Multiple half-quantum vortices in rotating superfluid ^{3}He

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Half-quantum vortices and ordinary vortices in a rotating thin-film superfluid ^{3}He under a strong magnetic field are considered. It is shown that $2n+1$ half-quantum vortices interpolate between n singular vortices and $n+1$ singular vortices as the angular velocity is changed when the external magnetic field is strong enough. The phase diagram of the vortex configurations in the angular velocity–magnetic field space is obtained for $p=0$.

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I. INTRODUCTION

Superfluid ^{3}He exhibits extremely exotic and interesting properties due to its complex order parameter [1–3], which has attracted much attention not only from condensed matter physicists but also from particle theorists. One of the manifestations of such exotic properties is a vortex having a half-amount of vortex quantum, called a half-quantum vortex (HQV), whose existence was predicted first by Volovik and Mineev in 1976 [4]. An HQV is also expected to be present in BECs of alkali atoms [5–7] and spin-triplet superconductors [8–10], among other physical systems.

In spite of extensive theoretical [11–13] and experimental [14,15] research on HQVs in superfluid ^{3}He, since then, their existence has yet to be experimentally confirmed. Recently, we investigated a rotating superfluid ^{3}He in a slab geometry under a strong magnetic field [16], in which we have shown that an HQV is energetically stable compared to a singular vortex (SV) in the A_2 phase in the vicinity of the A_1-A_2 phase boundary. In this part of the phase diagram, a HQV will nucleate first as the angular velocity of the rotation is increased from 0 when the external magnetic field is strong enough.

In this paper, we investigate textures arising when the angular velocity is further increased. It will be shown that HQVs are arranged on two concentric circles with different radii supporting two different types of HQVs, respectively, when a small number of HQVs exist. When the angular velocity is further increased, an HQV appears at the center of the slab geometry. From these data, we construct a phase diagram of possible textures in the magnetic field–angular velocity plane.

We summarize the result of [16] in the next section, to make this paper self-contained and to establish notations and convention. In Sec. III, we evaluate the free energies of multiple HQVs and multiple SVs and obtain the phase diagram of vortices realized. Section IV is devoted to a summary and discussion.

II. HALF-QUANTUM VORTEX

Consider a rotating thin film of superfluid ^{3}He in the A_2 phase in a cylindrical slab geometry under a strong magnetic field H. In the presence of a magnetic field, the superfluid has different populations between the spin $\uparrow\uparrow$-condensate, which is called the $(+)$-condensate for simplicity, and the spin $\downarrow\downarrow$-condensate, called the $(-)$-condensate, where the spin direction is measured with respect to the magnetic field. The angular velocity $\boldsymbol{\Omega}$ and the magnetic field $\boldsymbol{H}$ are taken parallel to the z axis and the film is perpendicular to the z axis. The thickness and the radius of the film are denoted d and R, respectively, where d must be less than the dipole coherence length so that the $\hat{\boldsymbol{d}}$ vector stays in the xy plane throughout the condensate. The boundary condition forces the condensate to have only $\hat{\boldsymbol{e}}_x + i\hat{\boldsymbol{e}}_y$ or $\hat{\boldsymbol{e}}_x - i\hat{\boldsymbol{e}}_y$ orbital component throughout the system. We assume that the vortices are embedded in a texture with $\hat{\boldsymbol{l}} = \hat{z}$ for definiteness in this paper.

We use the Ginzburg-Landau free energy [1] in our analysis. The bulk free energy is

$$F_B = -\alpha A^*_{\alpha i}A_{\alpha i} + \beta_1 A^*_{\alpha i}A^*_{\alpha i}A_{\beta j}A_{\beta j} + \beta_2 A^*_{\alpha i}A_{\alpha i}A^*_{\beta j}A_{\beta j} + \beta_3 A^*_{\alpha i}A^*_{\beta i}A_{\alpha j}A_{\beta j} + \beta_4 A^*_{\alpha i}A_{\beta i}A^*_{\beta j}A_{\alpha j} + \beta_5 A^*_{\alpha i}A_{\beta i}A_{\beta j}A^*_{\alpha j}, \quad (1)$$

where the strong coupling correction to the fourth-order coefficients β_i is taken into account following the Rainer and Serene formalism [17] as

$$\beta_1 = -(1+0.03)\beta_0, \quad \beta_2 = (2+0.0)\beta_0,$$
$$\beta_3 = (2-0.01)\beta_0, \quad \beta_4 = (2-0.05)\beta_0,$$
$$\beta_5 = -(2+0.09)\beta_0.$$

Here we employed deviations of β_i from BSC weak-coupling values (the integer parts of β_i above) at $p=0$ based on the Helsinki data given in [18]. We have chosen $p=0$ since our previous observation [16] shows that the HQV-stable region is sizable for a small paramagnon parameter δ, corresponding to low pressure.

The gradient energy takes the form [1]

$$F_G = K_1\partial_i A_{\alpha j}\partial_i A^*_{\alpha j} + K_2\partial_i A_{\alpha i}\partial_j A^*_{\alpha j} + K_3\partial_i A_{\alpha j}\partial_j A^*_{\alpha i}, \quad (2)$$

where $K_1 = K_2 = K_3 \equiv K$ in the weak-coupling limit. The coherence length in the absence of a magnetic field is

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defined as

$$\xi(t) = \sqrt{\frac{K}{\alpha(t)}}. \tag{3}$$

Instead of expanding the order parameter in terms of the standard Cartesian base $\{\boldsymbol{e}_i\} = \{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$, we expand it in terms of the $\{\boldsymbol{e}_\nu\} = \{\boldsymbol{e}_\pm, \boldsymbol{e}_0\}$ base defined by

$$\boldsymbol{e}_\pm = \mp\frac{1}{\sqrt{2}}(\boldsymbol{e}_x \pm i\boldsymbol{e}_y), \quad \boldsymbol{e}_0 = \boldsymbol{e}_z.$$

The boundary condition $\hat{\boldsymbol{l}} = \pm\hat{z}$ forces $A_{\nu\pm}$ to have nonvanishing values in the bulk. Here the first subscript to A is the spin index, while the second one is the orbital index. A strong magnetic field along the z axis further forces the order parameter to have only four nonvanishing components $A_{\pm\pm}$ in the bulk. Let $t = 1 - T/T_c$, T_c being the critical temperature with a vanishing magnetic field, and $\alpha(t) = \alpha' t$, where $\alpha(t)$ is the coefficient of the second-order term of the bulk free energy, (1), and define the scaled magnetic field h by $h = \eta H/\alpha' T_c$, where η is a constant coupling strength between H and the condensate. It turns out to be convenient to further scale h as $\hat{h} = h/t$. The bulk order parameter is found by minimizing the uniform Ginzburg-Landau free energy. Suppose there is a vortex at $r = 0$, the center of the cylinder. We assume that the vortex is embedded in an $\hat{\boldsymbol{l}} = +\hat{z}$ texture, for concreteness, and the order parameter has only nonvanishing components $A_{\pm+}$ at $r \gg 1$, where the length is scaled by the coherence length with vanishing external magnetic field. We parametrize the components as $A_{\mu\nu} = C_{\mu\nu}(r)e^{in_{\mu\nu}\phi}$ assuming the cylindrical symmetry, where ϕ is the azimuthal angle in the xy plane and $n_{\mu\nu} \in \mathbb{Z}$. It turns out that $n_{\mu\nu}$ satisfy the quantization condition $n_{\mu-} = n_{\mu+} + 2$ due to the coupling between $A_{\mu+}$ and $A_{\mu-}$ through the gradient free energy [16].

When the HQV order parameter is expanded in $\{\boldsymbol{e}_\nu\}$, it is found that the order parameter is a superposition of the (+)-condensate, with no winding number, and the (−)-condensate, with a unit winding number, or the other way around. Such an HQV has a free energy

$$F_{\rm HQV}^{(\pm)} = 2\pi\int_0^R r dr (F - F_0) = 4\pi\left(A_{\pm+}^{(0)}\right)^2(\ln R + C_\pm), \tag{4}$$

where F_0 is the bulk free energy without a vortex. Here $A_{\pm+}^{(0)}$ stands for the amplitude of the bulk order parameter with orbital state $l_z = +1$ of the (+)- or the (−)-condensate, while $C_\pm$ is the vortex core energy of the (±)-condensate. The parameters $C_\pm$ are obtained numerically as functions of the scaled external magnetic field $\hat{h}$ [16]. The coefficient of the second-order term in the bulk free energy is modified as $\alpha(t) \to (1 + \hat{h})\alpha(t)$ for the spin-up condensate, and $\alpha(t) \to (1 - \hat{h})\alpha(t)$ for the spin-down condensate, in the presence of an external magnetic field $\hat{h}$. Then Eq. (3) leads to an inequality, $\xi_+ < \xi_-$. As a result, the inequality $C_- < C_+$ is always satisfied, from which we find that an HQV carrying a vortex of unit winding number in the (−)-condensate has less energy compared to that with a vortex in the (+)-condensate. Let $L^{(-)} = 4\pi(2m/\hbar)(A_{-+}^{(0)})^2 R^2$ be the angular momentum of the (−)-condensate. By considering the free energy $F_{\rm vor}^{(-)} - \Omega L^{(-)}$ in the rotating frame, we find that an HQV nucleates in the (−)-condensate at $\Omega = \Omega_c^{(-)} = \ln R + C_-$, where Ω is scaled by $\hbar/2mR^2$.

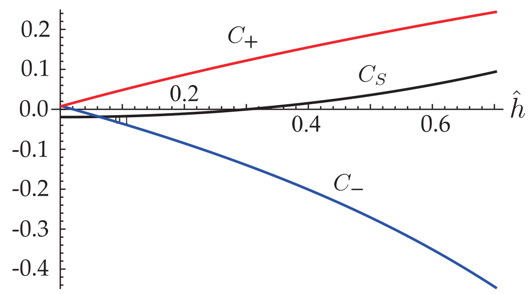


FIG. 1. (Color online) $\hat{h}$ dependence of the parameters C_+ [upper (red) line], C_- [lower (blue) line], and C_S (middle, black line) at vanishing pressure. See Eqs. (4) and (5) for definitions of these parameters.

In an SV, each of (+)- and (−)-condensates carries a vortex with a unit winding number and these components are superposed so that the vortex cores overlap exactly. An SV has the free energy

$$F_{\rm SV} = 4\pi\left[\left(A_{++}^{(0)}\right)^2 + \left(A_{-+}^{(0)}\right)^2\right](\ln R + C_S). \tag{5}$$

The parameter C_S is the SV core energy and is a function of $\hat{h}$. An SV nucleates at a critical angular velocity $\Omega_c^S = \ln R + C_S$.

Figure 1 depicts the parameters $C_\pm$ and C_S at $p = 0$ as functions of $\hat{h}$. Observe that, where $\hat{h}$ is large, there is a range in the diagram where $C_S > C_-$, which implies that a single HQV, having a phase factor $e^{i\phi}$ in the (−)-condensate, nucleates first as Ω is gradually increased from 0.

III. MULTIPLE HALF-QUANTUM VORTICES

In this paper, we consider the case in which the angular velocity Ω is further increased and investigate how many HQVs and SVs exist in the superfluid and the patterns of the stable configurations of these vortices. We again assume that $\hat{\boldsymbol{l}} = +\hat{z}$ at $r \gg 1$. The free energy of the system with more than one vortex is evaluated by making use of $C_\pm$ and C_S numerically obtained in Fig. 1. We again take $p = 0$ for numerical calculations, expecting a larger HQV-stable region [16]. Let $\boldsymbol{r}_i$ be the position of the ith vortex center. When the condition $|\boldsymbol{r}_i - \boldsymbol{r}_j| \gg 1$ is satisfied for all pairs $i \neq j$, the London approximation is valid and the energies of the vortices in the (+)-condensate and that of the vortices in the (−)-condensate may be evaluated independently since the coupling between two condensates appears only through the fourth-order free energy, which is insensitive to the relative positions of the vortices. The hydrodynamic energy associated with the flow around vortices has been evaluated previously for superfluid ^{4}He [19], which we employ in this paper.

Let us first consider SVs. Suppose the number n of SVs satisfies $n \leqslant 5$. Then the vortices are distributed uniformly on a circle with radius $r \approx \sqrt{(n-1)/2\Omega}\, R$ centered at the origin of the cylinder, where Ω is scaled by $\hbar/2mR^2$ as before. Then the free energy of n SVs in the A_2 phase of superfluid ^{3}He in the rotating frame takes the form

$$F_n^{(S)}(\Omega,u) = 4\pi\left[\left(A_{++}^{(0)}\right)^2 + \left(A_{-+}^{(0)}\right)^2\right]F_n(\Omega,u), \quad (6)$$

where $u = r/R$ and

$$F_n(\Omega,u) = n[\ln R + C_S + \ln(1-u^{2n}) - (n-1)\ln u - \ln n - \Omega(1-u^2)]. \quad (7)$$

Here the core energy of the vortices has been taken into account in the definition of F_n. It has been shown that the function $F_n(\Omega,u)$ has a minimum at u in the physical region (0,1) when Ω is greater than some critical value $\Omega_0(n)$ [19]. Let

$$f_n(\Omega) = \min_{u\in(0,1)} F_n(\Omega,u) \approx n(\ln R + C_S - \Omega) + \tfrac{1}{2}n(n-1)[1+\ln(2\Omega) - \ln(n-1)] \quad (8)$$

be the minimum value of $F_n(\Omega,u)$, where the approximate value $u \approx \sqrt{(n-1)/2\Omega}$ has been used. This approximation is verified numerically to be quite accurate in the given parameter range when $n \geqslant 2$. Then the energy of the stable configuration of n SVs is given by

$$F_n^{(S)}(\Omega) = 4\pi\left[\left(A_{++}^{(0)}\right)^2 + \left(A_{-+}^{(0)}\right)^2\right]f_n(\Omega) \quad (9)$$

for $n \leqslant 5$.

Next, let us consider n SVs, where $6 \leqslant n \leqslant 8$. It was shown for superfluid ^{4}He that a stable configuration for $6 \leqslant n \leqslant 8$ is $n-1$ vortices distributed uniformly in a circle centered at the origin plus a single vortex at the origin [19]. A vortex configuration with less symmetry is expected as the angular velocity is further increased beyond $n = 8$. These patterns are verified both experimentally [20] and by numerical simulation [21]. When $6 \leqslant n \leqslant 8$, $F_n(\Omega,u)$ in Eq. (6) takes the form [19]

$$F_n(\Omega,u) = n(\ln R + C_S) + (n-1)[\ln(1-u^{2n}) - n\ln u - \ln(n-1) - \Omega(1-u^2)] - \Omega. \quad (10)$$

With the same approximation employed to obtain Eq. (8), the free energy minimizing configuration is given by $u \approx \sqrt{n/2\Omega}$, which gives the minimum energy

$$f_n(\Omega) = \min_{u\in(0,1)} F_n(\Omega,u) = n(\ln R + C_S - \Omega) - (n-1)\ln(n-1) + \tfrac{1}{2}n(n-1)[1+\ln(2\Omega) - \ln n]. \quad (11)$$

Similarly, the free energies of n HQVs with vortices in the (+)-condensate and (−)-condensate are evaluated as

$$F_n^{(\pm)}(\Omega) = 4\pi\left(A_{\pm+}^{(0)}\right)^2(f_n(\Omega) + n\Delta C_\pm), \quad (12)$$

where $\Delta C_\pm = C_\pm - C_S$. Equation (12) applies to cases in which $2 \leqslant n \leqslant 8$, provided that Eq. (8) and Eq. (11) are used for $f_n(\Omega)$ for $2 \leqslant n \leqslant 5$ and $6 \leqslant n \leqslant 8$, respectively.

It is expected that HQVs appear in the vicinity of the parameter domain where the free energy difference between the n SVs and the $n+1$ SVs is small. We expect that there are n HQVs in the (+)-condensate and $n+1$ HQVs in the (−)-condensate in the process of the transition from n SVs to $n+1$ SVs. We denote this configuration of HQVs as HQV($n+1$,n),

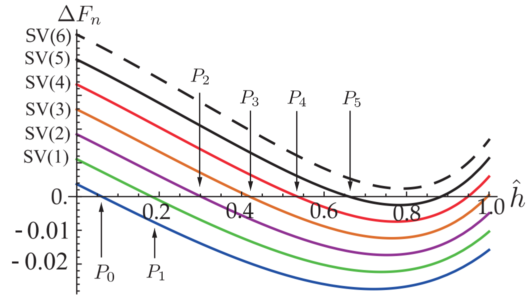


FIG. 2. (Color online) ΔF_n as a function of $\hat{h}$ for $n = 0$ (blue line), 1 (green line), 2 (purple line), 3 (orange line), 4 (red line), 5 (black line), and 6 (dashed black line) at $p = 0$. Singular vortices are formed when $\Delta F_n > 0$, while an HQV($n+1$,n) is formed when $\Delta F_n < 0$ and Ω is properly chosen. ΔF_n is always positive for $n \geqslant 6$.

while a configuration with n SVs is denoted SV(n). In a sense, HQV($n+1$,n) is roughly regarded as "SV($n+1/2$)". Note that the number of HQVs in the (−)-condensate should be larger than that of HQVs in the (+)-condensate due to the inequality $C_- < C_+$ (see Fig. 1); it is energetically favorable to have an extra vortex in the (−)-condensate rather than in the (+)-condensate.

The conditions under which HQV($n+1$,n) is stabilized are

$$F_{n+1}^{(-)}(\Omega) + F_n^{(+)}(\Omega) < F_n^{(S)}(\Omega) \quad (13)$$

and

$$F_{n+1}^{(-)}(\Omega) + F_n^{(+)}(\Omega) < F_{n+1}^{(S)}(\Omega) \quad (14)$$

simultaneously. More explicitly, these conditions are written as

$$\left(A_{-+}^{(0)}\right)^2(f_{n+1}(\Omega) - f_n(\Omega)) + \left(A_{++}^{(0)}\right)^2 n\Delta C_+ + \left(A_{-+}^{(0)}\right)^2(n+1)\Delta C_- < 0 \quad (15)$$

and

$$-\left(A_{++}^{(0)}\right)^2(f_{n+1}(\Omega) - f_n(\Omega)) + \left(A_{++}^{(0)}\right)^2 n\Delta C_+ + \left(A_{-+}^{(0)}\right)^2(n+1)\Delta C_- < 0, \quad (16)$$

from which the necessary condition for the existence of a stable HQV($n+1$,n) configuration is found to be

$$\Delta F_n \equiv \left(A_{++}^{(0)}\right)^2 n\Delta C_+ + \left(A_{-+}^{(0)}\right)^2(n+1)\Delta C_- < 0. \quad (17)$$

Figure 2 shows the magnetic-field ($\hat{h}$) dependence of ΔF_n for $n = 0,1,\ldots,6$. Note that $\Delta F_n < 0$ is a necessary condition for the existence of an HQV($n+1$,n), but not a sufficient condition. The stability of HQVs and SVs also depends on Ω as shown in the phase diagram (Fig. 3). It is observed that the stability of an HQV($n+1$,n) requires a larger magnetic field as n becomes larger. ΔF_n is always positive for $n \geqslant 6$, implying that there are no HQV($n+1$,n) phases with $n \geqslant 6$.

We plot the phase diagram of various vortex configurations in the $\hat{h}$-Ω plane in Fig. 3. For definiteness, we have taken $p = 0$, $R = 1000$, and $n = 0$, 1, 2, 3, 4, and 5. The dashed (red) line near P_0 shows the first critical angular velocity Ω_c^S

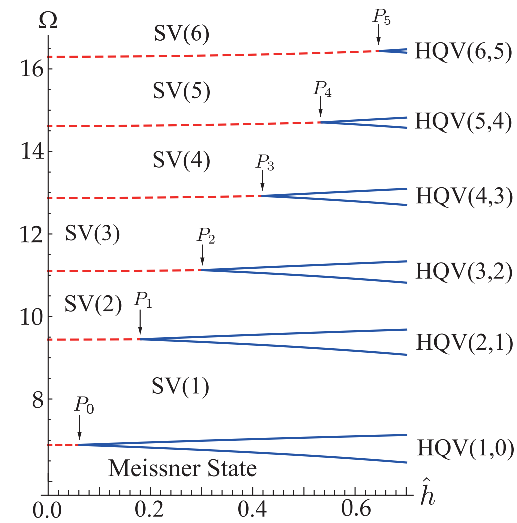
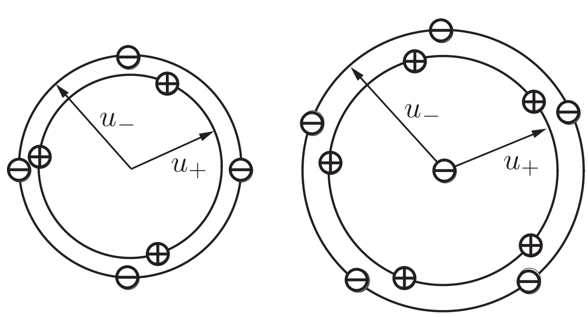


FIG. 3. (Color online) Phase diagram of various types of vortices for $\Omega > 0$ for $p = 0$. SV(n) is the domain where n SVs are the most stable configuration, while the wedge-shaped domain between the solid (blue) curves, denoted HQV($n+1,n$), is a region where a configurations of $n+1$ HQVs in the ($-$)-condensate and n HQVs in the (+)-condensate are most stable. A dashed (red) line is a boundary between two types of SVs, while a solid (blue) line is a boundary between SVs and HQVs. Point P_i denotes the point of the same symbol in Fig. 2.

for formation of an SV, while the lower (blue) solid line of domain HQV(1,0) is the first critical angular velocity $\Omega_c^{(-)}$ for formation of an HQV in the ($-$)-condensate.

Figure 4(a) depicts an HQV arrangement for $\hat{h} = 0.7$ and $\Omega = 12.9$, for which HQV(4,3) is the most stable configuration. The inner circle, of radius $u_+ = \sqrt{(3-1)/2\Omega} \approx 0.28$, supports three (+)-HQVs, while the outer circle, of radius $u_- = \sqrt{(4-1)/2\Omega} \approx 0.34$, supports four ($-$)-HQVs. Here the radius is scaled so that the wall of the cylinder is at $u = 1$.



(a) (b)

FIG. 4. (a) Configuration of HQV(4,3) for $\hat{h} = 0.7$ and $\Omega = 12.9$. The inner circle, of radius $u_+ \approx 0.28$, supports three (+)-HQVs, denoted by circles containing a cross, while the outer circle, of radius $u_- \approx 0.34$, supports four ($-$)-HQVs, denoted by circles containing a horizontal line. The relative orientation of (+)-HQVs and ($-$)-HQVs is arbitrary. The wall of the cylinder at $u = 1$ is not shown. (b) Configuration of an HQV(6,5) for $\hat{h} = 0.7$ and $\Omega = 16.4$. There are ($-$)-HQV at the center, five ($-$)-HQVs on a circle of radius $u_- \approx 0.43$, and five (+)-HQVs on a circle of radius $u_+ \approx 0.35$.

TABLE I. An HQV($n+1,n$) is stable between T_n (mK) and $T_{c2} \simeq 0.907$ mK. The difference $T_{c2} - T_n$ is given in the third row. The fourth row lists Ω (rad/s) for $R = 10^{-4}$ m.

	n					
	0	1	2	3	4	5
$1/\hat{h}_n$	16.5	5.51	3.32	2.39	1.87	1.51
t_n	0.351	0.117	0.0707	0.0507	0.0398	0.0323
$T_{c2} - T_n$ (mK)	0.304	0.0865	0.0435	0.0249	0.0148	0.00784
Ω (rad/s)	7.24	9.93	11.7	13.6	15.5	17.3

Figure 4(b) shows the HQV arrangement for $\hat{h} = 0.7$ and $\Omega = 16.4$, for which HQC(6,5) is stabilized. There is a single ($-$)-HQV in the center and five ($-$)-HQVs on the circle of radius $u_- = \sqrt{6/2\Omega} \approx 0.43$, while five (+)-HQVs are distributed uniformly on a circle of radius $u_+ = \sqrt{(5-1)/2\Omega} \approx 0.35$.

Suppose h is fixed. The critical temperature T_{c1} of the A_1 phase is set to $t = -h$ in our convention [16], namely, $T_{c1} - T_c = hT_c$. The transition temperature to the A_2 phase is then given by $t = -\beta_5/\beta_{245} \simeq 1.12h$, giving $T_c - T_{c2} \simeq 1.12hT_c$. Let $H = 5$ T as a concrete example. Then we find $h = \eta H/\alpha' T_c \simeq 2.13 \times 10^{-2}$ at $p = 0$ [22], from which we find $T_{c1} - T_c \simeq 0.0198$ mK, where we put $T_c \simeq 0.929$ mK. Similarly, we obtain $T_c - T_{c2} \simeq 0.0222$ mK.

The critical $1/\hat{h}_n$ and Ω for the appearance of an HQV($n+1,n$) is obtained from Fig. 3 as reported in Table I. Here T_n stands for the critical temperature, above which an HQV($n+1,n$) is stabilized and $t_n = 1 - T_n/T_c = h/\hat{h}_n$. Ω in Fig. 3 is scaled by $\hbar/2mR^2$ and the values of Ω (rad/s) in Table I are evaluated with $R = 10^{-4}$ m [$\hbar/2mR^2 \simeq 1.05$ (rad/s)]. For $R = r \times 10^{-4}$ cm, it scales as Ω/r^2. Thus a lower critical angular velocity is obtained for a larger cylinder.

IV. CONCLUSION

In summary, we have analyzed the stable textures of a thin film of rotating superfluid ^{3}He under a magnetic field by using the Ginzurg-Landau free energy. It was shown that n HQVs in the spin (+)-condensate and $n+1$ HQVs in the spin ($-$)-condensate interpolate between an n-SV texture and an $n+1$-SV texture as the scaled angular velocity Ω is increased for a sufficiently large scaled magnetic field $\hat{h}$. There is a re-entrant transition to the SV texture as the magnetic field is further increased. The phase diagram for $0 \leqslant n \leqslant 5$ has been plotted in the $\hat{h}$-Ω plane.

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[1] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Taylor & Francis, London, 1990).
[2] G. E. Volovik, *Exotic Properties of Superfluid* ^{3}*He* (World Scientific, Singapore, 1992).
[3] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, UK, 2003).
[4] G. E. Volovik and V. P. Mineev, Pis'ma Zh. Eksp. Teor. Fiz. **24**, 605 (1976)[JETP Lett. **24**, 561 (1976)].
[5] K. Kasamatsu, M. Tsubota, and M. Ueda, Int. J. Mod. Phys. B **19**, 1835 (2005).
[6] M. Eto, K. Kasamatsu, M. Nitta, H. Takeuchi, and M. Tsubota, Phys. Rev. A **83**, 063603 (2011).
[7] M. Cipriani and M. Nitta, Phys. Rev. Lett. **111**, 170401 (2013).
[8] S. B. Chung, H. Bluhm, and E.-A. Kim, Phys. Rev. Lett. **99**, 197002 (2007).
[9] S. B. Chung and S. A. Kivelson, Phys. Rev. B **82**, 214512 (2010).
[10] J. Jang, D. G. Ferguson, V. Vakaryuk, R. Budakian, S. B. Chung, P. M. Goldbart, and Y. Maeno, Science **331**, 186 (2011).
[11] M. C. Cross and W. F. Brinkman, J. Low Temp. Phys. **27**, 683 (1977).
[12] M. M. Salomaa and G. E. Volovik, Phys. Rev. Lett. **55**, 1184 (1985).
[13] V. Vakaryuk and A. J. Leggett, Phys. Rev. Lett. **103**, 057003 (2009).
[14] P. J. Hakonen, K. K. Nummila, J. T. Simola, L. Skrbek, and G. Mamniashvili, Phys. Rev. Lett. **58**, 678 (1987).
[15] R. Ishiguro, O. Ishikawa, M. Yamashita, Y. Sasaki, K. Fukuda, M. Kubota, H. Ishimoto, R. E. Packard, T. Takagi, T. Ohmi, and T. Mizusaki, Phys. Rev. Lett. **93**, 125301 (2004).
[16] K. Kondo, T. Ohmi, M. Nakahara, T. Kawakami, Y. Tsutsumi, and K. Machida, J. Phys. Soc. Jpn. **81**, 104603 (2012).
[17] D. Rainer and J. Serene, Phys. Rev. B **13**, 4745 (1976).
[18] L. Levin and O. T. Valls, Phys. Rep. **98**, 1 (1983).
[19] G. B. Hess, Phys. Rev. **161**, 189 (1967).
[20] R. E. Packard, Physica **109–110**, 1474 (1982).
[21] L. J. Campbell and R. M. Ziff, Phys. Rev. B **20**, 1886 (1979).
[22] U. E. Israelsson, B. C. Crooker, H. M. Bozler, and C. M. Gould, Phys. Rev. Lett. **53**, 1943 (1984).