Electronic relaxation rates in metallic ferromagnets

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We show that the magnon-exchange contribution to the single-particle and transport relaxation rates in ferromagnetic metals, which determine the thermal and electrical conductivities, respectively, at asymptotically low temperatures does not obey a power law as previously thought, but rather shows an exponential temperature dependence. The reason is the splitting of the conduction band, which inevitably results from a nonzero magnetization. At higher temperatures, there is a sizable temperature window where the transport rate shows a T^2 temperature dependence, in accord with prior results. This window is separated from the asymptotic regime by a temperature scale that is estimated to range from tens of millidegrees of Kelvin to tens of degrees of Kelvin for typical ferromagnets. We motivate and derive a very general effective theory for metallic magnets that we then use to derive these results. Comparisons with existing experiments are discussed, and predictions for future experiments at low temperatures are made.

I. INTRODUCTION

Electronic relaxation rates contain important information about the excitations in a metallic system. The single-particle relaxation rate $1/\tau$ determines the lifetime of quasiparticles as well as the thermal conductivity $\kappa = v_F^2 c_V \tau/3$; the transport relaxation rate $1/\tau_{tr}$, determines the electrical conductivity via the Drude formula $\sigma = n_e e^2 \tau_{tr}/m_e$. Here, m_e and n_e are the conduction electron effective mass and number density, respectively, v_F is the Fermi velocity, and c_V is the specific heat. There are various contributions to these relaxation rates, including those from the scattering of electrons by propagating, or particlelike, excitations. For instance, the coupling of longitudinal phonons to conduction electrons leads to the well-known Bloch T^5 behavior of the electrical resistivity; the corresponding effect in the single-particle relaxation rate is a T^3 law [1]. In magnetically ordered phases, the coupling of the conduction electrons to any magnetic Goldstone modes contributes to the relaxation rates. In isotropic Heisenberg ferromagnets, the Goldstone modes are the ferromagnons with a frequency-momentum relation $\omega \sim k^2$. They have been found to contribute a T^2 term to the transport relaxation rate [2,3]. In helimagnets [4,5], which have a helically modulated magnetic ground state, the corresponding Goldstone mode (the helimagnon) has been shown to lead to a term in the electrical resistivity that is proportional to $T^{5/2}$ in the low-temperature limit [6–8]. In antiferromagnets, the corresponding contribution is known to be proportional to T^3 [9]. These results all hold for three-dimensional systems, which is the only physical dimension in which long-range magnetic order exists. For later reference, we note, however, that the various power laws quoted above are dimensionality dependent. For instance, in a generic dimension $d > 2$, the contribution from ferromagnons to the resistivity is proportional to $T^{(d+1)/2}$.

In addition to the scattering by propagating excitations, there are contributions to the transport coefficients due to excitations with a continuous spectrum. The best known example is the one due to the Coulomb interaction between the electrons. In simple metals, it leads to a T^2 contribution to both the single-particle rate and the transport rate, i.e., a lower power than the phonon contribution. However, since the relevant energy scale is the Fermi energy ϵ_F or Fermi temperature T_F (we use units such that $\hbar = k_B = 1$), which is much larger than the Debye temperature, this dominates the phonon contribution only at very low temperatures [10]. In metals that display ferromagnetism, the latter statement is not necessarily true, due to Fermi surfaces that consist of multiple sheets, and the issue of both the temperature dependence and the prefactor of the Coulomb contribution to the electrical resistivity is complicated. These are old questions [1,11,12] that recently have been revisited in the context of quantum criticality and exotic metals [13,14]. Another example is the scattering of electrons in ferromagnets by both longitudinal magnetization fluctuations and the so-called Stoner excitations in the transverse channel [2]. The latter are dissipative, nonhydrodynamic transverse excitations in addition to the propagating spin waves. In a random-phase approximation, the contribution to the resistivity from these dissipative excitations, both longitudinal and transverse, was shown in Ref. [15] to result in a T^2 behavior with a prefactor that is inversely proportional to the magnetization. This is qualitatively the same behavior these authors found for the scattering by magnons, and it agrees roughly with the trend observed in Fe, Co, and Ni [16]. As we will see, this conclusion, as far as the magnons is concerned, is true only in a temperature window, but not at asymptotically low temperature. It should be stressed, however, that this similarity is somewhat accidental and approximation dependent even in the regime where it holds. For instance, the power law of the magnon contribution is dimensionality dependent, as mentioned above, while the contribution from the dissipative excitations is not. Also, the prefactor of the former is essentially determined by the dispersion relation of the magnons, which is governed by very general principles, whereas the latter is dependent on many nonuniversal details. Nevertheless, the fact that various contributions of very different nature to the relaxation rates show a T^2-temperature dependence makes the interpretation of the experimentally observed T^2 behavior of the electrical resistivity in many ferromagnetic materials

difficult [16]. At the same time, the electrical resistivity is a basic physical property that is very useful, for instance, for tracking and identifying magnetic phase transitions [17–21], and establishing its behavior in the ferromagnetic phase as a benchmark is important.

In this paper, we focus on the magnon contribution to the relaxation rates in ferromagnets and show that for this process the established result is qualitatively incorrect at asymptotically low temperatures; instead of a T^2-temperature dependence, the magnon contributions to both the electrical resistivity and the thermal resistivity display an exponential behavior. A problem with the established ferromagnetic result was first noted in Ref. [22], which showed that the results for the helimagnetic and ferromagnetic cases are not mutually consistent: if one considers the ferromagnetic limit of the helimagnetic ground state, by letting the wavelength of the helix go to infinity, one finds that the leading contribution to the relaxation rate, which would yield a power law, vanishes. What is left behind is an exponential behavior of the form [23]

$$1/\tau_{\rm tr} \propto (T^2/\lambda) \exp(-T_0/T) , \qquad (1.1)$$

where the temperature scale T_0 depends on the conduction band splitting or "Stoner gap" λ or, equivalently, the magnetization [24], and on the Fermi energy $\epsilon_{\rm F}$. This result is surprising, given that the relaxation rates due to magnetic Goldstone modes in both helimagnets and antiferromagnets show a power-law behavior. The purpose of this paper is to discuss this problem and to elaborate on the brief remarks that were given in Appendix D of Ref. [22]. We will show that the asymptotic low-temperature behavior of both the transport relaxation rates due to magnons is indeed exponential of the form shown in Eq. (1.1), with $T_0 \approx D k_{\rm F}^2 (\lambda/\epsilon_{\rm F})^2$ with D the spin-wave stiffness, which itself depends on λ, and $k_{\rm F}$ and $\epsilon_{\rm F}$ the Fermi wave number and Fermi energy, respectively. This result holds in an asymptotic regime defined by $T \ll T_0$. However, in a sizable preasymptotic temperature window given by $T_0 \ll T \ll D k_{\rm F}^2$, one recovers the T^2 behavior found previously. The reason for the exponential asymptotic result is the fact that, in a ferromagnet, the Goldstone modes are purely transverse, and therefore couple only quasiparticles in different Stoner bands. The effective electron-electron-interaction due to ferromagnon exchange therefore describes purely inter-Stoner-band scattering, which leads to an activated process. In contrast, in helimagnets and antiferromagnets, there is an intra-Stoner-band coupling, which leads to a power law. This vanishes as the characteristic wave number of the magnetic order goes to zero in the ferromagnetic limit.

These results are valid for all metallic ferromagnets, whether or not the magnetism is caused by the conduction electrons themselves or by localized electrons in a different band. We will refer to such systems as "itinerant ferromagnets" and "localized-moment ferromagnets," respectively. In the main body of the paper, we will consider a very general model that does not depend on which of these two cases is realized and that uses only very general properties of ferromagnets that follow from symmetry arguments. A more specific Stoner-type model for the case of itinerant ferromagnets is considered in Appendix A.

This paper is organized as follows. In Sec. II, we derive an effective action that describes an effective electron-electron interaction due to the exchange of ferromagnons. The effective action is valid for calculating relaxation rates to first order in the magnon propagator, and it holds for both itinerant and localized-moment ferromagnets. In Sec. III, we use this model to calculate the single-particle relaxation time, and in Sec. IV, we calculate the transport relaxation time, and hence the electrical conductivity, by evaluating the pertinent Kubo formula in an approximation that is equivalent to the Boltzmann equation. In Sec. V, we discuss our results. In Appendix A, we recall the Stoner-Moriya mean-field treatment of itinerant ferromagnets. In Appendix B, we recall the cases of electron-electron and electron-phonon scattering in nonmagnetic metals and cast them in a language that illustrates why our general method works even in the case of itinerant ferromagnets.

II. EFFECTIVE ACTION

In this section, we derive and motivate an effective action that is suitable for calculating the effects of long-range ferromagnetic order and the associated Goldstone modes, on the electronic relaxation rates in a metallic ferromagnet.

A. Coupling of magnetic fluctuations to conduction electrons

Let $S_0[\bar\psi, \psi]$ be an action for conduction electrons in terms of fermionic spinor fields $\bar\psi = (\bar\psi_\uparrow, \bar\psi_\downarrow)$ and $\psi = (\psi_\uparrow, \psi_\downarrow)$ that depend on a spin projection index $\sigma = (\uparrow, \downarrow) \equiv (+, -)$. The electronic spin density is given by

$$\boldsymbol{n}_{\rm s}(x) = \sum_{\sigma, \sigma'} \bar\psi_\sigma(x) \,\boldsymbol{\sigma}_{\sigma\sigma'} \,\psi_{\sigma'}(x). \qquad (2.1)$$

Here, $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ denotes the Pauli matrices, and $x = (\boldsymbol{x}, \tau)$ comprises the real-space position $\boldsymbol{x}$ and the imaginary-time variable τ. Now assume that the conduction electrons are subject to a magnetization $\boldsymbol{M}(x)$ of unspecified origin. The magnetization will act as an effective magnetic field that couples to the conduction electrons via a Zeeman term. The action then reads

$$S[\bar\psi, \psi] = S_0[\bar\psi, \psi] + \Gamma_{\rm t} \int dx \, \boldsymbol{M}(x) \cdot \boldsymbol{n}_{\rm s}(x), \qquad (2.2)$$

with $\Gamma_{\rm t}$ a coupling constant that dimensionally is an energy times a volume, or an inverse density of states. In a ferromagnetic state, the magnetization has a nonzero average value that we assume to be in the 3-direction, $\langle M_i(x) \rangle = \delta_{i3} \, m$. In a mean-field approximation that replaces $\boldsymbol{M}$ by its average value, the action then takes the form

$$S_\lambda[\bar\psi, \psi] = S_0[\bar\psi, \psi] + \lambda \int dx \, n_{{\rm s},3}(x), \qquad (2.3a)$$

where $\lambda = \Gamma_{\rm t} m$ is directly proportional to the average magnetization. Here, we have chosen the sign of the action such that the partition function is given by

$$Z_\lambda = \int D[\bar\psi, \psi] \, e^{S_\lambda[\bar\psi, \psi]}. \qquad (2.3b)$$

λ splits the conduction band into two subbands, one for each spin projection. We will refer to λ as the Stoner gap

[24], but we emphasize that the physical situation we are considering is much more general than the one considered in the Stoner model [25]. In particular, we do not necessarily assume that the conduction electrons themselves are the source of the magnetization.

Now consider fluctuations $\delta \boldsymbol{M}$ of the magnetization. The action, Eq. (2.2), then reads

$$S[\bar{\psi},\psi] = S_\lambda[\bar{\psi},\psi] + \Gamma_t \int dx \, \delta \boldsymbol{M}(x) \cdot \boldsymbol{n}_s(x). \quad (2.4a)$$

In addition, we need an action that governs $\delta \boldsymbol{M}$. If the latter is to describe the fluctuations of the physical magnetization, then this must be

$$S_{\text{fluct}}[\delta \boldsymbol{M}] = \frac{-1}{2} \int dx \, dy \, \delta M_{s,i}(x) \, \chi_{ij}^{-1}(x,y) \, \delta M_{s,j}(y), \quad (2.4b)$$

where $\chi_{ij}(x,y)$ is the physical magnetic susceptibility. In a ferromagnetic phase, the transverse ($i,j = 1,2$ with our choice for the magnetization direction) components of χ_{ij} contain the ferromagnons, which are the Goldstone modes associated with the ferromagnetic order. The transverse part of χ_{ij} is thus singular in the limit of small frequencies and wave numbers. Adding Eqs. (2.4a) and (2.4b) and integrating out $\delta \boldsymbol{M}$, we obtain a purely electronic effective action

$$S_{\text{eff}}[\bar{\psi},\psi] = S_\lambda[\bar{\psi},\psi] + S_{\text{ex}}[\bar{\psi},\psi], \quad (2.5a)$$

with

$$S_{\text{ex}}[\bar{\psi},\psi] = \frac{\Gamma_t^2}{2} \int dx\, dy \, \delta n_{s,i}(x) \, \chi_{ij}(x,y) \, \delta n_{s,j}(y). \quad (2.5b)$$

If we use only the singular, transverse, part of χ_{ij}, then S_{ex} describes an effective electron-electron interaction mediated by an exchange of magnons [26].

B. Effective action

In order to make the effective action given by Eq. (2.5) suitable for explicit calculations, we now specify S_0 and χ_{ij}. The former, in principle, describes interacting electrons in a conduction band. However, the electron-electron interaction is not of any qualitative importance for our purposes, and we therefore take S_0 to describe noninteracting electrons with an energy-momentum relation $\epsilon_{\boldsymbol{k}}$. We denote the chemical potential by μ and define $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$. S_λ then reads

$$S_\lambda[\bar{\psi},\psi] = \sum_k \sum_a [i\omega_n - \omega_\sigma(\boldsymbol{k})] \, \bar{\psi}_\sigma(k)\, \psi_\sigma(k), \quad (2.6a)$$

with

$$\omega_\pm(\boldsymbol{k}) = \xi_{\boldsymbol{k}} \mp \lambda. \quad (2.6b)$$

Here, we see explicitly that the magnetization splits the conduction band into two Stoner bands whose Fermi surfaces (FS) are defined by

$$\omega_\sigma(\boldsymbol{p})\big|_{\boldsymbol{p} \in \text{FS}_\sigma} = 0, \quad (2.7)$$

and we denote the density of states at the σ-Fermi surface and the corresponding Fermi wave number by N_F^σ and k_F^σ,

respectively. In the case of a parabolic band, we have

$$k_F^\pm = k_F\sqrt{1 \pm \lambda/\epsilon_F}, \quad N_F^\pm = k_F^\pm m_e/2\pi^2. \quad (2.8)$$

The Green functions for the two Stoner bands are

$$G_{\lambda,\sigma}(p) = 1/[i\omega_n - \omega_\sigma(\boldsymbol{p})] \quad (2.9)$$

with $\omega_n = 2\pi T(n + 1/2)$ (n integer) a fermionic Matsubara frequency.

The structure of the transverse magnetic susceptibility at small frequencies and wave numbers in an isotropic ferromagnet is entirely determined by symmetry arguments [27]. The Goldstone modes of the spontaneously broken rotational symmetry in spin space are ferromagnons with a resonance frequency

$$\omega_0(\boldsymbol{k}) = D(\lambda)\, \boldsymbol{k}^2. \quad (2.10)$$

The spin-stiffness coefficient D vanishes as $\lambda \to 0$. It has the dimensions of a diffusion coefficient and is given by a magnetic energy scale divided by a microscopic wave number scale squared, with the latter on the order of the Fermi wave number. In the Stoner-Moriya mean-field theory [2] of itinerant ferromagnets the former is given by λ, and for nearly free electrons, one obtains

$$D(\lambda) = \lambda/6k_F^2 \quad \text{(Stoner).} \quad (2.11a)$$

In a Heisenberg spin model with exchange energy J and lattice constant a, the corresponding result is [28,29]

$$D = J\, a^2 \quad \text{(Heisenberg).} \quad (2.11b)$$

If one takes into account mode-mode coupling effects that are not included in the mean-field theory one finds that $D(\lambda)$ is a nonanalytic function of λ [30,31]. The transverse magnetic susceptibility can be expressed in terms of simple poles that describe circularly polarized ferromagnons, viz.,

$$\chi_\pm(\boldsymbol{k},i\Omega) = \frac{K(\lambda)}{(2N_F\Gamma_t)^2} \frac{1}{\omega_0(\boldsymbol{k}) \pm i\Omega}. \quad (2.12)$$

The coefficient $K(\lambda)$ is dimensionally an inverse volume. It vanishes as $\lambda \to 0$; in the Stoner-Moriya mean-field theory, it is given by (see Appendix A)

$$K(\lambda) = 4N_F\lambda \quad \text{(Stoner).} \quad (2.13a)$$

In a Heisenberg spin model, one has [32]

$$K = 2m \quad \text{(Heisenberg).} \quad (2.13b)$$

The transverse susceptibility tensor takes the form

$$\chi_T(k) = \frac{1}{2} \begin{pmatrix} \chi_+(k) + \chi_-(k) & i[\chi_+(k) - \chi_-(k)] \\ -i[\chi_+(k) - \chi_-(k)] & \chi_+(k) + \chi_-(k) \end{pmatrix}, \quad (2.14a)$$

where $k \equiv (\boldsymbol{k},i\Omega)$. Explicitly, one has for small $\boldsymbol{k}$ and Ω,

$$\chi_T(k) = \frac{K(\lambda)}{(2N_F\Gamma_t)^2} \frac{1}{\omega_0(\boldsymbol{k})^2 - (i\Omega)^2} \begin{pmatrix} D(\lambda)\boldsymbol{k}^2 & -i(i\Omega) \\ i(i\Omega) & D(\lambda)\boldsymbol{k}^2 \end{pmatrix}. \quad (2.14b)$$

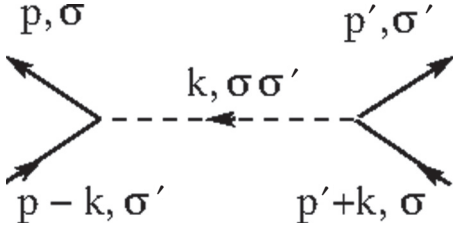In Appendix A, we show how this structure emerges in an explicit model calculation.

FIG. 1. Effective electron-electron interaction due to magnon exchange. The dashed line represents the effective potential $\mathcal{V}_{\sigma\sigma'}(k)$.

The magnon exchange interaction, Eq. (2.5b), can now be written:

$$S_{\text{ex}}[\bar{\psi},\psi] = \frac{1}{2}\sum_{\sigma,\sigma'}\int_k \delta n_{\sigma\sigma'}(k)\,\mathcal{V}_{\sigma'\sigma}(k)\,\delta n_{\sigma'\sigma}(-k). \quad (2.15a)$$

Here, $\int_k \equiv (1/V)\sum_{\boldsymbol{k}}T\sum_{i\Omega}$, and the effective potential is given by

$$\mathcal{V}_{\sigma\sigma'}(k) = V_{\sigma'\sigma}(k) + V_{\sigma\sigma'}(-k) \quad (2.15b)$$

with

$$V_{\sigma\sigma'}(k) = (1-\delta_{\sigma\sigma'})\,\Gamma_t^2\,\chi_{\sigma'}(k). \quad (2.15c)$$

This effective interaction is shown diagrammatically in Fig. 1. Notice that the exchange of magnons couples only electrons with opposite spin projections, i.e., it leads to inter-Stoner-band scattering only. This is in contrast to the case of helimagnets, where there is an intra-Stoner-band contribution whose prefactor is proportional to the square of the helical pitch wave number [6].

We add a few remarks concerning the validity of this effective action. We have assumed that the conduction electrons are subject to a magnetization and magnetic fluctuations of unspecified origin whose dynamics are governed by the physical magnetic susceptibility. Integrating out these fluctuations leads to an effective action that is purely electronic. Since the feedback of the conduction electrons on the magnetic susceptibility has already been built into the effective action, the latter must not be used in ways that constitute, directly or indirectly, a renormalization of the susceptibility; doing so would constitute double counting. However, it is safe to use the effective action for perturbative calculations of any observable to first order in the effective potential given by $\Gamma_t^2\chi$, and we will use it to calculate the quasiparticle and transport lifetimes to that order. We also note that the validity of this procedure is more obvious in cases where the magnetization is due to localized electrons in a band different from the conduction band than in the case of itinerant magnets. However, the coupling of the spin density to the magnetization fluctuations produced by the other electrons is still the same if all electrons are in the same band, and with the above caveats the effective action is still valid in that case. To illustrate this point, we consider the ordinary Fermi-liquid contribution to the electronic relaxation rate, as well as the one due to phonons, in Appendix B, where we demonstrate that a reasoning for density fluctuations that is analogous to the one given above

for magnetization fluctuations leads to the standard results for the relaxation rate in these cases.

C. Energy scales

Before we use the effective action to calculate the single-particle and transport relaxation rates, let us discuss the relevant energy scales and their relation to experimentally observable quantities. Here, we do so for the simple case of one conduction band; in Sec. V, we will discuss the more complicated, and more realistic, situation that arises from the presence of several bands.

The most obvious fundamental magnetic energy scale is the Stoner gap λ, or the closely related exchange splitting $\delta E_{\text{ex}} = 2\lambda$ [24]. It can be measured by photoemission, and also obtained by band-structure calculations. The smallest wave number that can be transferred by means of magnon exchange is $k_0 = \delta E_{\text{ex}}/v_{\text{F}}$. For a parabolic band, this corresponds to $k_0 = k_{\text{F}}^+ - k_{\text{F}}^-$, but the above expression is more general. The smallest energy that can be transferred by magnon exchange is thus

$$T_0 = D\,k_0^2 \approx \tfrac{1}{4}\,Dk_{\text{F}}^2(\delta E_{\text{ex}}/\epsilon_{\text{F}})^2. \quad (2.16)$$

The largest momentum transfer is given by $k_1 \approx 2k_{\text{F}}$, and we thus have another energy scale,

$$T_1 = 4Dk_{\text{F}}^2. \quad (2.17)$$

One expects T_1 to be close to the exchange splitting; within Stoner theory, one has $T_1 = 2\lambda/3 = \delta E_{\text{ex}}/3$. Finally, the microscopic energy scale is given by the Fermi energy ϵ_{F}, and and we have a hierarchy of energy scales, viz., $T_0 \ll T_1 \ll \epsilon_{\text{F}}$. In particular, the ratio T_0/T_1 is given in terms of the Stoner gap in units of the microscopic energy,

$$T_0/T_1 \approx \tfrac{1}{4}\,(\lambda/\epsilon_{\text{F}})^2. \quad (2.18a)$$

Alternatively, we can use $n_{\text{e}}/2N_{\text{F}}$ as the microscopic energy scale and express the ratio T_0/T_1 in terms of the magnetization m,

$$T_0/T_1 \approx \tfrac{1}{9}\,(m/n_{\text{e}})^2. \quad (2.18b)$$

Within Stoner theory this relation holds for $\lambda/\epsilon_{\text{F}} \ll 1$, see Eq. (A7b), but as an order-of-magnitude estimate it is expected to hold much more generally.

We finally mention that crystal-field effects break spin-rotational invariance, which gives the magnons a small gap and leads to yet another energy scale that affects the relaxation rates at very low temperatures. The magnitude of this effect is highly material dependent, and we neglect it for simplicity.

III. SINGLE-PARTICLE RELAXATION RATE

We now calculate the single-particle inelastic relaxation rate due to the exchange of magnons. To linear order in the effective potential, Eq. (2.15a) yields two contributions to the electronic
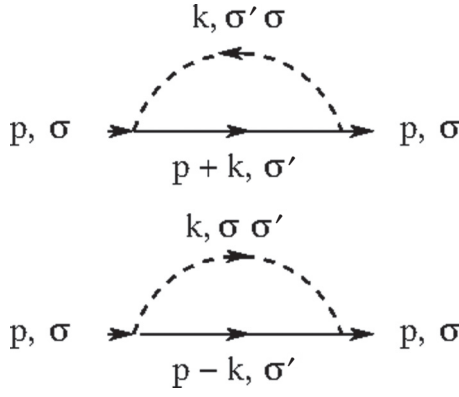
FIG. 2. Self-energy contributions $\Sigma_\sigma(p)$ for the σ-spin Green function.

self-energy Σ, which are shown in Fig. 2. Analytically, we have

$$
\Sigma_\sigma(p) = \int_k \sum_{\sigma'} \mathcal{V}_{\sigma\sigma'}(k) \, G_{\lambda,\sigma'}(p+k)
$$

$$
= 2\Gamma_t^2 \int_k \chi_\sigma(k) \, G_{\lambda,-\sigma}(p+k). \tag{3.1}
$$

Here, we have defined the self-energy such that the full Green function $\mathcal{G}$ is given by a Dyson equation:

$$
\mathcal{G}_\sigma^{-1}(p) = G_{\lambda,\sigma}^{-1}(p) - \Sigma_\sigma(p). \tag{3.2}
$$

Now we consider the single-particle relaxation rate Γ for a spin-σ quasiparticle, averaged over the Fermi surface:

$$
\Gamma_\sigma(\epsilon) = \frac{-1}{N_F^\sigma V} \sum_p \delta(\omega_\sigma(\boldsymbol{p})) \, \Sigma_\sigma''(\boldsymbol{p}, \epsilon), \tag{3.3}
$$

where $\Sigma_\sigma''(\boldsymbol{p}, \epsilon) = \operatorname{Im} \Sigma(\boldsymbol{p}, i\omega \to \epsilon + i0)$ is the spectrum of the self-energy. Using a spectral representation for the effective potential and performing the Matsubara frequency sum in Eq. (3.1), we find

$$
\Gamma_\sigma(\epsilon) = N_F^{-\sigma} \int du \, [n_B(u) + n_F(u+\epsilon)] \sum_{\sigma'} \bar{\mathcal{V}}_{\sigma\sigma'}''(u)
$$

$$
= 2\Gamma_t^2 N_F^{-\sigma} \int_{-\infty}^{\infty} du \, [n_B(u) + n_F(u+\epsilon)] \, \bar{\chi}_\sigma''(u), \tag{3.4a}
$$

where $n_B(u) = 1/(e^{u/T} - 1)$ and $n_F(u) = 1/(e^{u/T} + 1)$ are the Bose and Fermi distribution functions, respectively. Here, we have defined

$$
\bar{\mathcal{V}}_{\sigma\sigma'}''(u) = \frac{1}{N_F^\sigma N_F^{\sigma'} V^2} \sum_{\boldsymbol{k},\boldsymbol{p}} \delta[\omega_\sigma(\boldsymbol{k})] \, \delta[\omega_{\sigma'}(\boldsymbol{p})] \mathcal{V}_{\sigma\sigma'}''(\boldsymbol{k}-\boldsymbol{p}, u)
$$

$$
\tag{3.4b}
$$

and, analogously,

$$
\bar{\chi}_\sigma''(u) = \frac{1}{N_F^\sigma N_F^\sigma V^2} \sum_{\boldsymbol{k},\boldsymbol{p}} \delta[\omega_\sigma(\boldsymbol{k})] \, \delta[\omega_{\sigma'}(\boldsymbol{p})] \chi_\sigma''(\boldsymbol{k}-\boldsymbol{p}, u), \tag{3.4c}
$$

with

$$
\chi_\pm''(\boldsymbol{k}, u) = \frac{\mp K(\lambda)\pi}{(2N_F \Gamma_t)^2} \delta[\omega_0(\boldsymbol{k}) \mp u] \tag{3.4d}
$$

the spectra of the susceptibilities $\chi_\pm$ defined in Eq. (2.12). We note the symmetry relation

$$
N_F^+ \, \Gamma_+(\epsilon) = N_F^- \, \Gamma_-(-\epsilon), \tag{3.4e}
$$

which follows from the symmetry properties of $\bar{\chi}_\sigma''(u)$.

Notice that the wave vectors $\boldsymbol{k}$ and $\boldsymbol{p}$ in Eq. (3.4a) are pinned to different Fermi surfaces as a result of the pure inter-Stoner-band scattering mentioned after Eq. (2.15b). The spectrum $\bar{\chi}_\sigma''(u)$ will therefore be nonzero only for frequencies

$$
T_0 \leqslant |u| \leqslant T_1, \tag{3.5}
$$

with T_0 and T_1 given by Eqs. (2.16) and (2.17).

On the energy shell, $\epsilon = 0$, we obtain for the relaxation rate $1/\tau$ on the σ-Fermi surface,

$$
1/2\tau_\sigma \equiv \Gamma_\sigma(\epsilon = 0) = \frac{\pi K}{2N_F^\sigma T_1} T \int_{T_0/T}^{T_1/T} \frac{dx}{\sinh x}
$$

$$
= \frac{\pi K}{N_F^\sigma T_1}
\begin{cases}
T \, e^{-T_0/T} & \text{if } T \ll T_0 \\
\frac{1}{2} \, T \, \ln(T/T_0) & \text{if } T_0 \ll T \ll T_1 . \\
\frac{1}{2} \, \ln(T_1/T_0) \, T & \text{if } T \gg T_1
\end{cases}
$$

$$
\tag{3.6}
$$

For the thermal resistivity $\rho_{\text{th}} = 1/\kappa$, this implies

$$
\rho_{\text{th}} = \frac{6}{v_F^2 c_V} \frac{\pi K/N_F}{T_1}
\begin{cases}
T \, e^{-T_0/T} & \text{if } T \ll T_0 \\
\frac{1}{2} \, T \, \ln(T/T_0) & \text{if } T_0 \ll T \ll T_1 . \\
\frac{1}{2} \, \ln(T_1/T_0) \, T & \text{if } T \gg T_1
\end{cases}
$$

$$
\tag{3.7}
$$

In Eq. (3.7), the prefactor is valid in the limit $\lambda \to 0$; more generally, there are corrections of $O[(\lambda/\epsilon_F)^2]$. The second line in Eqs. (3.6) and (3.7) is valid to leading logarithmic accuracy only. We see that at asymptotically low temperatures, the relaxation rate is exponentially small, and that in the preasymptotic temperature window $T_0 \ll T \ll T_1$ there is a logarithmic correction to the linear behavior. We will further discuss these results in Sec. V.

IV. TRANSPORT RELAXATION RATE

We now turn to the transport relaxation rate, which determines the electrical resistivity. The latter is the inverse of the electrical conductivity, which is given by the Kubo formula [33]

$$
\sigma_{ij}(i\Omega) = \frac{i}{i\Omega} [\pi_{ij}(i\Omega) - \pi_{ij}(i\Omega = 0)], \tag{4.1a}
$$

where the tensor

$$
\pi_{ij}(i\Omega) = -e^2 T \sum_{n_1,n_2} \frac{1}{V} \sum_{\boldsymbol{k},\boldsymbol{p}} v_i(\boldsymbol{k}) \, v_j(\boldsymbol{p})
$$

$$
\times \langle \bar{\psi}_{n_1,\sigma}(\boldsymbol{k}) \, \psi_{n_1+n,\sigma}(\boldsymbol{k}) \, \bar{\psi}_{n_2,\sigma'}(\boldsymbol{p}) \, \psi_{n_2-n,\sigma'}(\boldsymbol{p}) \rangle
$$

$$
\tag{4.1b}
$$

is the current-current susceptibility or polarization function. Here, $\boldsymbol{v}(\boldsymbol{k}) = \partial\epsilon_k/\partial\boldsymbol{k}$, and the average is to be taken with the effective action, Eq. (2.5a). The four-fermion correlation function in Eq. (4.1b) is conveniently expressed in terms of the single-particle Green function

$$\mathcal{G}_\sigma(p) = 1/[G_{\lambda,\sigma}(p) - \Sigma_\sigma(p)] \quad (4.2)$$

and a vector vertex function $\boldsymbol{\Gamma}_\sigma$ with components Γ_σ^i:

$$\pi_{ij}(i\Omega) = -ie^2 T \sum_{i\omega} \frac{1}{V} \sum_{\boldsymbol{p},\sigma} \frac{p^i}{m_{\rm e}} \mathcal{G}_\sigma(\boldsymbol{p},i\omega)\mathcal{G}_\sigma$$
$$\times (\boldsymbol{p},i\omega - i\Omega)\Gamma_\sigma^j(\boldsymbol{p};i\omega,i\omega - i\Omega). \quad (4.3)$$

Here, we have assumed a quadratic dependence of ϵ_k on $\boldsymbol{k}$ for simplicity. It is important to calculate the vertex function $\boldsymbol{\Gamma}$ and the self-energy Σ in mutually consistent approximations [34]. We use the familiar procedure that consists of a self-consistent Born approximation for the self-energy, which to linear order in the potential V is represented by Eq. (3.1), and a ladder approximation for the vertex function,

$$\boldsymbol{\Gamma}_\sigma(\boldsymbol{p};i\omega,i\omega - i\Omega)$$
$$= i\frac{\boldsymbol{p}}{m_{\rm e}} + \frac{T}{V} \sum_{\boldsymbol{k},i\Omega'} \sum_{\sigma'} \mathcal{V}_{\sigma\sigma'}(\boldsymbol{k} - \boldsymbol{p},i\Omega')$$
$$\times \mathcal{G}_{\sigma'}(\boldsymbol{k},i\omega + i\Omega') \mathcal{G}_{\sigma'}(\boldsymbol{k},i\omega - i\Omega + i\Omega')$$
$$\times \boldsymbol{\Gamma}_{\sigma'}(\boldsymbol{k};i\omega + i\Omega',i\omega - i\Omega + i\Omega'). \quad (4.4)$$

We mention that umklapp processes, which are not explicitly considered here, are necessary in order to obtain a nonzero transport relaxation rate. In fact, in a Galilean invariant system the electrical resistivity vanishes due to momentum conservation and the contributions contained in our approximation are canceled by terms not included in the ladder approximation. However, the above approximation is effectively valid in the presence of umklapp processes, as is the case for Coulomb scattering [1]. If we define a scalar vertex function γ by $\boldsymbol{\Gamma}(\boldsymbol{p};i\omega,i\omega') = i(\boldsymbol{p}/m_{\rm e})\gamma(\boldsymbol{p};i\omega,i\omega')$, then the Bethe-Salpeter equation for the latter becomes

$$\gamma_\sigma(\boldsymbol{p};i\omega,i\omega - i\Omega)$$
$$= 1 + \frac{T}{V} \sum_{\boldsymbol{k},i\Omega'} \sum_{\sigma'} \mathcal{V}_{\sigma\sigma'}(\boldsymbol{p} - \boldsymbol{k},i\Omega')$$
$$\times \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{\boldsymbol{p}^2} \mathcal{G}_{\sigma'}(\boldsymbol{k},i\omega + i\Omega') \mathcal{G}_{\sigma'}(\boldsymbol{k},i\omega - i\Omega + i\Omega')$$
$$\times \gamma_{\sigma'}(\boldsymbol{k};i\omega - i\Omega',i\omega - i\Omega - i\Omega'). \quad (4.5)$$

The polarization and conductivity tensors are diagonal, $\sigma_{ij}(i\Omega) = \delta_{ij}\sigma(i\Omega)$, and the sum over Matsubara frequencies in Eq. (4.3) can be transformed into an integral along the real axis. In the limit of low temperature, the imaginary part of the self-energy, which yields the relaxation rate, goes to zero as we have seen in the preceding section. The real part just renormalizes the Fermi energy. The relevant limit is thus the one of a vanishing self-energy, and in this limit, the leading contributions to the integral come from terms where the frequency arguments of the two Green functions lie on different sides of the real axis. In the static limit, the

Kubo formula for the conductivity $\sigma = \lim_{\Omega\to 0} \mathrm{Re}\,\sigma(i\Omega \to \Omega + i0)$, thus becomes

$$\sigma = \frac{e^2}{3\pi m_{\rm e}^2} \int_{-\infty}^{\infty} \frac{d\epsilon}{4T} \frac{1}{\cosh^2(\epsilon/2T)} \frac{1}{V} \sum_{\boldsymbol{p}} \boldsymbol{p}^2$$
$$\times \sum_{\sigma} |\mathcal{G}_\sigma(\boldsymbol{p},\epsilon + i0)|^2 \gamma_\sigma(\boldsymbol{p};\epsilon + i0,\epsilon - i0). \quad (4.6)$$

The pole of the Green function ensures that the dominant contribution from the momentum integral comes from the momenta that obey $\omega_\sigma(\boldsymbol{p}) = \epsilon$. Furthermore, since ϵ scales as T, for the leading T dependence, we can neglect all ϵ dependencies that do not occur in the form ϵ/T. Equation (4.6) then reduces to

$$\sigma = \frac{e^2}{2m_{\rm e}} \int_{-\infty}^{\infty} \frac{d\epsilon}{4T} \frac{1}{\cosh^2(\epsilon/2T)} \sum_{\sigma} n_\sigma \frac{\Lambda_\sigma(\epsilon)}{\Gamma_\sigma(\epsilon)}. \quad (4.7a)$$

Here, n_σ is the density of the σ-spin electrons, Γ_σ is the single-particle rate defined by Eq. (3.3), and

$$\Lambda_\sigma(\epsilon) = \frac{1}{N_F^\sigma V} \sum_{\boldsymbol{p}} \delta\left[\omega_\sigma(\boldsymbol{p})\right]\gamma_\sigma(\boldsymbol{p};\epsilon + i0,\epsilon - i0). \quad (4.7b)$$

Using analogous arguments we find, from Eq. (4.5), that $\Lambda_\sigma(\epsilon)$ obeys an integral equation:

$$\Lambda_\sigma(\epsilon) = 1 + \sum_{\sigma'} N_F^{\sigma'} \int du\, W_{\sigma\sigma'}(u)[n_{\rm B}(u)$$
$$+ n_{\rm F}(u + \epsilon)] \frac{\Lambda_{\sigma'}(\epsilon + u)}{\Gamma_{\sigma'}(\epsilon + u)}, \quad (4.8a)$$

where

$$W_{\sigma\sigma'}(u) = \frac{1}{N_F^\sigma N_F^{\sigma'} V^2} \sum_{\boldsymbol{p},\boldsymbol{k}} \sum_{\sigma'\neq\sigma} \delta\left[\omega_{\sigma'}(\boldsymbol{k})\right]$$
$$\times \delta[\omega_\sigma(\boldsymbol{p})]\mathcal{V}_{\sigma\sigma'}''(\boldsymbol{k} - \boldsymbol{p},u)\,\boldsymbol{k}\cdot\boldsymbol{p}/\boldsymbol{p}^2 \quad (4.8b)$$

with $\mathcal{V}_{\sigma\sigma'}''$ the spectrum of the effective potential defined in Eq. (2.15b).

Now we exploit the fact that $\boldsymbol{k}$ and $\boldsymbol{p}$ are pinned to the respective Fermi surfaces, and use the resulting identity

$$\boldsymbol{k}\cdot\boldsymbol{p} = k_{\rm F}^2\left[1 - \omega_0(\boldsymbol{k} - \boldsymbol{p})/2Dk_{\rm F}^2\right]$$

to write

$$W_{\sigma\sigma'}(u) = \left(k_{\rm F}/k_{\rm F}^\sigma\right)^2\left[\bar{\mathcal{V}}_{\sigma\sigma'}''(u) - \bar{\mathcal{V}}_{\sigma\sigma'}''^{(2)}(u)\right] \quad (4.9)$$

with $\bar{\mathcal{V}}''$ from Eq. (3.4b), and

$$\bar{\mathcal{V}}_{\sigma\sigma'}''^{(2)}(u) = \frac{1}{N_F^\sigma N_F^{\sigma'} V^2} \sum_{\boldsymbol{k},\boldsymbol{p}} \delta[\omega_\sigma(\boldsymbol{k})]\,\delta[\omega_{\sigma'}(\boldsymbol{p})] \frac{\omega_0(\boldsymbol{k} - \boldsymbol{p})}{2Dk_{\rm F}^2}$$
$$\times \mathcal{V}_{\sigma\sigma'}''(\boldsymbol{k} - \boldsymbol{p},u). \quad (4.10)$$

Note that the magnon frequency ω_0 in Eq. (4.10) is equal to $\pm u$ on account of the spectrum, and therefore $\bar{\mathcal{V}}_{\sigma\sigma'}''^{(2)}(u)$ has an extra factor of u compared to $\bar{\mathcal{V}}''(u)$. $\bar{\mathcal{V}}''$ determines the single-particle rate Γ via Eq. (3.4a), and we define analogously

$$\Gamma_\sigma^{(2)}(\epsilon) = N_F^{-\sigma} \int du\, [n_B(u) + n_F(u + \epsilon)] \sum_{\sigma'} \bar{\mathcal{V}}_{\sigma\sigma'}''^{(2)}(u).$$
$$(4.11)$$

The integral equation for the vertex function Λ now reads

$$\Lambda_\sigma(\epsilon) = 1 + \left(\frac{k_F}{k_F^\sigma}\right)^2 \int du \sum_{\sigma'} N_F^{\sigma'} \left[\bar{\mathcal{V}}''_{\sigma\sigma'}(u) - \bar{\mathcal{V}}''^{(2)}_{\sigma\sigma'}(u)\right]$$

$$\times \left[n_B(u) + n_F(u+\epsilon)\right] \frac{\Lambda_{\sigma'}(u+\epsilon)}{\Gamma_{\sigma'}(u+\epsilon)}. \qquad (4.12)$$

For the case of a spin-independent potential, Eq. (4.12) reduces to the integral equation familiar from the electron-phonon scattering problem; only the u dependence of the kernel is different. This integral equation is usually solved in the seemingly uncontrolled approximation that replaces $\Lambda(u+\epsilon)/\Gamma(u+\epsilon)$ on the right-hand side by $\Lambda(\epsilon)/\Gamma(\epsilon)$, turning the integral equation into an algebraic equation. In Ref. [35], two of the present authors have shown that the integral equation can be solved asymptotically exactly, the exact solution yields a result for the conductivity that coincides with the lowest-order variational solution of the Boltzmann equation, and the simple approximation yields the same low-temperature dependence (albeit with a different prefactor) as the exact solution. The proof of these statements can be generalized to the current case of a two-by-two matrix equation. For the purpose of deriving the low-temperature behavior, we thus can employ the approximation, which turns Eq. (4.12) into two coupled algebraic equations for $\Lambda_\pm(\epsilon)$. Since the prefactor of the temperature dependence of the conductivity is approximation-dependent anyway, we can put $\epsilon = 0$ and use the temperature-dependent rates $\Gamma_\sigma \equiv \Gamma_\sigma(\epsilon = 0)$ and vertices $\Lambda_\sigma \equiv \Lambda_\sigma(\epsilon = 0)$ in the Kubo formula, Eq. (4.7a). Λ_σ then obeys

$$\Lambda_+ = 1 + (k_F/k_F^+)^2 [\Gamma_+ - \Gamma_+^{(2)}] \Lambda_-/\Gamma_-,$$
$$\Lambda_- = 1 + (k_F/k_F^-)^2 [\Gamma_- - \Gamma_-^{(2)}] \Lambda_+/\Gamma_+. \qquad (4.13)$$

Γ_σ is given by Eq. (3.6), and $\Gamma_\sigma^{(2)}$ is given by an analogous integral with an additional power of the frequency in the integrand. We find

$$\Gamma_\sigma^{(2)} = \frac{\pi K}{N_F^\sigma} \frac{T^2}{T_1^2} \int_{T_0/T}^{T_1/T} dx \frac{x}{\sinh x}$$

$$= \frac{\pi K}{N_F^\sigma} \frac{T}{T_1} \begin{cases} \frac{2T_0}{T_1}\left(1 + \frac{T}{T_0}\right) e^{-T_0/T} & \text{if} \quad T \ll T_0 \\ \frac{\pi^2}{4} T/T_1 & \text{if} \quad T_0 \ll T \ll T_1 \\ 1 & \text{if} \quad T \gg T_1 \end{cases}. \qquad (4.14)$$

Comparing with Eq. (3.6), we see that for asymptotically small T, $\Gamma^{(2)}$ is proportional to Γ with a small factor of proportionality $2T_0/T_1 \ll 1$, whereas for $T_0 \ll T \ll T_1$, it carries an additional factor of temperature.

We now solve the equations (4.13). Neglecting $\lambda/\epsilon_F \ll 1$ wherever it is not of qualitative importance, we find

$$\frac{\Lambda_\pm}{\Gamma_\pm} = \frac{\Gamma_+ + \Gamma_- - \Gamma_\pm^{(2)}}{-\frac{4T_0}{T_1}\Gamma_+\Gamma_- + \Gamma_+\Gamma_-^{(2)} + \Gamma_-\Gamma_+^{(2)} - \Gamma_+^{(2)}\Gamma_-^{(2)}}. \qquad (4.15)$$

Equation (3.4e) allows us to express $\Lambda_\sigma/\Gamma_\sigma$ entirely in terms of Γ_σ and $\Gamma_\sigma^{(2)}$. Neglecting all prefactors that just give small

corrections of $O(\lambda/\epsilon_F)$ to factors of $O(1)$, we finally obtain a transport relaxation time,

$$\tau_{tr} = \frac{\Gamma - \Gamma^{(2)}/2}{-\frac{4T_0}{T_1}(\Gamma)^2 + 2\Gamma\Gamma^{(2)} - (\Gamma^{(2)})^2}, \qquad (4.16a)$$

in terms of which the electrical conductivity is given by a Drude formula

$$\sigma = \frac{n_e e^2}{m_e} \tau_{tr}, \qquad (4.16b)$$

Here, $\Gamma \approx \Gamma_+ \approx \Gamma_-$ and $\Gamma^{(2)} \approx \Gamma_+^{(2)} \approx \Gamma_-^{(2)}$ are given by Eqs. (3.6) and (4.14), respectively, with N_F^σ replaced by N_F. Note that our approximations have affected overall prefactors only, but not the relative prefactors of the four terms in the denominator in Eq. (4.16a). Comparing Eq. (3.6) with (4.14) we see that for $T \ll T_0$, $\Gamma^{(2)}$ is proportional to Γ:

$$\Gamma^{(2)} = \frac{2T_0}{T_1}(1 + T/T_0)\Gamma, \qquad (4.17)$$

and an inspection reveals that the leading contributions among the three terms in the denominator in Eq. (4.16a) cancel, which leads to $1/\tau_{tr} \propto T^2 \exp(-T_0/T)$. At asymptotically low temperatures, τ_{tr} is therefore *not* proportional to $1/\Gamma^{(2)}$, but rather carries an extra factor of T_0/T. For the contribution, to the electrical resistivity $\rho_{el} = 1/\sigma$ due to magnon exchange, we finally obtain

$$\rho_{el}(T) = \frac{m_e}{n_e e^2} \frac{\pi K}{N_F T_1} \begin{cases} (4/T_1) T^2 e^{-T_0/T} & \text{if} \quad T \ll T_0 \\ (\pi^2/2T_1) T^2 & \text{if} \quad T_0 \ll T \ll T_1 \\ T & \text{if} \quad T \gg T_1 \end{cases}. \qquad (4.18)$$

We see that in the preasymptotic temperature window $T_0 \ll T \ll T_1$, we recover the T^2 behavior that was obtained in Ref. [15], but for asymptotically low temperatures, we obtain an exponentially small result that has the form of Eq. (1.1). We will discuss this result in the next section.

V. DISCUSSION

To summarize our results, we have presented a very general theory of electron relaxation due to the exchange of magnons in metallic ferromagnets. The theory is valid for both itinerant ferromagnets, where the magnetization is due to the conduction electrons themselves, and for localized-moment ferromagnets, where the magnetization is due to localized spins in a different band. We have found that at asymptotically low temperatures, below a temperature scale T_0, both the single-particle relaxation rate and the transport relaxation rate are exponentially small. This behavior carries over to the magnon-exchange contributions to the thermal and electrical resistivities, which are determined by these respective rates. The exponential temperature dependence is a direct consequence of the split conduction band in a metallic ferromagnet. In a preasymptotic temperature regime, $T_0 \ll T \ll T_1$, with T_1 close to the exchange splitting, we recover the T^2 behavior of the transport rate that was found in Ref. [15]. The single-particle rate is proportional to T in
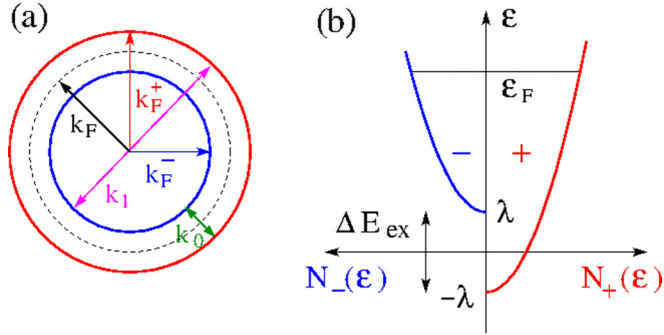
FIG. 3. (Color online) Fermi surfaces and associates Fermi wave numbers (a) and densities of states (b) for the up- and down-spin electrons. λ is the Stoner gap and ΔE_{ex} is the exchange splitting. k_0 and k_1 are the smallest and largest transferrable wave numbers, respectively. See the text for more explanation.

this regime. For $T \gg T_1$, the two rates both show a linear temperature dependence.

We start our discussion of these results by recalling the physical reason for the exponential dependence at low temperatures. Figure 3 schematically shows the split conduction band (a), and the densities of states for the up (+) and down (−) spin electrons (b), for the case of a spherical Fermi surface. Since the magnons couple only electrons with opposite spin, the smallest transferrable wave number is $k_0 = k_F^+ - k_F^- \approx \Delta E_{ex}/v_F$. Given the magnon dispersion relation, $\omega = Dk^2$, this translates into a smallest transferrable energy $T_0 = D k_0^2$, and since the magnon stiffness coefficient D is itself roughly proportional to ΔE_{ex}, we have $T_0 \propto (\Delta E_{ex})^3$. For temperatures $T \ll T_0$, the relaxation rates will thus show activated behavior with an activation energy T_0. The exponential behavior is multiplied by a power law that cannot be captured by elementary arguments. The largest momentum transfer is given by $k_1 = k_F^+ + k_F^- \approx 2k_F$, and the corresponding largest energy transfer is $T_1 = D k_1^2 \approx \Delta E_{ex}$. T_1 is the fundamental magnetic energy scale, analogous to the Debye temperature Θ_D in the case of electron-phonon coupling. T_0 has no analog in the electron-phonon problem. For $T \ll T_1$, the transport-relaxation rate is small compared to the single-particle rate by a factor of T/T_1. This is analogous to the electron-phonon case, where the corresponding factor is $(T/\Theta_D)^2$. The difference between our results and those of Ueda and Moriya, Ref. [15], can be traced to the fact that these authors neglected the exchange splitting in the final stages of their calculation. As a result, they obtained a T^2 behavior of the transport relaxation rate at low temperatures, which in fact is valid only for temperatures larger than T_0. Note that this discrepancy pertains to the magnon or spin-wave contribution to the electrical resistivity only. The contributions from dissipative excitations, which we have not discussed, have been found to be unaffected by the exchange splitting and proportional to T^2 even at asymptotically low temperatures [15].

For the power-law behavior at $T \gg T_0$, the quadratic spectrum of the magnons is important, and also the coupling of the electrons to the magnetic fluctuations. Comparing with the case of helical magnets [6], we notice one important difference with respect to the latter. In either case, the Goldstone mode is a phase fluctuation, but in the ferromagnon case, the electron

spin density couples directly to the phase, whereas in the helimagnon case, the coupling is to the gradient of the phase. This is because in the helimagnon case the dominant low-T contribution to the scattering rates comes from intra-Stoner-band scattering. Within a given band, the phase itself has no physical meaning, and the coupling therefore involves a gradient. In the ferromagnetic case, on the other hand, we deal with inter-Stoner-band scattering. The coupling therefore effectively is to the difference of two phases, which does have a physical meaning. We note in passing that this latter notion also manifests itself in a spin Josephson effect, see Ref. [36].

We now turn to estimates of the values of T_1 and T_0. To get an idea about the order of magnitude of these temperature scales, let us first consider the fictitious case of simple (i.e., single-conduction-band) metals with magnetic properties as in the classic "high-temperature" ferromagnets nickel, cobalt, and iron. The values of the exchange splitting in these materials, as determined by photoemission, are $\Delta E_{ex} \approx 0.25$, 1.0, and 2.0 eV, respectively [37,38]. Values for the spin-stiffness coefficient D in meV Å^2 obtained from neutron scattering are 364 for Ni, 500 for Co, and 281 for Fe [28]. With a generic value $k_F \approx 1$ Å^{-1} for the Fermi wave number, and $\epsilon_F \approx 10^5$ K for the Fermi energy, Eqs. (2.16) and (2.17) yield $T_1 \approx 10\,000$–$20\,000$ K for these materials, and $T_0 \approx 500$ mK for Ni, 10 K for Co, and 30 K for Fe. Estimates of the ratio T_0/T_1 using the relation (2.18b) yields similar results. Notice that the prefactor $\pi K/N_F T_1$ in Eq. (4.18) is of order unity, so the prefactor of the T^2 behavior of the resistivity is larger than the Fermi-liquid T^2 contribution [see Eq. (B7)] by roughly a factor of $\epsilon_F/T_1 \approx 10$ in a single-band model.

Also of interest are weak ferromagnets, such as MnSi [39] or Ni$_3$Al, where $D \approx 23.5$ meV Å^2 (MnSi) [40] and $D \approx 70$ meV Å^2 (Ni$_3$Al) [41], respectively. The magnetic moments, $0.4\mu_B$ per formula unit for MnSi [42] and $0.17\mu_B$ for Ni$_3$Al [41], are about two thirds and one third, respectively, of that of Ni. Given the observed near-linear correlation between the magnetic moment and the exchange splitting [37], this suggests $\Delta E_{ex} \approx 0.17$ eV for MnSi [43], and $\Delta E_{ex} \approx 0.07$ eV for Ni$_3$Al. If we use again $k_F \approx 1$ Å^{-1} and $\epsilon_F \approx 10^5$ K, this yields $T_1 \approx 1000$ K and $T_0 \approx 20$ mK for MnSi, and $T_1 \approx 2800$ K and $T_0 \approx 10$ mK for Ni$_3$Al.

In reality, all of these materials are transition metals, or compounds containing transition metals, with a complicated band structure and Fermi surfaces that consist of multiple sheets. One consequence of this is that the electron-electron scattering contribution to the electrical resistivity is likely much larger than a single-band model would imply, and it has been suggested that it makes the largest contribution to the observed T^2 behavior at low temperatures [16]. The reason is that different band edges have different distances from the common chemical potential, which in effect leads to different Fermi temperatures. Depending on whether or not the various scattering processes flip the electron spin, and whether or not they couple different sheets of the Fermi surface, the relaxation rates or the relaxation times may be additive, which leads a complicated structure of the overall resistivity. In addition, there are the contributions from the dissipative spin excitations, which also are proportional to T^2 [15]. As a result, the low-temperature transport rate in Fe, Ni, and Co is about 100 times larger than one would expect

from the Coulomb contribution in a single-band model with a single Fermi temperature of about 10^5 K [16]. T_1, on the other hand, is largely unaffected by a complicated band structure; it is given by D times the largest possible momentum transfer squared, see Sec. II C, and in a good metal the latter is on the order of $2\pi/a$, with a the lattice constant, which is close to the value of $2k_F$ for a single spherical Fermi surface that yields the same electron density. The estimates of the temperature scale T_1, which is the magnetic analog of the Debye temperature for phonons, given above are therefore model independent and depend only on the experimentally measured spin stiffness coefficient.

As a result, we expect the magnon contribution to the electrical resistivity in Fe, Ni, and Co at temperatures $T > T_0$ to be about an order of magnitude *less* than the combined contribution from the Coulomb interaction and the dissipative magnetic excitations. In MnSi and Ni$_3$Al, T_1 is much lower and the magnon scattering is accordingly stronger. However, the observed prefactors of the T^2 term in the resistivity of MnSi and Ni$_3$Al are orders of magnitude larger than even the ones in Fe, Ni, and Co, and the same is true for the weak ferromagnet ZrZn$_2$ [44,45]. The prefactor $\pi K/N_F T_1$ in Eq. (4.18) is expected to be of $O(1)$ not just in model calculations, but also in real materials, since both K and T_1 correlated roughly with the magnetization. Given the above discussion of the relatively narrow range of plausible values of T_1, we conclude that the experimental value of the prefactor of the T^2 term in the electrical resistivity of weak ferromagnets cannot possibly be explained by electron-magnon scattering. We emphasize again, however, that these considerations do not take into account the scattering of electrons by dissipative magnetization fluctuations, which lead to a T^2 contribution to the resistivity even at low temperatures and whose prefactor is not as universal as that of the magnon-exchange contribution. A corresponding statement holds for the Coulomb contribution.

For T_0, the influence of the band structure is more complicated. Consider the effective potential given by Eqs. (3.4b)–(3.4d). If the up-spin and down-spin electrons, respectively, belong to different bands with different effective masses, then there will be a lower cutoff for the frequency u even in the limit of a vanishing Stoner gap, $\lambda \to 0$. For magnon-exchange scattering between electrons in Stoner subbands of the same band, on the other hand, the structure of the calculations in Secs. III and IV is unchanged. We therefore expect different values of T_0 for the various scattering processes that involve electrons on different sheets of the Fermi surface.

The following picture now emerges. With decreasing temperature, contributions to the magnon-exchange part of the electronic scattering rate will sequentially freeze-out as the temperature drops below a sequence of temperature scales T_0. Rough estimates for the lowest of these temperature scales have been given above; estimating the higher ones requires a detailed analysis of the band structure. Below this lowest T_0 the magnon contribution to both the transport and the single-particle rates will be exponentially small, leaving the Coulomb contribution and the one from dissipative magnetization fluctuations as the most obvious candidates for a T^2 behavior. Experimentally, this is expected to manifest

itself in a distinct temperature dependence of the prefactor of the T^2 term in the electrical resistivity. It is desirable for the relevant temperature scales to be small enough that phonon contributions are negligible. In that respect, Fe, Ni, and Co are not ideal. In MnSi, the helical nature of the magnetic phase is expected to manifest itself on the temperature scale given by T_0. This leaves Ni$_3$Al, or other true weak ferromagnets as the most promising candidates for observing this consequence of the exchange splitting in a metallic ferromagnet. We stress, however, that according to the above discussion the magnon contribution to the electrical resistivity in weak ferromagnets is likely dwarfed by other contributions.

Another possible effect of a complicated band structure is that there may be points or lines in reciprocal space where the two Stoner band cross. This will weaken the exponential suppression of the relaxation rates, but the weakening will depend on the nature of the crossing.

We finally mention that the interplay of quenched disorder with the scattering processes discussed above constitutes an interesting problem that is likely important for a quantitative understanding of real materials. For fairly strong disorder, $\lambda\tau_{el} \ll 1$ with τ_{el} the elastic scattering time, the theory of Ref. [46] applies, and it is easy to see that there is no exponential suppression of the magnon contribution to the relaxation rates at low temperature. A complete discussion of disorder effects constitutes a separate problem.

APPENDIX A: STONER MODEL FOR ITINERANT FERROMAGNETS

In this Appendix, we show how to recover the Stoner-Moriya results [2] for itinerant ferromagnets within the present formalism. Our starting point is a fermionic action,

$$S[\bar\psi,\psi] = S_0[\bar\psi,\psi] + \frac{\Gamma_t}{2} \int dx\, \boldsymbol{n}_s(x) \cdot \boldsymbol{n}_s(x), \quad (A1)$$

with Γ_t the spin-triplet interaction amplitude that is responsible for ferromagnetism. Our notation is the same as in Sec. II.

1. Mean-field approximation

A simple mean-field approximation that describes ferromagnetic order, and its coupling to the electron spin density, consists of replacing one of the spin density fields in Eq. (A1) by its expectation value according to

$$\boldsymbol{n}_s^2 \approx 2\langle \boldsymbol{n}_s\rangle \cdot \boldsymbol{n}_s - \langle \boldsymbol{n}_s\rangle^2 . \quad (A2)$$

If we take the magnetic order to be in the 3-direction, $\langle n_{s,i}\rangle = \delta_{i3}\lambda/\Gamma_t$, then this approximation amounts to replacing the action S with an action S_λ that describes electrons with not

spin-triplet interaction subject to a magnetic field of strength λ in the 3-direction:

$$S_\lambda[\bar\psi,\psi] = S_0[\bar\psi,\psi] + \lambda \int dx \, n_{s,3}(x). \qquad \text{(A3)}$$

The remaining question pertains to the action one should use to calculate $\langle \boldsymbol{n}_s \rangle$; this choice determines λ. The usual self-consistent mean-field requirement stipulates that this average be determined by S_λ itself :

$$\lambda = \Gamma_t \langle n_{s,3}(x) \rangle_\lambda \equiv \frac{\Gamma_t}{Z_\lambda} \int D[\bar\psi,\psi] \, n_{s,3}(x) \, e^{S_\lambda[\bar\psi,\psi]}$$

$$= \Gamma_t \frac{d}{d\lambda} \ln Z_\lambda. \qquad \text{(A4)}$$

For simplicity, we take S_0 to describe free electrons. That is, we neglect all electron-electron interactions that are not crucial for magnetism, and we assume a parabolic band; a generalization to band electrons is straightforward. The Green function corresponding to S_0 then is

$$G_0(\boldsymbol{k},i\omega_n) = \frac{1}{i\omega_n - \xi_{\boldsymbol{k}}}, \qquad \text{(A5)}$$

and the self-consistency condition, Eq. (A4), takes the form

$$1 = -2\Gamma_t \int_p \frac{1}{G_0^{-2}(p) - \lambda^2}. \qquad \text{(A6)}$$

We recognize this as the equation of state of Stoner theory, with λ the Stoner gap. The condition for a nonzero solution for λ is $2N_F\Gamma_t > 1$, and by performing the integral, we find explicitly

$$\lambda = 2N_F\Gamma_t \frac{\epsilon_F}{3} [(1 + \lambda/\epsilon_F)^{3/2} - (1 - \lambda/\epsilon_F)^{3/2}]. \qquad \text{(A7a)}$$

If we recall that the magnetization is given as $m = \lambda/\Gamma_t$ [see Eqs. (2.2) and (2.3a)], we can write this result as

$$m = 2N_F\lambda \{1 + O[(\lambda/\epsilon_F)^2]\}. \qquad \text{(A7b)}$$

The action S_λ contains information about the long-range order, but does not contain any ferromagnetic fluctuations. It will serve as a building block for the effective action, and we will refer to it as the "reference ensemble."

We now determine the spin susceptibility

$$\chi_{\lambda,ij}(x,y) = \langle \delta n_{s,i}(x) \, \delta n_{s,j}(y) \rangle_{S_\lambda}, \qquad \text{(A8)}$$

associated with the reference ensemble. In terms of the reference-ensemble Green function

$$G_\lambda(k) = \frac{G_0^{-1}(k)}{G_0^{-2}(k) - \lambda^2} \sigma_0 - \frac{\lambda}{G_0^{-2}(k) - \lambda^2} \sigma_3, \qquad \text{(A9)}$$

χ_λ can be written

$$\chi_{\lambda,ij}(x,y) = -\text{tr}\,[\sigma_i \, G_\lambda(x,y) \, \sigma_j \, G_\lambda(y,x)], \qquad \text{(A10)}$$

where the trace is over the spin degrees of freedom. Evaluating the trace and performing a Fourier transform, we find

$$\chi_{\lambda,ij}(k) = \begin{pmatrix} f_1(k) & f_2(k) & 0 \\ -f_2(k) & f_1(k) & 0 \\ 0 & 0 & f_3(k) \end{pmatrix}, \qquad \text{(A11)}$$

where

$$f_1(k) = -2 \int_p \frac{G_0^{-1}(p)G_0^{-1}(p-k) - \lambda^2}{[G_0^{-2}(p) - \lambda^2][G_0^{-2}(p-k) - \lambda^2]}, \qquad \text{(A12a)}$$

$$f_2(k) = -2i\lambda \int_p \frac{G_0^{-1}(p) - G_0^{-1}(p-k)}{[G_0^{-2}(p) - \lambda^2][G_0^{-2}(p-k) - \lambda^2]}, \qquad \text{(A12b)}$$

$$f_3(k) = -2 \int_p \frac{G_0^{-1}(p)G_0^{-1}(p-k) + \lambda^2}{[G_0^{-2}(p) - \lambda^2][G_0^{-2}(p-k) - \lambda^2]}. \qquad \text{(A12c)}$$

We note that

$$f_1(k = 0) = 1/\Gamma_t, \qquad \text{(A13a)}$$

$$f_2(k = 0) = 0, \qquad \text{(A13b)}$$

with the first equality following from the equation of state, Eq. (A6).

2. Physical spin susceptibility and Goldstone modes

The reference ensemble does not reflect the magnons that are the Goldstone modes of the spontaneously broken symmetry in the ferromagnetic phase. To describe the magnons, we need a theory of fluctuations that is consistent with the treatment of the static magnetization. Quite generally, a Gaussian approximation for the order-parameter fluctuations is consistent with a mean-field treatment of the order parameter itself [47]. To determine the former, we first note that the reference ensemble spin susceptibility χ_λ corresponds to a Gaussian fluctuation action

$$\mathcal{A}_{\lambda,\text{fluct}}[\delta\boldsymbol{n}_s] = \frac{-1}{2} \int dx\,dy \, \delta n_{s,i}(x) \, \chi_{\lambda,ij}^{-1}(x,y) \, \delta n_{s,j}(y) \qquad \text{(A14a)}$$

that generates χ_λ via

$$\chi_{\lambda,ij}(x,y) = \int D[\delta\boldsymbol{n}_s] \, \delta n_{s,i}(x) \, \delta n_{s,j}(y) \, e^{-\mathcal{A}_{\lambda,\text{fluct}}[\delta\boldsymbol{n}_s]}. \qquad \text{(A14b)}$$

To this we need to add the fluctuation contribution from the original spin-triplet interaction in Eq. (A1). The Gaussian fluctuation action then reads

$$\mathcal{A}_{\text{fluct}}[\delta\boldsymbol{n}_s] = \frac{-1}{2} \int dx\,dy \, \delta n_{s,i}(x) \, \chi_{ij}^{-1}(x,y) \, \delta n_{s,j}(y) \qquad \text{(A15a)}$$

with the physical spin susceptibility χ given by

$$\chi_{ij}^{-1}(x,y) = \chi_{\lambda,ij}^{-1}(x,y) - \delta_{ij} \, \Gamma_t. \qquad \text{(A15b)}$$

Focusing on the transverse (T) channel ($i = 1,2$), and performing a Fourier transform, we have

$$\chi_{\mathrm{T}}^{-1}(\boldsymbol{k},i\Omega_n) = \begin{pmatrix} f_1(\boldsymbol{k},i\Omega_n)/N(\boldsymbol{k},i\Omega_n) - \Gamma_{\mathrm{t}} & -f_2(\boldsymbol{k},i\Omega_n)/N(\boldsymbol{k},i\Omega_n) \\ f_2(\boldsymbol{k},i\Omega_n)/N(\boldsymbol{k},i\Omega_n) & f_1(\boldsymbol{k},i\Omega_n)/N(\boldsymbol{k},i\Omega_n) - \Gamma_{\mathrm{t}} \end{pmatrix}, \quad \text{(A16a)}$$

where

$$N(\boldsymbol{k},i\Omega_n) = [f_1(\boldsymbol{k},i\Omega_n)]^2 + [f_2(\boldsymbol{k},i\Omega_n)]^2. \quad \text{(A16b)}$$

From Eq. (A.13), we see that χ_{T}^{-1} at zero frequency and wave number has two zero eigenvalues. These reflect the two Goldstone modes. Expanding to linear order in $i\Omega$ and to second order in $\boldsymbol{k}$, we find explicitly

$$\chi_{\mathrm{T}}^{-1}(\boldsymbol{k},i\Omega_n) = \frac{(2N_{\mathrm{F}}\Gamma_{\mathrm{t}})^2}{2N_{\mathrm{F}}} \begin{pmatrix} \hat{\boldsymbol{k}}^2 f_{\boldsymbol{k}}(\lambda)/3 & 2i(i\hat{\Omega}_n)f_\Omega(\lambda)\epsilon_{\mathrm{F}}/\lambda \\ -2i(i\hat{\Omega}_n)f_\Omega(\lambda)\epsilon_{\mathrm{F}}/\lambda & \hat{\boldsymbol{k}}^2 f_{\boldsymbol{k}}(\lambda)/3 \end{pmatrix}, \quad \text{(A17a)}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/2k_{\mathrm{F}}$, $\hat{\Omega} = \Omega/4\epsilon_{\mathrm{F}}$, and

$$f_{\boldsymbol{k}}(\lambda) = \frac{-4\epsilon_{\mathrm{F}}^3}{5\lambda^3} \left[\left(1 - \frac{3\lambda}{2\epsilon_{\mathrm{F}}}\right)\left(1 + \frac{\lambda}{\epsilon_{\mathrm{F}}}\right)^{3/2} - \left(1 + \frac{3\lambda}{2\epsilon_{\mathrm{F}}}\right)\left(1 - \frac{\lambda}{\epsilon_{\mathrm{F}}}\right)^{3/2} \right], \quad \text{(A17b)}$$

$$f_\Omega(\lambda) = \frac{\epsilon_{\mathrm{F}}}{3\lambda} \left[\left(1 + \frac{\lambda}{\epsilon_{\mathrm{F}}}\right)^{3/2} - \left(1 - \frac{\lambda}{\epsilon_{\mathrm{F}}}\right)^{3/2} \right]. \quad \text{(A17c)}$$

Physically, the Stoner gap is always small compared to the Fermi energy, and it therefore is useful to consider the limit of weak ferromagnets, $2N_{\mathrm{F}}\Gamma_{\mathrm{t}} \approx 1$ and $\lambda/\epsilon_{\mathrm{F}} \ll 1$, where we have

$$f_{\boldsymbol{k}}(\lambda \to 0) = f_\Omega(\lambda \to 0) = 1 + O(\lambda^2). \quad \text{(A18)}$$

Inverting Eq. (A17a), we obtain the transverse physical spin susceptibility in the form given in Eq. (2.14b), with the mean-field values for $K(\lambda)$ and $D(\lambda)$ as quoted in the main text. We note that the spin precession effect, which is represented by the off-diagonal matrix elements in Eq. (A17a) and leads to the characteristic $\omega \propto \boldsymbol{k}^2$ dispersion relation of ferromagnetic magnons, appears in a rather elementary way in this treatment of itinerant electrons. In spin models, by contrast, it emerges from a topological contribution to the action [48].

APPENDIX B: SINGLE-PARTICLE SCATTERING RATE IN A FERMI LIQUID DUE TO COULOMB AND ELECTRON-PHONON INTERACTIONS

As a further illustration of our arguments leading to an effective action for calculating relaxation rates, let us consider the well-known case of quasiparticle relaxation due to density fluctuations. To this end, we consider the very simple case of spinless, noninteracting electrons with action S_0, and add a statically screened Coulomb interaction

$$S_{\mathrm{int}} = \int_k \delta n(k)\, v_{\mathrm{sc}}(\boldsymbol{k})\, \delta n(-k)\,. \quad \text{(B1)}$$

Here, $v_{\mathrm{sc}}(\boldsymbol{k}) = 4\pi e^2/(\boldsymbol{k}^2 + \kappa^2)$, with κ the screening wave number, and $n(k)$ is the Fourier transform of the electron number density $n(x) = \bar{\psi}(x)\psi(x)$. A finite average density is already built into S_0 via the chemical potential, so S_0 serves the purpose of the reference ensemble action S_λ in Sec. II or Appendix A. Now we follow the logic of Sec. II A. A number density fluctuation δN will couple to the field $\delta n(x)$ via the

interaction v_{sc} to produce an action,

$$S[\bar{\psi},\psi] = S_0[\bar{\psi},\psi + \int dx\, dy\, \delta N(x)\delta(\tau_x - \tau_y) \\ \times v_{\mathrm{sc}}(\boldsymbol{x} - \boldsymbol{y})\delta n(y), \quad \text{(B2)}$$

and the density fluctuations are governed by a Gaussian action

$$S_{\mathrm{fluct}}[\delta N] = \frac{-1}{2} \int dx\, dy\, \delta N(x)\, \chi^{-1}(x - y)\, \delta N(y), \quad \text{(B3)}$$

with χ the physical density susceptibility. Integrating out the density fluctuations, we obtain an effective action

$$S_{\mathrm{eff}}[\bar{\psi},\psi] = S_0[\bar{\psi},\psi] + \frac{1}{2} \int_k \delta n(k)\, V(k)\, \delta n(-k) \quad \text{(B4a)}$$

with an effective potential

$$V(k) = [v_{\mathrm{sc}}(\boldsymbol{k})]^2\, \chi(k). \quad \text{(B4b)}$$

Now we calculate the single-particle relaxation rate as in Sec. III. We obtain

$$\frac{1}{2\tau} = \Gamma(\epsilon = 0) = 2N_{\mathrm{F}} \int_{-\infty}^{\infty} du\, \bar{V}''(u)\, \frac{1}{\sinh(u/T)}, \quad \text{(B5a)}$$

where

$$\bar{V}''(u) = \frac{1}{(N_{\mathrm{F}}V)^2} \sum_{\boldsymbol{k},\boldsymbol{p}} \delta(\xi_{\boldsymbol{k}})\,\delta(\xi_{\boldsymbol{p}})\, V''(\boldsymbol{k} - \boldsymbol{p},u). \quad \text{(B5b)}$$

From Eq. (B4b), we see that the spectrum of the potential V is given by the spectrum of the density susceptibility, which to lowest order in the screened Coulomb interaction is just the Lindhard function χ_0. For $|u| < (2k_{\mathrm{F}}|\boldsymbol{k}| - \boldsymbol{k}^2)/2m_{\mathrm{e}}$, the spectrum of the latter is

$$\chi_0''(\boldsymbol{k},u) = \pi N_{\mathrm{F}}\, u/v_{\mathrm{F}}|\boldsymbol{k}|, \quad \text{(B6)}$$

with v_{F} the Fermi velocity. For the relaxation rate due to the electron-electron interaction we thus obtain the well-known

Fermi-liquid result

$$\frac{1}{2\tau_{\text{e-e}}} = \frac{\pi}{4}\frac{T^2}{\epsilon_{\text{F}}}. \tag{B7}$$

The above derivation is similar in spirit to the arguments given in Ref. [49]. The point of this exercise is to demonstrate that our heuristic method of coupling density fluctuations to the appropriate fermion fields that we employed in Sec. II still works in this case where the density fluctuations are produced by the very electrons they couple to. To put the result for the effective interaction, Eq. (B4a), in context, consider a bare Coulomb interaction, $v_{\text{c}}(\boldsymbol{k}) = 4\pi e^2/\boldsymbol{k}^2$ and perform an RPA resumption to produce a dynamically screened Coulomb interaction

$$V_{\text{sc}}(k) = \frac{v_{\text{c}}(\boldsymbol{k})}{1 + v_{\text{c}}(\boldsymbol{k})\chi_0(k)} . \tag{B8}$$

To linear order in the frequency, the spectrum of the effective potential V coincides with the spectrum of V_{sc}, and V therefore suffices to produce the leading low-temperature dependence of the relaxation rate. Our effective action thus captures the leading effects of the soft modes in the system (here, the soft particle-hole excitations that are reflected in the spectrum of the Lindhard function; in Secs. II and III, the magnons). Note that it does *not* suffice to produce static screening, which requires taking into account massive modes, which is why the above argument starts with a statically screened interaction. Also note that the effective interaction V is quadratic in the bare interaction v_{sc}, in accordance with Fermi's golden rule. Analogously, the effective interaction in Sec. II, Eq. (2.15b), is quadratic in the coupling constant Γ_{t}. We also mention that the

T^2 result, Eq. (B7), holds for any short-ranged interaction, with the prefactor proportional to the potential strength squared.

The above considerations assumed an electronic density fluctuation δn interacting with a density fluctuation δN created by all other electrons, in analogy with magnetization fluctuations in the case of an itinerant magnet. However, there is no reason why δN cannot be a density fluctuation extraneous to the electron system, in analogy to magnetization fluctuations due to electrons in a band other than the conduction band. For instance, if δN is an ionic density fluctuation, it will still couple to δn via a statically screened Coulomb interaction. Equations (B2)–(B5) remain formally valid, except that the susceptibility χ now describes ionic density fluctuations, i.e., phonons. If we consider longitudinal phonons, the susceptibility is the same as in a fluid and given by [32]

$$\chi''(\boldsymbol{k},u) = \pi\,\rho^2\,\kappa\,u^2\,\delta\big[u^2 - \omega_{\text{L}}^2(\boldsymbol{k})\big], \tag{B9}$$

with $\omega_{\text{L}}(\boldsymbol{k}) = c|\boldsymbol{k}|$ the longitudinal phonon frequency. Here, ρ is the ionic number density, c is the longitudinal speed of sound, and $\kappa = -(\partial V/\partial p)/V$, with V the system volume and p the pressure, is the compressibility. We thus have

$$\bar{V}''(u) = \frac{\pi}{16}\frac{\rho^2\kappa}{c^2 k_{\text{F}}^2 N_{\text{F}}^2}\,u^2\,\text{sgn}\,u, \tag{B10}$$

which leads to the familiar T^3 result for the single-particle scattering due to the electron-phonon interaction in metals,

$$1/\tau_{\text{e-ph}} = \frac{7\pi}{3}\,\zeta(3)\,\frac{\rho^2\kappa}{n_{\text{e}}\,m_{\text{e}}\,c^2}\,T^3. \tag{B11}$$

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