Polaronic effects and thermally enhanced weak superconductivity

A. V. Parafilo,[1] I. V. Krive,[1,2,3,4] R. I. Shekhter,[2] Y. W. Park,[4] and M. Jonson[2,5,6]

[1]*B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Lenin Avenue, Kharkov 61103, Ukraine*

[2]*Department of Physics, University of Gothenburg, SE-412 96 Göteborg, Sweden*

[3]*Physical Department, V. N. Karazin National University, Kharkov 61077, Ukraine*

[4]*Department of Physics and Astronomy, Seoul National University, 599 Gwanak-ro, Gwanak-gu, Seoul 151-747, Korea*

[5]*SUPA, Institute of Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom*

[6]*Department of Physics, Division of Quantum Phases and Devices, Konkuk University, Seoul 143-701, Korea*

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We theoretically analyze the d.c. Josephson current between two superconductors connected by a vibrating weak link. Assuming the weak link to be a "quantum dot", we predict that the critical current is thermally enhanced at low temperatures if the electron-vibron interaction is strong and has an anomalous temperature dependence in a large temperature interval. We estimate that this unusual behavior, which occurs because the current through the weak link is carried by "polaronic" Andreev states, is measurable in experiments on, e.g., carbon-nanotube-based Josephson junctions.

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I. INTRODUCTION

The question of how a vibrating weak link affects the supercurrent through a Josephson junction has been addressed in a number of theoretical studies over the last few years [1–5] (see also the brief review in Ref. [6]). Experiments have largely been lacking until very recently [7,8], when steps in the I-V characteristics of a nanoelectromechanical Josephson junction device comprising a flexible InAs nanowire were interpreted as the result of a resonant coupling between a.c. Josephson oscillations and vibration modes of the nanowire [7]. Encouraged by this development, we have theoretically investigated how the critical d.c. Josephson current in a nanoelectromechanical Josephson junction (where the weak link is a vibrating quantum dot) is affected by a strong electron-vibron coupling.

Vibrational effects are well known to influence *normal* electron transport in molecular transistors [9]. They typically give rise to sideband peaks in traces of the differential conductance and a modified elastic (zero bias) peak. Recent transport experiments involving suspended single-wall carbon nanotubes have shown that the electron-vibron interaction in nanotube-based molecular transistors can be strong, with a coupling constant g of order 1–2 [10]. Such a strong electron-vibron interaction leads to a Franck-Condon blockade (exponential suppression) of nonresonant tunneling at low temperatures [11]. The theoretical prediction [12] (supported by subsequent experiments [13]) of a nonmonotonic temperature dependence of the conductance as the Franck-Condon blockade is lifted at higher temperatures is relevant for the discussion to follow. The temperature scale associated with this anomalous behavior is related to the "polaron" energy $\varepsilon_p \simeq g^2\hbar\omega_0$, where ω_0 is the angular frequency of the vibrating quantum dot (QD). The Franck-Condon blockade is most pronounced when an electron occupying the QD has sufficient time to form a polaron (i.e., when $t_p \sim \hbar/\varepsilon_p \ll \hbar/\Gamma_0$, where Γ_0 is the width of the QD energy level ε_0) at bias voltages V and temperatures T such that $\varepsilon_p \gg eV, k_BT \gg \Gamma_0$. This is the regime of sequential electron tunneling. In the opposite limit, which corresponds to resonant tunneling, the Franck-Condon blockade is lifted by the emission and reabsorbtion of virtual vibrons. In this case vibrational effects do not influence the peak values of the conductance, although they shift the level position and modify the widths of the zero-bias and sideband conductance peaks [14,15].

Turning to superconductive transport, we note that when Cooper pairs tunnel through a single-level QD, one electron of each pair has to occupy a virtual energy level within the superconducting gap Δ_0 (for the short time $t_s \sim \hbar/\Delta_0$ allowed by Heisenberg's uncertainty relation). Therefore one would expect a Franck-Condon blockade of the Josephson current only if the polaron energy exceeds the superconducting gap (and hence the polaron formation time t_p is shorter than t_s). This is indeed the case, as was shown in Ref. [1] by a direct calculation of the d.c. Josephson current through a vibrating single-level QD. To lowest order in the QD level width the critical current was there found to be strongly suppressed if the electron-vibron coupling constant g was large: $J_c \propto \Gamma_L\Gamma_R \exp(-2g^2)$, where L/R stands for the left/right electrode. However, this effect is "hidden" in a multiplicative renormalization of the unknown bare partial level widths Γ_L, Γ_R. Furthermore, the Franck-Condon blockade cannot be lifted by increasing the temperature without destroying superconductivity itself. From the point of view of an experimental verification it is therefore more interesting to study the interplay of mechanical vibrations and superconductivity in the "soft" vibron limit, $\varepsilon_p \ll \Delta_0$. In this case our theoretical analysis below shows that there are large deviations from the standard temperature dependence of the critical current at temperatures of order $T_p \sim \varepsilon_p/k_B < T_c$, for which thermal fluctuations start to destroy the polaronic states (T_c is the critical temperature of the relevant superconductors) [16].

Before we introduce a specific model system and present a rigorous mathematical theory for this anomalous temperature dependence in Sec. II, we would like to note here that a qualitative understanding can be obtained by using a semiclassical description of the soft vibrational subsystem. In this approach which we will elaborate a bit further at the end of Sec. II, the impact of soft vibrons on electrons in an superconductor-normal metal-superconductor (SNS) junction is to form Andreev "polarons". These are bound Andreev subgap states which couple to the classical vibron field and have to be determined self-consistently. If the vibrons are modeled as a harmonic oscillator and the electron-vibron interaction is linear in the oscillator displacement, the result is a temperature-dependent shift x_c of the equilibrium position of the oscillator accompanied by a polaronic shift of the Andreev energy levels. At $T = 0$ the polaronic shift only leads to a power-law suppression of the critical current, $J_c \propto 1/g^2$ (see below). This is in striking contrast to the exponential suppression (Franck-Condon blockade) induced by zero-point fluctuations of "hard" vibrons [1]. However, soft vibrons are (obviously) more sensitive to temperature, and by increasing the temperature we find that thermal fluctuations diminish the QD-oscillator shift so that $|x_c(T)| < |x_c(0)|$. At low temperatures, $T \ll \varepsilon_p/k_B$, this leads to a thermally enhanced critical current, $J_c(T) > J_c(0)$. At higher temperatures, $T \gg \varepsilon_p/k_B$, we find a crossover to the standard asymptotic $1/T$ decay of the critical current as the oppositely directed partial currents carried by the negative- and positive-energy Andreev polaronic levels increasingly cancel each other. We will, however, show that even in the high-T region the temperature corrections to this standard behavior are determined by vibrational effects, which are due to thermal fluctuations of the oscillator in the symmetric state ($x_c = 0$). We stress once more that our predictions hold for the range of model parameters $\Gamma_0 \ll \varepsilon_0, \hbar\omega_0 \ll \Delta_0$, and $g \gtrsim 1$ for which the approximations used in our calculations are valid.

The qualitative arguments given above have to be justified by a rigorous theory for a specific model system. That is the objective of the Sec. II. Our conclusions follow in Sec. III.

II. ANOMALOUS TEMPERATURE DEPENDENCE OF CRITICAL CURRENT

We consider a simple model comprising a single-electron-level quantum dot that oscillates in a harmonic potential and is weakly coupled to two bulk superconductors (see Fig. 1). The Hamiltonian of the "normal part" of this nanoelectromechanical Josephson junction takes the form

$$H_{QD} = \sum_{\sigma=\uparrow,\downarrow} \varepsilon_0 d_\sigma^\dagger d_\sigma - \frac{g\hbar\omega_0}{\sqrt{2}}(\hat{n}_\uparrow + \hat{n}_\downarrow)(b^\dagger + b) + \hbar\omega_0 b^\dagger b + U_c \hat{n}_\uparrow \hat{n}_\downarrow, \tag{1}$$

where ε_0 is the energy of the QD level relative to the Fermi level, $d_\sigma^\dagger$ (d_σ) is an operator that creates (annihilates) a bare electron in the dot, $\hat{n}_\sigma = d_\sigma^\dagger d_\sigma$, g is a dimensionless electron-vibron coupling constant, $b^\dagger$ (b) is the vibron creation (destruction) operator, ω_0 is the vibron frequency, and U_c is the electron-electron correlation energy. The second term in Eq. (1) describes the electron-vibron interaction, which occurs due to the electrostatic interaction of the electron on the quantum dot with a gate electrode.

The superconducting electrodes are described by the standard BCS Hamiltonian with equal order parameter Δ_0 and different phases $\varphi_{L,R}$. A weak coupling between the QD and the superconductors is modeled by a standard tunnel Hamiltonian with energy-independent tunnel matrix elements.

In the limit of interest, where Δ_0 is the largest energy in the problem, the supercurrent is fully determined by the current carried by the subgap Andreev bound states. Without electron-vibron interactions, i.e., for noninteracting electrons, the nonresonant ($|\varepsilon_0| \gg \Gamma_L,\Gamma_R$) Josephson current can be expressed as $J(T,\varphi) = J_c(T)\sin\varphi$, where $\varphi = \varphi_R - \varphi_L$ is the superconducting phase difference and $J_c(T) = (e\Gamma_L/2\hbar)(\Gamma_R/\varepsilon_0)\tanh(\varepsilon_0/2k_B T)$ [17]. In this case the critical current is a monotonically decreasing function of temperature; it is essentially temperature independent at low temperatures ($T \ll \varepsilon_0/k_B$) and decays as $1/T$ at high temperatures. The same temperature dependence follows from the Ambegaokar-Baratoff formula for the critical current [18].

The Josephson current through a quantum dot of electrons that do interact with vibrons can be expressed in a Meir-Wingreen-like form [19,20] by using retarded Green's functions for the dot levels in the Nambu representation, $G_\pm^{(r)}$ [21]. The corresponding formula for the nonresonant critical current through a single-level QD in the considered limit $\Delta_0 \to \infty$ takes the form

$$J_c = \frac{2e\Gamma_L\Gamma_R}{h}\int_{-\infty}^{\infty} d\varepsilon f(\varepsilon) Im\{G_+^{(r)}(\varepsilon)G_-^{(r)}(\varepsilon)\}, \tag{2}$$

where $\Gamma_{L,R}$ are the partial widths of the quantum dot's energy level and $f(\varepsilon)$ is the Fermi distribution function. To lowest order in the tunneling rates the Green's functions can be evaluated by perturbation theory using only the QD Hamiltonian, which, after a standard unitary transformation [22] [$U = \exp(-ig\hat{p}[d_\uparrow^\dagger d_\uparrow + d_\downarrow^\dagger d_\downarrow])$, where $\hat{p} = i(b^\dagger - b)/\sqrt{2}$ is a dimensionless dot momentum operator], takes the convenient polaron form

$$\tilde{H}_{QD} = \sum_{\sigma=\uparrow,\downarrow} \varepsilon_{ps}\hat{n}_\sigma^{(p)} + \hbar\omega_0 b^\dagger b + U_{\rm eff}\hat{n}_\uparrow^{(p)}\hat{n}_\downarrow^{(p)}. \tag{3}$$

Here $\hat{n}_\sigma^{(p)} = D_\sigma^\dagger D_\sigma$, $D_\sigma = d_\sigma \exp(-ig\hat{p})$, where $\varepsilon_{ps} = \varepsilon_0 - g^2\hbar\omega_0/2$ is the polaronic shift, and $U_{\rm eff} = U_c - g^2\hbar\omega_0$. In what follows we neglect correlation effects by fine-tuning the model parameters so that $U_{\rm eff} \leqslant \Gamma_L + \Gamma_R$. In this case Eq. (3) describes noninteracting electrons and vibrons. The polaron

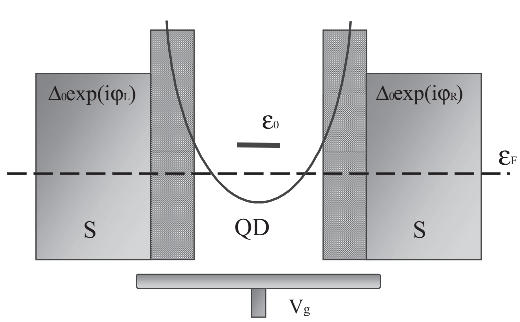

FIG. 1. Schematic representation of the model system discussed in the text. A gated quantum dot (QD) with a single electron level (ε_0) is vibrating in a harmonic potential while in tunneling contact with two bulk superconducting electrodes (S).

Green's functions can be evaluated analytically [see Eq. (24) of Ref. [15]] with the result that

$$G_{\pm}^{(r)}(\varepsilon) = \exp[-g^2(1+2n_B)] \times \sum_{n=-\infty}^{\infty} \frac{\exp\left(-\frac{n\hbar\omega_0}{2k_BT}\right) I_n(2g^2\sqrt{n_B(1+n_B)})}{\varepsilon \mp \varepsilon_{ps} \pm n\hbar\omega_0 + i0^+}. \quad (4)$$

Here $n_B = [\exp(\hbar\omega_0/k_BT) - 1]^{-1}$ is the Bose-Einstein distribution function, and $I_n(z)$ is the modified Bessel function of the first kind.

It is now straightforward to evaluate the temperature dependence of the critical current using Eqs. (2) and (4). In particular, one finds the low- and high-temperature asymptotic results. At low temperatures we can use an expansion of the Bessel function valid for small arguments: $I_n(z \to 0) \approx (z/2)^{|n|}/\Gamma(n+1)$ [23]. At zero temperature the quantum dot spectral function, $A(\varepsilon) = -2\mathrm{Im}[G^r(\varepsilon)]$, has a set of δ-function peaks, the heights of which satisfy a Poisson-like distribution,

$$A_{\pm}(\varepsilon)|_{T=0} = 2\pi \exp(-g^2) \times \sum_{n=0}^{\infty} \frac{g^{2n}}{n!(\varepsilon \mp \varepsilon_{ps} \mp n\hbar\omega_0)} \delta(\varepsilon \mp \varepsilon_{ps} \mp n\hbar\omega_0). \quad (5)$$

For a strong electron-vibron interaction, $g \gg 1$, many terms (up to a maximum n value of order g^2) contribute to the sum in Eq. (5). Thus the importance of the exponentially small factor $\exp(-g^2)$ in Eq. (5) is largely compensated. Using the same reasoning in the calculation of the imaginary part of the product of two Green's functions in Eq. (2), we obtain the low-temperature asymptotics of the critical current as

$$J_c(T \to 0) \simeq \frac{e\Gamma_L\Gamma_R}{2\hbar} \frac{1}{|\varepsilon_{ps} + g^2\hbar\omega_0|}. \quad (6)$$

The high-temperature asymptotics ($T \gg g^2\hbar\omega_0/k_B$) can be found by using the well-known generating function for Bessel functions $\sum_{k=-\infty}^{\infty} t^k I_k(z) = e^{z/2(t+1/t)}$ (see, e.g., Ref. [24]).

The result is

$$J_c(T \gg g^2\hbar\omega_0/k_B) \simeq \frac{e\Gamma_L\Gamma_R}{4\hbar k_BT}\left(1 - \frac{g^2\hbar\omega_0}{3k_BT}\right). \quad (7)$$

We conclude that strong electron-vibron interactions ($g \gg 1$) result in a polaronic suppression of the zero-temperature supercurrent, $J_c(0) \propto 1/g^2$. Furthermore, as shown Fig. 2, the critical current grows with increasing temperature at low temperatures since the temperature corrections to Eq. (6) are positive. At high temperatures Eq. (7) reveals that the temperature correction to the leading $1/T$ term is negative and totally determined by vibrational effects. The nonmonotonic nature of this anomalous temperature dependence is illustrated in Fig. 3.

Now we proceed to a perhaps more physically transparent derivation of the obtained results by further elaborating the semiclassical description of the soft vibrational subsystem that was sketched in Sec. I. In this approach, which does not involve any Green's functions, the dimensionless position operator $\hat{x} = (b^\dagger + b)/\sqrt{2}$ for the vibrating QD is replaced by the classical coordinate x. This coordinate measures the deviation of the center of mass of the QD oscillator from its equilibrium position in the absence of electron-vibron interactions. Furthermore we introduce the notation x_c for the QD equilibrium position measured in units of the amplitude $x_0 = \sqrt{\hbar/m\omega_0}$ of its zero-point fluctuations and note that x_c, which may differ from zero in the presence of electron-vibron interactions, has to be large to justify a classical approach.

In equilibrium $x_c = x_c(T,\varphi)$ is determined by minimizing the total thermodynamical potential $\Omega_t = F_v(x_c) + \Omega_A(x_c,\varphi)$. Here F_v is the free energy of the vibrational subsystem,

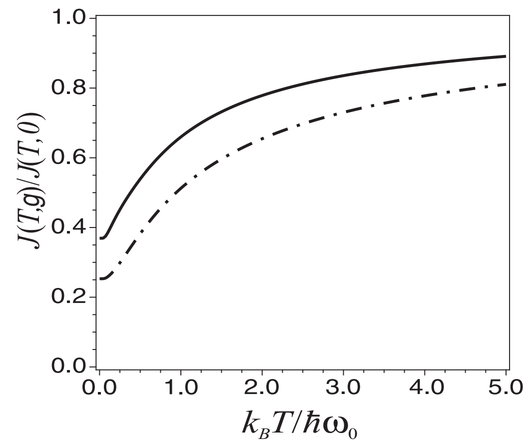

FIG. 2. Temperature dependence of the critical Josephson current normalized to its value without electron-vibron interactions ($g = 0$) and evaluated for $\Gamma_0/\hbar\omega_0 = 0.2$ and $\varepsilon_0/\hbar\omega_0 = 0.7$. The polaronic suppression of the critical current for $g = 1.4$ (solid curve) and $g = 2.0$ (dash-dotted curve) at $T = 0$ is reduced as the temperature is increased, hence leading to a thermally enhanced Josephson current in the range of temperatures shown.

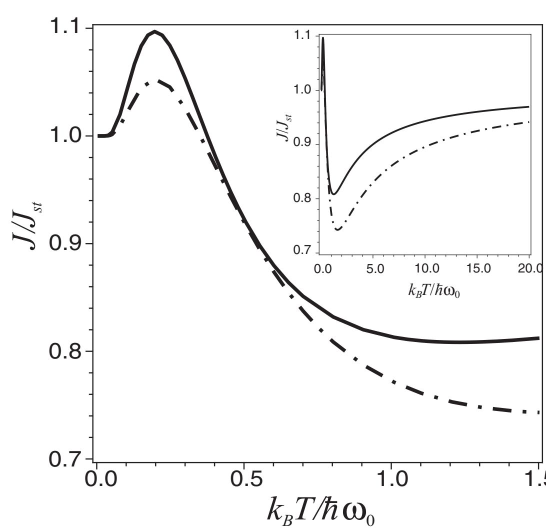

FIG. 3. Temperature dependence of the ratio between the critical Josephson current and the "standard" result for the critical current, $J_{st} = (e\Gamma_0^2/2\hbar\tilde{\varepsilon})\tanh(\tilde{\varepsilon}/2T)$. In each case (solid curve: $g = 1.4$, dash-dotted curve: $g = 2.0$) $\tilde{\varepsilon}$ was chosen to make the ratio unity at $T = 0$. Deviations from unity at low and intermediate temperatures indicate an anomalous temperature dependence of the Josephson current. The plotted ratio approaches unity (from below) only asymptotically at high temperatures (see inset).

and Ω_A is the thermodynamic potential of the two Andreev polaron states $E_A(x_c,\varphi) = \pm\sqrt{(\varepsilon_0 + g\hbar\omega_0 x_c)^2 + \Gamma_0^2\cos\varphi}$ (for simplicity we consider a symmetric junction, $\Gamma_L = \Gamma_R = \Gamma_0$). Off resonance, i.e., when $\Gamma_0 \ll |\varepsilon_0|, g^2\hbar\omega_0$, the supercurrent takes the standard Josephson form $J(T,\varphi) = J_c(T,\varphi)\sin\varphi$, where

$$J_c(T,\varphi) = \frac{e\Gamma_0^2}{2\hbar}\frac{1}{\varepsilon_p(T,\varphi)}\tanh\left(\frac{\varepsilon_p(T,\varphi)}{2k_BT}\right) \tag{8}$$

is the critical current and $\varepsilon_p(T,\varphi) = |\varepsilon_0 + g\hbar\omega_0 x_c(T,\varphi)|$.

By minimizing the total energy at $T = 0$, when the free energy $F_v(x_c)$ of the vibrational system reduces to $\hbar\omega_0 x_c^2/2$, one readily finds that in the strong-coupling limit ($g \gg 1$, $g^2 \gg \Gamma_0/\hbar\omega_0$) the equilibrium dot position $x_c \simeq \pm g$, while $x_c = 0$ corresponds to a local energy maximum. In this case Eq. (8) reproduces the low-T asymptotic result Eq. (6) obtained by the Green's function method. The polaronic suppression of the Josephson current can be explained as a result of a displacement of the classical QD harmonic oscillator $[x^2 \to (x - x_c)^2]$ and the corresponding shift of Andreev energy levels (associated with the formation of Andreev polarons).

The finite-T corrections for $T \ll \varepsilon_p/k_B$ can be evaluated by minimizing the total potential Ω_t, where the bosonic free energy now is the sum of an elastic (mechanical) energy and the free energy of mixed vibron-Andreev-polaron excitations. The latter are elementary excitations from the shifted ground state x_c, with an energy quantum $\hbar\omega_p = \hbar\omega_0 [1 - g^2\hbar\omega_0\Gamma_0^2\cos(\varphi/2)/E_A^3(x_c)]$.

The temperature dependence of the equilibrium coordinate $x_c(T,\varphi)$ is shown in Fig. 4. One notes that an increase of temperature decreases x_c, so that $x_c(T) < x_c(0)$. It is easy to show that the temperature-induced shift of supercurrent $\delta J_c(T) = J_c(T) - J_c(T = 0)$ is proportional to the average number of excited vibrons, $\delta J_c(T) \propto +n_B(T)$. As a consequence, the critical current is enhanced by temperature up to a "critical" temperature $T_p^* \lesssim \varepsilon_p/k_B$ at which the system "jumps" to the symmetric state $x_c = 0$ and the polaronic suppression is lifted [25].

Unlike the Green's function method, our semiclassical approach is not valid in the symmetric state where $\langle\hat{x}\rangle = 0$ unless the thermal fluctuations of the QD are large. This is the case if $T \gg \varepsilon_p/k_B$, when a classical description of the QD vibrations is again possible. Here one may estimate the temperature corrections to the leading term for the critical current, which is proportional to $1/T$, by replacing $\varepsilon_p(T,\varphi)$ by $\sqrt{\varepsilon_0^2 + (g\hbar\omega_0)^2\langle\hat{x}^2\rangle}$ in Eq. (8). In the high-T limit considered $\langle\hat{x}^2\rangle \simeq n_B(T) \simeq k_BT/\hbar\omega_0$, from which it follows that the high-T expression for the critical current in Eq. (7), which was evaluated in the Green's function approach, is recovered. Note that in the absence of vibrations the temperature-correction factor scales as $(1/T^2)$ rather than as $(1/T)$. Therefore thermal fluctuations of the QD coordinate suppress the critical current more at high temperatures than would be the case without any electron-vibron interaction.

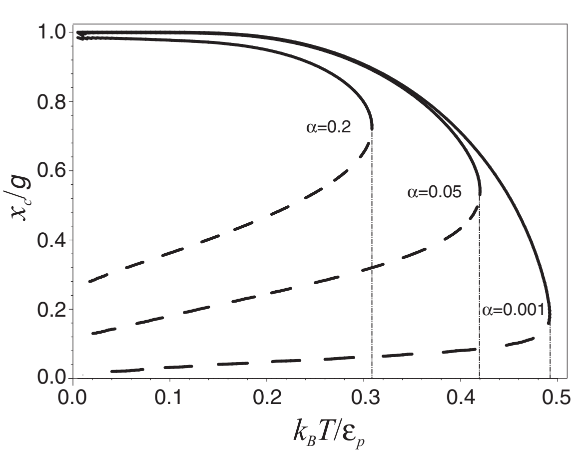

FIG. 4. Equilibrium position x_c of the weak-link quantum dot (QD) plotted (solid lines) as a function of temperature T for fixed superconducting phase difference, $\varphi = 1$, and electron-vibron coupling strength, $g = 3$, but different values of the QD level width Γ_0 ($\alpha = \Gamma_0/g^2\hbar\omega_0$). The dimensionless coordinate x_c is zero for $g = 0$. Note that $x_c(T) < x_c(0)$ and that x_c jumps to zero at a parameter-dependent critical temperature T_p^* (thin vertical lines).

III. CONCLUSION

In summary we have shown that the temperature dependence of the critical Josephson current through a vibrating quantum dot may be used as an indicator of soft vibrational effects. At zero-temperature the critical Josephson current is suppressed by electron-vibron interactions independently of whether hard ($\varepsilon_p \gtrsim \Delta_0$) or soft ($\varepsilon_p \ll \Delta_0$) vibrons are involved. These effects are, however, hidden since the unsuppressed values are not experimentally known. In normal (nonsuperconducting) electron transport the hidden vibrational effects may be disclosed by increasing the bias voltage or the temperature, which results in steplike features in traces of the conductance [10] or in an anomalous temperature dependence of the conductance. [13] In contrast, the interaction between a d.c. Josephson current and a hard vibrational subsystem is not influenced at all at temperatures where the system remains superconducting. For soft vibrons, $\Gamma_0 \ll \hbar\omega_0 \ll \Delta_0$ and $g \gtrsim 1$ [26]; however, we have shown that the situation is different since by increasing the temperature the critical current is enhanced at low temperatures, while at high temperatures it is suppressed compared with the standard temperature dependence of the critical Josephson current, which is given by the function $\tanh(\varepsilon/2k_BT)$. This means that the temperature dependence of the Josephson current through a weak link in the form of a softly vibrating quantum dot is anomalous in a large temperature interval. For existing carbon-nanotube-based molecular transistors (see, e.g., Ref. [10]) this temperature interval is well within the superconducting gap ($\Delta_0 < 10$ K) of (low-T) superconductors commonly used as electrodes in SNS junctions.

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