdpqruvw)$, and can also depend on the labels of adjoining links $(a'b'c'd'p'q'r'u'v'w')$ (but can not change these). Explicitly, the matrix element between a state with plaquette links $(abcdpqruvw)$ and $(a''b''c''d''p''q''r''u''v''w'')$ is

$$\begin{aligned} B^{s,a,\ldots,w}_{p,a'',\ldots,w''} = {}& R^{q'b}_{q}\left(R^{c'r}_{c}\right)^*\left(R^{q'b''}_{q''}\right)^* R^{c'r''}_{c''} \\ & \times \left[F^{a''sp}_{a'}\right]_{ap''}\left[F^{p''sq}_{p'}\right]_{pq''}\left[F^{q''sb}_{q'}\right]_{qb''}\left[F^{b''sc}_{b'}\right]_{bc''} \\ & \times \left[F^{c''sr}_{c'}\right]_{cr''}\left[F^{r''su}_{r'}\right]_{ru''}\left[F^{u''sd}_{u'}\right]_{ud''}\left[F^{d''sv}_{d'}\right]_{dv''} \\ & \times \left[F^{v''sw}_{v'}\right]_{vw''}\left[F^{w''sa}_{w'}\right]_{wa''}. \end{aligned}$$

The intuition behind this complicated-looking term is that it fuses in the loop s to the skeleton of the plaquette using multiple F moves, but in the process of doing so must use R moves to temporarily displace certain links (c' and q' in Fig. 6). It is possible to check that all of these terms commute, and the resulting ground state is a superposition of string nets as given in Eq. (17). Following the proof in [25,26], we can see that the model has the bulk and surface deconfined excitations as desired.

In order to realize time-reversal symmetry on this model, we need to add two spin-$\frac{1}{2}$ degrees of freedom to each link and put them into Haldane chains or trivial chains along different strings. At each vertex listed in Eqs. (24) and (25), there is an unpaired spin $\frac{1}{2}$. We add an up-pointing magnetic field to this spin if it is at a vertex in Eq. (24) and a down-pointing

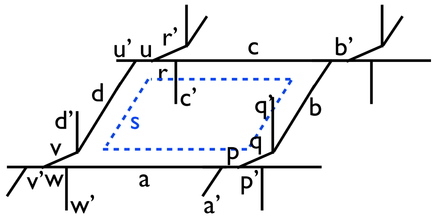

FIG. 6. (Color online) Plaquette term in Walker-Wang Hamiltonian (taken from [25]).

magnetic field if it is at one in Eq. (25). The plaquette term should then be modified correspondingly to act only between the low-energy configurations. It can then be checked that the Hamiltonian is indeed time-reversal invariant.

In order to add charge to this model, we can use similar tricks and attach fractional charges to the end of each link. When the links connect according to the fusion rules of the theory, the charges cancel at the vertex. When excitations are created by end of strings, extra fractional charges are present at the position of the excitation.

There is one subtle point about this construction. The Walker-Wang model as defined above is a spin model instead of a fermion model as one would expect for a 3D topological insulator. The fermions in the bulk appear as gauge fermions coupled to a Z_2 gauge field. Therefore, the model is topologically ordered not only on the surface, but also in the bulk. Apart from the deconfined fermionic point excitations, there are also Z_2 flux loop excitations in the bulk around which the gauge fermions pick up a -1 phase factor.

A simple way to remove the topological order in the bulk and obtain a real fermionic model is to couple the above construction with a trivial fermionic insulator and condense the fermion pairs. Because the gauge fermion transforms as $T^2 = -1$ under time reversal, it comes in two species $\psi_{4\uparrow}$ and $\psi_{4\downarrow}$. It is coupled to a fermionic insulator made also of $T^2 = -1$ fermions $c_\uparrow$, $c_\downarrow$. One can then turn on the tunneling $c_\uparrow^\dagger \psi_{4\uparrow} + c_\downarrow^\dagger \psi_{4\downarrow} + \mathrm{H.c.}$ and condense the particle-hole pair. The Z_2 flux loops are then confined because they have nontrivial statistics with the condensate. Therefore, the bulk topological order is removed and one obtains a short-range entangled fermionic model as desired.

VII. PFAFFIAN-ANTISEMION STATE

The process of removing topological order by breaking charge conservation but not time-reversal symmetry is realized in a simpler way in the Pfaffian antisemion model [23,24]. In this section, we show this model can be realized as the surface state of a $\mathcal{T}$ and U(1) conserving bulk phase and discuss its properties compared to the free-fermion TI surface.

The Pfaffian-antisemion state is a product of the antisemion state and the Pfaffian state. We label the anyons as $\{I_k, I_k s, \sigma_k, \sigma_k s, \psi_k, \psi_k s\}$. The topological spins for the anyons are

	0	1	2	3	4	5	6	7
I	1		i		1		i	
Is	$-i$		1		$-i$		1	
σ		$e^{i\pi/4}$		$-e^{i\pi/4}$		$-e^{i\pi/4}$		$e^{i\pi/4}$
σs		$e^{-i\pi/4}$		$-e^{-i\pi/4}$		$-e^{-i\pi/4}$		$e^{-i\pi/4}$
ψ	-1		$-i$		-1		$-i$	
ψs	i		-1		i		-1	

(28)

The Ising and antisemion parts are both neutral while the U(1) part has charge $k\frac{e}{4}$. Time-reversal symmetry exchanges the following pairs of anyons $I_0 s$ and $\psi_0 s$, I_2 and ψ_2, $I_4 s$ and $\psi_4 s$, I_6 and ψ_6, σ_k and $\sigma_k s$. Moreover, using the rules in Sec. IV, we can determine the local time-reversal-symmetry action on anyons which do not change type under time reversal: $I_2 s$, I_4, $I_6 s$, ψ_0, $\psi_2 s$, ψ_4, $\psi_6 s$.

Because I_2 and ψ_2 map into each other under time reversal and braid with a -1, $T^2 = -1$ for their fusion product ψ_4. Therefore, the Pfaffian-antisemion state is the surface state of a $T^2 = -1$ topological insulator. Similarly, σ_1 and $\sigma_1 s$ are time-reversal partners. In fusion channel $I_2 s$, they have trivial mutual statistics, therefore, $T^2 = 1$ for $I_2 s$ (also for $I_6 s$). In fusion channel $\psi_2 s$, they have mutual semionic statistics, therefore, $T^2 = -1$ for $\psi_2 s$ (also for $\psi_6 s$). Moreover, $I_0 s$ and $\psi_0 s$ are time-reversal partners and have mutual semionic statistics. Therefore, $T^2 = -1$ for their fusion product ψ_0. Finally, from the fusion of ψ_0 and ψ_4 into I_4 we find that $T^2 = 1$ for I_4.

A Walker-Wang model can be written explicitly for the Pfaffian-antisemion state as well. The model is defined on a 3D trivalent lattice similar to the T-Pfaffian case while the dimension of the Hilbert space on each link is doubled. Basis states on each link correspond to the 24 quasiparticles in the theory and the vertex term and plaquette term enforces the fusion and braiding rules. On the surface of the system, there are deconfined excitations corresponding to all 24 quasiparticles in the theory while in the bulk only the ψ_4 fermion is deconfined. Time-reversal symmetry acts as complex conjugation, permutation of link basis states, and phase factors at each vertex. The phase factors involved are such that vertices such as $\{\sigma_1, \sigma_1 s, \psi_2 s\}$ transform as $T^2 = -1$. Such vertex transformation laws can be understood as coming from the Kramer degeneracy of ψ_0, ψ_4, $\psi_2 s$, and $\psi_6 s$.

A. Breaking charge conservation, Z_2 ness, and breaking $\mathcal{T}$

A property of this model, which is different from T-Pfaffian, is that if we condense $I_2 s$, the topological order can be removed without breaking time-reversal symmetry. One may also observe that the fundamental $hc/2e$ vortices carry Majorana zero modes. To see this, first notice that, naively, a charge $e/2$ condensate would entail vorticity in multiples of $2hc/e$. A vortex of strength $hc/2e$ would induce a Berry phase of $\pi/2$ on the condensed particles. However, this can be rectified by attaching a σ (σs) particle to the vortex (to the antivortex), which has the opposite braiding statistics That is, we can obtain the superconducting surface state of the free-fermion TI starting from the Pfaffian-antisemion state.

Now, we can take two copies of this state and see if it can be made trivial. We label the anyons in the two copies as $\{I_k, I_k s, \sigma_k, \sigma_k s, \psi_k, \psi_k s\}$ and $\{\tilde{I}_k, \tilde{I}_k\tilde{s}, \tilde{\sigma}_k, \tilde{\sigma}_k\tilde{s}, \tilde{\psi}_k, \tilde{\psi}_k\tilde{s}\}$. First, we set aside the antisemion parts of the theory $\{I,s\}$ and $\{I,\tilde{s}\}$ and analyze the two Pfaffian parts. We can again condense the following set of composite bosonic particles without breaking time reversal or charge conservation:

$$I_2\tilde{\psi}_6,\ \psi_2\tilde{I}_6,\ I_6\tilde{\psi}_2,\ \psi_6\tilde{I}_2,\ I_4\tilde{I}_4,\ \psi_0\tilde{\psi}_0,\ \psi_4\tilde{\psi}_4. \tag{29}$$

The particles that remain in the two Pfaffian states include (up to the condensed particles)

$$\sigma_1\tilde{\sigma}_3,\ \sigma_1\tilde{\sigma}_7,\ I_4,\ \psi_0,\ \psi_4. \tag{30}$$

This is very similar to the T-Pfaffian case except that the $\sigma_1\tilde{\sigma}_3$ particle has topological spin $-i$ and the $\sigma_1\tilde{\sigma}_7$ particle has topological spin i. In the resulting theory, these quantum dimension two particles split into two Abelian particles s_1, s_2 (semions), which fuse to ψ_0:

$$s_1 \times s_2 = \psi_0. \tag{31}$$

Combined with the antisemion part of the theory, the total theory after condensation is a product of two semion theories, two antisemion theories, and a trivial electron

$$\{I,s_1\} \times \{I,s_2\} \times \{I,s\} \times \{I,\tilde{s}\} \times \{I,\psi_4\}. \tag{32}$$

The semion and antisemions parts are all neutral. Only the electron is charged. We can further condense $s_1 s$ and its time-reversal partner $s_2\tilde{s}$ without breaking any symmetry, which confines everything except the electron ψ_4. Therefore, the Pfaffian-antisemion state likely corresponds to a Z_2 topological insulator.

In order to check what happens between time-reversal symmetry-breaking domains, we can couple the the Pfaffian-antisemion surface state to 2D realizations of the Pfaffian-antisemion state, which must break time-reversal symmetry. Following similar analysis as above, we find that if we couple a 2D Pfaffian-antisemion state on the left half of the plane, we can remove all topological order. Similarly, if we couple a time-reversed 2D Pfaffian-antisemion state on the right half of the plane, with time-reversed coupling, we can also remove all topological order. Between these two domains, a chiral edge state with $c_- = 1$, $\sigma_{xy} = 1$ is left behind, which is expected for the free-fermion topological insulator surface state.

B. Connecting the Pfaffian antisemion state to the T-Pfaffian topological order

We can directly show that one of the two T-Pfaffians can be connected to the Pfaffian-antisemion state, i.e., is surface equivalent to it. Again, the procedure is not constructive, so we do not know which of the two it is. We do not make any assumption about the classification of interacting topological insulators (Ref. [31]) but utilize a related construction.

Consider combining the T-Pfaffian state with Pfaffian-antisemion state and attempting to remove all the topological order while preserving both T and charge U(1) symmetry. To help this process, we introduce a Z_8 gauge theory topological order of Cooper pairs, where the gauge charges p_m $m = 0,1,\ldots,7$ carry charges $q_m = m\frac{2e}{8}$, and the gauge fluxes v_n, where $n = 0,1,\ldots,7$ are exchanged under time-reversal symmetry $n \to (8-n)\mathrm{mod}8$ under T. v_4 maps to itself under time reversal and we choose it to transform as a Kramers doublet. Consider condensing $I_2\tilde{\psi}_6 v_2$ and its time-reversed conjugate $\psi_2\tilde{I}_6 v_6$. Note, here the first particle is in the Pfaffian-antisemion theory while the second is in the T-Pfaffian (with tilde). These condensates are self- and mutual bosons with charge 0 and preserve time-reversal symmetry. The last is particularly crucial since in the absence of the v_2 factors, the square of these condensates $I_4\tilde{I}_4$ would break T symmetry since one of these bosons is a Kramers doublet. However, here it appears as $I_4\tilde{I}_4 v_4$ which is a total time-reversal singlet and can be condensed. As $I_2\tilde{\psi}_6 v_2$ is a strength-2 flux of the whole theory, all the surviving excitations after this condensation have integer electric charge in units of e. We can separate all the quasiparticles after this condensate into two sets, a charge neutral set and a charge e set which are related to each other through combination with an electron. The neutral set forms a closed modular topological theory X which is T symmetric (note, T symmetry does not interchange particles differing by an electron since they carry different charges) [31] and the whole theory can be written as $X \times \{1, f\}$, where the electric charge is only carried by the electron f. Therefore, the T-Pfaffian state and the Pfaffian-antisemion state is surface equivalent up to either the surface of a 3D bosonic SPT of neutral boson protected by Z_2^T or a 2D T-symmetric topological order. Again, one can exclude the three-fermion state since the combined theory has no nontrivial edge states even when realized in 2D. So, the question is whether they differ by a bosonic SPT which is $e\mathrm{T}m\mathrm{T}$ (both e, m transforming as Kramers doublets). Now, since the two T-Pfaffians differ by precisely this theory, one of them is equivalent to the Pfaffian-antisemion state.

VIII. CONCLUSIONS

We have discussed the possibility of a 3D topological insulator with a symmetric gapped surface state and non-Abelian surface topological order. We constructed a model for a 3D topological insulator, with magnetoelectric response $\theta = \pi$, with a surface state given by T-Pfaffian topological order. We find that the symmetry transformation on the T-Pfaffian state can take two different forms. One of them is consistent with being the surface state of the free-fermion TI while the other differs from the free-fermion TI by a neutral bosonic topological superconductor. One remaining question is which specific T-Pfaffian state is connected to the free-fermion surface state. We can not yet answer this question due to the lack of a simple way to smoothly connect the T-Pfaffian state to the superconducting surface state of the free-fermion TI, and is left for future work. We also constructed an exactly soluble 3D lattice model for a somewhat more complicated topological order, the Pfaffian-antisemion theory [23,24] with twice as many particles, which can be smoothly connected to the superconducting surface state. These finds are consistent with the classification result obtained in [31].

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APPENDIX A: GAUGING THE T-PFAFFIAN THEORY

In order to see, with broken U(1) symmetry, whether the T-Pfaffian theory can be realized in two dimensions with time-reversal symmetry, we try to gauge the Z_2 fermion parity symmetry in the theory and examine whether the resulting theory can be time-reversal invariant or not. There are different ways (at least 16) to consistently gauge the Z_2 fermion parity symmetry and we need to consider them all.

One obvious way to gauge the Z_2 fermion parity in the T-Pfaffian theory is into the full Ising* $\times$ U$(1)_8$ theory. The topological spins of all 24 particles are listed as follows:

	0	1	2	3	4	5	6	7
I	1	$e^{i\frac{\pi}{8}}$	i	$-e^{i\frac{\pi}{8}}$	1	$-e^{i\frac{\pi}{8}}$	i	$e^{i\frac{\pi}{8}}$
σ	$e^{-i\frac{\pi}{8}}$	1	$e^{i\frac{3\pi}{8}}$	-1	$e^{i\frac{7\pi}{8}}$	-1	$e^{i\frac{3\pi}{8}}$	1
ψ	-1	$-e^{i\frac{\pi}{8}}$	$-i$	$e^{i\frac{\pi}{8}}$	-1	$e^{i\frac{\pi}{8}}$	$-i$	$-e^{i\frac{\pi}{8}}$

(A1)

The braiding of the I_1 particle correctly measures the (fractional) Z_2 quantum number associated with the original anyons. Therefore, I_1 is the Z_2 flux and the full table is obtained by combining I_1 with all the original anyons. This theory is obviously not time-reversal invariant.

There are (at least) 15 other different ways to gauge the Z_2 fermion parity symmetry in this theory. They can be obtained by combining one of Kitaev's 16-fold way to gauge free-fermion theory together with the full Ising* $\times$ U$(1)_8$ theory and condense the fermion-fermion pair. For example, if we consider the zeroth one in Kitaev's 16-fold way, the toric code model with I, $\tilde{e}$, $\tilde{m}$, $\tilde{\psi}$, then the anyon types in the resulting theory come in two sets: the set of anyons in the T-Pfaffian theory and the set of anyons in $\left[\text{Ising}^* \times \text{U}(1)_8 - \text{T-Pfaffian}\right] \times \tilde{m}$. and topological spins in the resulting theory are [here, we are still denoting the anyons with the label in Ising* $\times$ U$(1)_8$ but half of them are actually combined with $\tilde{m}$ in the toric code theory]

	0	1	2	3	4	5	6	7
I	1	$e^{i\frac{\pi}{8}}$	i	$-e^{i\frac{\pi}{8}}$	1	$-e^{i\frac{\pi}{8}}$	i	$e^{i\frac{\pi}{8}}$
σ	$e^{-i\frac{\pi}{8}}$	1	$e^{i\frac{3\pi}{8}}$	-1	$e^{i\frac{7\pi}{8}}$	-1	$e^{i\frac{3\pi}{8}}$	1
ψ	-1	$-e^{i\frac{\pi}{8}}$	$-i$	$e^{i\frac{\pi}{8}}$	-1	$e^{i\frac{\pi}{8}}$	$-i$	$-e^{i\frac{\pi}{8}}$

(A2)

which is the same as the Ising* $\times$ U$(1)_8$ theory, as we expected.

If we consider an even number element (νth) in the 16-fold way, with anyons $I, \tilde{e}, \tilde{m}, \tilde{\psi}$ and topological spins $1, e^{i\frac{\pi}{8}\nu}, e^{i\frac{\pi}{8}\nu}$, -1, then the anyon content in the resulting theory is similar to the previous case and the topological spins are (with slight abuse of notation for anyon label)

	0	1	2	3
I	1	$e^{i\frac{\pi}{8}(\nu+1)}$	i	$-e^{i\frac{\pi}{8}(\nu+1)}$
σ	$e^{i\frac{\pi}{8}(\nu-1)}$	1	$e^{i\frac{\pi}{8}(\nu+3)}$	-1
ψ	-1	$-e^{i\frac{\pi}{8}(\nu+1)}$	$-i$	$e^{i\frac{\pi}{8}(\nu+1)}$
	4	5	6	7
I	1	$-e^{i\frac{\pi}{8}(\nu+1)}$	i	$e^{i\frac{\pi}{8}(\nu+1)}$
σ	$e^{i\frac{\pi}{8}(\nu+7)}$	-1	$e^{i\frac{\pi}{8}(\nu+3)}$	1
ψ	-1	$e^{i\frac{\pi}{8}(\nu+1)}$	$-i$	$-e^{i\frac{\pi}{8}(\nu+1)}$

(A3)

The theory can not be time-reversal invariant for any even ν.

Now, let us check the case for odd ν. With odd ν, the gauged free-fermion theory has a non-Abelian Z_2 flux $\tilde{\sigma}$ with topological spin $e^{i\frac{\pi}{8}\nu}$. The total theory is obtained by combining the Ising* theory, the U$(1)_8$ theory, and the νth gauged free-fermion theory and condensing $\psi\tilde{\psi}_4$, where the subscript 4 denotes the U(1) charge. The resulting theory contains the following anyons:

$$\begin{array}{llll} II_0 = \psi\psi_4 & II_2 = \psi\psi_6 & II_4 = \psi\psi_0 & II_6 = \psi\psi_2 \\ \psi I_0 = I\psi_4 & \psi I_2 = I\psi_6 & \psi I_4 = I\psi_0 & \psi I_6 = I\psi_2 \\ \sigma I_1 = \sigma\psi_5 & \sigma I_3 = \sigma\psi_7 & \sigma I_5 = \sigma\psi_1 & \sigma I_7 = \sigma\psi_3 . \\ I\sigma_1 = \psi\sigma_5 & I\sigma_3 = \psi\sigma_7 & I\sigma_5 = \psi\sigma_1 & I\sigma_7 = \psi\sigma_3 \\ \sigma\sigma_0 = \sigma\sigma_4 & \sigma\sigma_2 = \sigma\sigma_6 & & \end{array}$$

Let us use the shorthand notation

$$\begin{array}{llll} I_0 & I_2 & I_4 & I_6 \\ \psi_0 & \psi_2 & \psi_4 & \psi_6 \\ \sigma_1^a & \sigma_3^a & \sigma_5^a & \sigma_7^a \\ \sigma_1^b & \sigma_3^b & \sigma_5^b & \sigma_7^b \\ \Omega_0 & \Omega_2 & & \end{array} .$$

The I_k and ψ_k particles have quantum dimension 1, the σ_k particles have quantum dimension $\sqrt{2}$, and the Ω_k particles have quantum dimension 2. The Ω particles could split into two quantum dimension-1 particles. Here,

$$\Omega_0 \times \Omega_0 = I_0 + I_4 + \psi_0 + \psi_4. \tag{A4}$$

Ω_0 fuses with itself into four different particles, therefore Ω_0 does not split. Neither does Ω_2.

The topological spins are

$$\begin{array}{llll} 1 & i & 1 & i \\ -1 & -i & -1 & -i \\ 1 & -1 & -1 & 1 \\ e^{i\frac{\nu+1}{8}\pi} & -e^{i\frac{\nu+1}{8}\pi} & -e^{i\frac{\nu+1}{8}\pi} & e^{i\frac{\nu+1}{8}\pi} \\ e^{i\frac{\nu-1}{8}\pi} & e^{i\frac{\nu+3}{8}\pi} & & \end{array} .$$

When $\nu = -1$, the anyon theory could potentially be time-reversal invariant with topological spins

$$\begin{array}{llll} 1 & i & 1 & i \\ -1 & -i & -1 & -i \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & & \end{array}.$$

The fusion and braiding all follow from the parent Ising* × Ising* × U(1)$_8$ theory.

Under time reversal, Ω_0 would map into Ω_2 and vice versa. Moreover, time reversal maps between the following pairs of particles:

$$I_2 \leftrightarrow \psi_2,\ I_6 \leftrightarrow \psi_6,\ \sigma_1^b \leftrightarrow \sigma_7^b,\ \sigma_3^b \leftrightarrow \sigma_5^b,\ \Omega_0 \leftrightarrow \Omega_2. \quad \text{(A5)}$$

We can check that this time-reversal-symmetry action is consistent with the fusion rule of the theory.

This is in contrast to the semion-fermion theory where one of the gauged theories does have time-reversal pairs of topological spins but time-reversal-symmetry action is not consistent with the fusion rule. In that case, the gauged theory is obtained from Ising* × Ising* × semion by condensing the $\psi\tilde{\psi}$ pair. The two-dimensional particle $\sigma\tilde{\sigma}$ needs to split into two one-dimensional particles in order to make the theory unitary which causes the inconsistency between time-reversal-symmetry action and the fusion rule. However, in the gauged T-Pfaffian case, the $\sigma\sigma$ particle does not split and time reversal can be consistent with fusion rule.

So, now we need more powerful tools to determine whether T-Pfaffian or its gauged version can be realized in 2D with time reversal. The gauged T-Pfaffian theory is a bosonic nonchiral theory. Assuming it has time-reversal symmetry, it should either be (1) realizable in 2D or (2) realizable on the surface of a 3D bosonic topological superconductor (the Z_2 class within group cohomology).

Note, ψ_4 is a time-reversal doublet since it is composed by fusing a pair of particles ψ_2, I_2 which are mutual semions and go into each other under $\mathcal{T}$ symmetry. This implies I_4 is a Kramers doublet too, and in that case we can not condense I_4. However, we can combine this theory with the changeless T-Pfaffian [i.e., having broken U(1) symmetry], and condense the product of the ψ_4 particles ($C = \langle \psi_4^{\text{gauged}} \psi_4 \rangle \neq 0$) in the two theories. This boson can be condensed and confines the flux excitations in the gauged theory resulting in two copies of T-Pfaffian, which can be confined. Thus, the changeless T-Pfaffian can be converted into a confined Sc without topological order, by combining it with this state. Hence, assuming T symmetry of the gauged theory, the changeless T-Pfaffian is either trivial (realizable in 2D with T symmetry) or equivalent to the surface of a 3D $\mathcal{T}$-symmetric bosonic SPT phase.

APPENDIX B: LOCAL TIME-REVERSAL-SYMMETRY ACTION FROM WALKER-WANG CONSTRUCTION

In this section, we show how to obtain the rules for determining the local action of time-reversal symmetry given in Sec. IV from the Walker-Wang construction. We apply the procedure to the case of T-Pfaffian and find which anyons transform as $T^2 = -1$. We discuss how various gauge choices in the problem can affect the solution.

In the Walker-Wang construction, in order to have a time-reversal-invariant Hamiltonian and ground state, we need to find vertex phase factors α_k^{ij} which satisfy

$$\left[F_l^{ijk}\right]^*_{m,n} = \left[F_{\bar{l}}^{\bar{i}\bar{j}\bar{k}}\right]_{\bar{m},\bar{n}} \frac{\alpha_m^{ij}\alpha_l^{mk}}{\alpha_n^{jk}\alpha_l^{in}},$$

$$\left(R_k^{ij}\right)^* = R_{\bar{k}}^{\bar{i}\bar{j}} \frac{\alpha_k^{ij}}{\alpha_k^{ji}}.$$

T^2 at each vertex is then obtained by

$$(T^2)_k^{ij} = \left(\alpha_k^{ij}\right)^* \alpha_{\bar{k}}^{\bar{i}\bar{j}}, \quad \text{(B1)}$$

which is equal to the combination of T^2 for i, j, and k:

$$(T^2)_k^{ij} = T_i^2 T_j^2 T_k^2. \quad \text{(B2)}$$

Of course, for anyon types i which change under time reversal, T^2 is not well defined. However, we have the constraint that

$$\left(T_i^2\right)^* = T_{\bar{i}}^2. \quad \text{(B3)}$$

In particular, if $i = \bar{i}$, $T_i^2 = \pm 1$.

From these equations, we can derive the rules given in Sec. IV. Moreover, we shall see that the arbitrariness of T^2 for i which is not equal to $\bar{i}$ is taken into account naturally in these equations.

When i, j, and k are all invariant under time reversal and i, j fuse into k,

$$(T^2)_k^{ij} = \left(\alpha_k^{ij}\right)^* \alpha_k^{ij} = 1. \quad \text{(B4)}$$

Therefore,

$$T_k^2 = T_i^2 \times T_j^2 \quad \text{(B5)}$$

as given in the first rule in Sec. IV. Next, when $i \neq \bar{i}$ and i and $\bar{i}$ fuse into k ($k = \bar{k}$),

$$\left(R_k^{i\bar{i}}\right)^* = R_k^{\bar{i}i} \frac{\alpha_k^{i\bar{i}}}{\alpha_k^{\bar{i}i}}. \quad \text{(B6)}$$

Therefore,

$$(T^2)_k^{i\bar{i}} = \left(\alpha_k^{i\bar{i}}\right)^* \alpha_k^{\bar{i}i} = R_k^{\bar{i}i} R_k^{i\bar{i}} = s_k^{i\bar{i}}, \quad \text{(B7)}$$

where $s_k^{i\bar{i}}$ is the phase factor coming from a full braid of i around $\bar{i}$ in fusion channel k. Moreover,

$$(T^2)_k^{i\bar{i}} = T_i^2 T_{\bar{i}}^2 T_k^2 = T_k^2. \quad \text{(B8)}$$

Therefore,

$$T_k^2 = s_k^{i\bar{i}} \quad \text{(B9)}$$

as given in the second rule in Sec. IV.

In particular in the case of T-Pfaffian, solving the equations in Eq. (24) with the F and R matrices given in Sec. VI A, we obtain a set of solutions for α_k^{ij}. Pick one possible solution and we can determine the T^2 transformation law for each vertex $\{i,j,k\}$ from

$$(T^2)_k^{ij} = \left(\alpha_k^{ij}\right)^* \alpha_{\bar{k}}^{\bar{i}\bar{j}}. \quad \text{(B10)}$$

We find that the following vertices have $T^2 = -1$:

$$(\sigma_1,\sigma_5,I_2),\ (\sigma_1,\sigma_5,\psi_2),\ (\sigma_3,\sigma_7,I_6),\ (\sigma_3,\sigma_7,\psi_6), \\ (I_2,I_2,I_4),\ (\psi_2,\psi_2,I_4),\ (I_2,\psi_2,\psi_4),\ (\psi_2,I_2,\psi_4), \\ (I_6,I_6,I_4),\ (\psi_6,\psi_6,I_4),\ (I_6,\psi_6,\psi_4),\ (\psi_6,I_6,\psi_4), \tag{B11}$$

and permutations of them. The T^2 of each vertex is determined by the T^2 of all three anyons involved:

$$(T^2)^{ij}_k = T_i^2 T_j^2 T_k^2. \tag{B12}$$

From this, we find that the possible set of Kramers doublet anyons has two possibilities:
(i) I_4, ψ_4, σ_3 and σ_5,
(ii) I_4, ψ_4, σ_1 and σ_7.
$T^2 = 1$ for all other anyons. These assignments satisfy

$$(T_i^2)^* = T_{\bar{i}}^2. \tag{B13}$$

In fact, this Kramers doublet assignment can be simply determined from the rules given in Sec. IV.

The F and R matrices can change by gauge β^{ij}_k as

$$[F^{ijk}_l]_{m,n} \to [F^{ijk}_l]_{m,n} \frac{\beta^{ij}_m \beta^{mk}_l}{\beta^{jk}_n \beta^{in}_l}, \\ (R^{ij}_k) \to R^{ij}_k \frac{\beta^{ij}_k}{\beta^{ji}_k}.$$

Under this change, the α's change by

$$\alpha^{ij}_k \to \alpha^{ij} \beta^{ij}_k \beta^{\bar{i}\bar{j}}_{\bar{k}}. \tag{B14}$$

This does not affect the value of $(T^2)^{ij}_k$ for each vertex but can change the value of α^{ij}_k to be anything consistent with the $(T^2)^{ij}_k$ value. Using this degree of freedom, we can make α's to be i for vertices in Eq. (24), $-i$ for vertices in Eq. (25), and 1 for all other vertices.

The α's satisfying Eq. (24) are not unique. In particular, if they change by

$$\tilde{\alpha}^{ij}_k = \alpha^{ij}_k \frac{a_i a_j}{a_k}, \tag{B15}$$

then, obviously, the equations for F and R are still satisfied. Such a change would lead to a change in T^2_{ijk}. If α^{ij}_k changes by $\frac{a_i a_j}{a_k}$, then $(T^2)^{ij}_k$ changes by $\frac{a_{\bar{i}} a_{\bar{j}} a_k}{a_{\bar{k}} a_i a_j}$. Correspondingly, such a change leads to the change in T_i^2 as

$$T_i^2 \to T_i^2 \frac{a_{\bar{i}}}{a_i}. \tag{B16}$$

If $i = \bar{i}$, then T_i^2 does not change. If $i \neq \bar{i}$, T_i^2 does change. However, T_i^2 is not well defined for $i \neq \bar{i}$ and the degree of freedom given by $\frac{a_{\bar{i}}}{a_i}$ reflects exactly this arbitrariness. Note that with arbitrary a_i, we have

$$(T_i^2)^* = T_{\bar{i}}^2. \tag{B17}$$

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] M. Z. Hasan and J. E. Moore, Annu. Rev. Condens. Matter Phys. **2**, 55 (2011).
[4] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).
[5] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[6] A. M. Essin, J. E. Moore, and D. Vanderbilt, Phys. Rev. Lett. **102**, 146805 (2009).
[7] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
[8] X. Chen, Z. Gu, Z.-X. Liu, and X. G. Wen, Phys. Rev. B **87**, 155114 (2013).
[9] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[10] M. Levin and Z. Gu, Phys. Rev. B **86**, 115109 (2012).
[11] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[12] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[13] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B **88**, 035131 (2013).
[14] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).
[15] C. Wang and T. Senthil, Phys. Rev. B **87**, 235122 (2013).
[16] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath, arXiv:1302.7072.
[17] M. B. Hastings, Phys. Rev. B **69**, 104431 (2004).
[18] S. Wolgast, C. Kurdak, K. Sun, J. W. Allen, D.-J. Kim, and Z. Fisk, Phys. Rev. B **88**, 180405(R) (2013).
[19] D. J. Kim, S. Thomas, T. Grant, J. Botimer, Z. Fisk, and J. Xia, Sci. Rep. **3**, 3150 (2013).
[20] X. Zhang, N. P. Butch, P. Syers, S. Ziemak, R. L. Greene, and J. Paglione, Phys. Rev. X **3**, 011011 (2013).
[21] G. Moore and N. Seiberg, Phys. Lett. B **220**, 422 (1989).
[22] This state was independently identified by P. Bonderson, C. Nayak, and X.-L. Qi, J. Stat. Mech. (2013) P09016.
[23] C. Wang, A. C. Potter, and T. Senthil, Phys. Rev. B **88**, 115137 (2013).
[24] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, arXiv:1306.3286.
[25] K. Walker and Z. Wang, Front. Phys. **7**, 150 (2012).
[26] C. W. von Keyserlingk, F. J. Burnell, and S. H. Simon, Phys. Rev. B **87**, 045107 (2013).
[27] This is strictly true when the surface topological order is modular, and contains anyons with nontrivial mutual statistics with at least one other particle.
[28] A. Kitaev, Ann. Phys. (NY) **321**, 2 (2006).
[29] B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Phys. Rev. B **83**, 195139 (2011).
[30] M. Levin and A. Stern, Phys. Rev. B **86**, 115131 (2012).
[31] C. Wang, A. C. Potter, and T. Senthil, Science **343**, 629 (2014).
[32] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[33] In fact, there is no gauge choice of F and R such that simple exchange and complex conjugation leave them invariant.