Linear response of a one-dimensional conductor coupled to a dynamical impurity with a Fermi edge singularity

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I study the dynamical correlations that a quantum impurity induces in the Fermi sea to which it is coupled. I consider a quantum transport setup in which the impurity can be realized in a double quantum dot. The same Hamiltonian describes tunneling states in metallic glasses, and can be mapped onto the Ohmic spin-boson model. It exhibits a Fermi edge singularity, i.e., many fermion correlations result in an impurity decay rate with a nontrivial power-law energy dependence. I show that there is a simple relation between temporal impurity correlations on the one hand and the linear response of the Fermi sea to external perturbations on the other. This results in a power-law singularity in the space and time dependence of the nonlocal polarizability of the Fermi sea, which can be detected in transport experiments.

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I. INTRODUCTION

Often, when a Fermi sea couples to the localized degree of freedom of a quantum impurity, the dynamics and thermodynamics of the impurity are nontrivially affected [1–4]. In turn, the impurity induces correlations between the otherwise noninteracting electrons in the Fermi sea [5–10]. At present, we have a more complete understanding of the dissipative dynamics of the impurity than we have of impurity-induced spatial and temporal correlations induced in the Fermi sea [9]. Motivated by this relative lack of understanding, I study a quantum transport setup where an impurity interacts with electrons in a one-dimensional conductor. The internal dynamics of the impurity is restricted to a two-dimensional Hilbert space. In the past, the model has been used to account for the low-temperature thermodynamics of metallic glasses, in terms of tunneling atoms coupled to the conduction electrons [11]. However, as in the case of the Kondo effect [12], a realization of the model in a quantum transport setup allows for greater tunability [13,14] and a wider variety of possible measurements. Whereas the electron-electron correlations that I study would be hard to detect in a metallic glass, they are imminently observable in the setup I study.

The following is known about the system. In the weak tunneling limit, the exponential decay rate $W(\varepsilon)$ associated with the relaxation of the impurity has a power-law singularity [15,16] $\sim\varepsilon^{2\alpha-1}$. Here, ε is the energy bias between the two impurity states and α is the coupling strength between the impurity and the Fermi sea. The power-law form of $W(\varepsilon)$ is known as a Fermi edge singularity [17–19]. It is a nontrivial many-body effect involving a sum to infinite order of an expansion in α. In this expansion, higher powers imply more particle-hole excitations in the conductor. The involvement of this multitude of particle-hole excitations in the impurity decay process is known as Fermi sea shakeup [20]. The results that I report in this paper reveal a simple relation between charge fluctuations associated with Fermi sea shakeup and the decay rate $W(\varepsilon)$.

Further insight into the dynamics of the impurity is obtained by bosonizing the Fermi sea [21,22]. This maps the system onto the Ohmic spin-boson model [23–26]. It reveals that the impurity undergoes a localization-delocalization quantum phase transition at $\alpha = 1$ [27]. The Ohmic spin-boson model can in turn be mapped onto an anisotropic Kondo model [26,28,29]. In the language of the Kondo model, the point $\alpha = 1$ separates the antiferromagnetic ($\alpha < 1$) and ferromagnetic ($\alpha > 1$) regimes.

The analysis of the localization-delocalization transition is typical of many studies into open quantum systems, in that the bath degrees of freedom are traced out at an early stage [30]. This is the appropriate approach for addressing fundamental questions regarding dissipation and decoherence in quantum mechanics.

In this paper, I take a different perspective. I study the correlations that the two-level system induces between the otherwise noninteracting degrees of freedom of the bath. This is in the same spirit as recent studies on the screening cloud around an impurity that displays a Kondo effect [5–10]. In a previous work, I considered static density correlations among electrons in the system's ground state [25]. (Static here means correlations between densities at equal times but at different points in space.) Long-range correlations were found. Thanks to the Fermi edge singularity, these correlations have a power-law dependence on ε, the power-law exponent being $2\alpha - 3$. In this work, I take the logical next step and investigate dynamic density correlations. I also generalize to arbitrary temperatures. It is important to ask whether the electron-electron correlations that I study are observable. In many open systems, bath degrees of freedom are not directly accessible to outside observers, but only indirectly through their effect on the impurity. I will show that in the setup I consider, a striking signal is produced in the electron transport through the conductor.

I calculate the conductor's nonlocal polarizability, which measures the linear response of the electron density to a potential fluctuation. In the language of the spin-boson model, this corresponds to a retarded single-particle Green's function associated with the bosonic degrees of freedom. In the language of the anisotropic Kondo model, it corresponds to the $z-z$ component of the nonlocal magnetic susceptibility

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of the itinerant electrons. The polarizability is affected by the interaction with the impurity as follows. When the system is perturbed by means of a potential fluctuation, a charge fluctuation is generated. The charge fluctuation propagates towards the impurity. As it passes the impurity, it sets it in motion. The original fluctuation is distorted by this excitation process but continues to propagate towards the detector. The excited impurity then acts back on the electrons in the conductor, creating further charge fluctuations. These are also picked up by the detector. One of the main results of this work is that the polarizability has a power-law singularity as a function of time after first arrival of the signal at the detector. This can be traced back to the Fermi edge singularity. The power-law exponent is found to be -2α, the same as that of the Fourier transform of $W(\varepsilon)$.

The plan for the rest of the paper is as follows. In Sec. II, the Hamiltonian for the model that I study is presented. The connection to the Fermi edge singularity and the mapping to the spin-boson model are explained. The questions that will be answered in the rest of the text are formulated precisely. In Sec. III, an exact relation between electron and impurity correlations is derived. The implications that known results for impurity correlations have for electron correlations are discussed. These include short-time asymptotics derived from the impurity's equations of motion, and general features revealed by exact numerical solution of the corresponding Kondo model. The rest of the paper is devoted to obtaining further analytical results for the electron polarizability. In Sec. IV, an exact expression for the polarizability is derived at a special value of the coupling, where an exact expression for the impurity Green's function is known. Based on this expression, a regime is identified where perturbation theory in the impurity tunneling amplitude is valid. In Sec. V, the leading-order term in this expansion is calculated for arbitrary coupling. Section VI contains a discussion of results.

II. MODEL

I study a setup in which electrons in a one-dimensional conductor interact with a two-level impurity. The nature of the interaction is as follows. The impurity creates a local electrostatic potential that scatters the electrons propagating in the conductor. The shape of the potential, and hence the scattering matrix of the conductor, depends on the state of the impurity [15,16,24,25]. When the impurity is held fixed in, respectively, state $|+\rangle$ or $|-\rangle$, the scattering matrix of the conductor is $S_+ = e^{-iU_+}$ or $S_- = e^{-iU_-}$. I denote the dimension of the scattering matrices by M and refer to this matrix structure as channel space. Furthermore, there is a tunneling amplitude Δ and an energy bias ε_0 between impurity states $|+\rangle$ and $|-\rangle$. An impurity of this type can be realized by a single electron trapped in a double quantum dot [31,32]. Here, $|\pm\rangle$ corresponds to the electron being localized in the ground state of the one or the other of the two dots. The trapped electron produces an electrostatic potential that is felt by the electrons in the conductor. This potential depends on which one of the states $|+\rangle$ or $|-\rangle$ the trapped electron occupies. Thus, the conductor in the vicinity of the double dot acts as a quantum point contact with a constriction profile that is determined by the state of the electron in the double dot. This is depicted in Fig. 1.

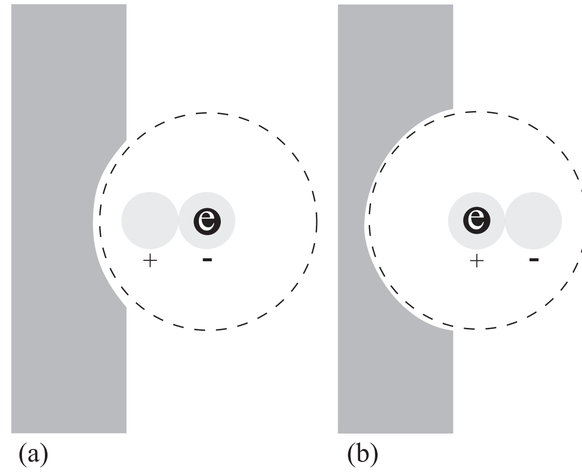

FIG. 1. A physical realization of the studied model. A one-dimensional conductor is coupled to a double quantum dot in which a single electron is trapped. Conduction electrons traverse the region that is shaded dark gray. The repulsive potential of the electron trapped in the double quantum dot produces a point contact in this conductor. (The dashed circles in the figures schematically represent an equipotential line of this potential.) The profile of the point contact depends on the state of the electron in the double dot: In (a) on the left, this electron is in the further of the two dots (the $|-\rangle$ state). As a result, the point contact constriction is less narrow than in (b) on the right, where the electron is in the nearer of the two dots (the $|+\rangle$ state).

An effective low-energy description is provided by the Hamiltonian

$$H = H_+P_+ + H_-P_- + \frac{\varepsilon_0}{2}\sigma_z + \frac{\Delta}{2}\sigma_x, \tag{2.1}$$

with Pauli matrices $\sigma_x = |+\rangle\langle -| + |-\rangle\langle +|$ and $\sigma_z = |+\rangle\langle +| - |-\rangle\langle -|$ acting on the impurity's internal degree of freedom, and $P_\pm = (1 \pm \sigma_z)/2$ projection operators onto impurity states $|\pm\rangle$. The fermion Hamiltonians

$$H_\pm = \int dx\, \boldsymbol{\phi}^\dagger(x)\,[-i\partial_x + U_\pm f(x)]\,\boldsymbol{\phi}(x) \tag{2.2}$$

describe the electrons in the conductor when the impurity is held fixed in the state $|\pm\rangle$. I use units were the Fermi velocity equals unity. In the last equation, $\boldsymbol{\phi}(x)$ is an M-dimensional column vector of fermion annihilation operators, the vector structure referring to channel space. To arrive at this form, the dispersion relation around the Fermi energy was linearized. In the usual one-dimensional Fermi gas, that contains both left- and right-moving electrons, (2.2) is obtained by the standard trick of unfolding of the scattering channels (see Fig. 2). This entails splitting up the electronic wave functions close to the Fermi energy into left- and right-moving components. A parity transformation, that replaces the coordinate x with $-x$, is then applied to the left-moving electrons. In the transformed system, all propagation is from left to right [33]. Thus, $x < 0$ refers to electron amplitudes incident on the impurity, while $x > 0$ refers to outgoing amplitudes. The diagonal matrix elements of $U_\pm$ describe forward scattering off the impurity in

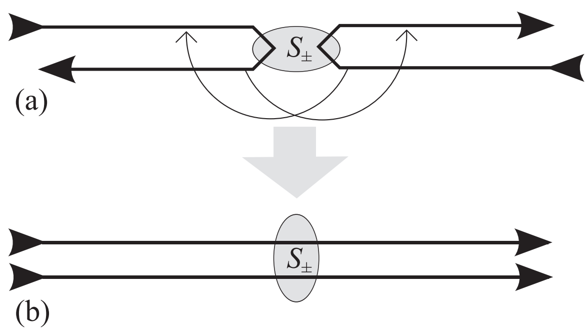

FIG. 2. Unfolding of scattering channels. (a) A usual one-dimensional conductor containing both left- and right-moving electrons. The thick black contours schematically indicate the paths of reflected particles. Also indicated by arrows is how these paths transform when the system is unfolded. (b) The unfolded picture in which the Hamiltonians $H_\pm$ take the form (2.2). Reflection in (a) maps onto intrachannel scattering in (b) while transmission in (a) maps onto interchannel scattering in (b).

the unfolded representation, i.e., intrachannel reflection in the physical or "folded" picture. The off-diagonal elements of $U_\pm$ describe interchannel scattering in the unfolded representation, which corresponds to transmission or interchannel reflection in the physical picture. It is also possible to realize a Fermi gas in which there is only a right mover, say. An example is a quantum Hall edge channel. For these realizations, (2.2) follows without unfolding. The function $f(x)$ is sharply peaked at $x = 0$, the position of the impurity, with $\int dx\, f(x) = 1$, i.e., it is delta-function-like.

Taking the impurity interactions to be of the form $U_\pm f(x)$, i.e., factorizable into position-independent factors $U_\pm$ and a short-range function $f(x)$, is an approximation. It is valid for a sufficiently small impurity level splitting. In this case, the wavelengths associated with particle-hole excitations created by the impurity are long compared to the range of the interaction between the impurity and the conductor. As a result, impurity-induced excitations in the conductor can not resolve the spatial features of the impurity potential, and all potentials associated with the same scattering matrices are equivalent [24].

The Hamiltonian H results from integrating out those high-energy degrees of freedom for which the linear dispersion and the truncation of the impurity Hilbert space to two dimensions breaks down. As a result, the length scale $1/\Lambda$ on which $f(x)$ varies is longer than the actual scale on which the impurity interaction varies. It is at least a few times the Fermi wavelength. In other words, the ultraviolet scale Λ is at most a fraction of the Fermi energy. Below, $f(x)$ will be replaced by a delta function. When an ultraviolet regularization is required, the delta function will be taken as a Lorentzian

$$f(x) = \frac{1}{2\pi\Lambda}\frac{1}{x^2 + (1/2\Lambda)^2}. \tag{2.3}$$

I will only be concerned with physics at length scales larger than $1/\Lambda$.

The Hamiltonian can be replaced by a form that is diagonal in channel space as follows. Define new fermion operators

$$\boldsymbol{\varphi}(x) = e^{iU_+ g(x)}\boldsymbol{\phi}(x), \tag{2.4}$$

with

$$g(x) = \int_0^x dx'\, f(x'). \tag{2.5}$$

In terms of these operators, $H_\pm$ read as

$$H_+ = \int dx\, \boldsymbol{\varphi}^\dagger(x)(-i\partial_x)\boldsymbol{\varphi}(x),$$
$$H_- = \int dx\, \boldsymbol{\varphi}^\dagger(x)[-i\partial_x + \tilde{U}(x)f(x)]\boldsymbol{\varphi}(x), \tag{2.6}$$

with $\tilde{U}(x) = e^{iU_+g(x)}[U_- - U_+]e^{-iU_+g(x)}$. Now, consider the single-particle Schrödinger equation associated with H_-, namely,

$$E\boldsymbol{\varphi}_1(x) = [-i\partial_x + \tilde{U}(x)f(x)]\boldsymbol{\varphi}_1(x), \tag{2.7}$$

where $\boldsymbol{\varphi}_1(x)$ is an M-component single-particle wave function. This equation is solved by

$$\boldsymbol{\varphi}_1(x) = e^{iEx}e^{iU_+g(x)}e^{-iU_-g(x)}\boldsymbol{\varphi}_-, \tag{2.8}$$

where $\boldsymbol{\varphi}_-$ is the wave function at $x \to -\infty$, up to a phase. Utilizing the fact that $g(x)$ approaches 1 to the right of the impurity, we identify the scattering matrix associated with (2.7) as $S_+^\dagger S_-$. For the low-energy physics I am interested in, the short-wavelength structure of the potential in (2.6) and (2.7) is irrelevant. I therefore replace the potential with a simpler potential that produces the same scattering matrix, namely $Vf(x)$, where $V = i\ln S_+^\dagger S_-$ [13]. The branch of the logarithm is fixed by the requirement that V should evolve continuously from zero as the overall coupling strength between the conductor and the impurity is varied from zero to full strength. Finally, the unitary transformation $\boldsymbol{\psi}(x) = Q^\dagger e^{iVg(x)/2}\boldsymbol{\varphi}(x)$, with Q the matrix that diagonalizes V, leads to the form

$$H = \sum_l \int dx\, \psi_l^\dagger(x)[-i\partial_x + \gamma_l\delta(x)\sigma_z]\psi_l(x) + \frac{\varepsilon}{2}\sigma_z + \frac{\Delta}{2}\sigma_x. \tag{2.9}$$

Here, γ_l is the phase shift in channel l of the combined scattering matrix $S_+^\dagger S_-$, i.e., γ_l are the eigenvalues of $-i\ln(S_+^\dagger S_-)/2$. Also, as advertised above, $f(x)$ has been replaced with a delta function. From here on, I will use the representation (2.9) of the Hamiltonian, referring to it as the representation in the ψ basis, as opposed to the ϕ basis, of (2.2).

In this derivation, a subtle issue was glossed over. It involves a contribution to the impurity bias and is reflected in (2.9) by the replacement of ε_0 with ε. It has to do with the singular nature of a density operator such as $\phi_l^\dagger(x)\phi_l(x)$ for fermions with a linear dispersion, and hence an infinitely deep Fermi sea. Naively, it would seem that for such fermions, a Hamiltonian containing a scalar potential $v(x)$ can be transformed into a free Hamiltonian, using a gauge transformation $\phi(x) = \exp -i\int_0^x dx'\, v(x')\tilde{\phi}(x)$, that leaves the density unaffected. This would mean that no potential could trap any charge. Careful regularization of the problem, for instance by means of point splitting or bosonization [22], shows that this is not the case. The transformed density operator turns out to be $\tilde{\rho}(x) = \rho(x) - v(x)/2\pi$, thereby accounting for the missing charge.

The only effect of implementing a regularized version of the derivation of (2.9) is to change ε_0 to $\varepsilon = \varepsilon_0 +$ offset, with the offset depending in the short-distance details of the impurity interaction. Such a contribution to the bias is clearly required to account for the interaction with the "missing" charge after the transformation from ϕ to ψ. From here on, I treat ε as a phenomenological parameter that can be adjusted by varying the external bias between impurity states $|+\rangle$ and $|-\rangle$. In Sec. V, where I trace out the fermions, a further contribution to ε, that again depends on the short-distance details of the impurity interaction, will be found. At that point, I will simply redefine ε to incorporate the new contribution too, rather than using a different symbol. A further point to note is that the density of $\sum_l \psi_l^\dagger(x)\psi_l(x)$ of ψ fermions in (2.9) is in general not the same as the density of ϕ fermions in (2.2). However, as indicated in the previous paragraph, the two operators only differ where the impurity interaction, which is proportional to $f(x)$, is nonzero. Thus, away from the impurity, the subtle issue of the difference between ϕ and ψ densities can be ignored.

As mentioned in the Introduction, in its original fermionic incarnation, the system displays a Fermi edge singularity. In this context, the following is relevant. Consider the regime of sufficiently large ε and the system initialized with the impurity in the state $|+\rangle$ and the conductor in the ground state of H_+. As a function of time, the expectation value $\langle\sigma_z(t)\rangle$ will decay from 1 at $t = 0$ to a value of $-1 + O(\Delta_r/\varepsilon)$, where Δ_r is the effective tunneling amplitude of the impurity. Its precise definition (2.13) is deferred for two paragraphs, until I have introduced some concepts related to the spin-boson model. For times larger than $1/\Lambda$, the decay is exponential $\sim\exp[-W(\varepsilon)t]$. For $\varepsilon \ll \Lambda$, the decay rate is given by [24]

$$W(\varepsilon) = \frac{\pi\Delta_r}{2\Gamma(2\alpha)}\left(\frac{\Delta_r}{\varepsilon}\right)^{1-2\alpha}, \tag{2.10}$$

with

$$\alpha = -\frac{1}{2}\mathrm{tr}\frac{(\ln S_+^\dagger S_-)^2}{4\pi^2}, \tag{2.11}$$

and $\Gamma(x)$ is the standard gamma functiuon (not to be confused with the energy Γ defined in Sec. IV). One of the main results I derive in Sec. V is a relation between the rate $W(\varepsilon)$ and the charge associated with the system's response to a perturbation in the external electrostatic potential.

With the aid of bosonization, the Hamiltonian of (2.9) maps onto the spin-boson model with an Ohmic bath, i.e., at low frequencies, the bath spectrum is linear [23,26]. At frequencies larger than Λ, the scale set by $f(x)$, the bath spectral density falls of to zero. The precise detail of how it does so depends on the shape (but not the overall magnitude) of $f(x)$ [24], but only affects the correlations I study at short times and distances ($<1/\Lambda$). The Lorentzian regularization (2.3) of $f(x)$ corresponds to the conventional choice of a spectral density that decays exponentially at large frequencies [27]. The bath spectral density is then given by

$$J(\omega) = 2\pi\alpha\omega e^{-\omega/\Lambda}. \tag{2.12}$$

(A redundant coupling constant between the bath and the impurity, conventionally denoted as q_0 in the spin-boson model, has been set equal to 1.) The parameter α defined in (2.11) is identical to the parameter α of Ref. [27] and K of Refs. [2,23]. It characterizes the coupling strength between the bath and the impurity, and hence also the dissipation strength. As another indication of the nontrivial effect that the bath has on the impurity, I mention in passing that the Ohmic spin-boson model undergoes a quantum phase transition at $\alpha = 1$. For $\alpha < 1$, a system that is initially prepared with the impurity in one of the states $|\pm\rangle$ and the bath in the corresponding ground state of $H_\pm$ eventually relaxes so that the reduced density matrix of the impurity approaches its equilibrium form, even when $\varepsilon = 0$. For $\alpha > 1$ and $\varepsilon = 0$, however, the impurity never relaxes but remains stuck in the state it is initialized in.

It is useful to define an effective impurity tunneling amplitude

$$\Delta_r = \Delta\left(\frac{\Delta}{\Lambda}\right)^{\alpha/(1-\alpha)}. \tag{2.13}$$

It turns out that physical quantities that are not ultraviolet divergent (i.e., insensitive to physics at length scales $1/\Lambda$ or shorter) only depend on Δ and Λ through Δ_r [2,27]. It should further be noted that Δ itself is an effective tunneling amplitude that emerges after the impurity Hilbert space has been truncated to two dimensions by integrating out high-energy excited states. In Refs. [2,27] it is shown that this leads to a dependence $\Delta \sim \Lambda^\alpha$ so that Δ_r is in fact independent of Λ. Rather, the ultraviolet scale that Δ_r is sensitive to is the one at which restricting the dynamics of the impurity to a two-dimensional Hilbert space breaks down. (This scale is larger than Λ.) I will treat Δ_r as a phenomenological parameter characterizing the effective impurity tunneling amplitude, and express final answers in terms of it. A final fact to note about the Ohmic spin-boson model is that it is exactly solvable for $\alpha = \frac{1}{2}$ [23]. This will allow me to calculate electronic correlation functions exactly in Sec. IV for this specific value of the dissipation parameter. Apart from being illuminating in their own right, these exact results support the perturbative analysis that is done in Sec. V for arbitrary α.

I calculate the polarizability in the ψ basis

$$\chi_{ll'}(x,y,t) = -i\theta(t)\,\langle[\rho_l(x,t),\rho_{l'}(y,0)]\rangle\,, \tag{2.14}$$

where $\rho_l(x) = \psi_l^\dagger(x)\psi_l(x)$ is the density at x in channel l. The linear response at time t and position x of the density in channel l to a potential perturbation $v = \sum_{l'}\int dx'\, v_{l'}(x',t)\rho_{l'}(x')$ is

$$\langle\Delta\rho_l(x,t)\rangle = \sum_{l'}\int dy\int dt'\,\chi_{ll'}(x,y,t-t')v_{l'}(y,t'). \tag{2.15}$$

The impurity-induced part of $\chi_{ll'}(x,y,t)$ (with $x<0$ and $y>0$) consists of two parts, namely, a delta pulse $-Q_{ll'}\delta(y+t-x)$, followed by a decaying tail. Different physical processes are responsible for these two contributions. The delta pulse results from the original excitation of the impurity, while the tail results from the subsequent interaction of the excited impurity with the conductor. The two contributions are, however, not completely independent. Owing to charge conservation, $\int_{x-y+0^+}^{\infty} dt\,\chi_{ll'}(x,y,t) = Q_{ll'}$. I call $Q_{ll'}$ the response charge, and calculate it in the following, it being a single number that characterizes the strength of the impurity-induced electron-electron response.

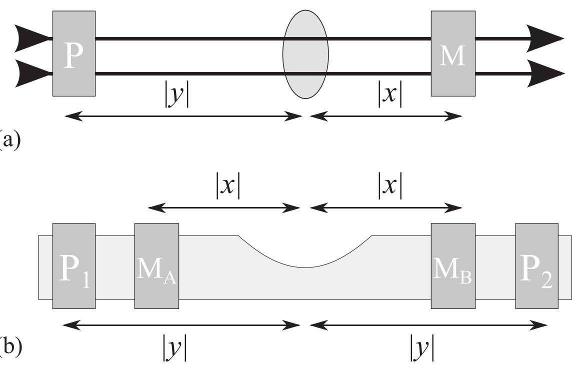

FIG. 3. Measurement of $\chi(x,y,t)$ in a conductor containing both left- and right-moving electrons. (a) In the unfolded picture, which contains only right movers, $\chi(x,y,t)$ describes the response at M to a perturbation at P. (b) To measure $\chi(x,y,t)$ in a physical (folded) system containing both left and right movers, one first perturbs at P_1 and then measures at M_A and M_B. Then, one perturbs at P_2 and again measures the signal at M_A and M_B. To obtain $\chi(x,y,t)$, these signals are added to those of the first experiment.

How would one measure the polarizability $\chi_{ll'}(x,y,t)$? Recall that it is defined in the so-called ψ basis for channel space, that is convenient for calculation. [See the discussion following Eq. (2.9).] Since perturbing or measuring in individual channels of the ψ basis does not seem realistic, it makes sense to sum over l and l', thereby obtaining the total linear response $\chi(x,y,t)$ at x to a perturbation at y. It is important to remember here that x and y refer to positions in the unfolded channels. A perturbation or measurement that is local (i.e., occurs at a single position) in the unfolded representation, is nonlocal, occurring simultaneously at two points, in the folded, physical representation. This, however, by no means implies that $\chi(x,y,t)$ is unobservable. In a system containing both left and right movers, one simply has to perform four kinds of measurements (see Fig. 3). First, one would perturb the potential at a distance $|y|$ to the left of the impurity (at P_1 in Fig. 3), and measure the fluctuation in the densities at a distance $|x|$ both to the left and to the right of the impurity (M_A and M_B in Fig. 3), taking care to subtract the component of the fluctuation that reaches the left detector before interacting with the impurity. Then, one would repeat the procedure, now with an identical potential perturbation at a distance $|y|$ to the right of the impurity (at P_2 in Fig. 3). Finally, one would add the four measured density fluctuations. It is this total density fluctuation that is described by the polarizability $\chi(x,y,t)$. Note also that in realizations where all the electrons move in the same direction (such as a quantum Hall edge channel), these considerations do not apply, and $\chi(x,y,t)$ can be obtained from a single measurement.

III. RELATION BETWEEN IMPURITY AND ELECTRON-ELECTRON CORRELATIONS

In this section, I derive a simple relation between generating functionals for electron and for impurity correlations. I use this to prove a linear relation between the polarizability $\chi_{ll'}(x,y,t)$ and the retarded Green's function of the impurity observable σ_z. The results of this section are nonperturbative and exact. It is interesting to note that real-space correlations between electrons in the Kondo model have similarly been related to impurity correlations [9].

The generating functional for imaginary-time density-density correlations at inverse temperature β is

$$F_\rho[V] = \ln \mathrm{tr}\left\{\mathcal{T} e^{-\int_0^\beta d\tau\, [H+\sum_l \int dx\, V_l(x,\tau)\rho_l(x)]}\right\}. \tag{3.1}$$

Here, $\mathcal{T}$ time-orders the exponential with the smallest time argument to the right, and $\rho_l(x) = \psi_l^\dagger(x)\psi_l(x)$ is the (Schrödinger picture) density operator at point x in channel l. Functional derivatives with respect to V, evaluated at $V=0$, generate density correlation functions.

The generating functional can be expressed as a path integral. The electronic degrees of freedom are represented by Grassmann fields $\bar{\psi}$ and ψ. The impurity's internal degree of freedom can be represented by a complex field via for instance the SU(2) coherent state construction [34]. This will have the effect of replacing Pauli matrices σ_x and σ_z with complex scalar functions $\sigma_x(\tau)$ and $\sigma_z(\tau)$. Our analysis does not require an explicit expression for the action of the noninteracting impurity or for the integration measure associated with the impurity degree of freedom, and these will therefore simply be denoted as, respectively, $S_0[\sigma]$ and $\mathcal{D}[\sigma]$. The path-integral expression for $F_\rho[V]$ then reads as

$$F_\rho[V] = \ln \int \mathcal{D}[\sigma]\, e^{-S_0[\sigma]}\langle e^{-S_1[\sigma,V]}\rangle_0, \tag{3.2}$$

where

$$\langle \ldots \rangle_0 = \frac{1}{Z_0}\int \mathcal{D}\bar{\psi}\, D\psi \ldots e^{-\bar{\psi} g^{-1}\psi}, \tag{3.3}$$

and g refers to the free-electron Green's function $g_k(i\Omega_n) = (i\Omega_n - k)^{-1}$. Ω_n is a fermionic Matsubara frequency and I use the shorthand notation

$$\bar{\psi} g^{-1}\psi = \frac{1}{\beta}\sum_{ln}\int \frac{dk}{2\pi}\, \bar{\psi}_{lk}(\Omega_n)(i\Omega_n - k)\psi_{lk}(\Omega_n). \tag{3.4}$$

The functional

$$S_1[\sigma,V] = \frac{1}{\beta}\sum_{lm}\int \frac{dk}{2\pi}\, h_{l-k}(-\omega_m)\rho_{lk}(\omega_m), \tag{3.5}$$

with

$$h_{lk}(\omega_n) = \gamma_l\, \sigma_z(\omega_m) + V_{lk}(\omega_m), \tag{3.6}$$

contains the coupling of the electrons to the impurity and to the field V. In the above expression, ω_m is a bosonic Matsubara frequency, and

$$\rho_{lk}(\omega_m) = \frac{1}{\beta}\sum_n \int \frac{dq}{2\pi}\, \bar{\psi}_q(\Omega_n)\psi_{q+k}(\Omega_n+\omega_m) \tag{3.7}$$

is the Grassmann representation of the (k,ω_n) component of the Fourier transform of the electron density in channel l. In (3.3), $Z_0 = \int \mathcal{D}\bar{\psi}\, D\psi\, e^{-\bar{\psi} g^{-1}\psi}$ refers to the partition function of the electrons in the absence of the impurity. A term $+\ln Z_0$ has been dropped from (3.2), because it does not depend on V, and hence does not show up in correlators calculated using $F_\rho[V]$.

The generating functional for imaginary-time correlators for the impurity observable σ_z is defined as

$$F_\sigma[B] = \ln \mathrm{tr}\{\mathcal{T} e^{-\int_0^\beta d\tau\,[H+B(\tau)\sigma_z]}\}. \tag{3.8}$$

Expressed as a path integral, it reads as

$$F_\sigma[B] = \ln \int \mathcal{D}[\sigma]\, e^{-S_0[\sigma]} \langle e^{-S_1[\sigma,0]}\rangle_0 e^{\frac{1}{\beta}\sum_n B(-\omega_n)\sigma_z(\omega_n)}. \tag{3.9}$$

I will now show that there exists a simple relation between $F_\sigma[B]$ and $F_\rho[V]$ and hence between impurity correlations and the density-density correlations the impurity induces in the electron gas. The starting point is the path-integral expression (3.2) for $F_\rho[V]$. The factor $\langle e^{-S_1[\sigma,V]}\rangle_0$ is the ratio between two Gaussian integrals and can therefore easily be evaluated. Defining two operator kernels

$$\begin{aligned} h_{lkn,l'k'n'} &= \delta_{ll'} h_{lk-k'}(\Omega_n - \Omega_{n'}),\\ g_{lkn,l'k'n'} &= 2\pi\beta\delta_{ll'}\delta(k-k')\delta_{nn'} g_k(i\Omega_n), \end{aligned} \tag{3.10}$$

one finds

$$\begin{aligned} \langle e^{-S_1[\sigma,V]}\rangle_0 &= \frac{\det[-g^{-1}+h]}{\det[-g^{-1}]}\\ &= \exp \mathrm{tr} \ln(1-gh). \end{aligned} \tag{3.11}$$

Remarkably, when the logarithm in (3.11) is expanded in gh, it is found that all terms higher than second order are identically zero. This is known as the Dzyaloshinskii-Larkin theorem [35,36]. Thus,

$$\mathrm{tr} \ln(1-gh) = -\mathrm{tr}(gh) - \tfrac{1}{2}\mathrm{tr}(ghgh). \tag{3.12}$$

Explicitly evaluating the first-order term, one finds

$$\mathrm{tr}(gh) = \sum_l h_{lk=0}(\omega_m = 0)\left[\frac{1}{\beta}\sum_n \int \frac{dp}{2\pi} g_p(i\Omega_n)\right]. \tag{3.13}$$

From the fundamental definition of the free fermionic Green's function

$$g_p(i\Omega_n) = \int_0^\beta d\tau\, e^{i\Omega\tau} \int dx\, e^{-ikx} \langle \psi_l(x,\tau)^\dagger \psi_l(x,0)\rangle_0 \tag{3.14}$$

follows that the term in square brackets in (3.13) equals $\bar{\rho}$, the average density of electrons per channel in the absence of the impurity. In the second-order term in (3.12), one of the frequency sums and one of the momentum integrals can be done explicitly, yielding

$$\mathrm{tr}(ghgh) = \frac{1}{\beta}\sum_{ln}\int dp\, \frac{p}{i\omega_n - p}\frac{h_{lp}(\omega_n)h_{l-p}(-\omega_n)}{(2\pi)^2}. \tag{3.15}$$

The next step is to substitute the explicit expression for $h_{lp}(\omega_n)$ from (3.5) into (3.13) and (3.15), and to separate the resulting expressions into terms that only contain σ_z, terms that only contain V, and terms that contain both. Putting it all back into (3.11), one finds

$$\begin{aligned} \langle e^{-S_1[\sigma,V]}\rangle_0 &= \langle e^{-S_1[\sigma,0]}\rangle_0\\ &\quad\times \exp\left\{F_\rho^{(0)}[V] - \frac{1}{\beta}\sum_n B_V(-\omega_n)\sigma_z(\omega_n)\right\}, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} F_\rho^{(0)}[V] &= \sum_l V_{l0}(0)\bar{\rho}\\ &\quad + \frac{1}{2\beta}\sum_{ln}\int dp\, \frac{p}{i\omega_n - p}\frac{V_{lp}(\omega_n)V_{l-p}(-\omega)}{(2\pi)^2} \end{aligned} \tag{3.17}$$

is the generating functional for density correlations in the absence of the impurity, and

$$B_V(\omega_n) = \sum_l \int dp\, \frac{p}{i\omega_n - p}\frac{\gamma_l}{2\pi}\frac{V_{lp}(\omega_n)}{2\pi}. \tag{3.18}$$

Substituting this back into (3.2) and comparing to the expression (3.8) for $F_\sigma[B]$, one obtains the simple relation

$$F_\rho[V] = F_\rho^{(0)}[V] + F_\sigma[B_V]. \tag{3.19}$$

Now, consider the Matsubara Green's functions

$$\begin{aligned} \mathcal{G}_\rho(l,p,l',q,i\omega_n) &= -\int_0^\beta d\tau\, e^{i\omega_n\tau}\int dx\,dy\, e^{-ipq-iqy}\\ &\quad\times \langle \Delta\rho_l(x,\tau)\Delta\rho_{l'}(y,0)\rangle \end{aligned} \tag{3.20}$$

and

$$\mathcal{G}_\sigma(i\omega_n) = -\int_0^\beta d\tau\, e^{i\omega_n\tau}\, \langle \Delta\sigma_z(\tau)\Delta\sigma_z(0)\rangle, \tag{3.21}$$

where $\Delta\rho = \rho - \langle\rho\rangle$ and similarly for $\Delta\sigma_z$. These can be obtained from the generating functionals $F_\rho[V]$ and $F_\sigma[B]$ by means of the appropriate functional derivatives

$$\begin{aligned} &\mathcal{G}_\rho(l,p,l',q,i\omega_n)\\ &\quad = -\frac{(2\pi\beta)^2}{\beta}\frac{\delta}{\delta V_{l-p}(-\omega_n)}\frac{\delta}{\delta V_{l'-q}(\omega_n)}F_\rho[V]|_{V=0}, \end{aligned} \tag{3.22}$$

and similarly for G_σ. Owing to (3.19), these two Green's functions are related to each other, i.e.,

$$\begin{aligned} \mathcal{G}_\rho(l,p,l',q,i\omega_n) &= \mathcal{G}_\rho^{(0)}(l,p,l',q,i\omega_n)\\ &\quad - \frac{\gamma_l\gamma_{l'}}{(2\pi)^2}\frac{p}{i\omega_n - p}\frac{q}{i\omega_n + q}\mathcal{G}_\sigma(i\omega_n), \end{aligned} \tag{3.23}$$

where

$$\mathcal{G}_\rho^0(l,p,l',q,i\omega_n) = \delta_{l,l'}\delta(p+q)\frac{p}{i\omega_n - p} \tag{3.24}$$

is the density-density Green's function in the absence of the impurity.

The polarizability $\chi_{ll'}(x,y,t)$ of (2.14) can be obtained by analytically continuing $\mathcal{G}_\rho(l,p,l',q,i\omega_n)$ to real frequencies and Fourier transforming to space and time. Similarly, the retarded impurity Green's function

$$G_\sigma(t) = -i\theta(t)\,\langle[\sigma_z(t),\sigma_z(0)]\rangle \tag{3.25}$$

can be calculated by analytically continuing $\mathcal{G}_\sigma(i\omega_n)$ to real frequencies and then Fourier transforming to time. Thus, one finds

$$\chi_{ll'}(x,y,t) = \chi^{(0)}_{ll'}(x,y,t) - \frac{\gamma_l\gamma_{l'}}{(2\pi)^2}\partial_x\partial_y\,[\theta(x)\theta(-y)G_\sigma(t+y-x)], \tag{3.26}$$

where $\chi^{(0)}_{ll'}(x,y,t) = -\delta_{l,l'}\partial_x\,[\theta(x-y)\delta(t+y-x)]$ is polarizability in the absence of the impurity, in which case a delta pulse in the potential in the incoming channels at y produces a density fluctuation $\langle\Delta\rho_l(x,t)\rangle = \partial_t\delta(t+y-x)$. The fact that this fluctuation travels at the Fermi velocity without spreading is due to the linear dispersion of the electrons. The structure $\partial_t\delta(t+y-x)$ is consistent with charge conservation: since a potential pulse can not create charge, we must have $\int_0^\infty dt\,\chi_{ll'}(x,y,t) = 0$.

The response measured for outgoing electrons ($x>0$) to a perturbation of the incoming electrons ($y<0$) is given by

$$\chi_{ll'}(x,y,t) = \chi^{(0)}_{ll'}(x,y,t) + \frac{\gamma_l\gamma_{l'}}{(2\pi)^2}G''_\sigma(t+y-x), \tag{3.27}$$

where $G''_\sigma \equiv \partial_t^2 G_\sigma$ contains the correlations induced by the impurity. From (3.25) follows that $G''_\sigma(t)$ can be written as

$$G''_\sigma(t) = i\theta(t)R(t) - i\delta(t)\int_0^\infty dt\,R(t), \tag{3.28}$$

where $R(t) = -\partial_t^2\,\langle[\sigma_z(t),\sigma_z(0)]\rangle$. The total response charge in channel l due to a delta potential pulse in channel l' is

$$Q_{ll'} = i\frac{\gamma_l\gamma_l'}{(2\pi)^2}\int_0^\infty dt\,R(t). \tag{3.29}$$

The task of calculating $\chi_{ll'}(x,y,t)$ has now been reduced to calculating $G''_\sigma(t+y-x)$. Short-time asymptotics ($1/\Lambda \ll t \ll 1/\varepsilon,\ 1/\Delta_r$) for $G''_\sigma(t)$ can be derived from the equations of motion obeyed by σ_z [37,38]. It is found that $G_\sigma(t) \propto t^{-2\alpha}$. (The $t\to 0$ divergence is cut off at the scale below $1/\Lambda$.) This implies that for $\alpha < \frac{1}{2}$ (weak damping), the response charge $Q_{ll'}$ is finite, while for $\alpha > \frac{1}{2}$ it diverges in the $\Lambda\to\infty$ limit as $(\Lambda/\Delta_r)^{2\alpha-1}$.

In the context of the spin-boson model, further analytical results have been obtained by a method known as the noninteracting blip approximation [27]. This yields an expression for the quantity $C(t) = \langle\{\sigma_z(t),\sigma_z(0)\}\rangle/2$. $C(t)$ is related to $G_\sigma(t)$ via the fluctuation-dissipation theorem. However, $G_\sigma(t)$ can not be calculated reliably using this approximate expression for $C(t)$. The reason is that the relation between $G_\sigma(t)$ and $C(t)$ is nonlocal in time, while the approximate expression for $C(t)$ is not valid for large times [2]. (It also breaks down for $\varepsilon \neq 0$.) Thus, a different method is required to calculate $G_\sigma(t)$.

At times of order $1/\Delta_r$ or $1/\varepsilon$, one intuitively expects damped oscillating or overdamped behavior for $G_\sigma(t)$. Exact numerical results confirm this [39,40]. The results were obtained with the aid of the numerical renormalization group together with the mapping to the anisotropic Kondo model or by performing a real-time renormalization analysis on the spin-boson model. For fixed α, the damping rate increases if Δ_r or T increases. For ε small compared to Δ_r, oscillatory behavior is only observed for $\alpha \lesssim \frac{1}{3}$. For larger ε, oscillatory behavior survives up to larger α.

The rest of this paper is devoted to obtaining analytical results for $G_\sigma(t)$ that go beyond the above asymptotics, circumvent the problems associated with the noninteracting blip approximation, and complement the discussed numerical results.

IV. EXACT EXPRESSIONS FOR $\alpha = \frac{1}{2}$

For $\alpha = \frac{1}{2}$, an exact result is available for the Fourier transform

$$R(\omega) = \int_{-\infty}^{\infty} dt\,e^{i\omega t}R(t), \tag{4.1}$$

of R as defined following (3.28) [23]. The low-temperature regime is the most interesting. (The role of temperature is merely to produce exponential decay with a rate π/β at large times.) I therefore specialize to zero temperature, where one has

$$R(\omega) = \frac{4\Gamma}{\pi}\frac{\omega^2}{\omega^2+\Gamma^2}\left\{\arctan\frac{2(\omega+\varepsilon)}{\Gamma} + \arctan\frac{2(\omega-\varepsilon)}{\Gamma} + \frac{\Gamma}{2\omega}\ln\left[\frac{[\Gamma^2+4(\omega+\varepsilon)^2][\Gamma^2+4(\omega-\varepsilon)^2]}{(\Gamma^2+4\varepsilon^2)^2}\right]\right\}, \tag{4.2}$$

with $\Gamma = \pi\Delta_r/2$.

Figure 4 shows the function $R(\omega)$ for different values of Γ. As ω tends to $\pm\infty$, $R(\omega)$ tends to $\pm 4\Gamma$. For $\Gamma\to 0$, $R(\omega)/\Gamma$ is piecewise constant, with a step from -4 to 0 at $\omega = -|\varepsilon|$ and a step from 0 to 4 at $\omega = |\varepsilon|$. As Γ is increased, these steps become smoothed out.

Extracting the small- and large-time asymptotics of G''_σ is, however, straightforward. The small-time behavior of G''_σ is determined by the large frequency behavior of $R(\omega)$. From (4.2) it follows that

$$G''_\sigma(t\to 0^+) \simeq \frac{4\Gamma}{\pi t}. \tag{4.3}$$

The $1/t$ divergence in $G''_\sigma(t)$ implies that at $\alpha = \frac{1}{2}$, the response charge $Q_{ll'}$ suffers from a logarithmic ultraviolet divergence.

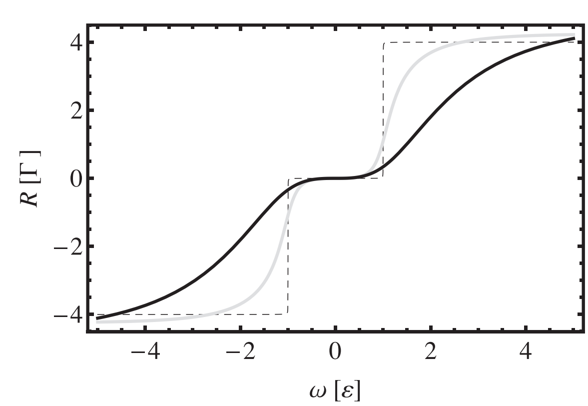

FIG. 4. The function $R(\omega)$, for different values of Γ. The black curve corresponds to $\Gamma = 2\varepsilon$, the gray curve to $\Gamma = \varepsilon/2$, and the thin dashed curve to the limiting case $\Gamma/\varepsilon \to 0$.

A damping factor that kicks in when $|\omega| > \Lambda$, and that is omitted from (4.2), regularizes the $1/t$ singularity in (4.3) and the logarithmic divergence in $Q_{ll'}$ at the scale of $1/\Lambda$.

The long-time behavior of G''_σ is determined by the analyticity structure of $R(\omega)$ in the complex ω plane [41]. The singularities in $R(\omega)$ at $\omega = \pm\varepsilon \pm i\Gamma/2$ and the analyticity and boundedness of $R(\omega)$ for $|\mathrm{Im}(\omega)| < \Gamma/2$ imply that for $t \gg \Gamma$,

$$G''_\sigma(t) \sim e^{-\Gamma t/2}. \tag{4.4}$$

By numerically performing the Fourier transform, I have found that

$$A(t) = \frac{4\Gamma}{\pi t} e^{-\Gamma t/2} \tag{4.5}$$

excellently describes the envelope of $G''_\sigma(t)$, also for intermediate times. This can be seen in Fig. 5. In the figure, $G''_\sigma(t)/A(t)$, with $G''_\sigma(t)$ numerically calculated from (4.2), is plotted for two different values of Γ. It is seen that $G''_\sigma(t)/A(t)$ oscillates with an amplitude approaching 1 at times larger than a few times $2\pi/\varepsilon$. I have found that, for $\Gamma < \varepsilon$, and $t > 8\pi\varepsilon$, $G''_\sigma(t)/A(t)$ equals $\cos(\varepsilon t + \gamma/\varepsilon)$ with an error less than 1%.

Expanding $R(\omega)$ in Γ and then performing the Fourier transform term by term, one obtains

$$G''_\sigma(t) = \frac{4\Gamma}{\pi} \frac{\cos|\varepsilon|t}{t} + O(\Gamma^2). \tag{4.6}$$

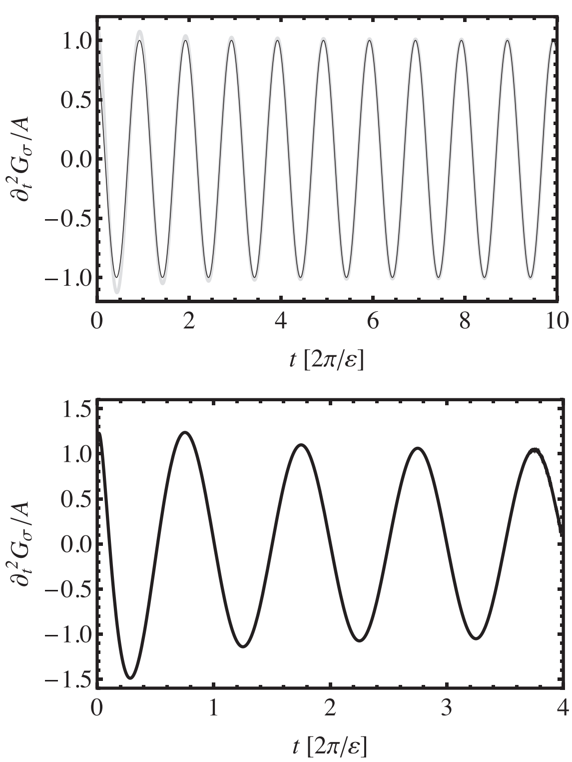

FIG. 5. The function $G''_\sigma(t)/A(t)$ for different values of Γ. Top: $\Gamma = \varepsilon/2$. The thin black line shows the approximation $\cos(\varepsilon t + \Gamma/\varepsilon)$. In the time interval shown, the envelope function $A(t)$ decreases by 9 orders of magnitude. Bottom: $\Gamma = 2\varepsilon$. In the time interval shown, the envelope function $A(t)$ decreases by 13 orders of magnitude. In both cases, the first turning point, close to $t = 0$, is somewhat sharper than the rest, and hence the behavior of the function at $t \ll \varepsilon$ is poorly resolved. Close inspection of the data (not shown) confirms that $\lim_{t\to 0^+} G''_\sigma(t)/A(t) = 1$ as expected.

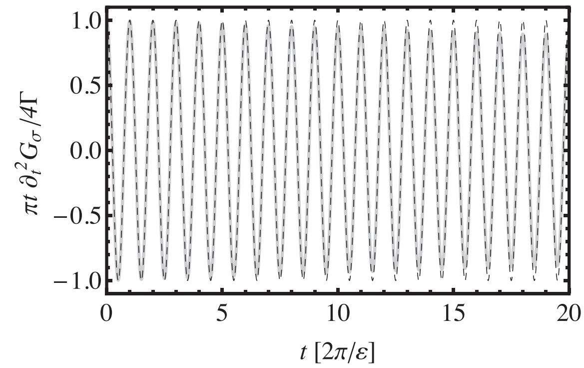

FIG. 6. The function $\pi t G''_\sigma(t)/4\Gamma$ for $\Gamma/\varepsilon = 2 \times 10^{-3}$ (solid line), compared to the $\Gamma \ll \varepsilon, 1/t$ limiting case $\pi t G''_\sigma(t)/4\Gamma = \cos(|\varepsilon| t)$ (dashed line) of (4.6).

In view of the asymptotics derived above, the status of the above expression is clear: The leading-order term in the $\Gamma \propto \Delta^2$ expansion provides an accurate approximation to $G''_\sigma(t)$ when $\Gamma \ll \varepsilon$ for times $t \ll 1/\Gamma$. It correctly captures the oscillatory behavior and power-law envelope that governs $G''_\sigma(t)$ for $1/\Lambda < t < 1/\Gamma$, but not the eventual exponential decay at large times. The accuracy of (4.6) for $\Gamma \ll \varepsilon$ and $t \ll 1/\Gamma$ is confirmed in Fig. 6, where (4.6) is compared to $G''_\sigma(t)$, numerically obtained from (4.2), for $\Gamma/\varepsilon = 2 \times 10^{-3}$.

V. PERTURBATIVE ANALYSIS AT ARBITRARY α

For $\alpha \neq \frac{1}{2}$, no exact solution is available. In this section, I therefore calculate $G''_\sigma(t) = -i\partial_t^2 \{\theta(t)\langle[\sigma_z(t),\sigma_z(0)]\rangle\}$ and hence $\chi_{ll'}(x,y,t)$, to second order in the impurity tunneling amplitude Δ. Here, the expectation value is with respect to the thermal density matrix $\exp(-\beta H)/\mathrm{tr}\exp(-\beta H)$ at inverse temperature β and $\sigma_z(t) = \exp(iHt)\sigma_z \exp(-iHt)$. This is the usual limit in which the Fermi edge singularity is considered. Based on the results of the previous section, I expect the expansion to be accurate for Δ sufficiently smaller than ε and $t < 1/\Delta_r$. In other words, this section contains short-time results. This, however, does not mean that the goal is merely to recover the $t \ll \Delta_r^{-1}, \varepsilon^{-1}$ asymptotic result $G''_\sigma(t) \propto t^{-2\alpha}$ discussed in Sec. III. In particular, no assumption is made about the relative size of t compared to ε^{-1}. Thus, one still has access to a regime of damped oscillations. Also, an expression for the $Q_{ll'}$ can be obtained that is valid in the limit of small Δ_r/ε.

Expanding the operators $\exp(-\beta H)$ and $\exp(\pm iHt)$ in Δ, using the interaction picture, and tracing out the impurity degree of freedom, one finds

$$G_\sigma(t) = -\Delta^2\theta(t)\int_0^\beta d\tau \int_0^t dt'\, P(\tau - it'), \tag{5.1}$$

where

$$P(z) = \frac{\zeta(z) + \zeta(\beta - z)}{\zeta(0) + \zeta(\beta)},$$
$$\zeta(z) = e^{(\beta/2 - z)\varepsilon}\frac{\mathrm{tr}[e^{-(\beta-z)H_-}e^{-zH_+}]}{Z_0}, \tag{5.2}$$

and $H_\pm$ are defined in (2.6). In order to calculate the polarizability (3.27), we need the second time derivative of

$G_\sigma(t)$, which from (5.1) is given by

$$G''_\sigma(t) = -\Delta^2 \left\{ -i\theta(t)[P(\beta - it) - P(-it)] + \delta(t)\int_0^\beta d\tau\, P(\tau)\right\}. \tag{5.3}$$

The analyticity structure of $P(\tau)$ allows us to replace the integral from 0 to β in the second term by two integrals, one along the imaginary axis from 0 to $i\infty$ and the other along the line $\beta + it'$ with t' from ∞ to 0. From the definition (5.2) of $P(t)$ it is seen that $P(\beta - it) = P(it) = P(-it)^*$, so that (5.3) becomes

$$G''_\sigma(t) = -2\Delta^2\left[\theta(t)\mathrm{Im}\, P(it) - \delta(t)\int_0^\infty dt\, \mathrm{Im}\, P(it)\right]. \tag{5.4}$$

The task is now to calculate $P(z)$. This is done in the Appendix for real arguments $z = \tau$. The final result is

$$P(\tau) = \frac{\cosh\left(\frac{\varepsilon\beta}{2} - \varepsilon\tau\right)}{\cosh\left(\frac{\varepsilon\beta}{2}\right)} \times \left[\frac{\Gamma\left(1 + \frac{1}{\Lambda\beta} - \frac{\tau}{\beta}\right)\Gamma\left(\frac{1}{\Lambda\beta} + \frac{\tau}{\beta}\right)}{\Gamma\left(1 + \frac{1}{\Lambda\beta}\right)\Gamma\left(\frac{1}{\Lambda\beta}\right)}\right]^{2\alpha}, \tag{5.5}$$

with α as defined in (2.11).

Now, analytical continuation of $P(\tau)$ to complex arguments is straightforward. Further simplification is possible if one uses the identity $\Gamma(1+z) = z\Gamma(z)$ to make the arguments of all Γ functions nonzero in the $\Lambda \to \infty$ limit. This allows one to take $\Lambda \to \infty$ in the arguments of the Γ functions. Finally, I employ the identity $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin\pi z$, and substitute the result into (5.4) to obtain for $t > x - y$ my main result

$$\chi_{ll'}(x,y,t) = \frac{\gamma_l\gamma_{l'}}{(2\pi)^2}G''_\sigma(t + y - x),$$
$$G''_\sigma(t) = -2\Delta_r^2\mathrm{Im}\left[\frac{\cosh\left(\frac{\varepsilon\beta}{2} - i\varepsilon t\right)}{\cosh\left(\frac{\varepsilon\beta}{2}\right)} \times \left[\left(\frac{1}{\Lambda} + it\right)\frac{\beta\Delta_r}{\pi t}\sinh\frac{\pi t}{\beta}\right]^{-2\alpha}\right]. \tag{5.6}$$

For nonzero temperatures, the result of (5.6) implies exponential damping at a rate $2\pi\alpha/\beta$ for times larger than β. The $\cosh(\varepsilon\beta/2 - i\varepsilon t)$ prefactor implies damped coherent oscillations with angular frequency ε. Regardless of temperature, the short-time behavior is a power law $\sim t^{-2\alpha}$, regularized at the scale $1/\Lambda$. This agrees with the general asymptotic result discussed in Sec. III in the regime $1/\Lambda \ll t \ll 1/\Delta_r, 1/\varepsilon$, as it should.

At zero temperature, (5.6) reduces to

$$G''_\sigma(t) = -2\Delta_r^2\mathrm{Im}\left[e^{-i|\varepsilon|t}\left(\frac{\Delta_r}{\Lambda} + i\Delta_r t\right)^{-2\alpha}\right]. \tag{5.7}$$

At $\alpha = \frac{1}{2}$ and times $t \gg 1/\Lambda$, this agrees with the $\Delta_r \ll \varepsilon, 1/t$ limit of the exact $\alpha = \frac{1}{2}$ solution, again as it should. As was found in the case of $\alpha = \frac{1}{2}$, I expect (5.7) to break down when t becomes comparable to or larger than $1/\Delta_r$. From (5.7), the response charge in channel l due to a perturbation in channel l' at zero temperature and $\alpha < \frac{1}{2}$ is

$$Q_{ll'} = 2\Gamma(1-2\alpha)\frac{\gamma_l\gamma_{l'}}{(2\pi)^2}\Delta_r\left(\frac{\Delta_r}{\varepsilon}\right)^{1-2\alpha} = \frac{\gamma_l\gamma_{l'}}{\pi^2}\mathrm{cosec}(2\pi\alpha)W(\varepsilon), \tag{5.8}$$

where $W(\varepsilon)$ is the impurity transition rate, known from the theory of the Fermi edge singularity.

How does the response charge $Q_{ll'}$ compare to the response of the conductor in the absence of the impurity? To answer this question, consider a brief voltage pulse of magnitude V applied to a section of length L of channel l', for a time duration $\tau \ll L/v_F$ (with v_F the Fermi velocity). In the absence of the impurity, the conductance of the channel is e^2/h. As a result, the voltage pulse produces two propagating pulses of opposite charge, a distance L apart, each containing a charge $Q_0 = e^2V\tau/h$. The impurity-induced response charge associated with this pulse is (after reinstating units) $Q_R = e^2Q_{ll'}VL\tau/\hbar^2 v_F$. The ratio between the impurity-induced response charge and Q_0 is

$$\frac{Q_R}{Q_0} = \frac{2\gamma_l\gamma_{l'}}{\pi}\mathrm{cosec}(2\pi\alpha)\frac{W(\varepsilon)L}{\hbar v_F}. \tag{5.9}$$

The quantities γ_l and α are related to phase shifts of the impurity potential, and therefore readily tunable in a quantum transport realization of the model such as discussed in Sec. II. This means that the prefactor $\gamma_l\gamma_l'\,\mathrm{cosec}(2\pi\alpha)$ in (5.9) can be varied from zero at weak coupling to order unity. (Close to $\alpha = \frac{1}{2}$ the result is unreliable.) Q_R/Q_0 is therefore bounded by $W(\varepsilon)L/\hbar v_F$. Since the natural high-energy cutoff scale is the Fermi energy, which is of the order of an electron volt, $W(\varepsilon)$ has to be an order of magnitude or two smaller than that. Fermi velocities can be of the order 10^6 m/s. In order to resolve the oscillatory behavior of $\chi_{ll'}(x,y,t)$, L has to be smaller than the wavelength $\sim\hbar v_F/\varepsilon$ of the oscillations. On the other hand, L must be larger than the lattice constant. With $\varepsilon \sim 10^{-2}$ eV, the wavelength is $\sim 10^2$ nm, and $L \sim 1$ nm is therefore suitable. These considerations imply the rough estimate $Q_R/Q_0 \sim 10^{-2}$–10^{-1}. As explained above, the scale for Q_0 is set by the conductance quantum e^2/h. Since responses on this scale are routinely measured in quantum transport experiments, and the impurity response is only an order of magnitude or two smaller, its detection should be feasible with current technology.

VI. SUMMARY AND DISCUSSION

I studied a fermionic realization of the Ohmic spin-boson model. The system consists of a two-level impurity coupled to a one-dimensional conductor. My aim was to characterize the dynamical correlations that the impurity induces between electrons in the conductor. The technical result from which the rest follows is a simple relation (3.19) between a generating functional for electron-electron correlations and one for impurity correlations. This relation implies that for $x > 0$ and $y < 0$, in the unfolded coordinates (cf. Sec. II), the impurity contribution to the electronic polarizability $\chi_{ll'}(x,y,t)$ defined in (2.14) is [cf. Eq. (3.27)]

$$\frac{\gamma_l \gamma_{l'}}{(2\pi)^2} G''_\sigma(t+y-x), \tag{6.1}$$

where $G_\sigma(t)$ is the retarded Green's function for the impurity operator σ_z. Its second derivative consists of a delta spike at $t=0$ followed by a decaying tail [cf. Eq. (3.29)]. Due to charge conservation, the area under the tail equals minus the weight of the delta spike. The polarization response of the conductor is as follows. A potential perturbation $v(x,t)=\delta(x-y)\delta(t)$ in channel l' applied to electrons incident on the impurity ($y<0$) produces a charge fluctuation $\partial_t\delta(t+y-x)$ among the incident electrons in channel l'. This fluctuation is modified when it reaches the impurity at time $t=-y$. The modification $-Q_{ll'}\delta(x-t-y)$ in channel l is due to the excitation of the impurity by the incident charge fluctuation. Behind it follows an oppositely charged decaying tail, produced by the subsequent interaction between the excited impurity and the electrons in the conductor. The response charge $Q_{ll'}$ is a measure of the strength of the impurity-induced correlations in the conductor. For $1/\Lambda \ll t \ll 1/\Delta_r, 1/\varepsilon$, $G''_\sigma(t)$ is known to behave as $t^{-2\alpha}$. For $\alpha > \frac{1}{2}$, this implies that the response charge is ultraviolet divergent as $\Lambda^{2\alpha-1}$, whereas it is finite for $\alpha < \frac{1}{2}$.

For a dissipation strength $\alpha = \frac{1}{2}$, the available exact solution of the spin-boson model provides an exact expression (4.2) for the Fourier transform of $G''_\sigma(t)$. From this I extracted the following behavior of $\chi_{ll'}(x,y,t)$ [cf. Eqs. (4.3) and (4.4)]. Coherent damped oscillations are observed. In the weak tunneling limit, the angular frequency of these oscillations is ε. For large times, χ decays as $e^{-\Gamma t/2}$, where $\Gamma = \pi\Delta_r/2$. For $x-y<t\ll 1/\Gamma$ it behaves as a power law $(t+y-x)^{-1}$, as is expected from general arguments. I also showed that, for $t \ll 1/\Gamma$ and $\Gamma \ll \varepsilon$, accurate results are obtained by expanding the exact result to leading order in Δ or equivalently Δ_r (cf. Fig. 6).

I used this last insight to investigate the polarizability response of the conductor for $\alpha \neq \frac{1}{2}$, where no exact solution is available. I obtained a result (5.6) to leading order in Δ. It is expected to hold for sufficiently large ε/Δ (but with ε still sufficiently smaller than the cutoff scale Λ) and for times $t \ll 1/\Delta_r$. Again, there are damped coherent oscillations with angular frequency ε. The power-law singularity $G''_\sigma(t)\sim t^{-1}$ found for $\alpha=\frac{1}{2}$ generalizes to $G''_\sigma(t)\sim t^{-2\alpha}$ for $\alpha\neq\frac{1}{2}$, as it should. The $t\to 0^+$ divergence can be understood in terms of the severe shakeup that the excited impurity causes in the Fermi sea of the conductor. The interpretation is confirmed by calculating the response charge $Q_{ll'}$ for $\alpha<\frac{1}{2}$, where it is not ultraviolet divergent. This reveals a simple relation (5.8) between $Q_{ll'}$ and the impurity decay rate $W(\varepsilon)$, that due to Fermi sea shakeup, displays a Fermi edge singularity. For $\alpha \geqslant \frac{1}{2}$, the Fermi sea shakeup induced by the excited impurity is so severe that the response charge diverges if the ultraviolet cutoff Λ is sent to infinity. The divergence reflects a strong dependence of $Q_{ll'}$ on short length scale $\sim 1/\Lambda$ physics for $\alpha>\frac{1}{2}$. Due to the mapping between the studied system and the spin-boson model, results obtained in this paper extend known analytical results for the latter model. In particular, the presented analytical expression for $G_\sigma(t)$ in Sec. V is obtained by studying a regime (that of finite ε and small Δ) that, as far as I know, has not yet been considered in the spin-boson model context.

Regarding measurement, the magnitude of the impurity-induced response is estimated at the end of Sec. V. It is found that the impurity contribution is only an order of magnitude or two less than the response of the conductor without the impurity. The scale for spatial variations such as oscillations and damping in the response is set by $\hbar v_F/\varepsilon$ and $\hbar v_F/\Delta_r$. A spatial scale of 10–100 nm is attainable, without jeopardizing robustness of the effect.

A possible line for future enquiry is the generalization of this work to situations where the electrons in the conductor are described by a nonequilibrium distribution function. This is known to affect the impurity decay rate $W(\varepsilon)$ in a nontrivial way [42,43] and may therefore be expected to modify electron-electron correlations in a similarly interesting manner. In order to study these correlations, one will have to solve the following technical problem. Due to the difference in Fermi energy between left and right incident electrons in a voltage-biased conductor, channel space becomes entangled with the impurity state in such a way that the diagonal representation of the Hamiltonian derived in Sec. II is of no use [44].

APPENDIX: CALCULATION OF $\zeta(z)$ AND $P(z)$ AT REAL ARGUMENTS

In this appendix, $\zeta(z)$ and hence $P(z)$ that appear in (5.2) are calculated at real $z=\tau$. The results can then subsequently be continued analytically to complex z. For $\tau\in(0,\beta)$, $\zeta(\tau)$ can be written as a path integral

$$\zeta(\tau) = e^{(\frac{\beta}{2}-\tau)\varepsilon}\langle e^{-S_1[\sigma,0]}\rangle_0, \tag{A1}$$

in the notation of Sec. II, with

$$\begin{aligned}\sigma_{lk}(\omega_n) &= \gamma_l\int_0^\beta d\tau' \,\mathrm{sign}(\tau-\tau')e^{i\omega_n\tau'}\\ &= 2\gamma_l\left[\delta_{n,0}\left(\tau-\frac{\beta}{2}\right)+(1-\delta_{n,0})\frac{e^{i\omega_n\tau}-1}{i\omega_n}\right].\end{aligned} \tag{A2}$$

Thanks again to the Dzyaloshinskii-Larkin theorem, the path integral evaluates to a Gaussian functional in σ:

$$\begin{aligned}\ln\zeta(\tau) &= \tilde{c}_0+\tilde{c}_1\tau-\frac{1}{2}\int dp\,\frac{1}{\beta}\sum_{ln}\frac{p}{i\omega_n-p}\left|\frac{\sigma_{lp}(\omega_n)}{2\pi}\right|^2\\ &= c_0+c_1\tau-\sum_l\left(\frac{\gamma_l}{\pi}\right)^2\\ &\quad\times\underbrace{\int dp\,\frac{1}{\beta}\left[\sum_n\frac{p}{i\omega_n-p}\,\frac{1-\cos\omega_n\tau}{\omega_n^2}-\frac{\tau^2}{2}\right]}_{=\mathcal{A}}.\end{aligned} \tag{A3}$$

I will not require explicit expressions for the constants c_0 or c_1, which depend on microscopic detail at the scale of $1/\Lambda$. Using the explicit expression (A2) for $\sigma_{lk}(\omega_n)$, the frequency sum can be evaluated by converting it into a contour integral. Each term in the sum is associated with a pole along the imaginary axis

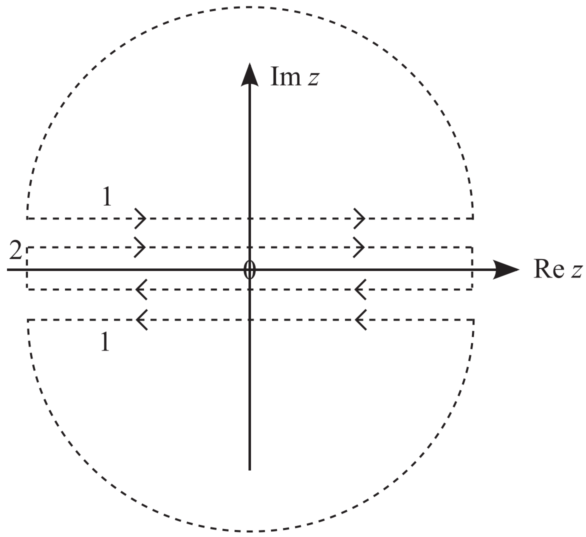

FIG. 7. Integration contours for evaluating the frequency sum in (A3).

at $i\omega_n, n \neq 0$:

$$\mathcal{A} = -\frac{\tau^2}{2\beta} + \frac{1}{4\pi i}\int_C dz\,\frac{1-e^{z\tau}}{2z^2} \times \left(\frac{p}{z-p} - \frac{p}{z+p}\right)\left(\coth\frac{\beta z}{2} - 1\right). \quad \text{(A4)}$$

The contour can be deformed from contour 1 in Fig. 7 to contour 2, which contains poles at 0 and $\pm p$ along the real line. With the aid of this deformation, one finds

$$\mathcal{A} = -\frac{\tau}{2} + \left[\frac{1-e^{p\tau}}{4p}\left(\coth\frac{\beta p}{2} - 1\right) + \frac{1-e^{-p\tau}}{4p}\left(\coth\frac{\beta p}{2} + 1\right)\right]. \quad \text{(A5)}$$

The p integral in (A3) formally suffers from an ultraviolet divergence because I used a delta function for the impurity potential. Using the Lorentzian regularization of (2.3) and integrating (A5) over p leads to

$$\begin{aligned} -\int_{-\infty}^{\infty} dp\, e^{-|p|/\Lambda}&\left[\frac{1-e^{p\tau}}{4p}\left(\coth\frac{\beta p}{2} - 1\right) + \frac{1-e^{-p\tau}}{4p}\left(\coth\frac{\beta p}{2} + 1\right)\right] \\ &= \int_0^{\infty}\frac{dp}{p}e^{-\frac{p}{\Lambda}}\left[\frac{e^{\tau p}-1}{e^{\beta p}-1} + \frac{e^{-\tau p}-1}{1-e^{-\beta p}}\right] \\ &= \int_{\frac{1}{\Lambda\beta}}^{\infty} du \int_0^{\infty} dv\, e^{-uv}\left[\frac{e^{\tau v/\beta}-1}{e^{v}-1} + \frac{e^{-\tau v/\beta}-1}{1-e^{-v}}\right]. \end{aligned} \quad \text{(A6)}$$

The integrals involved in the last line of (A6) can be done with the help of the identity

$$\int_0^{\infty} dx\,\frac{e^{px}-e^{qx}}{e^x-1} = \psi(1-q) - \psi(1-p), \quad \text{(A7)}$$

with $\psi(x) = \partial_x \ln\Gamma(x)$. The v integral, evaluated at the upper boundary, vanishes so that one obtains

$$\zeta(\tau) = \zeta(0)e^{-\varepsilon\tau}\left[\frac{\Gamma\left(1+\frac{1}{\Lambda\beta}-\frac{\tau}{\beta}\right)\Gamma\left(\frac{1}{\Lambda\beta}+\frac{\tau}{\beta}\right)}{\Gamma\left(1+\frac{1}{\Lambda\beta}\right)\Gamma\left(\frac{1}{\Lambda\beta}\right)}\right]^{2\alpha}, \quad \text{(A8)}$$

with α as defined in (2.11). Here, I have incorporated all the linear in τ terms into another redefinition $\varepsilon \to \varepsilon-$ offset of the impurity bias energy. Substitution into (5.2) gives

$$P(\tau) = \frac{\cosh\left(\frac{\varepsilon\beta}{2} - \varepsilon\tau\right)}{\cosh\left(\frac{\varepsilon\beta}{2}\right)}\left[\frac{\Gamma\left(1+\frac{1}{\Lambda\beta}-\frac{\tau}{\beta}\right)\Gamma\left(\frac{1}{\Lambda\beta}+\frac{\tau}{\beta}\right)}{\Gamma\left(1+\frac{1}{\Lambda\beta}\right)\Gamma\left(\frac{1}{\Lambda\beta}\right)}\right]^{2\alpha}. \quad \text{(A9)}$$

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The last term in Eq. (2.1) of the reference should read as $(g_0/N)\sum_{k,k',\sigma}\sigma c^{\dagger}_{k,\sigma}c_{k',\sigma}e^{i(k-k')r}$. Without the factor of σ in the sum, the model would have been trivial.
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