Same universality class for the critical behavior in and out of equilibrium in a quenched random field

Ivan Balog,[1,2,*] Matthieu Tissier,[1,†] and Gilles Tarjus[1,‡]
[1]*LPTMC, CNRS-UMR 7600, Université Pierre et Marie Curie, Boîte 121, 4 Place Jussieu, 75252 Paris Cédex 05, France*
[2]*Institute of Physics, P.O. Box 304, Bijenička cesta 46, HR-10001 Zagreb, Croatia*
(Received 5 November 2013; revised manuscript received 5 February 2014; published 7 March 2014)

The random-field Ising model (RFIM) is one of the simplest statistical-mechanical models that captures the anomalous irreversible collective response seen in a wide range of physical, biological, or socioeconomic situations in the presence of interactions and intrinsic heterogeneity or disorder. When slowly driven at zero temperature, it can display an out-of-equilibrium phase transition associated with critical scaling ("crackling noise"), while it undergoes at equilibrium, under either temperature or disorder-strength changes, a thermodynamic phase transition. We show that the out-of-equilibrium and equilibrium critical behaviors are in the same universality class: they are controlled, in the renormalization-group (RG) sense, by the same zero-temperature fixed point. We do so by combining a field-theoretical formalism that accounts for the multiple metastable states and the exact (functional) RG. As a spin-off, we also demonstrate that critical fluids in disordered porous media are in the same universality class as the RFIM, thereby unifying a broad spectrum of equilibrium and out-of-equilibrium phenomena.

DOI: 10.1103/PhysRevB.89.104201 PACS number(s): 11.10.Hi, 75.40.Cx

I. INTRODUCTION

In the presence of both interactions and intrinsic heterogeneity or quenched disorder, a wide spectrum of systems display, when slowly driven by an external solicitation, discrete collective events, bursts, shocks, jerks, or avalanches that span a broad range of sizes. The signature appears as a "crackling noise" [1] and it can be found in quite different situations [1–3], from Barkhausen noise in disordered magnets [4] to a variety of social and economic phenomena [5] in passing by capillary condensation in mesoporous materials [6,7] or hysteresis and noise in disordered electron nematics in high-T_c superconductors [8,9].

It has been shown that a simple system such as the random-field Ising model (RFIM) already has all the required ingredients to display crackling noise [10–12]. In this case, the latter results from the presence of an out-of-equilibrium critical point in the hysteretic response of the system to an infinitely slowly changed external field at zero temperature. The critical point separates a phase characterized by finite-size avalanches and a continuous hysteresis curve from a phase with a macroscopic avalanche and a discontinuous hysteresis curve. It requires tuning two control parameters, namely the disorder strength and the external magnetic field.

On the other hand, for decades the RFIM has been a model for equilibrium phase behavior in the presence of quenched disorder [13,14]. For dimensions greater than 2, the RFIM shows a phase transition and a critical point at fixed disorder strength when changing temperature or at fixed temperature when changing disorder strength [14].

The puzzle we address and solve in the present work is the following: despite the fact that one is at equilibrium and the other is not, one is at zero external field and the other is not, and that they take place at different values of the disorder strength, the two types of critical points are characterized by critical exponents and scaling functions that have been found to be very close in numerical simulations, within numerical accuracy [15–17]. (A similar observation concerning the critical exponents can be made from experiments, but the uncertainties are much bigger.)

The theoretical tool for a proper resolution of this puzzle is the renormalization group (RG). The critical behaviors in and out of equilibrium are the same, and are therefore in the same universality class, *if and only if* they are controlled by the same RG "fixed point." (A first piece of information is that the fixed points associated with both types of criticality occur at zero temperature where sample-to-sample fluctuations dominate over thermal ones [18,19]; however, this is a necessary but not sufficient condition [20].) A first attempt through an RG formalism was proposed on the basis of perturbation theory [12]. However, the latter is known to seriously fail in the RFIM [14,21] and cannot provide a useful method. The route we follow here is based on the exact RG and builds on our recent work on the equilibrium behavior of the RFIM [22–26].

Our demonstration relies on a field-theoretical setting and on the nonperturbative functional RG. The approach is powerful, but it involves a somewhat abstract formalism. However, most of it having been detailed in our previous publications, we will try to maintain the exposition of formal manipulations at a minimal level.

The demonstration proceeds in several steps. The first one is to replace the *a priori* complex problem of following a history-dependent evolution among configurations, which results from the dynamics of the slowly driven RFIM, by one that is more readily tackled by statistical mechanical methods. The limit of interest is the adiabatic, or quasistatic, one in which the driving rate is vanishingly slow so that the system reaches a stationary state before being evolved again [10]. The trick is that, due to the ferromagnetic nature of the interactions in the RFIM and the properties of the zero-temperature relaxational dynamics (and the associated

*balog@ifs.hr
†tissier@lptmc.jussieu.fr
‡tarjus@lptmc.jussieu.fr

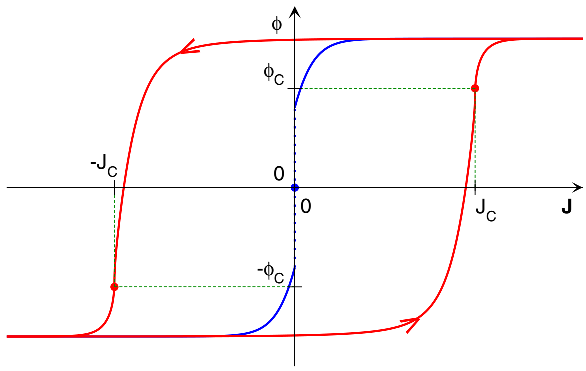

FIG. 1. (Color online) RFIM at zero temperature: Schematic illustration of the hysteresis loop (red) and of the equilibrium curve (blue) in the magnetization (ϕ) vs applied magnetic field (J) representation. For the chosen value of the disorder strength, the system has two (symmetric) out-of-equilibrium critical points (red dots), but a first-order transition in equilibrium.

"no-passing rule" [10,27]), the configurations visited along the hysteresis loop correspond to *extremal* states [28,29]: for a given value of the applied magnetic field (in the language of magnetic systems), they correspond to the stationary states that have the largest local magnetization at each point (for the "descending" branch obtained by decreasing the magnetic field from a fully positively magnetized configuration; see Fig. 1) or the smallest one (for the "ascending branch" obtained by increasing the field from a fully negatively magnetized configuration). When the distribution of the random fields is continuous, which we shall consider, these extremal states are unique for a given realization of the disorder (with exceptional degeneracies) [28]. One can then formulate *a statistical mechanical treatment of the extremal states with no reference to dynamics and history*.

II. MODEL AND FORMALISM

The model that we consider is the field-theoretical version of the RFIM with short-ranged interactions. The associated "bare action" (microscopic Hamiltonian) is

$$S[\varphi;h+J] = S_B[\varphi] - \int_x [h(x)+J(x)]\varphi(x)\,,$$
$$S_B[\varphi] = \int_x \left\{\frac{1}{2}[\partial\varphi(x)]^2 + \frac{r}{2}\varphi(x)^2 + \frac{u}{4!}\varphi(x)^4\right\}, \tag{1}$$

where $\int_x \equiv \int d^dx$, $h(x)$ is a random "source" (a random magnetic field), and $J(x)$ is an external source (a magnetic field); the quenched random field is taken with a Gaussian distribution characterized by a zero mean and a variance $\overline{h(x)h(y)} = \Delta_B\delta^{(d)}(x-y)$.

At zero temperature, the driven dynamics associated with the hysteresis curve is described by the following equation of motion:

$$\partial_t\varphi_t(x) = -\frac{\delta S_B[\varphi]}{\delta\varphi_t(x)} + h(x) + J_t. \tag{2}$$

As discussed above, in the quasistatic limit where the external source drives the system infinitely slowly, the stationary states that are relevant for the hysteresis curve are the two extremal solutions (Fig. 1) of the stochastic field equation,

$$\frac{\delta S_B[\varphi]}{\delta\varphi(x)} = h(x) + J, \tag{3}$$

which is obtained by setting $\partial_t\varphi_t(x) = 0$ and $J_t = J$ in Eq. (2).

The general recipe to build a generating functional from which one can derive all the needed correlation functions describing the extremal states is (i) to introduce a weighting factor with an auxiliary source linearly coupled to the φ field to select the magnetization, and (ii) to consider copies or replicas of the original disordered system, each being independently coupled to distinct external sources [24,25]. The associated generating functional is then

$$\overline{\mathcal{Z}_h[\{\hat{J}_a,J_a\}]} = \int\prod_a \mathcal{D}\varphi_a\,\overline{\delta\left[\frac{\delta S_B[\varphi_a]}{\delta\varphi_a} - h - J_a\right]}$$
$$\times\det\left[\frac{\delta^2 S_B[\varphi_a]}{\delta\varphi_a\delta\varphi_a}\right]\exp\int_x \hat{J}_a(x)\varphi_a(x), \tag{4}$$

where the overline denotes an average over the Gaussian random field $h(x)$, and square brackets generically indicate functionals.

Note that the above generating functional *a priori* includes contributions from all solutions of the stochastic field equation (for each copy a). However, in the limit where all auxiliary sources $\hat{J}_a$ go to infinity, *the dominant contribution is that of the extremal states*, with maximum magnetization when $\hat{J}_a \to +\infty$ and minimum one when $\hat{J}_a \to -\infty$. The correlation functions are then obtained by first differentiating $\log(\overline{\mathcal{Z}_h[\{\hat{J}_a,J_a\}]})$ with respect to the $\hat{J}_a$'s and then taking the latter to infinity while considering all J_a's equal to J.

It is worth pointing out the difference with the equilibrium situation at $T=0$. There, the properties of the system are obtained from the ground state, i.e., the solution with minimal action (energy). The ground state can be selected through the introduction of a Boltzmann-like weighting factor with an auxiliary temperature in the limit where the latter is taken to zero [24,25]. The selection of the extremal states is thus quite different from that of the ground state.

To proceed further, as explained in detail in Ref. [24], the above functional can be reexpressed with the help of auxiliary fields through standard field-theoretical techniques [30,31] (see also Appendix A). This leads to a "superfield theory" with a large group of symmetries and supersymmetries [24].

The next step consists in applying an exact RG formalism to this superfield theory. This can be done by progressively including the contribution of the fluctuations of the superfield on longer length scales, or alternatively with shorter momenta [32]. Technically, this can be implemented through the addition to the bare action of an "infrared (ir) regulator" depending on a running ir scale k; its role is to suppress, in the generating functional derived from Eq. (4), the integration over modes with momentum $|q| \lesssim k$ [22,33,34].

The central quantity of our RG approach is the k-dependent "effective average action" [33,34], Γ_k. This functional exactly interpolates between the bare action at the microscopic (or uv) scale $k=\Lambda$, which then corresponds to the mean-field approximation where no fluctuations are accounted for, and the exact effective action (Gibbs free energy) when $k = 0$. The latter is the generating functional of the so-called "one-particle irreducible" (1PI) correlation functions [30], and its knowledge entails a full description of the statistical properties of the extremal states, hence of the out-of-equilibrium hysteresis behavior of the RFIM. Expanding Γ_k in increasing numbers of unrestricted sums over copies (or replicas) generates a cumulant expansion for the renormalized disorder at the scale k [24].

The RG flow of Γ_k is generated by continuously decreasing the ir scale k. This leads to an *exact functional RG equation* [33,34], from which one derives an exact hierarchy of coupled functional RG equations for the cumulants of the renormalized disorder (see Appendix A).

III. AN IDENTITY FROM THE EXACT RG

An important simplification occurs in the situation of interest here. As already mentioned, the hysteresis loop corresponds to the limit of infinite auxiliary source, $\hat{J} \to \pm\infty$, or in the Legendre transformed setting, the limit of infinite auxiliary field, $\hat{\phi} \to \pm\infty$ ($\hat{\phi}_a$ is the average of the auxiliary field introduced as a conjugate of the source J_a to reexpress Eq. (4) [24]). The main point is that *the uniqueness of the extremal states* (for each branch separately) translates in the present superfield framework into the formal property of the random generating functional, which we called "Grassmannian ultralocality" [24] and which greatly simplifies the formalism. A discussion of this property and technical details are provided in Appendixes A, B, C, and D.

After some algebra, we end up with exact RG functional equations for the cumulants of the renormalized disorder, $\Gamma_{k1}[\phi_a]$, $\Gamma_{k2}[\phi_a,\phi_b]$, etc., or more precisely for the cumulants of the renormalized random field, $\Gamma^{(1)}_{k1;x_1}[\phi_a]$, $\Gamma^{(1,1)}_{k2;x_1,x_2}[\phi_a,\phi_b]$, etc, with the *physical* fields ϕ_a only as arguments (superscripts denote functional differentiation with respect to the arguments). More details are given in Appendix A. As an illustration, the equations for the first two cumulants read

$$\partial_t \Gamma^{(1)}_{k1;x_1}[\phi_1] = -\frac{1}{2}\tilde{\partial}_t \frac{\delta}{\delta\phi_1(x_1)} \int_{x_2x_3} \widehat{P}_{k;x_2x_3}[\phi_1]\big(\Gamma^{(11)}_{k2;x_2,x_3}[\phi_1,\phi_1] - \widetilde{R}_{k;x_2x_3}\big) \tag{5}$$

and

$$\partial_t \Gamma^{(11)}_{k2;x_1,x_2}[\phi_1,\phi_2] = \frac{1}{2}\tilde{\partial}_t \frac{\delta^2}{\delta\phi_1(x_1)\delta\phi_2(x_2)} \int_{x_3x_4} \Big\{ -\widehat{P}_{k;x_3x_4}[\phi_1]\Gamma^{(101)}_{k3;x_3,.,x_4}[\phi_1,\phi_2,\phi_1] + \widetilde{P}_{k;x_3x_4}[\phi_1,\phi_1]\Gamma^{(20)}_{k2;x_3x_4,.}[\phi_1,\phi_2] + \frac{1}{2}\widetilde{P}_{k;x_3x_4}[\phi_1,\phi_2]\big(\Gamma^{(11)}_{k2;x_3,x_4}[\phi_1,\phi_2] - \widetilde{R}_{k;x_3x_4}\big) + \text{perm}(12)\Big\}, \tag{6}$$

where $t = \log(k/\Lambda)$, and $\widehat{R}_k$ and $\widetilde{R}_k$ are ir regulators: in Fourier space, $\widehat{R}_k(q^2)$ gives a mass $\sim k^2$ for modes with $|q| \lesssim k$ and is essentially zero for modes with $|q| \gtrsim k$, while $\widetilde{R}_k(q^2)$ suppresses fluctuations of the random field and is related to $\widehat{R}_k(q^2)$ in a way that is compatible with the underlying supersymmetry of the theory [24,25]. The short-hand notation $\widetilde{\partial}_t$ indicates a derivative with respect to t that acts on the cutoff functions only (i.e., $\widetilde{\partial}_t \equiv \partial_t \widehat{R}_k\, \delta/\delta\widehat{R}_k + \partial_t \widetilde{R}_k\, \delta/\delta\widetilde{R}_k$), and perm(12) denotes the expression obtained by permuting ϕ_1 and ϕ_2. Finally, the propagators $\widehat{P}_k$ and $\widetilde{P}_k$ are defined as

$$\widehat{P}_k[\phi] = \big(\Gamma^{(2)}_{k1}[\phi] + \widehat{R}_k\big)^{-1} \tag{7}$$

and

$$\widetilde{P}_k[\phi_1,\phi_2] = \widehat{P}_k[\phi_1]\big(\Gamma^{(11)}_{k2}[\phi_1,\phi_2] - \widetilde{R}_k\big)\widehat{P}_k[\phi_2]. \tag{8}$$

Note that the auxiliary fields have completely dropped out of the equations.

One finds that Eq. (5) coincides with the derivative with respect to $\phi_1(x)$ of the exact RG flow equation followed by $\Gamma_{k1}[\phi_1]$ for the RFIM at equilibrium [see Eq. (7) of Ref. [25]]. Similarly, Eq. (6) coincides with the derivative with respect to $\phi_1(x_1)$ and $\phi_2(x_2)$ of the exact RG flow equation followed by $\Gamma_{k2}[\phi_1,\phi_2]$ for the RFIM at equilibrium [see Eq. (8) of Ref. [25]]. It is easily derived that this generalizes to all higher-order cumulants, so that the exact hierarchies of RG flow equations for the cumulants of the renormalized random field $\Gamma^{(1\cdots1)}_{kp;x_1,\dots,x_p}[\phi_1,\dots,\phi_p]$ for the RFIM in and out of equilibrium are *identical*. This is the central result of the present work. The key physical ingredients underlying this result are on the one hand that out-of-equilibrium and equilibrium behaviors can be studied as zero-temperature phenomena and therefore involve exploration of specific stationary states, and on the other hand that the selected stationary states, be they extremal states for the hysteresis or ground states for the equilibrium case, are unique [35].

IV. SYMMETRY CONSIDERATIONS

We have therefore shown that the out-of-equilibrium hysteresis behavior and the ground-state physics are described by the same exact RG equations. As a result, *they share the same set of fixed-point solutions*. For the equilibrium case, one knows that there exists a fixed point associated with critical behavior. This fixed point has a Z_2 symmetry, i.e., all functions are symmetric under the inversion of the fields (local magnetizations). It has two relevant directions, one corresponding to a Z_2 symmetric perturbation and associated with the disorder strength that must be fine-tuned to be exactly at criticality, and the other being non-Z_2 symmetric and associated with the external source (which in some sense is also tuned to be zero, which amounts to staying in the Z_2 symmetric subspace).

The out-of-equilibrium critical points, on the other hand, have no Z_2 symmetry: they take place at nontrivial values of the external source (magnetic field) and of the field (magnetization): $J_c > 0$, $\phi_c \neq 0$ for the ascending branch of the hysteresis loop, and $-J_c$, $-\phi_c$ for the descending branch (see Fig. 1). This implies that the initial condition of the exact RG flow equations, i.e., the mean-field description at the microscopic scale, has no Z_2 symmetry around a given critical point. (This is akin to the situation encountered when relating the liquid-gas critical point of a genuine fluid that has no particle-hole symmetry to that of the simple Ising model with Z_2 symmetry.)

To show that non-Z_2-symmetric initial conditions appropriate for describing out-of-equilibrium criticality can flow under RG transformation to the already characterized Z_2 symmetric equilibrium fixed point, we consider the nonperturbative approximation scheme for the effective average action that we have already introduced in our previous work on the RFIM at equilibrium [24,25]. It combines a truncation in the "derivative expansion," i.e., an expansion in the number of spatial derivatives of the fundamental fields for approximating the long-distance behavior of the 1PI correlation functions, and a truncation in the expansion in cumulants of the renormalized disorder. The scheme also ensures that the symmetries and supersymmetries of the theory are not explicitly violated, which turns out to be an important issue for a proper description of "dimensional reduction" and its breakdown [24,25]. The approximation scheme then leads to a closed set of coupled nonperturbative functional RG equations that can be solved numerically.

When formulated at the level of the cumulants of the renormalized random field, the ansatz takes the form

$$\Gamma^{(1)}_{k1;x_1}[\phi] = U'_k[\phi(x_1)] + \frac{\delta}{\delta\phi(x_1)}\left\{\frac{1}{2}Z_k[\phi(x_1)][\partial\phi(x_1)]^2\right\}$$
$$\Gamma^{(11)}_{k2;x_1,x_2}[\phi_1,\phi_2] = \Delta_k(\phi_1(x_1),\phi_2(x_2)), \tag{9}$$

with the higher-order cumulants set to zero. For concreteness, we focus on the critical point along the ascending branch of the hysteresis loop, with $J_c > 0$. After insertion in the hierarchy of exact RG equations [Eqs. (5) and (6)], the above ansatz provides three coupled flow equations for $U'_k(\phi)$, which describes the renormalized source as a function of magnetization, the so-called "field-renormalization" function $Z_k(\phi)$, and the second cumulant of the renormalized random field $\Delta_k(\phi_1,\phi_2)$.

These flow equations are supplemented by an initial condition at the microscopic (uv) scale $k = \Lambda$. It corresponds to a mean-field approximation where only some coarse-graining over short-ranged fluctuations has been carried out (see, e.g., [29]). The crucial point is that the bare action has no Z_2 symmetry around the out-of-equilibrium critical point. The initial condition can then be taken with the same form as in Eq. (9) with

$$Z_\Lambda = 1\,,\ \Delta_\Lambda = \Delta_B \tag{10}$$

and $U'_\Lambda(\phi)$ generically given by

$$U'_\Lambda(\phi) = J_\Lambda + r_\Lambda\phi + \frac{\lambda_\Lambda}{2}\phi^2 + \frac{u_\Lambda}{3!}\phi^3 + \frac{v_\Lambda}{4!}\phi^4\,. \tag{11}$$

$U''_\Lambda(\phi)$ is then nonsymmetric and cannot be symmetrized by a shift in ϕ.

To cast the RG flow equations in a dimensionless form that allows one to investigate the critical physics at long length scales, one must introduce scaling dimensions. This is the second operation of any RG transformation. Near a zero-temperature fixed point, the renormalized temperature is irrelevant and is characterized by an exponent $\theta > 0$ [18,19]. One then has the following scaling dimensions [22–25]: $Z_k \sim k^{-\eta}$, $\phi - \phi_c \sim k^{(d-4+\bar\eta)/2}$, $U'_k - J_c \sim k^{(d-2\eta+\bar\eta)/2}$, $\Delta_k \sim k^{-(2\eta-\bar\eta)}$, where ϕ_c and J_c, respectively, denote the values of the magnetization and the magnetic field at the out-of-equilibrium critical point (see above), and the exponents θ, η, and $\bar\eta$ are related through $\theta = 2 + \eta - \bar\eta$.

Due to the lack of Z_2 symmetry, two relevant parameters must be fine-tuned to reach the critical point. In practice, we account for the additional condition by defining a displaced field variable $\tilde\phi = \phi - \phi_{r,k}$, where $\phi_{r,k}$ is fixed such that the third derivative of the renormalized potential is zero all along the flow: $U'''_k(\phi_{r,k}) = 0$. If indeed the critical system flows to a fixed point where Z_2 symmetry is restored, then $\phi_{r,k}$ flows to the critical value ϕ_c and $U'_k(\phi_{r,k})$ flows to J_c in the limit $k \to 0$.

Using lower-case letters, $u''_k, z_k, \delta_k, \tilde\varphi$, to denote the dimensionless counterparts of $U''_k, Z_k, \Delta_k, \tilde\phi$, the dimensionless form of the flow equations can be symbolically written as

$$\partial_t u''_k(\tilde\varphi) = \beta_{u'',k}(\tilde\varphi),\ \partial_t z_k(\tilde\varphi) = \beta_{z,k}(\tilde\varphi),$$
$$\partial_t \delta_k(\tilde\varphi_1,\tilde\varphi_2) = \beta_{\delta,k}(\tilde\varphi_1,\tilde\varphi_2), \tag{12}$$

where the beta functions on the right-hand sides themselves depend on u''_k, z_k, δ_k and their derivatives. As already stressed above, these flow equations are *the same* as for the RFIM at equilibrium; they are given in Ref. [25] and not reproduced here.

The nonperturbative RG equations can be solved for any spatial dimension d and any initial condition (yet two parameters must be fine-tuned to reach the critical fixed point). In all cases, we find that the flow leads to the Z_2 symmetric fixed point already derived for the equilibrium critical point. We illustrate the outcome for two cases (see

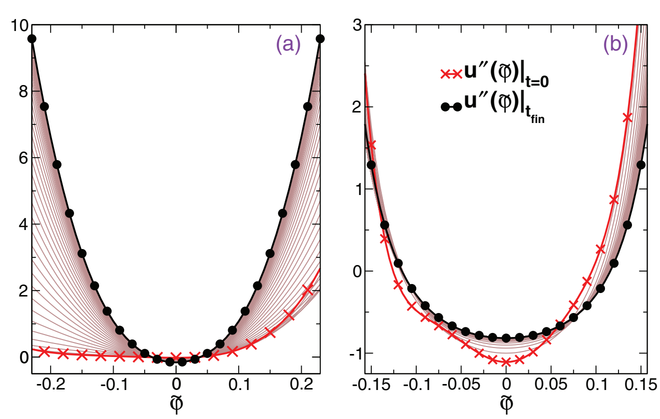

FIG. 2. (Color online) Nonperturbative RG evolution of the dimensionless "mass" function $u''_k(\tilde\varphi)$ for $d = 5.5 > d_{DR} \simeq 5.1$ (a) and $d = 4 < d_{DR}$ (b). The initial condition (red) is asymmetric, but the asymmetry gradually decreases along the flow and vanishes at the fixed point (thick black curve). Furthermore, the fixed-point function for $u''_k(\tilde\varphi)$ is identical to that for the equilibrium critical point.

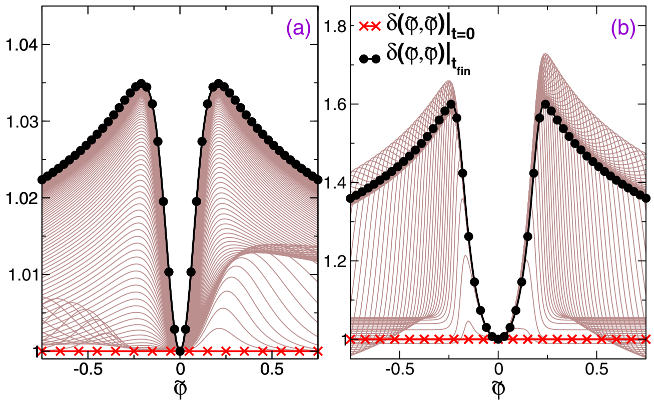

FIG. 3. (Color online) Same as Fig. 2 for the nonperturbative RG evolution of the dimensionless second cumulant of the renormalized random field $\delta_k(\tilde\varphi,\tilde\varphi)$ for $d = 5.5 > d_{\rm DR} \simeq 5.1$ (a) and $d = 4 < d_{\rm DR}$ (b). The initial condition (red) is a constant and is therefore symmetric, but the asymmetry first builds up along the flow before decreasing and finally vanishing at the fixed point (thick black curve). The fixed-point function is identical to that for the equilibrium critical point.

Figs. 2 and 3): one is above the critical dimension for dimensional-reduction breakdown, $d_{\rm DR} \simeq 5.1$ [24,25], and is therefore exactly described by the $d \to d-2$ dimensional-reduction property; the other is below $d_{\rm DR}$ and does not follow dimensional reduction. In both situations, one can clearly see that the asymmetry of the functions $u_k''(\tilde\varphi)$ and $\delta_k(\tilde\varphi,\tilde\varphi)$ eventually decreases and vanishes when reaching the fixed point. [The same is observed for the other function $z_k(\tilde\varphi)$ but is not displayed here.] The Z_2 symmetry is thus asymptotically restored and the fixed point *exactly* coincides with that found for the equilibrium criticality.

V. CONCLUSION

We have shown that the critical behaviors of the RFIM in and out of equilibrium are in the same universality class, with the same critical exponents, the same scaling functions, and the same avalanche-size distribution. This gives a solid theoretical foundation to the empirical numerical findings.

Along the way, the above developments also help to prove that the in- and out-of-equilibrium critical behaviors of fluids in a disordered porous material, which are both described by non-Z_2 symmetric theories [38], are in this same universality class [39]. Our present work, therefore, unifies a very large class of collective phenomena in and out of equilibrium that involve interactions and disorder.

APPENDIX A: GRASSMANNIAN ULTRALOCALITY AND EXACT FUNCTIONAL RG EQUATIONS

As explained in detail in Ref. [24], the generating functional in Eq. (4) of the main text can be reexpressed through standard field-theoretical techniques [30,31] as that of a "superfield theory" with a large group of symmetries and supersymmetries. In a nutshell, one introduces auxiliary (bosonic) fields $\hat\varphi_a$ to "exponentiate" the δ functional, pairs of auxiliary (fermionic) fields $\psi_a,\bar\psi_a$ to "exponentiate" the determinant, and one averages over the Gaussian random field [31]. By a Legendre transform, one then obtains the "effective action" (Gibbs free energy), $\Gamma[\{\Phi_a\}]$, where the Φ_a's are now "superfields" leaving in a "superspace" spanned by the d-dimensional Euclidean coordinate x and two anticommuting Grassmann coordinates $\theta,\bar\theta$ [24,30,31]:

$$\Phi_a(x,\theta,\bar\theta) = \phi_a(x) + \bar\theta\Psi_a(x) + \bar\Psi_a(x)\theta + \bar\theta\theta\hat\phi_a(x), \quad \text{(A1)}$$

where $\phi_a(x),\Psi_a(x),\bar\Psi_a(x),\hat\phi_a(x)$ denote the averages of the physical field and of the associated auxiliary fields in copy a.

After having introduced infrared (ir) regulators, one may define an effective average action $\Gamma_k[\{\Phi_a\}]$ which is the effective action of the system at the scale k [33,34]. Its expansion in increasing numbers of unrestricted sums over copies generates (modulo some inessential subtleties [24]) the cumulant expansion for the renormalized disorder:

$$\Gamma_k[\{\Phi_a\}] = \sum_a \Gamma_{k1}[\Phi_a] - \frac{1}{2}\sum_{a,b}\Gamma_{k2}[\Phi_a,\Phi_b] + \cdots, \quad \text{(A2)}$$

where Γ_p is essentially the pth cumulant of the renormalized disorder [24]. Such an expansion in increasing numbers of free sums over copies led to systematic algebraic manipulations that we have used repeatedly.

As recalled in the main text, the evolution of $\Gamma_k[\{\Phi_a\}]$ with k is described by an exact functional renormalization-group (RG) equation,

$$\partial_t\Gamma_k[\{\Phi_a\}] = \frac{1}{2}\mathrm{Tr}\left\{\partial_t\mathcal{R}_k\left(\Gamma_k^{(2)}[\{\Phi_a\}] + \mathcal{R}_k\right)^{-1}\right\}, \quad \text{(A3)}$$

where $t = \log(k/\Lambda)$, the trace involves summing over copy indices and integrating over superspace, and $\Gamma_k^{(2)}[\{\Phi_a\}]$ is the second functional derivative of the effective average action with respect to the superfields; $\mathcal{R}_k$ denotes the infrared regulator, which satisfies

$$\mathcal{R}_{k,a_1a_2}(x_1\underline{\theta}_1,x_2\underline{\theta}_2) = I_{(a,\theta_1,\bar\theta_1)(b,\theta_2,\bar\theta_2)}\widehat{R}_k(|x_1-x_2|) + \widetilde{R}_k(|x_1-x_2|), \quad \text{(A4)}$$

with $I_{(a,\theta_1,\bar\theta_1)(b,\theta_2,\bar\theta_2)} = \delta_{ab}\delta_{\bar\theta_1,\bar\theta_2}\delta_{\theta_1,\theta_2}$, where δ_{ab} is the Kronecker symbol and, due to the anticommuting properties of the Grassmann variables [30], $\delta_{\bar\theta_1,\bar\theta_2}\delta_{\theta_1,\theta_2} = (\bar\theta_1-\bar\theta_2)(\theta_1-\theta_2)$; $\widehat{R}_k(q^2)$ and $\widetilde{R}_k(q^2)$ are ir cutoff functions that are chosen such that the integration over modes with momentum $|q| \ll k$ is suppressed [22,24,34].

From the above equation, one can derive a hierarchy of exact functional RG equations for the cumulants. For the sake of illustration, we give below the exact RG equation for the first two cumulants:

$$\partial_t \Gamma_{k1}[\Phi_1] = \frac{1}{2}\tilde{\partial}_t \mathrm{Tr}\big\{ \log(\Gamma_{k1}[\Phi_1] + \widehat{R}_k I) + \big(\Gamma_{k1}^{(2)}[\Phi_1] + \widehat{R}_k I\big)^{-1}\big(\Gamma_{k2}^{(11)}[\Phi_1,\Phi_1] - \widetilde{R}_k I\big)\big\}, \tag{A5}$$

$$\begin{aligned}\partial_t \Gamma_{k2}[\Phi_1,\Phi_2] = \frac{1}{2}\tilde{\partial}_t \mathrm{Tr}\big\{ &- \Gamma_{k3}^{(101)}[\Phi_1,\Phi_2,\Phi_1]\big(\Gamma_{k1}^{(2)}[\Phi_1] + \widehat{R}_k I\big)^{-1} + \Gamma_{k2}^{(20)}[\Phi_1,\Phi_2]\big(\Gamma_{k1}^{(2)}[\Phi_1] + \widehat{R}_k I\big)^{-1}\big(\Gamma_{k2}^{(11)}[\Phi_1,\Phi_1] - \widetilde{R}_k I\big)\\ &+ \frac{1}{2}\big(\Gamma_{k2}^{(11)}[\Phi_1,\Phi_2] - \widetilde{R}_k I\big)\big(\Gamma_{k1}^{(2)}[\Phi_2] + \widehat{R}_k I\big)^{-1}\big(\Gamma_{k2}^{(11)}[\Phi_2,\Phi_1] - \widetilde{R}_k I\big)\big(\Gamma_{k1}^{(2)}[\Phi_1] + \widehat{R}_k I\big)^{-1} + \mathrm{perm}(12)\big\},\end{aligned} \tag{A6}$$

where superscripts denote functional differentiation with respect to the superfield arguments, I is the identity (defined above), $\widetilde{\partial}_t$ is a short-hand notation indicating a derivative with respect to t that acts on the cutoff functions only (see also the main text), and perm(12) denotes the expression obtained by permuting Φ_1 and Φ_2.

The above RG equations, and the whole hierarchy for higher-order cumulants, is exact but too formal to be useful as such. A major simplification, however, occurs when the generating functional is built from a unique stationary state (in each replica), which is the case here in the limit of infinite auxiliary source, $\hat{J} \to \pm\infty$, or in the Legendre transformed setting, i.e., the limit of infinite auxiliary field, $\hat{\phi} \to \pm\infty$. The uniqueness of the extremal states indeed translates in the present superfield framework in the formal property of the random generating functional that we called "Grassmannian ultralocality" [24]. The cumulants are then "ultralocal," i.e.,

$$\begin{aligned}\Gamma_{k1}[\Phi_1] &= \int_{\underline{\theta}_1} \Gamma_{k1}[\Phi_1(\underline{\theta}_1)],\\ \Gamma_{k2}[\Phi_1,\Phi_2] &= \int_{\underline{\theta}_1}\int_{\underline{\theta}_2} \Gamma_{k2}[\Phi_1(\underline{\theta}_1),\Phi_2(\underline{\theta}_2)]\,,\end{aligned} \tag{A7}$$

etc. $\Gamma_{k1},\Gamma_{k2},\ \dots$ on the right-hand sides only depends on the superfields at the explicitly displayed "local" Grassmann coordinates, hence the name "Grassmannian ultralocality." (On the other hand, the dependence on the Euclidean coordinates, which is left implicit, is *not* purely local.)

The property of Grassmannian ultralocality is also true for the equilibrium case, where the generating functional is dominated by the ground state, which is also unique for a given sample (except, again, for a set of conditions of measure zero); it then greatly simplifies the exact functional RG equations [24,25]. In the present case, however, one must proceed differently. We first differentiate the exact RG equations, such as Eqs. (A5) and (A6), in order to obtain RG equations for the cumulants of the *renormalized random field*, $\Gamma^{(1)}_{k1;x_1\underline{\theta}_1}[\Phi_1]$, $\Gamma^{(11)}_{k2;x_1\underline{\theta}_1;x_2\underline{\theta}_2}[\Phi_1,\Phi_2]$, etc. We next evaluate the latter equations at the (external) Grassmann coordinates $\theta_a = \bar{\theta}_a = 0$, for $a = 1,2,\dots$. Then, e.g., $\Phi_a(x,\underline{\theta_a}) = \phi_a(x)$, $\partial_t\Gamma^{(1)}_{k1;x_1\underline{\theta}_1}[\Phi_1]|_{\theta_1=\bar{\theta}_1=0} = \partial_t\Gamma^{(1)}_{k1;x_1}[\phi_1]$, and $\partial_t\Gamma^{(11)}_{k2;x_1\underline{\theta}_1;x_2\underline{\theta}_2}[\Phi_1,\Phi_2]|_{\theta_1=\bar{\theta}_1=\theta_2=\bar{\theta}_2=0} = \partial_t\Gamma^{(11)}_{k2;x_1,x_2}[\phi_1,\phi_2]$. After some straightforward algebra, we end up with exact RG functional equations for the cumulants $\Gamma^{(1\cdots 1)}_{kp;x_1,\dots,x_p}$ with the *physical* fields ϕ_a only as arguments. For instance, the equation for the first two cumulants is given in Eqs. (5) and (6) of the main text. These equations coincide exactly with those obtained for the same quantities, after using there the very same property of Grassmannian ultralocality, in the equilibrium case [24].

APPENDIX B: CORRECTIONS TO "GRASSMANNIAN ULTRALOCALITY" IN THE FUNCTIONAL RENORMALIZATION GROUP

In the derivation outlined in the preceding appendix, we actually used a shortcut that needs justification. Indeed, we have taken the limit $\hat{\phi} \to \infty$ before a full account of the fluctuations and the limit $k \to 0$. The correct procedure is instead to solve the exact RG flow down to $k = 0$ for $\hat{\phi}$ very large but finite and then take $\hat{\phi}$ to infinity. For large but finite $\hat{\phi}$, there are corrections to the Grassmannian ultralocality. It can, however, be checked that these corrections become irrelevant as one approaches the fixed point when $k \to 0$ and therefore give rise to only subdominant contributions. This is what we discuss now.

We now illustrate the structure of the functional RG flow in the presence of "non-ultralocal" components by looking at the corrections in the first cumulant and assuming that all other cumulants are purely "ultralocal" in both Grassmann and Euclidean coordinates. More specifically, we consider

$$\begin{aligned}\Gamma_{k1}[\Phi_1] = \int_{\underline{\theta}_1} \big(&\Gamma^{\mathrm{UL}}_{k1}[\Phi_1(\underline{\theta}_1)] + \Gamma^{\mathrm{NUL}}_{k1}[\Phi_1(\underline{\theta}_1),\\ &\partial_{\theta_1}\Phi_1(\underline{\theta}_1),\partial_{\bar{\theta}_1}\Phi_1(\underline{\theta}_1),\partial_{\theta_1}\partial_{\bar{\theta}_1}\Phi_1(\underline{\theta}_1)]\big),\end{aligned} \tag{B1}$$

where $\Gamma^{\mathrm{NUL}}_{k1}$ is non-ultralocal in the Grassmann coordinates (i.e., it depends on the derivatives) but ultralocal in the Euclidean coordinates, and for $p \geqslant 2$,

$$\begin{aligned}&\Gamma_{kp}[\Phi_1,\dots,\Phi_p]\\ &\quad= \int_{\underline{\theta}_1}\cdots\int_{\underline{\theta}_p}\int_{x_1}\cdots\int_{x_p} \Gamma^{\mathrm{UL}}_{kp}[\Phi_1(x_1,\underline{\theta}_1),\dots,\Phi_p(x_p,\underline{\theta}_p)].\end{aligned} \tag{B2}$$

By virtue of the supersymmetries of the theory, the non-ultralocal part of the first cumulant can be rewritten in terms of components in the following form:

$$\begin{aligned}\Gamma^{\mathrm{NUL}}_{k1}[\Phi_1] = \int_x \big[&\hat{\phi}_1(x)\,\gamma_{k1}(\phi_1(x),\hat{\phi}_1(x))\\ &+ \Psi_1(x)\bar{\Psi}_1(x)\,\gamma^{(10)}_{k1}(\phi_1(x),\hat{\phi}_1(x))\big].\end{aligned} \tag{B3}$$

In principle, all manipulations should involve the fermionic fields $\Psi_1,\bar{\Psi}_1$, but it turns out that supersymmetries again lead to simplifications and that the same results are obtained by setting these fields to zero, which we do here to simplify the presentation.

The second functional derivative of the effective average action $\Gamma^{(2)}_{k;(a\underline{\theta}_1)(a\underline{\theta}_2)}$ that enters in the functional RG equations can be decomposed as [24]

$$\Gamma^{(2)}_{k;(a\underline{\theta}_1)(b\underline{\theta}_2)} = \widehat{\Gamma}^{(2)}_{k;a\underline{\theta}_1\underline{\theta}_2}\delta_{ab} + \widetilde{\Gamma}^{(2)}_{k;(a\underline{\theta}_1)(b\underline{\theta}_2)}. \tag{B4}$$

After adding the ir regulators, the "hat" and "tilde" components have the following general structure:

$$\begin{aligned}\widehat{\Gamma}^{(2)}_{k;a\underline{\theta}_1\underline{\theta}_2}&[\{\phi_a,\hat{\phi}_a\}] + \widehat{R}_k\delta_{\underline{\theta}_1\underline{\theta}_2}\\ &= \widehat{A}_{k;a} + \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2\widehat{B}_{k;a} + (\bar{\theta}_1\theta_1 + \bar{\theta}_2\theta_2)\widehat{C}_{k;a}\\ &\quad - (\bar{\theta}_1\theta_2 + \bar{\theta}_2\theta_1)\widehat{E}_{k;a},\end{aligned} \tag{B5}$$

$$\begin{aligned}\widetilde{\Gamma}^{(2)}_{k;(a\underline{\theta}_1)(b\underline{\theta}_2)}&[\{\phi_a,\hat{\phi}_a\}] + \widetilde{R}_k\delta_{\underline{\theta}_1\underline{\theta}_2}\\ &= \widetilde{A}_{k;ab} + \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2\widetilde{B}_{k;ab} + \bar{\theta}_1\theta_1\widetilde{C}_{k;ab} + \bar{\theta}_2\theta_2\widetilde{D}_{k;ab},\end{aligned} \tag{B6}$$

and to the lowest order of the expansions in increasing number of free sums over copies [24] (leaving implicit the dependence on the Euclidean coordinates),

$$\begin{aligned}\widehat{A}_{k;a}[\phi_a,\hat{\phi}_a] &= 2\gamma^{(01)}_{k1}[\phi_a,\hat{\phi}_a] + \hat{\phi}_a\gamma^{(02)}_{k1}[\phi_a,\hat{\phi}_a],\\ \widehat{B}_{k;a}[\phi_a,\hat{\phi}_a] &= \hat{\phi}_a\widehat{\Gamma}^{\mathrm{UL}(3)}_{k1}[\phi_a],\\ \widehat{C}_{k;a}[\phi_a,\hat{\phi}_a] &= \widehat{\Gamma}^{\mathrm{UL}(2)}_{k1}[\phi_a] + \widehat{R}_k + \gamma^{(01)}_{k1}[\phi_a,\hat{\phi}_a]\\ &\quad + \hat{\phi}_a\gamma^{(11)}_{k1}[\phi_a,\hat{\phi}_a],\\ \widehat{E}_{k;a}[\phi_a,\hat{\phi}_a] &= \widehat{\Gamma}^{\mathrm{UL}(2)}_{k1}[\phi_a] + \widehat{R}_k + \gamma^{(10)}_{k1}[\phi_a,\hat{\phi}_a],\end{aligned} \tag{B7}$$

and

$$\begin{aligned}\widetilde{A}_{k;ab}[\phi_a,\hat{\phi}_a,\phi_b,\hat{\phi}_b] &= -\Gamma^{\mathrm{UL}(11)}_{k2}[\phi_a,\phi_b] + \widetilde{R}_k,\\ \widetilde{B}_{k;ab}[\phi_a,\hat{\phi}_a,\phi_b,\hat{\phi}_b] &= -\Gamma^{\mathrm{UL}(22)}_{k2}[\phi_a,\phi_b],\\ \widetilde{C}_{k;ab}[\phi_a,\hat{\phi}_a,\phi_b,\hat{\phi}_b] &= -\hat{\phi}_a\Gamma^{\mathrm{UL}(21)}_{k2}[\phi_a,\phi_b],\\ \widetilde{D}_{k;ab}[\phi_a,\hat{\phi}_a,\phi_b,\hat{\phi}_b] &= -\hat{\phi}_b\Gamma^{\mathrm{UL}(12)}_{k2}[\phi_a,\phi_b].\end{aligned} \tag{B8}$$

The full propagator $\mathcal{P}_{k;(a_1\underline{\theta}_1)(a_2\underline{\theta}_2)}$, which is the inverse of $\Gamma^{(2)}_k + \mathcal{R}_k$ (where $\mathcal{R}_k$ collects the two ir regulators), has thesame structure as in Eqs. (B4)–(B6) with

$$\begin{aligned}\widehat{\mathcal{P}}_{k;a\underline{\theta}_1\underline{\theta}_2}[\{\phi_a,\hat{\phi}_a\}] &= \widehat{Q}_{k;a} + \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2\widehat{S}_{k;a} + (\bar{\theta}_1\theta_1 + \bar{\theta}_2\theta_2)\widehat{P}_{k;a}\\ &\quad - (\bar{\theta}_1\theta_2 + \bar{\theta}_2\theta_1)\widehat{T}_{k;a}\end{aligned} \tag{B9}$$

and an expression similar to Eq. (B6) for $\widetilde{\mathcal{P}}_{k;(a\underline{\theta}_1)(b\underline{\theta}_2)}[\{\phi_a,\hat{\phi}_a\}]$.

At the lowest order of the expansion in increasing free sums over copies, the components of $\widehat{\mathcal{P}}_k$ and $\widehat{\Gamma}^{(2)}_k$ are related by

$$\begin{aligned}\widehat{Q}_k[\phi_a,\hat{\phi}_a] &= -(\widehat{C}_k\widehat{C}_k - \widehat{A}_k\widehat{B}_k)^{-1}\widehat{A}_k,\\ \widehat{S}_k[\phi_a,\hat{\phi}_a] &= -(\widehat{C}_k\widehat{C}_k - \widehat{A}_k\widehat{B}_k)^{-1}\widehat{B}_k,\\ \widehat{P}_k[\phi_a,\hat{\phi}_a] &= (\widehat{C}_k\widehat{C}_k - \widehat{A}_k\widehat{B}_k)^{-1}\widehat{C}_k,\\ \widehat{T}_k[\phi_a,\hat{\phi}_a] &= \widehat{E}_k^{-1},\end{aligned} \tag{B10}$$

where one should keep in mind that the components are operators in Euclidean space.

On the other hand, the "tilde" components of the propagator are obtained at the lowest order of the expansion in free sums over copies from

$$\begin{aligned}\widetilde{\mathcal{P}}_{k;(a\underline{\theta}_1)(b\underline{\theta}_2)}&[\phi_a,\hat{\phi}_a,\phi_b,\hat{\phi}_b]\\ &= -\int_{\underline{\theta}_3}\int_{\underline{\theta}_4}\widehat{\mathcal{P}}_{k;\underline{\theta}_1\underline{\theta}_3}[\phi_a,\hat{\phi}_a]\big(\widetilde{\Gamma}^{(2)}_{k;\underline{\theta}_3\underline{\theta}_4}[\phi_a,\hat{\phi}_a,\phi_b,\hat{\phi}_b] + \widetilde{R}_k\big)\\ &\quad\times\widehat{\mathcal{P}}_{k;\underline{\theta}_4\underline{\theta}_2}[\phi_b,\hat{\phi}_b].\end{aligned} \tag{B11}$$

The algebraic manipulations are straightforward, but the resulting expressions are too lengthy to be reproduced here. We stress that *no* approximations are involved in deriving results at the lowest order of the expansion in free sums over copies. The higher orders are not needed.

We can now collect the above results and insert them in the exact RG equation for the first cumulant, Eq. (A5). This leads to

$$\begin{aligned}\partial_t\Gamma_{k1}[\Phi_1]\\ = \frac{1}{2}\int_{x_1x_2}\int_{\underline{\theta}_1,\underline{\theta}_2}&\big\{\partial_t\widetilde{R}_k(x_1-x_2)\widehat{\mathcal{P}}_{k;(x_1,\underline{\theta}_1)(x_2,\underline{\theta}_2)}[\phi_1,\hat{\phi}_1]\\ &+ \partial_t\widehat{R}_k(x_1-x_2)\,\delta_{\underline{\theta}_1,\underline{\theta}_2}\big(\widehat{\mathcal{P}}_{k;(x_1,\underline{\theta}_1)(x_2,\underline{\theta}_2)}[\phi_1,\hat{\phi}_1]\\ &+ \widetilde{\mathcal{P}}_{k;(x_1,\underline{\theta}_1)(x_2,\underline{\theta}_2)}[\phi_1,\hat{\phi}_1,\phi_2,\hat{\phi}_2]\big)\big\}.\end{aligned} \tag{B12}$$

After taking a functional derivative with respect to $\Phi_1(x,\underline{\theta})$, evaluating the outcome for $\theta = \bar{\theta} = 0$, and using Eqs. (B1)–(B3), one obtains an explicit RG flow equation for $\Gamma^{(1)\mathrm{UL}}_{k1;x}[\phi_1] + \gamma_{k1}(\phi_1(x),\hat{\phi}_1(x)) + \hat{\phi}_1(x)\gamma^{(01)}_{k1}(\phi_1(x),\hat{\phi}_1(x))$. To keep the presentation in a reasonable format, we further evaluate the equation for spatially uniform fields $\phi_1(x)\equiv\phi_1,\hat{\phi}_1(x)\equiv\hat{\phi}_1$, so that it simplifies to

$$\begin{aligned}\partial_t U^{(1)}_k(\phi_1) &+ \partial_t\big[\gamma_{k1}(\phi_1,\hat{\phi}_1) + \hat{\phi}_1\gamma^{(01)}_{k1}(\phi_1,\hat{\phi}_1)\big]\\ &= \frac{1}{2}\frac{\delta}{\delta\hat{\phi}_1}\int_q\big\{\partial_t\widetilde{R}_k(q^2)\widehat{S}_k(q^2) + \partial_t\widehat{R}_k(q^2)\big(2[\widehat{P}_k(q^2) - \widehat{T}_k(q^2)] + 2\widehat{S}_k(q^2)\big[\Gamma^{(11)}_{k2}(q^2;\phi_1,\phi_1) - \tilde{R}_k(q^2)\big]\widehat{P}_k(q^2)\\ &\quad + 2\widehat{Q}_k(q^2)\hat{\phi}_1^2\Gamma^{(22)}_{k2}(q^2;\phi_1,\phi_1)\widehat{P}_k(q^2) + \big[\widehat{Q}_k(q^2)\widehat{S}_k(q^2) + \widehat{P}_k(q^2)^2\big]\hat{\phi}_1\big[\Gamma^{(21)}_{k2}(q^2;\phi_1,\phi_1) + \Gamma^{(12)}_{k2}(q^2;\phi_1,\phi_1)\big]\big)\big\},\end{aligned} \tag{B13}$$

where $U_k(\phi_1)$ is the effective average potential, i.e., the component of the first cumulant that is ultralocal in both Euclidean and Grassmann coordinates; $\widehat{P}_k(q^2),\widehat{Q}_k(q^2),\widehat{S}_k(q^2),\widehat{T}_k(q^2)$ are functions of ϕ_1 and $\hat{\phi}_1$.

Since we are interested in showing that the non-ultralocal corrections give subdominant corrections near the fixed point in the limit $|\hat{\phi}|\to+\infty$, it is sufficient to consider an expansion in $1/|\hat{\phi}|$. For convenience, we choose to study the descending branch of the hysteresis loop with $\hat{J} > 0$ and $\hat{\phi} > 0$. The non-ultralocal component $\gamma_{k1}(\phi,\hat{\phi})$ has an expansion of the form

$$\gamma_{k1}(\phi,\hat{\phi}) = \frac{1}{\hat{\phi}^p}\left(X_{k0}(\phi) + \frac{X_{k1}(\phi)}{\hat{\phi}} + \cdots\right) \tag{B14}$$

with $p > 1$.

It is easily realized that when the above expansion is inserted in the functional RG equation, Eq. (B13), the right-hand side can also be expanded in powers of $1/\hat{\phi}$ and the flow of the ultralocal function $U_k^{\rm UL}(\phi_1)$ is *not* affected by the non-ultralocal contributions. This property generalizes to the higher cumulants and to the case in which the fields are not uniform in the Euclidean space. This is different from what is encountered in the equilibrium case when studying the asymptotic dominance of the ground state [24,25]. Along the same lines, the flow for any $X_{kn}(\phi)$ is independent of the higher-order terms of the expansion. For instance, the flow of $X_{k0}(\phi)$ reads

$$\begin{aligned}\partial_t X_{k0}(\phi)|_\phi &\equiv \beta_{X_0,k}(\phi)\\
&= \frac{1}{2(p-1)}\tilde{\partial}_t \int_q \{X''_{k0}(\phi)\big[\Gamma^{(11)}_{k2}(q^2;\phi,\phi) - \tilde{R}_k(q^2)\big]\widehat{P}_k(q^2)^2\\
&\quad + 2X'_{k0}(\phi)\widehat{P}_k(q^2)^2\big(2\widehat{P}_k(q^2)\Gamma^{(3)\rm UL}_{k1}(q^2;\phi)\big[\Gamma^{(11)}_{k2}(q^2;\phi,\phi) - \tilde{R}_k(q^2)\big]\\
&\quad - \big[\Gamma^{(21)}_{k2}(q^2;\phi,\phi) + \Gamma^{(12)}_{k2}(q^2;\phi,\phi)\big]\big) + 3X_{k0}(\phi)\widehat{P}_k(q^2)^2\big(2\widehat{P}_k(q^2)^2\Gamma^{(3)\rm UL}_{k1}(q^2;\phi)^2\big[\Gamma^{(11)}_{k2}(q^2;\phi,\phi) - \tilde{R}_k(q^2)\big] - \widehat{P}_k(q^2)\\
&\quad \times\big[\Gamma^{(21)}_{k2}(q^2;\phi,\phi) + \Gamma^{(12)}_{k2}(q^2;\phi,\phi)\big] + 2\Gamma^{(22)}_{k2}(q^2;\phi,\phi)\big)\}.\end{aligned} \tag{B15}$$

If X_{k0} is equal to zero at the microscopic scale Λ, which is the initial condition for the RG flow ($X_{\Lambda 0} = 0$), then it is obvious from the above equation that it stays zero all along the flow. The power p of the leading behavior in $1/\hat{\phi}$ in Eq. (B14) is thus fixed by the initial condition. The latter is a mean-field-like description, which amounts to an effective zero-dimensional model. In the following, we therefore make a detour to study a toy model, namely the $d = 0$ version of the out-of-equilibrium RFIM considered here. This will also prove instructive to elucidate the physics behind the non-ultralocal corrections.

APPENDIX C: ZERO-DIMENSIONAL RFIM MODEL

We consider the $d = 0$ version of the ϕ^4 theory in a quenched random field defined by Eqs. (1)–(4) of the main text, i.e.,

$$S(\phi; h + J) = \frac{r}{2}\phi^2 + \frac{u}{4!}\phi^4 - (J + h)\phi, \tag{C1}$$

where $r < 0$, so that the extremization equation $\partial S(\phi; h + J)/\partial\phi = -|r|\phi + (u/3!)\phi^3 - (J + h) = 0$ has three solutions for a range of $h + J$ around zero. The partition function in the presence of an auxiliary field $\hat{J}$ has contributions from the three solutions (when present):

$$\mathcal{Z}_h(\hat{J}, J) = \sum_{\alpha=1}^{3}(-1)^{n_\alpha}\mathcal{C}_\alpha(h + J)e^{\hat{J}\phi_\alpha(h+J)}, \tag{C2}$$

where $\mathcal{C}_\alpha(h + J)$ is the characteristic function of the interval of $h + J$ over which ϕ_α exists, and n_α is the index of the αth solution (here, $+1$ for a maximum and -1 for a minimum).

Consider again for illustration the descending branch of the hysteresis characterized by the extremal state with maximum magnetization $\phi_M(h + J)$, which is obtained when $\hat{J} \to +\infty$. When $\hat{J}$ is large but not infinite, the generating functional in Eq. (C2) is dominated by $\exp[\hat{J}\phi_M(h + J)]$ (the maximal state is a minimum). Corrections that do not vanish exponentially with $\hat{J}$ can only occur for the range of $h + J$ where a second solution has a magnetization, say ϕ_S, that is within $1/\hat{\phi}$ of ϕ_M. This takes place in the vicinity of the point $h + J = J^*$ and $\phi = \phi^*$, where the extremal state (minimum) collapses with the nearby saddle-point (maximum). Then, the disorder average of the logarithm of the generating functional, $W_1(\hat{J}, J) = \overline{\ln\mathcal{Z}_h(\hat{J}, J)}$, is given at leading orders in $\hat{J}$ by

$$\begin{aligned}\mathcal{W}_1(\hat{J}, J) &- \hat{J}\,\overline{\phi_M(h + J)}\\
&\sim \int d(\delta h)\frac{e^{-\frac{(\delta h - J + J^*)^2}{2\Delta_B}}}{\sqrt{2\pi\,\Delta_B}}\ln\left(1 - e^{-2\hat{J}\sqrt{\frac{2|\delta h|}{u\phi^*}}}\right),\end{aligned} \tag{C3}$$

where the integral over $\delta h = h + J - J^*$ is restricted to a finite range around 0. When $\hat{J} \to \infty$, this leads to

$$\mathcal{W}_1(\hat{J}, J) - \hat{J}\,\overline{\phi_M(h + J)} \sim \frac{e^{-\frac{(J-J^*)^2}{2\Delta_B}}}{\sqrt{2\pi\,\Delta_B}}\frac{1}{\hat{J}^2}. \tag{C4}$$

From the above behavior, one immediately obtains that $\hat{\phi} = \mathcal{W}_1^{(01)}(\hat{J}, J) \sim \hat{J} + O(1/\hat{J}^2)$, $\phi = \mathcal{W}_1^{(10)}(\hat{J}, J) \simeq \overline{\phi_M(h + J)} + O(1/\hat{J}^3)$, and that $\Gamma_1(\phi,\hat{\phi}) = -\mathcal{W}_1(\hat{J}, J) + \hat{J}\phi + J\hat{\phi}$ is given by

$$\Gamma_1(\phi,\hat{\phi}) = \hat{\phi}\,J_M(\phi) + \frac{Y(\phi)}{\hat{\phi}^2} + l\,O(1/\hat{\phi}^3) \tag{C5}$$

when $\hat{\phi}, \hat{J} \to \infty$, where $J_M(\phi)$ is the inverse function of $\overline{\phi_M(h + J)}$.

The first term on the right-hand side of Eq. (C5) is the contribution that is ultralocal in the Grassmann coordinates, and the second one is the dominant non-ultralocal correction (the fermionic fields have been set to zero for simplicity). The latter, therefore, behaves like $1/\hat{\phi}^2$ when $\hat{\phi} \to \infty$. The same result is valid for the mean-field approximation in general dimension d as it essentially amounts to considering a self-consistent zero-dimensional effective system. This shows that the non-ultralocal contribution at the uv scale, $\gamma_{k=\Lambda 1}(\phi,\hat{\phi})$ [see Eqs. (B3) and (B14) above], behaves as $1/\hat{\phi}^3$ at large $\hat{\phi}$, i.e., $p = 3$.

APPENDIX D: IRRELEVANCE OF NON-ULTRALOCAL CORRECTIONS AT LARGE DISTANCE

To investigate the long-distance physics in the vicinity of the out-of-equilibrium critical point, we must cast the functional RG flow equations in dimensionless form by using scaling dimensions appropriate for a zero-temperature fixed point. This is described in the main text (and in more detail in Ref. [25]). Accordingly, we define a dimensionless non-ultralocal contribution from $X_{k0}(\phi) = k^{\kappa}\chi_{k0}(\tilde{\varphi})$; the associated β function in Eq. (B15) similarly scales as $\beta_{X_0,k}(\phi) = k^{\kappa}\beta_{\chi_0,k}(\tilde{\varphi})$, so that in dimensionless form,

$$\partial_t \chi_{k0}(\tilde{\varphi})|_{\tilde{\varphi}} = -\kappa\chi_{k0}(\tilde{\varphi}) + \tfrac{1}{2}(d-4+\bar{\eta})\tilde{\varphi}\chi'_{k0}(\tilde{\varphi}) + \beta_{\chi_0,k}(\tilde{\varphi}). \tag{D1}$$

The naive expectation for the scaling of $\hat{\phi}$ is that it behaves like $k^{(d+2\eta-\bar{\eta})/2}$. However, $\hat{\phi}$ should rather be adjusted so that $\hat{J}$ can go to infinity even at the fixed point since this is the way to select the extremal state. As

$$\hat{J} = \frac{\partial\Gamma_{k1}[\phi,\hat{\phi}]}{\partial\hat{\phi}} \simeq U_k^{(2)}(\phi)\,\hat{\phi} \tag{D2}$$

with $U_k^{(2)}(\phi) \sim k^{2-\eta}$ near the fixed point, $\hat{\phi}$ should scale as $k^{-(2-\eta)}$. More precisely, we define a constant $\hat{\phi}_0$ which asymptotically behaves as $\hat{J}$ and such that $\hat{\phi}$ evolves under the RG flow close to the fixed point as $k^{-(2-\eta)}\hat{\phi}_0$. The relevant non-ultralocal quantity to be compared with the ultralocal one, $U_k^{(1)}(\phi) = k^{(d-2\eta+\bar{\eta})/2}u'_k(\tilde{\varphi})$, can thus be expressed as

$$\frac{X_{k0}(\phi)}{\hat{\phi}^3} = k^{\omega}\,\frac{\chi_{k0}(\tilde{\varphi})}{\hat{\phi}_0^3} \tag{D3}$$

with $\omega = \kappa + 3(2-\eta)$.

We solve Eq. (D1) as an eigenvalue equation by setting $\partial_t\chi_{*0}(\tilde{\varphi}) = 0$ and by using the ultralocal functions already found for the fixed point. The 1-replica functions $u''_*(\tilde{\varphi})$, $z_*(\tilde{\varphi})$, and $\chi_{*0}(\tilde{\varphi})$ are discretized on a grid of $2\times 110+1$ points with a mesh of $\delta\tilde{\varphi} = 0.005$, thus giving the range of the field $\tilde{\varphi}$ from -0.55 to 0.55. The 2-replica function, i.e., the second cumulant of the renormalized random field, $\delta_*(x,y)$ with $x = (\tilde{\varphi}_1+\tilde{\varphi}_2)/2$ and $x = (\tilde{\varphi}_1-\tilde{\varphi}_2)/2$, is discretized on a trapezoidal grid with a base identical to the domain of the 1-replica functions and a height of 36 points. The mesh in the second field y is identical to that of the field x, $\delta y = \delta x$. (We checked that by doubling the resolution of the mesh, our results change on the fourth digit, and by changing the range of fields, the change is on the sixth digit.)

There are two nontrivial solutions of the eigenvalue equation, which are illustrated in Fig. 4 for the dimension $d = 4$. The eigenvalues $\kappa_{1,2}$ are monotonically increasing functions of the dimension, and they reach the upper critical dimension $d = 6$ values that can be analytically derived: $\kappa_1(d=6) = 0$ and $\kappa_2(d=6) = 1$. The eigenvalue ω defined in Eq. (D3) is then simply obtained by adding $3(2-\eta)$. The result is plotted as a function of dimension in Fig. 5. One can clearly see that the exponent of the non-ultralocal contribution (whether obtained from the symmetric or the antisymmetric solution) is much larger than that of the ultralocal term. This proves that the corrections to Grassmann ultralocality are irrelevant at large distance, i.e., that the selection of the extremal states is properly ensured.

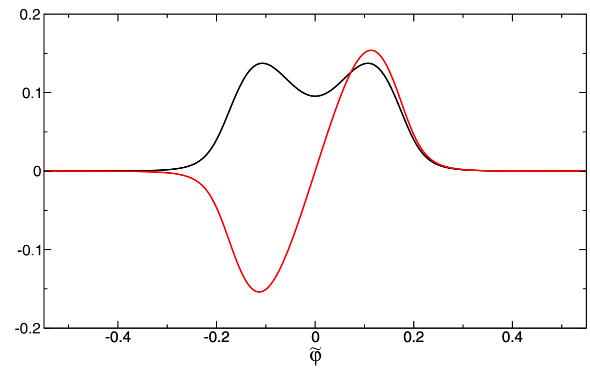

FIG. 4. (Color online) The two nontrivial solutions $\chi_{*0}(\tilde{\varphi}) = 0$ in $d = 4$. One is Z_2 symmetric and the other is antisymmetric. In this particular spatial dimension, the corresponding eigenvalues κ_1 and κ_2 are almost identical.

We conclude this appendix by briefly discussing the physics behind the non-ultralocal corrections. A first hint is given by the zero-dimensional model. As seen in Appendix C, the most significant contribution associated with the corrections comes from rare situations in which the extremal state almost coincides (within $1/\hat{J}$ when $\hat{J} \to +\infty$) with a nearby saddle point.

The reasoning can be carried over to the general case. The non-ultralocal corrections are due to rare events in which there is an almost degeneracy (within $1/|\hat{J}|$) between the relevant extremal state and a nearby stationary state, the solution of the stochastic field equation $\delta S[\varphi; h+J]/\delta\varphi(x) = 0$, with a very different configuration yet a very close total magnetization. These rare instances make the non-ultralocal contributions vanish at large distance as a power law rather than the naively anticipated exponential decay. This is somewhat reminiscent of the role of power-law rare "droplet" excitations near the ground state at low but nonzero temperature in the equilibrium case [18,19,23].

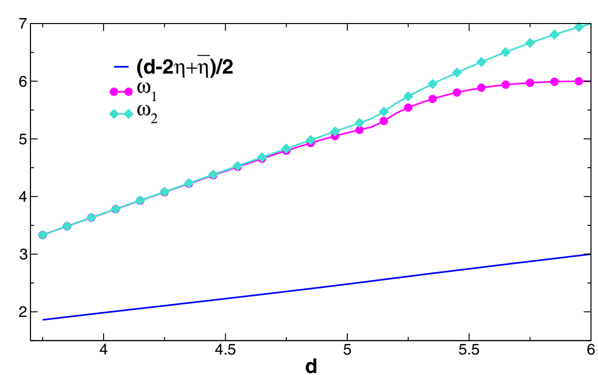

FIG. 5. (Color online) Exponents ω_1 (for the Z_2 symmetric eigenfunction) and ω_2 (for the Z_2 antisymmetric eigenfunction) of the non-ultralocal correction as a function of dimension d. We also plot the scaling exponent $(d-2\eta+\bar{\eta})/2$ of the corresponding ultralocal term (blue line). It is clear that the non-ultralocal terms are subdominant and do not affect then the universal behavior.

[1] J. P. Sethna, K. A. Dahmen, and C. R. Myers, Nature (London) **410**, 242 (2001).
[2] D. S. Fisher, Phys. Rep. **301**, 113 (1998).
[3] K. A. Dahmen and Y. Ben-Zion, in *Extreme Environmental Events*, edited by R. A. Meyers (Springer, New York, 2011), p. 5021.
[4] G. Bertotti, *Hysteresis in Magnetism* (Academic, New York, 1998).
[5] J.-P. Bouchaud, J. Stat. Phys. **151**, 567 (2013).
[6] M. P. Lilly, A. H. Wootters, and R. B. Hallock, Phys. Rev. Lett. **77**, 4222 (1996).
[7] F. Detcheverry *et al.*, Langmuir **20**, 8006 (2004).
[8] J. A. Bonetti, D. S. Caplan, D. J. Van Harlingen, and M. B. Weissman, Phys. Rev. Lett. **93**, 087002 (2004).
[9] E. W. Carlson, K. A. Dahmen, E. Fradkin, and S. A. Kivelson, Phys. Rev. Lett. **96**, 097003 (2006); E. W. Carlson and K. A. Dahmen, Nat. Commun. **3**, 915 (2012).
[10] J. P. Sethna, K. Dahmen, S. Kartha, J. A. Krumhansl, B. W. Roberts, and J. D. Shore, Phys. Rev. Lett. **70**, 3347 (1993).
[11] J. P. Sethna, K. A. Dahmen, and O. Perkovic, in *The Science of Hysteresis*, edited by G. Bertotti and I. Mayergoyz (Elsevier, Amsterdam, 2005), p. 107.
[12] K. Dahmen and J. P. Sethna, Phys. Rev. B **53**, 14872 (1996).
[13] Y. Imry and S. K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[14] For a review, see T. Nattermann, *Spin Glasses and Random Fields* (World Scientific, Singapore, 1998), p. 277.
[15] A. Maritan, M. Cieplak, M. R. Swift, and J. R. Banavar, Phys. Rev. Lett. **72**, 946 (1994).
[16] F. J. Perez-Reche and E. Vives, Phys. Rev. B **70**, 214422 (2004).
[17] Y. Liu and K. A. Dahmen, Phys. Rev. E **79**, 061124 (2009); Europhys. Lett. **86**, 56003 (2009).
[18] J. Villain, Phys. Rev. Lett. **52**, 1543 (1984); J. Phys. **46**, 1843 (1985).
[19] D. S. Fisher, Phys. Rev. Lett. **56**, 416 (1986).
[20] P. Le Doussal, K. J. Wiese, and P. Chauve, Phys. Rev. B **66**, 174201 (2002).
[21] J. Bricmont and A. Kupiainen, Phys. Rev. Lett. **59**, 1829 (1987).
[22] G. Tarjus and M. Tissier, Phys. Rev. Lett. **93**, 267008 (2004); Phys. Rev. B **78**, 024203 (2008).
[23] M. Tissier and G. Tarjus, Phys. Rev. Lett. **96**, 087202 (2006); Phys. Rev. B **78**, 024204 (2008).
[24] M. Tissier and G. Tarjus, Phys. Rev. Lett. **107**, 041601 (2011); Phys. Rev. B **85**, 104202 (2012).
[25] M. Tissier and G. Tarjus, Phys. Rev. B **85**, 104203 (2012).
[26] M. Baczyk, M. Tissier, G. Tarjus, and Y. Sakamoto, Phys. Rev. B **88**, 014204 (2013).
[27] A. A. Middleton, Phys. Rev. Lett. **68**, 670 (1992).
[28] M. Guagnelli, E. Marinari, and G. Parisi, J. Phys. A **26**, 5675 (1993).
[29] M. L. Rosinberg and G. Tarjus, J. Stat. Mech. (2010) P12011.
[30] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed. (Oxford University Press, New York, 1989).
[31] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[32] K. G. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1974).
[33] C. Wetterich, Phys. Lett. B **301**, 90 (1993).
[34] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).
[35] Accordingly, the result is expected to extend to other situations in the RFIM where a unique state is selected by a specific and well-defined process, e.g., for the so-called “demagnetized state” [36, 37].
[36] J. H. Carpenter and K. A. Dahmen, Phys. Rev. B **67**, 020412 (2003).
[37] F. Colaiori, M. J. Alava, G. Durin, A. Magni, and S. Zapperi, Phys. Rev. Lett. **92**, 257203 (2004).
[38] G. Tarjus *et al.*, Mol. Phys. **109**, 2863 (2013).
[39] I. Balog, M. Tissier, and G. Tarjus (unpublished).