



# Relaxation of optically excited carriers in graphene: Anomalous diffusion and Lévy flights

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We present a theoretical analysis of the relaxation cascade of a photoexcited electron in graphene in the presence of screened electron-electron interaction in the random phase approximation. We calculate the relaxation rate of high energy electrons and the jump-size distribution of the random walk constituting the cascade which exhibits fat tails. We find that the statistics of the entire cascade are described by Lévy flights with constant drift instead of standard drift diffusion in energy space. The Lévy flight manifests nontrivial scaling relations of the fluctuations in the cascade time, which is related to the problem of the first passage time of Lévy processes. Furthermore we determine the transient differential transmission of graphene after an excitation by a laser pulse taking into account the fractional kinetics of the relaxation dynamics.

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## I. INTRODUCTION

The fabrication of graphene [1] launched a new era of two-dimensional (2D) materials in condensed matter physics, giving access to fundamentally different phenomena and systems realized for the first time in a solid state environment [2–5]. Graphene promises to be an attractive platform for electronic [6] and in particular optoelectronic applications [7–9], where research reaches from lasing [10] to energy conversion [11,12]. The nature of interactions and their interplay will limit the intrinsic properties of graphene devices and has therefore attracted interest from the application oriented as well as fundamental standpoint. For the latter, neutral or intrinsic graphene embodies the paradigm of a marginal Fermi liquid (FL) [13–15]. While graphene in the presence of electron-electron interactions (EEI) establishes a finite Fermi surface at high doping, it crosses over to a relativistic Dirac liquid at lower densities and manifests non-FL relaxation rates [16–18] and transport characteristics [6,15,19]. Another interesting interaction-dominated transport phenomenon is Coulomb drag in graphene double layer systems [20–22] which is determined by the peculiar interaction-induced interlayer relaxation [23–28]. In the last years it became feasible to examine the interactions even on very short time scales by means of ultrafast pump-probe measurements [29–31]. They revealed that EEI in graphene dominates over phonon interaction at an early stage of relaxation processes making graphene a highly efficient material for thermoelectric applications [32,33]. On the other hand the relaxation of high energy electrons follows again a non-FL scheme as electrons relax via a cascade of small steps in energy space [34].

So far theoretical work focused on the relaxation rates of thermal electrons using static screening or dynamical screening in the random phase approximation (RPA) [14–19,35–38]. Comprehensive numerical studies elucidated the interplay of EEI and phonon interactions [39,40] as well as the importance of different scattering channels in particular in the context of carrier multiplication via Auger processes

[41]. The influence of flexural phonons [42,43] in free-standing graphene and combined effects of phonons and disorder [44] have been studied in detail. The relaxation of optically excited carriers in doped graphene [34,45] was theoretically studied [46] at zero temperature and is consistent with the cascade picture.

In this work we present an analysis of the relaxation cascade at finite temperature. We consider the first stage of the relaxation process dominated by electron-electron collisions and neglect phonon and disorder effects. In Sec. II we study a single cascade step for undoped as well as for doped graphene and calculate the relaxation rates of high energy electrons in graphene using the random phase approximation (RPA). The main result of Sec. II is the distribution of the size of a single jump in the random walk describing the relaxation cascade. In Sec. III we infer the characteristics of the whole cascade on the basis of the results presented in Sec. II, with emphasis on the fluctuations on top of the particle's drift in energy space. The cascade process manifest the unique Dirac nature of carriers in graphene as it is described by Lévy flights [47]. Finally, in Sec. IV we determine the transient differential transmission of a graphene sample after excitation with a laser pulse in the presence of EEI.

## II. SINGLE CASCADE STEP

We are going to discuss the relaxation of carriers excited by a laser pulse with central frequency  $\omega_{\text{pump}}$ . We focus on the dynamics of the excited electrons rather than the questions associated with the equilibration of the low energy thermal electrons. We restrict our analysis to the earliest stage dominated by EEI, in which the energy remains entirely in the electronic system. For moderate pump fluence the phase space density of the excited electrons is much lower than the one of thermal electrons. Scattering and energy relaxation of a high energy excited electron is therefore predominantly due to interaction with thermal electrons. We neglect the mutual scattering of high energy electrons and assume that the low energy electrons remain thermal with temperature  $T$ . For small

fluences we also neglect the change in  $T$  due to illumination. In this sense the excited electrons with an energy of the order  $\omega_{\text{pump}}/2$  are relaxing in consecutive steps due to the interaction with a thermal bath of low energy electrons at equilibrium.

In the following we label the eigenstates  $|\lambda, \vec{k}\rangle$  of the graphene Hamiltonian  $H_0 = v_F \vec{\sigma} \cdot \vec{k}$  with energy  $\varepsilon_{\lambda k} = \lambda v_F k$  by the momentum  $\vec{k}$  and band index  $\lambda = \pm 1$ . In the following we set  $v_F = \hbar = 1$ . We define the relaxation rate via the semiclassical Boltzmann equation,

$$\partial_t f_\lambda(\vec{k}) = St[f_\lambda(\vec{k})]. \quad (1)$$

Here  $f_\lambda(\vec{k})$  is the occupation of the state  $|\lambda, \vec{k}\rangle$ . The collision integral  $St[f]$  describes the electron-electron scattering. Based on the approximations mentioned above we follow the evolution of a single excited electron starting at momentum  $\vec{p}$  as it relaxes due to scattering with the thermal electrons with energies  $\varepsilon \ll \varepsilon_p$ . We make the ansatz,

$$f_\lambda(\vec{k}) = f_T(\lambda k) + \delta f_\lambda(\vec{k}), \quad \delta f_\lambda(\vec{k}) = \delta_{\lambda,+1} \delta_{\vec{k}, \vec{p}}. \quad (2)$$

Here  $f_T(\varepsilon) = 1/[1 + \exp((\varepsilon - \mu)/T)]$  is the Fermi-Dirac distribution. With the ansatz (2), the relaxation rate of the high energy electron is determined by the outscattering rate in the collision integral,

$$St[f_{+1}(\vec{p})] = - \sum_{2,3,4} W_{12,34} f_3(1 - f_4)(1 - f_2). \quad (3)$$

In Eq. (3) we used the short-hand notation  $i = (\varepsilon_i, \vec{k}_i)$ . The transition rate  $W_{12,34}$  is given in Appendix A. Here, we only want to point out that in the case of Dirac particles it contains the overlap of the eigenstates  $\langle \lambda_j, \vec{k}_j | \lambda_i, \vec{k}_i \rangle$  that leads to a suppression of backscattering, in addition to the semiclassical matrix element of Coulomb scattering. In terms of the transferred energy  $\omega$  and momentum  $q$ ,  $\varepsilon_2 = \varepsilon_p - \omega$ ,  $\varepsilon_3 = \varepsilon_4 - \omega$  and  $\vec{k}_2 = \vec{p} - \vec{q}$ ,  $\vec{k}_3 = \vec{k}_4 - \vec{q}$ , due to the conservation of energy and momentum [see inset in Fig. 1(b)].

We can classify the possible scattering processes in terms of interband,  $|\omega| > q$  and intraband scattering,  $|\omega| < q$ . Collinear scattering occurs exactly at  $|\omega| = q$ .

Combining Eqs. (1) and (3) we obtain an expression for the relaxation rate  $\Gamma(p)$  of the photoexcited electron, defined by the Boltzmann equation,

$$\partial_t f_{+1}(\vec{p}) = -\Gamma(p) = St[f_{+1}(\vec{p})], \quad (4)$$

which is written as

$$\Gamma = \int_{-\infty}^{+\infty} d\omega P(\omega). \quad (5)$$

Here  $P(\omega)$  is the scattering rate per frequency interval  $(\omega, \omega + d\omega)$ . On the other hand it defines the distribution of the transferred energy in a single scattering event or cascade step. We thus refer to  $P(\omega)$  as the jump-size distribution (JSD) of the relaxation cascade.

As long as  $\omega < \varepsilon_p$  the excited electron is scattered within the conduction band, which implies  $q > |\omega|$ . Since the particle number in the conduction and valence band are separately conserved in pair collisions, the thermal electron that scatters with the high energy electron also performs an intraband transition [48]. We find that the contribution for  $\omega > \varepsilon_p$  corresponding to interband transitions is negligible for the

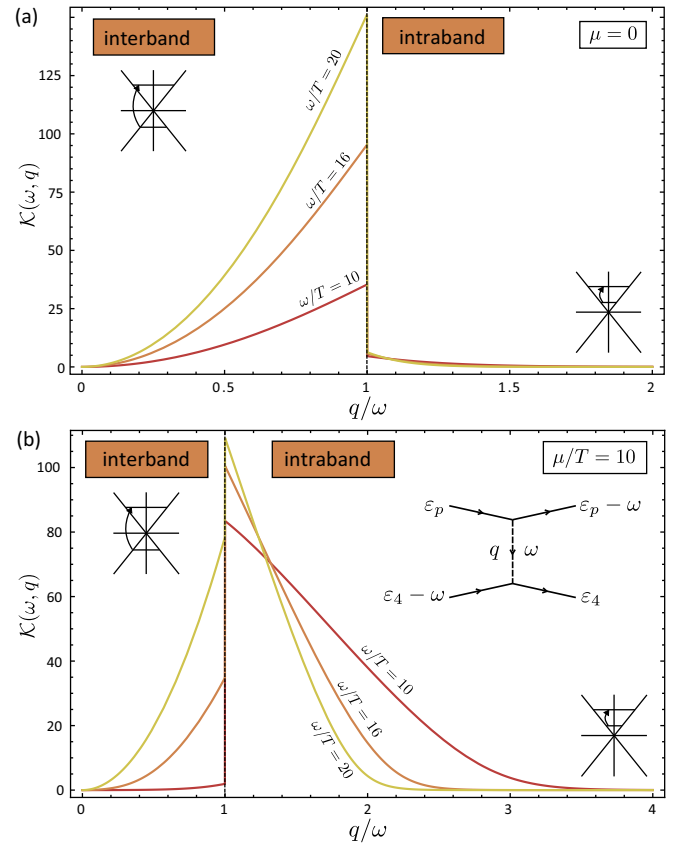


FIG. 1. (Color online) The kernel  $\mathcal{K}(\omega, q)$ , Eqs. (9) and (C1), determining the phase space of scattering for thermal electrons for different frequencies  $\omega$  and (a)  $\mu = 0$ , (b)  $\mu/T = 10$ . The regions of intraband ( $q > |\omega|$ ) and interband ( $q < |\omega|$ ) scattering are separated by the dashed line.

relaxation rate  $\Gamma$ , Eqs. (4) and (5), as well as for the statistics of the entire cascade (see Sec. III). Moreover, calculation shows that the relevant transferred energies satisfy  $|\omega| \ll \varepsilon_p$ . Scattering in this case is predominantly in forward direction, which simplifies the overlap functions,

$$|\langle \lambda_2, \vec{k}_2 | +1, \vec{p} \rangle|^2 = \frac{1 + \lambda_2(\vec{p} \cdot \vec{k}_2)/pk_2}{2} \simeq 1. \quad (6)$$

Taking into account that  $f_2 \simeq 0$  for  $|\varepsilon_p - \omega| \gg \max(|\mu|, T)$  in Eq. (3), we obtain the compact expression for the JSD,

$$P(\omega) = \int_{|\omega|}^{\infty} dq q \frac{N|V(\omega, q)|^2}{|q^2 - \omega^2|} \mathcal{K}(\omega, q). \quad (7)$$

Here we assumed  $\varepsilon_p \gg \max(|\mu|, T)$  and as a consequence  $P(\omega)$  is independent of the particle energy  $\varepsilon_p$ . In Eq. (7) the RPA-screened matrix element of Coulomb scattering,

$$V(\omega, q) = V_0(q)/\varepsilon(\omega, q), \quad (8)$$

where the dielectric function  $\varepsilon(\omega, q) = 1 + V_0(q)N\Pi(\omega, q)$ . The RPA polarization operator  $\Pi(\omega, q)$  is given in Appendix B and the bare Coulomb interaction  $V_0(q) = 2\pi\alpha_g/q$ . The number of flavors  $N = 4$  and the coupling constant in graphene  $\alpha_g = e^2/\epsilon\hbar v_F$  in our notations is  $\alpha_g = e^2/\epsilon$ . Note that in the presence of a dielectric

environment with dielectric constant  $\epsilon \gg 1$  the coupling constant can be small,  $\alpha_g \ll 1$ , which we assume in the following. The kernel,

$$\mathcal{K}(\omega, q) = \int_{-\infty}^{+\infty} d\epsilon_4 \sqrt{(\omega - 2\epsilon_4)^2 - q^2} \times f_T(\epsilon_4 - \omega)[1 - f_T(\epsilon_4)], \quad (9)$$

expresses the phase space (for  $q > |\omega|$ ) of the thermal electrons that scatter with the high energy photoexcited electron.

Let us briefly comment on the validity of the RPA. For small frequencies, the RPA sums up the leading logarithmically divergent diagrams [18]. For  $|\omega| > \max(T, |\mu|)$ , however, the RPA is justified by a large  $N$  expansion. By the same degree of approximation we also neglected the exchange term in the collision integral.

We observe that the denominator of the integrand in Eq. (7) is singular in the case of collinear scattering  $|\omega| = q$ , which in the absence of screening would lead to the logarithmically divergent Coulomb scattering integral [15,49,50]. However, the polarization operator in RPA is also divergent in the case of collinear scattering, thus the total scattering amplitude remains finite. The singular nature of the scattering of Dirac particles with linear dispersion also manifests itself in the phase space kernel (9). Figure 1 shows  $\mathcal{K}$  for intrinsic graphene ( $|\mu| \ll T$ ) as well as for  $|\mu| \gg T$ . In either case  $\mathcal{K}$  exhibits a jump at collinear scattering. One observes that for  $\mu = 0$  [Fig. 1(a)] the phase space of intraband processes is strongly suppressed and controlled by  $T$ . On the contrary, for  $|\mu| \gg T$  [Fig. 1(b)]  $\mathcal{K}$  is dominated by intraband processes.

Below we discuss the JSD separately for  $T \gg |\mu|$  and  $|\mu| \gg T$ .

### A. The limit $T \gg |\mu|$

For  $T \gg |\mu|$ , there are two important scattering processes. The first one is intraband scattering with small momentum transfer  $q < 2T$ , which leads to a logarithmic divergence in the JSD for frequencies  $|\omega| < \alpha_g T$ , depicted as the dash-dotted line in Fig. 2(b). The logarithm occurs due to the failure of

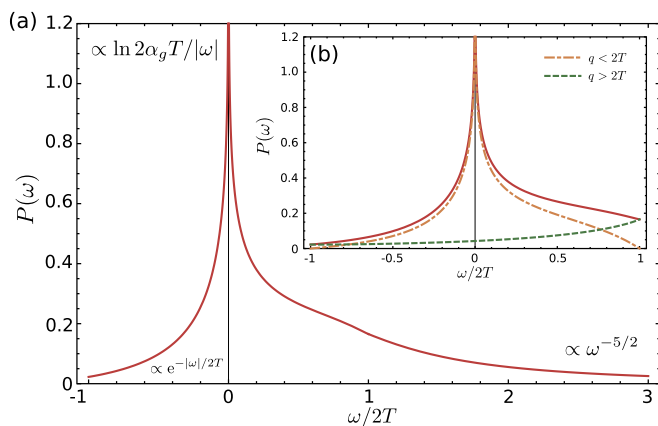


FIG. 2. (Color online) The jump-size distribution (7) for  $T \gg |\mu|$ . The inset (b) shows the contributions of  $q > 2T$  (dashed line) and  $q < 2T$  (dash-dotted line) to  $P(\omega)$  (solid line) for  $|\omega| < 2T$ . Both curves are calculated for  $\alpha_g = 0.75$ .

screening at small frequencies and momenta which enables resonant forward scattering. It is the only surviving feature of the logarithmic divergence of the unscreened Coulomb scattering integral typical for 2D systems. The contribution of scattering with  $q < 2T$  decreases monotonically with increasing frequency and vanishes for  $|\omega| \geq 2T$  since  $|\omega| > q$  forbids intraband scattering.

The second kind of process is intraband scattering with large momentum transfer  $q > 2T$ . This contribution increases with increasing frequency up to  $\omega = 2T$ . It dominates over scattering with small momentum transfer for  $\omega \sim 2T$  and higher frequencies. For frequencies  $\omega > 2T$  it decreases monotonically. Specifically, we find that at large  $\omega$  the JSD falls as  $\omega^{-5/2}$ , shown in Fig. 2(a). There is a finite probability for the excited electron to gain energy from the bath of thermal electrons. However, negative frequencies are exponentially suppressed as shown in Fig. 2(a). The slow decay of the JSD for large frequencies has important implications for the fluctuations of  $\omega$  as discussed in Sec. III. In particular it is different from the JSD of a FL which is flat in the range  $0 < \omega < \epsilon_p$ . Thus an electron in a FL would lose most of its energy by a single jump. The FL regime is realized under the conditions  $|\mu| \gg T$  and  $\epsilon_p \ll |\mu|$ .

It turns out that for the scattering rate (5) the region  $|\omega| < 2T$  is most important and

$$\Gamma = \kappa \alpha_g T, \quad |\mu| \ll T, \quad \alpha_g \ll 1, \quad (10)$$

where  $\kappa = 4\pi^2(1 + \ln 2 + G/2) \simeq 84.92$  and  $G \simeq 0.916$  is the Catalan constant. The linear dependence on  $T$  is a characteristic feature of intrinsic graphene that distinguishes it from the FL [14]. Furthermore, due to screening the rate (10) is independent of the number of flavors  $N$  and linear in  $\alpha_g$  contrary to the golden rule result  $\Gamma \propto \alpha_g^2 T$  [18]. The rate (10) is also independent of the particle energy  $\epsilon_p \gg \max(|\mu|, T)$ .

### B. The limit $|\mu| \gg T$

For  $|\mu| \gg T$  the JSD is dominated by the region  $|\omega| < 2|\mu|$  as can be seen in Fig. 3(a) while the weight of the tail is strongly reduced. In particular the mean jump size will be of the order  $|\mu|$ . At the lowest frequencies  $|\omega| < \alpha_g T$ , the JSD  $P(\omega)$  shows a logarithmic divergence due to unscreened collinear scattering. Here the JSD recovers the FL form  $P(\omega) \propto (T/|\mu|) \ln |\mu/\omega|$  (see Ref. [16]) in contrast to the result for  $T \gg |\mu|$ , where we obtain  $P(\omega) \propto \ln(\alpha_g T/|\omega|)$ . In the  $T = 0$  limit the logarithmic divergence at small energies vanishes [see Fig. 3(b)]. In this case  $P(\omega)$  reproduces the result of Ref. [46].

The dominant process for  $|\omega| < 2|\mu|$  is the intraband scattering with small momentum transfer,  $q < 2|\mu|$ . Similar to the case  $T \gg |\mu|$ , such small-momentum scattering is not possible for  $\omega > 2|\mu|$  where scattering with  $q > 2|\mu|$  leads to the fat tail  $\propto \omega^{-5/2}$ . The contribution of negative frequencies  $P(\omega < 0) \propto \exp(\omega/2T)$  is exponentially small.

In the case  $T \gg |\mu|$ , the relaxation rate was determined by  $|\omega| < 2T$ . The total rate for  $|\mu| \gg T$ , is dominated by  $0 < \omega < 2|\mu|$  and is given by

$$\Gamma = 8\alpha_g \pi^2 |\mu|, \quad |\mu| \gg T, \quad \alpha_g \ll 1. \quad (11)$$

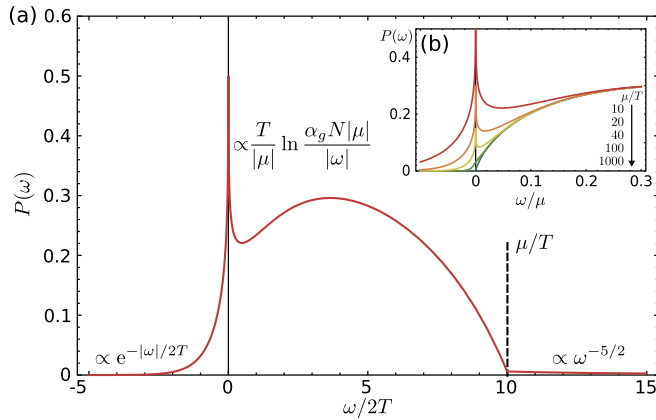


FIG. 3. (Color online) The JSD (7) for  $\mu/T = 10$ . In the region  $|\omega| < 2|\mu|$  processes with  $q < 2|\mu|$  are dominant. For  $|\omega| > 2|\mu|$  processes with  $q > 2|\mu|$  determine the fat tail of the JSD. The inset (b) illustrates the evolution of the forward scattering resonance with lowering temperature. Both curves are calculated for  $\alpha_g = 0.75$ . For details of the calculation, see Appendix A.

The rates (11) and (10) are calculated in the ballistic regime  $T\tau_{\text{dis}} \gg 1$ , where we neglect the influence of disorder with the characteristic scattering time  $\tau_{\text{dis}}$ . In the FL case it is known that the presence of disorder has strong influence on the inelastic relaxation of particles in the diffusive regime  $T\tau_{\text{dis}} \ll 1$  [51–53]. However, even in the diffusive regime the tails of the JSD  $\propto \omega^{-5/2}$  are preserved for  $\omega\tau_{\text{dis}} \gg 1$ , since they emerge due to scattering with large momentum transfer.

We finish this section with a short discussion of corrections to the results above due to nonlinearity of the spectrum at high energies  $\varepsilon^* \lesssim \Lambda$ , where  $\Lambda$  is the cutoff energy. The nonlinear correction to the dispersion relation reads  $\varepsilon_\lambda(p) - \lambda k \propto k^2 \sin \varphi_k / \Lambda$ , where  $\varphi_k$  is the angle of the direction of  $\vec{k}$ . The parameter that controls violations of the linear dispersion relation is therefore  $\varepsilon^* / \Lambda$ . Here  $\varepsilon^* \sim \omega_{\text{pump}}$  is a characteristic energy. A positive curvature of the spectrum opens a phase space for Auger processes (see Appendix D). Auger processes thus also contribute to the tail of the JSD. From a simple estimate (see Appendix D) we obtain that Auger processes dominate over intraband processes for  $\omega \gtrsim T(T^{1/3}\Lambda^{2/3}/\varepsilon^*)^2$ . This region is irrelevant if  $\varepsilon^* \lesssim \Lambda(T/\Lambda)^{5/9}$ . Under this condition the nonlinearity does not modify the tail of  $P(\omega)$ . For room temperature and the cutoff  $\Lambda = 1$  eV, even near infrared to visible light is within the range of validity of the results of this section. Since positive curvature only occurs in certain directions, Auger processes should be even weaker than in the simple estimate above. We want to stress that a negative curvature prevents Auger processes. Negative curvature appears due to intrinsic band curvature and due to renormalization of the electron spectrum.

### III. RELAXATION CASCADE: LÉVY FLIGHTS

We have seen that the JSD of a high energy electron with energy  $\varepsilon_p \gg \max(|\mu|, T)$  in graphene implies an average jump size of the order of either temperature or chemical potential. This is in contrast to the FL result where the JSD is flat up to the particle's energy. In graphene, the excited carriers relax in

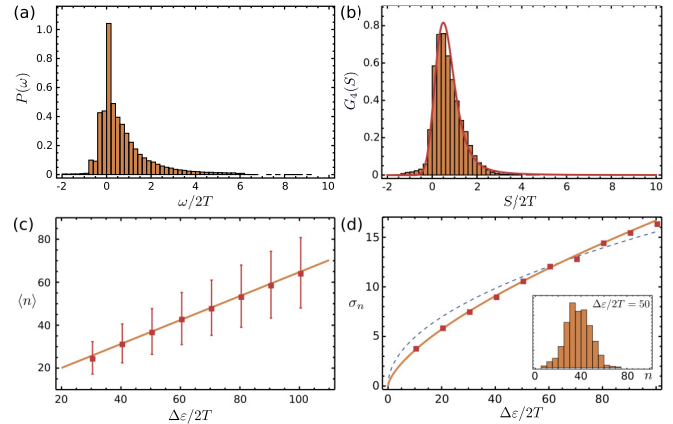


FIG. 4. (Color online) (a) Sample of the JSD for  $T \gg |\mu|$  (see Fig. 2). (b) Sample of the cascade variable  $S_n = \omega_1 + \dots + \omega_n$  from the JSD for  $n = 4$  with a high energy cutoff for the JSD given by the particle energy  $\varepsilon_p/T = 100$ . The solid line is the stable distribution with  $\alpha = 3/2$  and  $\beta = 1$ . (c) The average number of steps sampled from the JSD as a function of the cascade length  $\Delta\varepsilon$ . The error bars show the typical fluctuations  $\sigma_n$  of the number of cascade steps. (d) The fluctuation  $\sigma_n$  as a function of the cascade length  $\Delta\varepsilon$ . The solid line is the  $\Delta\varepsilon^{2/3}$  law (19). The dashed line illustrates Gaussian fluctuations for comparison. The inset shows a typical distribution of cascade steps for  $\Delta\varepsilon/2T = 50$ .

a cascade, with on average  $\langle n \rangle \sim \varepsilon_p / \langle \omega \rangle$  jumps, where  $\langle \dots \rangle$  is the average according to the JSD. The time scale of the cascade is then  $t \sim n / \Gamma$  [46].

The above conclusion concerns the mean number of steps in the cascade as well as the average cascade time. We now discuss the statistics of the random walk modeling the relaxation cascade in more detail with an emphasis on the fluctuations of the number of cascade steps.

Due to the fact that the JSD exhibits the fat tail  $P(\omega) \propto \omega^{-5/2}$ , it does not possess a second moment. Therefore, the fluctuations of the number of cascade steps should show an unusual behavior. The particle energy provides a natural cutoff for the JSD, rendering its variance finite. But on an intermediate scale, before the electron energy reaches  $\max(|\mu|, T)$ , the distribution behaves as if it possessed no finite variance. This is demonstrated in Figs. 4(a) and 4(b) by numerical sampling the JSD [Fig. 4(a)] and the cascade  $S_n = \omega_1 + \dots + \omega_n$  [Fig. 4(b)], where  $\omega_i$  are independent and identically distributed. For not too large  $n$ , a finite cutoff in the JSD does not change the distribution of  $S_n$  in Fig. 4(b).

As a consequence, the large- $n$  limit of the distribution of the cascade  $S_n$  does not approach the normal distribution. It rather lies in the domain of attraction of an  $\alpha$ -stable law  $G_n(S_n)$ . These are generalized limiting distributions for random processes with stationary and independent jumps including fat-tailed distributions as well as the normal distribution ( $\alpha = 2$ ) [47]. Their characteristic function (excluding the case  $\alpha = 1$  irrelevant for us),

$$\Phi_n(\alpha, \delta, \beta, c; z) = e^{izn\delta - nc|z|^\alpha(1 - i\beta\text{sign}(z)\tan(\alpha\pi/2))}, \quad (12)$$

is fully parametrized by four parameters. The index of stability  $\alpha = 3/2$  follows from the condition that  $G_n(S_n)$  lies in the



electrons and we can neglect the mutual interaction of the excited carriers. Second, we calculate the isotropic part of the distribution function at high energies  $\varepsilon > \omega_0$  [see Fig. 5(b)], thus we can neglect secondary electrons. Furthermore we neglect the exponential tail of the thermal electrons since  $\varepsilon \gg \max(|\mu|, T)$ . Therefore the isotropic part of the transient distribution function will be given by the distribution of downstream electrons, denoted  $F(\varepsilon, t)$ .

### A. Fractional kinetics

In the previous section we showed that the statistics of the relaxation dynamics is given by Lévy flights. In terms of the distribution function the relaxation will be described by the fractional Fokker-Planck equation (FFPE) [58],

$$\partial_t W(\varepsilon, t) = \Gamma \langle \omega \rangle \partial_\varepsilon W(\varepsilon, t) + D \nabla_{(\beta)}^\alpha W(\varepsilon, t). \quad (21)$$

Here  $W(\varepsilon, t)$  with  $W(\varepsilon, t = 0) = \delta(\varepsilon)$  is the propagator of the FFPE which will be given below. We also introduced the Riesz-Feller fractional derivative [59], which is defined by its Fourier transform,

$$\nabla_{(\beta)}^\alpha f(\varepsilon) = \int \frac{dz}{2\pi} \ln[\Phi_1(\alpha, 0, \beta, 1; z)] f(z) e^{iz\varepsilon}, \quad (22)$$

where  $\Phi_1$  is the characteristic function of the underlying stochastic process. In our case it is a Lévy  $\alpha$ -stable law with  $\alpha = 3/2$  and  $\beta = 1$  [see Eq. (12)]. In the FFPE (21) we also introduced the average energy loss rate  $\Gamma \langle \omega \rangle$  and the anomalous diffusion constant  $D = \Gamma c$ , where  $c$  is the scale parameter of the Lévy process [see Eqs. (12) and (14)]. From these formulas we obtain

$$D = \frac{2^\alpha 128 \sqrt{2\pi}}{N/4} T^{\alpha+1}. \quad (23)$$

The emergence of the fractional kinetics expressed by the FFPE (21) can be understood on the basis of a Langevin-type rate equation for the electron energy,

$$\partial_t \varepsilon(t) = -\Gamma \langle \omega \rangle + \eta(t), \quad (24)$$

where  $\eta(t)$  is a random variable which is distributed according to an  $\alpha$ -stable law and describes the interaction of the high energy electron with the bath of thermal electrons.

The general solution  $F(\varepsilon, t)$  of the FFPE with initial conditions  $F(\varepsilon, t = 0) = f(\varepsilon)$  is obtained with the propagator according to

$$F(\varepsilon, t) = \int d\varepsilon' W(\varepsilon - \varepsilon', t) f(\varepsilon'). \quad (25)$$

In our case we choose the initial probability density to be

$$f(\varepsilon) = n_0 \delta(\varepsilon - \omega_{\text{pump}}/2). \quad (26)$$

Here  $n_0$  is the integrated flux density of the pump pulse [60]. We have  $F(\varepsilon, t) = n_0 W(\varepsilon, t)$ , where

$$\varepsilon_t = \varepsilon - \omega_{\text{pump}}/2 + \Gamma \langle \omega \rangle t, \quad (27)$$

is the running energy. The propagator  $W(\varepsilon, t)$  and thus the solution  $F(\varepsilon, t)$  in our case of  $\alpha = 3/2$  and  $\beta = 1$  can be

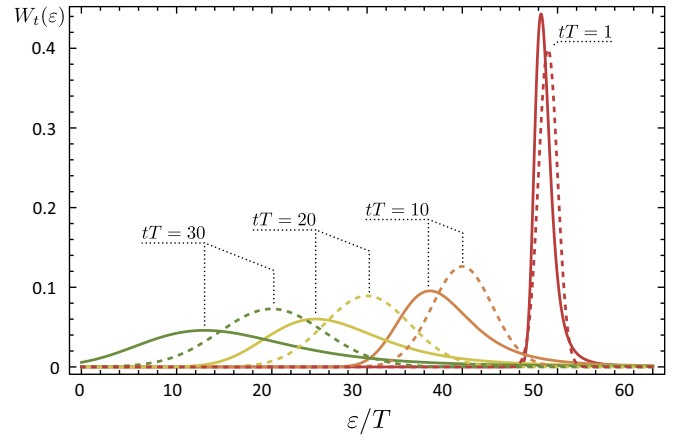


FIG. 6. (Color online) The solution [see Eqs. (28)–(30)] of the FFPE (21) (solid line) as a function of energy in comparison to the result obtained for Gaussian diffusion (dashed line) for different times.

calculated explicitly [61]. We obtain

$$W(\varepsilon, t) = \frac{\pi T}{\alpha} (Dt)^{-1/\alpha} K(s) \quad (28)$$

for the propagator in terms of the dimensionless variable

$$s = \varepsilon_t / (Dt)^{1/\alpha}. \quad (29)$$

In Eq. (28) the function  $K(s)$  is given by,

$$K(s) = -e^{\frac{s^3}{3}} \left[ \sqrt[3]{2/3} s \text{Ai} \left( \frac{s^2}{\sqrt[3]{486}} \right) + \sqrt[3]{12} \text{Ai}' \left( \frac{s^2}{\sqrt[3]{486}} \right) \right]. \quad (30)$$

Here  $\text{Ai}(z)$  is the Airy function and  $\text{Ai}'(z)$  its derivative. In particular,  $W$  has the following asymptotics for large times,

$$W(\varepsilon, t) \simeq \frac{T Dt}{\sqrt{2\pi\alpha}} |\varepsilon - \varepsilon_0 + \Gamma \langle \omega \rangle t|^{-(\alpha+1)}. \quad (31)$$

Using Eq. (23) and the results from Sec. II we obtain,

$$W(\varepsilon, t) \sim t^{-\alpha} \begin{cases} T(T/\mu^2)^{\alpha+1}, & |\mu| \gg T \\ T^{-\alpha}, & T \gg |\mu| \end{cases}. \quad (32)$$

We see that the tail of  $F$  for large times but fixed  $\varepsilon$  is proportional to  $t^{-3/2}$  and scales as  $T^{-3/2}$  for  $T \gg |\mu|$  and as  $T(T/\mu^2)^{5/2}$  for  $|\mu| \gg T$ .

The evolution of the probability distribution  $W(\varepsilon, t)$  due to the fractional kinetics is illustrated in Fig. 6. The solid line depicts the solution of the FFPE (21), given by Eqs. (28)–(30), while the dashed lines show the Gaussian solution of the usual Fokker-Planck equation. The fractional kinetics leads to a strong asymmetry, compared to the Gaussian drift diffusion, since the fluctuations in the underlying Lévy process are single sided, i.e.,  $\beta = 1$  in Eqs. (12) and (21).

### B. Transient change in transmission

We outline the consequences of the fractional kinetics for the transient differential transmission of the sample. The latter is determined by the change in the dynamic conductivity which

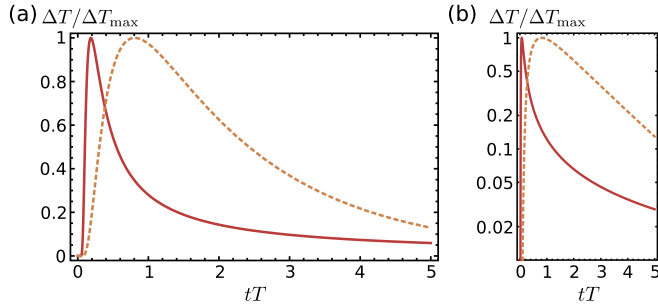


FIG. 7. (Color online) The normalized differential transmission  $\Delta T/\Delta T_{\max}$  as a function of the dimensionless time  $tT$ . Here  $\Delta T_{\max}$  denotes the maximum value of  $\Delta T$ . (b) Shows the results on a logarithmic scale. The solid curves are calculated according to Eq. (34) and Eqs. (28) and (30), for  $\Delta\varepsilon/T = (\omega_{\text{pump}} - \omega_{\text{probe}})/T = 25$  and  $\Gamma\langle\omega\rangle/T^2 = 20$  as well as  $D/T^{\alpha+1}$  from Eq. (23). The dashed lines in (a) and (b) illustrate the result for usual diffusion in comparison to the fractional kinetics (solid line).

is given by

$$\Delta\sigma(t)/\sigma_0 = -[F(\omega_{\text{probe}}/2, t) - F(-\omega_{\text{probe}}/2, t)]. \quad (33)$$

Given the particle hole symmetry of the correction to the distribution function at high energies, i.e.,  $F(-\varepsilon, t) = -F(\varepsilon, t)$ , we finally have for the relative differential transmission,

$$\frac{\Delta T(t)}{T_0} = 2n_0 W(\omega_{\text{probe}}/2, t), \quad (34)$$

where  $n_0$  is the integrated flux density.

The behavior of  $\Delta T$  as a function of time, Eq. (34), is illustrated in Fig. 7. The solid line depicts the result (28) due to the fractional kinetics in graphene, while the dashed line is the expected result for conventional Gaussian drift diffusion. We see that the diffusion in the case of Lévy flights (solid line) is stronger due to the fact that the  $\alpha$ -stable law is single sided, i.e.,  $\beta = 1$ . Therefore fluctuations enhance the drift in energy space (see also Fig. 6). Furthermore the transient differential transmission shows power-law behavior with time and temperature according to Eq. (32), instead of exponential decay in the case of usual diffusion [see Fig. 7(b)].

## V. CONCLUSION

We have provided an analysis of the relaxation cascade of photoexcited electrons in graphene at finite temperature. We calculated the relaxation rates of high energy electrons in the case of doped as well as undoped graphene. We find  $\Gamma \sim \alpha_g \max(|\mu|, T)$ , which distinguishes graphene from the FL. The  $\alpha_g$  dependence deviates distinctively from the golden rule result  $\propto \alpha_g^2$  and is due to the peculiar screening in graphene [18]. Furthermore the rates are independent of the particle energy  $\varepsilon_p$ . The entire relaxation cascade is determined by the distribution of the transferred energy in a single jump. This jump-size distribution exhibits logarithmic divergencies at small energy transfer due to resonant forward scattering which is very pronounced in graphene having truly linear spectrum. Specifically, we find  $P(\omega) \sim \ln \alpha_g T/|\omega|$  for  $|\mu| \ll T$  and small frequencies  $|\omega| \ll \alpha_g T$  which crosses over into the usual FL result  $P(\omega) \sim (T/|\mu|) \ln \alpha_g |\mu/\omega|$  at  $|\mu| \gg T$ .

Remarkably, the JSD exhibits fat tails that fall off as  $(\omega/T)^{-5/2}$  at large frequencies  $\omega > \max(2|\mu|, 2T)$  for both  $|\mu| \gg T$  and  $T \gg |\mu|$ .

Owing to the fat-tailed JSD, the relaxation cascade is described by an  $\alpha$ -stable distribution with a mean drift determined by either  $T$  or  $|\mu|$ : The fluctuation on top of the drift is described by Lévy flights with index of stability  $\alpha = 3/2$ . As a consequence, the fluctuations  $\sigma_t$  of the cascade time  $t$  exhibit characteristic scaling relations with the frequency  $\omega_{\text{pump}} \gg \omega_{\text{probe}}$  of the pump pulse,  $\sigma_t \sim \omega_{\text{pump}}^{1/\alpha}$ , as well as temperature. Specifically,  $\sigma_t \sim T^{1/3}$  for  $|\mu| \gg T$  and  $\sigma_t \sim T^{-5/3}$  for  $T \gg |\mu|$ . These scaling relations serve a clear imprint of the forward scattering resonance and related fractal nature of the relaxation cascade in graphene. The observed [34] variation of the average cascade time with  $\omega_{\text{pump}}$  is consistent with theoretical predictions for the energy drift made in Ref. [46] for the regime  $|\mu| \gg T$ . Using the experimental setup similar to that used in Refs. [34,45], it should be possible to detect the traces of the Lévy flights as well. Specifically, the width of the rise time in the measured change in transmission as depicted in Fig. 5(c) provides a direct measure of the fluctuation of the cascade time (20) [see also Fig. 4(d)].

Furthermore, the JSD is the distribution of the created electron-hole pairs during the cascade. We find that within the energy window  $\max(|\mu|, T) \ll \varepsilon < \omega_0$  a significant amount of secondary electrons is created according to  $P(\omega) \simeq \omega^{-5/2}$ . We find  $\omega_0 \sim T(\omega_{\text{pump}}/T)^{2/5}$  for  $T \gg |\mu|$  and  $\omega_0 \sim T(\omega_{\text{pump}}/|\mu|)^{2/5}$  for  $|\mu| \gg T$ . Probes in the mentioned energy interval should also reveal the tail of the JSD.

We predict the time evolution of the differential change in transmission in the presence of electron electron interactions. The transmission is directly measured in pump probe experiments and we obtain an analytical expression for the differential transmission from a fractional Fokker-Planck equation. The latter is suited to capture the fractional kinetics emerging from the Lévy flight statistics of the relaxation process.

The results of this work extend the study of relaxation dynamics of thermal electrons in graphene [18] to the case of high energy electrons also at finite chemical potential and should be relevant for future studies of the nonequilibrium steady states in irradiated graphene. This prospect includes the question of thermalization in driven graphene, the possibility of a population inversion [10,62] as well as frequency conversion [7]. It should also be interesting to extend it to the nonlinear regime of pumping where saturation effects become important. All these questions necessitate the full solution of the kinetic equation. In this context the present work sheds new light on the unique character of the interaction in graphene that controls the formation of such nonequilibrium states that might also be probed in future experiments.

## ACKNOWLEDGMENTS

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**APPENDIX A: CALCULATION OF THE RELAXATION RATE AND THE JSD FROM THE BOLTZMANN EQUATION**

In this section we derive Eq. (7) for the JSD and the relaxation rates from the main text. We start from a generic fermionic collision integral,

$$St[f(\lambda, \vec{p})] = \sum_{\lambda_3} \int \frac{d^2 p_3}{(2\pi)^2} \{W(\lambda, \vec{p} | \lambda_3, \vec{p}_3) f_{\lambda_3}(\vec{p}_3) [1 - f_\lambda(\vec{p})] - W(\lambda_3, \vec{p}_3 | \lambda, \vec{p}) f_\lambda(\vec{p}) [1 - f_{\lambda_3}(\vec{p}_3)]\}, \quad (\text{A1})$$

where the transition rates for the Coulomb interaction,

$$W(\vec{p}_2, \lambda_2 | \vec{p}_1, \lambda_1) = (2\pi)^{-1} \sum_{\lambda_3, \lambda_4} \int d\vec{p}_{3,4} \int d\vec{q} d\omega \delta(\lambda_2 p_2 + \omega - \lambda_1 p_1) \delta(\lambda_4 p_4 - \omega - \lambda_3 p_3) \\ \times \delta(\vec{p}_2 + \vec{q} - \vec{p}_1) \delta(\vec{p}_4 - \vec{q} - \vec{p}_3) K(\vec{q}, \omega, \{\lambda_i\}, \{\vec{v}_i\}) f(\vec{p}_3, \lambda_3) [1 - f(\vec{p}_4, \lambda_4)]. \quad (\text{A2})$$

Here the interaction kernel,

$$K(q, \omega, \{\lambda_i\}, \{\vec{v}_i\}) = N |V(\omega, q)|^2 \Theta_{1,2} \Theta_{3,4}, \quad (\text{A3})$$

contains the RPA screened Coulomb matrix element (see Appendix B),

$$|V(\omega, q)|^2 = \frac{4\pi^2 \alpha_g^2}{(q + 2\pi \alpha_g N \text{Re} \Pi)^2 + (2\pi \alpha_g N \text{Im} \Pi)^2}, \quad (\text{A4})$$

as well as the Dirac factors ( $\vec{v}_i = \lambda_i \vec{k}_i / k_i$ )  $\Theta_{1,2} = (1 + \vec{v}_1 \cdot \vec{v}_2) / 2$ . Upon inserting the ansatz (2) into the collision integral (A1), we obtain the explicit expression for the relaxation rate,

$$\Gamma = \sum_{\lambda_1} \int \frac{d^2 k}{(2\pi)^2} W_0(\lambda_1, \vec{k} | +1, \vec{p}) [1 - f_{\lambda_1}(\vec{k})] + W_0(+1, \vec{p} | \lambda_1, \vec{k}) f_{\lambda_1}(\vec{k}). \quad (\text{A5})$$

For  $\omega < \varepsilon_p$ , where interband processes are forbidden, the second term in Eq. (A6) can be dropped. Using Eqs. (A2) and (A3) we then obtain

$$\Gamma = (2\pi)^2 \sum_{\lambda_{1,3,4}} \int \frac{d^2 q}{(2\pi)^2} \frac{d\omega}{2\pi} \frac{d^2 k_4}{(2\pi)^2} \delta(\lambda_2 |\vec{p} - \vec{q}| + \omega - p) \delta(\lambda_4 k_4 - \omega - \lambda_3 |\vec{k}_4 - \vec{q}|) \\ \times N |V_{\text{RPA}}(\omega, q)|^2 \Theta_{1,2} |_{1=(\lambda_1, \vec{p}-\vec{q})} \Theta_{3,4} |_{3=(\lambda_3, \vec{k}_4-\vec{q})} f_T(\lambda_4 k_4 - \omega) [1 - f_T(\lambda_4 k_4)]. \quad (\text{A6})$$

Next we perform the angular integration in the integrals over  $\vec{k}_4$  and  $\vec{q}$ . The arising functional determinants are ( $\lambda = +1$ ),

$$\left| \frac{\partial}{\partial \varphi_q} \lambda_2 |\vec{p} - \vec{q}| \right| = \frac{pq |\sin(\varphi_q - \varphi_p)|}{|\vec{p} + \vec{q}|} = \frac{\sqrt{q^2 - \omega^2} [(\omega - 2\lambda p)^2 - q^2]^{1/2}}{2|\lambda p - \omega|}, \quad (\text{A7})$$

$$\left| \frac{\partial}{\partial \varphi_4} \lambda_3 |\vec{k}_4 - \vec{q}| \right| = \frac{k_4 q |\sin(\varphi_4 - \varphi_q)|}{|\vec{k}_4 - \vec{q}|} = \frac{\sqrt{q^2 - \omega^2} [(\omega - 2\lambda_4 k_4)^2 - q^2]^{1/2}}{2|\lambda_4 k_4 - \omega|}. \quad (\text{A8})$$

The corresponding Dirac factors are ( $\lambda_1 = \lambda = +1$ )

$$\Theta_{1,2} = \frac{1}{2} \left( 1 + \frac{\lambda_1 \lambda_2 \vec{k}_1 \cdot (\vec{p} - \vec{q})}{k_1 |\vec{p} - \vec{q}|} \right) = \frac{|\omega - 2\lambda p|^2 - q^2}{4p|\lambda p - \omega|}, \quad (\text{A9})$$

$$\Theta_{3,4} = \frac{1}{2} \left( 1 + \frac{\lambda_2 \lambda_3 \vec{k}_4 \cdot (\vec{k}_4 - \vec{q})}{k_4 |\vec{k}_4 - \vec{q}|} \right) = \frac{|\omega - 2\lambda_4 k_4|^2 - q^2}{4k_4 |\lambda_4 k_4 - \omega|}. \quad (\text{A10})$$

**1. The JSD  $P(\omega)$** 

Putting together Eqs. (A6)–(A10) we finally obtain the JSD,

$$P(\omega) = \int_0^\infty dq q \frac{\text{Re} \sqrt{\text{sgn}(q^2 - \omega^2) [(\omega - 2\lambda p)^2 - q^2]}}{2p} \frac{N |V(\omega, q)|^2}{|q^2 - \omega^2|} \mathcal{K}(\omega, q). \quad (\text{A11})$$

Here the kinetic kernel is given by Eq. (C1). If we assume  $p \gg \omega, q$  we obtain the result (7) stated in the main text,

$$P(\omega) = \int_{|\omega|}^\infty dq q \frac{N |V(\omega, q)|^2}{|q^2 - \omega^2|} \mathcal{K}(\omega, q). \quad (\text{A12})$$



TABLE I. The asymptotics of the polarization operator in graphene for  $|\tilde{\mu}| \gg 1$  in the different regimes from Eqs. (B2) and (B3). Here  $I_\eta(z)$  denotes the modified Bessel function of the first kind.

	$ \beta  < 1$		$ \beta  > 1$	
	$Q \ll \tilde{\mu}$	$Q \gg \tilde{\mu}$	$Q \ll \tilde{\mu}$	$Q \gg \tilde{\mu}$
$\text{Re}\Pi^R$	$\frac{ \mu }{2\pi}$	$\frac{TQ}{16\sqrt{1-\beta^2}}$	$-\frac{T}{8\pi} \frac{I_1(Q)}{\beta^2} \frac{\tilde{\mu}^2}{Q}$	$-\frac{T}{4\pi\beta^2 Q}$
$\text{Im}\Pi^R$	$\frac{ \mu }{2\pi} \frac{\Omega}{\sqrt{Q^2-\Omega^2}}$	$\frac{T}{4\sqrt{2\pi}Q} e^{-(1-\beta)Q}$ , for $(1-\beta)Q \gg 1$	$-\frac{T}{16} \frac{Q^2}{\sqrt{\Omega^2-Q^2}} \frac{\sinh(\Omega)}{\cosh(\Omega)+\cosh(\tilde{\mu})}$	$-\frac{T}{16} \frac{Q^2 \tanh \Omega}{\sqrt{\Omega^2-Q^2}}$

In the following we use the dimensionless variables  $\Omega = \omega/2T$ ,  $Q = q/2T$ ,  $\beta = \omega/q$ , and  $\tilde{\mu} = \mu/T$ . Using the asymptotics from Appendixes B and C we obtain limiting expressions for the JSD  $P(\Omega)$  presented below.

**a. The limit  $T \gg |\mu|$  for  $|\Omega| < 1$**

The contribution for small momentum transfer ( $Q < 1$ ) reads

$$P(\Omega)|_{Q<1} = 4 \ln 2 \alpha_g^2 \pi^2 N e^\Omega \int_{|\Omega|}^1 dQ \frac{Q}{|Q^2 - \Omega^2|(Q + \alpha_g N \ln 2)^2 + (\alpha_g N \ln 2 \Omega)^2} = \frac{4\pi^2}{N \ln 2} \ln \frac{\alpha_g N \ln 2}{|\Omega|}. \quad (\text{A13})$$

Here the last equality is valid for  $|\Omega| < \alpha_g N \ln 2$ . The contribution to the JSD with large momentum transfer ( $Q > 1$ ) for frequencies  $|\Omega| < 1$  is

$$P(\Omega)|_{Q>1} = 2\alpha_g^2 \pi^2 N e^\Omega \int_1^\infty dQ \frac{\sqrt{2\pi} Q^{3/2} e^{-Q}}{(\sqrt{Q^2 - \Omega^2} Q + \alpha_g \pi N Q^2/16)^2 + (\alpha_g \pi N e^{-Q} \sqrt{Q/2\pi})^2}. \quad (\text{A14})$$

The latter can be neglected for  $|\Omega| < \alpha_g$ .

**b. The limit  $T \gg |\mu|$  for  $|\Omega| > 1$  ( $Q > 1$ )**

For  $|\Omega| > 1$ , where only intraband transitions with  $Q > 1$  are possible, the JSD reads

$$P(\Omega) = 2\alpha_g^2 \pi^2 N e^\Omega \int_{|\Omega|}^\infty dQ \frac{\sqrt{2\pi} Q^{3/2} e^{-Q}}{(\sqrt{Q^2 - \Omega^2} Q + \alpha_g \pi N Q^2/16)^2 + (\alpha_g \pi N e^{-Q} \sqrt{Q/2\pi})^2} \simeq \frac{2^9 \sqrt{2\pi}}{N} |\Omega|^{-5/2}, \quad (\text{A15})$$

where the asymptotics is valid for  $|\Omega| \gg 1$ .

**c. The limit  $|\mu| \gg T$  for  $|\Omega| < |\tilde{\mu}|$  ( $Q < |\tilde{\mu}|$ )**

$$P(\Omega) = 4\alpha_g^2 \pi^2 N \Omega |\tilde{\mu}| (1 + \coth(\Omega)) \int_{|\Omega|}^{|\tilde{\mu}|} dQ \frac{Q}{(Q^2 - \Omega^2)(Q + \alpha_g N |\tilde{\mu}|/2)^2 + (\alpha_g N \tilde{\mu} \Omega/2)^2}. \quad (\text{A16})$$

Equation (A16) can be integrated analytically, similar to Eq. (A13), yielding a lengthy expression. For brevity we give the limit for  $|\Omega| \ll \alpha_g |\mu|$ ,

$$P(\Omega) \simeq \frac{1}{32\pi} \frac{\ln \alpha_g N |\mu/2\Omega|}{|\mu|^2}. \quad (\text{A17})$$

**d. The limit  $|\mu| \gg T$  for  $|\Omega| > |\tilde{\mu}|$  ( $Q > |\tilde{\mu}|$ )**

As in the case  $|\tilde{\mu}| \ll 1$ , here for  $|\Omega| > |\tilde{\mu}|$  the JSD is determined by scattering with large momentum transfer,

$$P(\Omega) = 2\alpha_g^2 \pi^2 N e^{+\Omega} \int_{|\Omega|}^\infty dQ \frac{\sqrt{2\pi} Q^{3/2} e^{-Q}}{(\sqrt{Q^2 - \Omega^2} Q + \alpha_g \pi N Q^2/16)^2} \simeq \frac{2^9 \sqrt{2\pi}}{N} |\Omega|^{-5/2}. \quad (\text{A18})$$

## 2. The relaxation rate $\Gamma$

**a. The limit  $T \gg |\mu|$**

We first calculate the relaxation rate for  $T \gg |\mu|$ . We find that the contribution from the region with  $Q > 1$  is of order  $\alpha_g^2$ , whereas  $|\Omega| < 1$  yields the leading contribution  $\propto \alpha_g$ :

$$\Gamma/2T = \int_0^{\alpha N \ln 2} d\Omega P(\Omega)|_{Q<1} + \int_{\alpha N \ln 2}^1 d\Omega P(\Omega)|_{Q<1}. \quad (\text{A19})$$

TABLE II. The asymptotics of the kinetic kernel (C1) expressing the phase space for the thermal electrons participating at the scattering event.

	$\beta < 1$		$\beta > 1$	
$ \tilde{\mu}  \gg 1$				
$\mathcal{K}$	$Q \ll \tilde{\mu}$ $4T^2 \Omega  \tilde{\mu}  (1 + \coth \Omega)$	$Q \gg \tilde{\mu}$ $2T^2 \sqrt{2\pi} Q e^{-(1-\beta)Q}$	$Q \ll \tilde{\mu}$ $T^2 \pi Q^2 e^{-(1-\text{sign}(\Omega)) \Omega }$	$Q \gg \tilde{\mu}$ $T^2 \pi Q^2 e^{-(1-\text{sign}(\Omega)) \Omega }$
	$\beta < 1$		$\beta > 1$	
$ \tilde{\mu}  \ll 1$				
$\mathcal{K}$	$Q \ll 1$ $4T^2 \ln 2 e^{\beta Q}$	$Q \gg 1$ $2T^2 \sqrt{2\pi} Q e^{-(1-\beta)Q}$	$Q \ll 1$ $T^2 \pi Q^2 e^{-(1-\text{sign}(\Omega)) \Omega }$	$Q \gg 1$ $T^2 \pi Q^2 e^{-(1-\text{sign}(\Omega)) \Omega }$

Here,  $P(\Omega)|_{Q < 1}$  is given by Eq. (A13) and we anticipate that the integrand contains the scale  $\alpha_g N \ln 2$  that separates the logarithmic divergence at small frequency from the rest. The first part in Eq. (A19) yields

$$\begin{aligned} \int_0^{\alpha_g N \ln 2} d\Omega P(\Omega)|_{Q < 1} &= 4 \ln 2 \alpha_g^2 \pi^2 N \int_0^{\alpha_g N \ln 2} d\Omega \left\{ \frac{1}{(\alpha_g N \ln 2)^2} \ln \frac{\alpha_g N \ln 2}{|\Omega|} + \int_{\alpha_g N \ln 2}^1 dQ \frac{1}{Q(Q^2 - \Omega^2)} \right\} \\ &= 4\alpha_g \pi^2 (1 + \ln 2). \end{aligned} \quad (\text{A20})$$

The second part in Eq. (A19) yields

$$\int_{\alpha_g N \ln 2}^1 d\Omega P(\Omega)|_{Q < 1} = 4 \ln 2 \alpha_g^2 \pi^2 N \int_{\alpha_g N \ln 2}^1 d\Omega \frac{\text{arccot}(\alpha_g N \ln 2 \Omega) - \arctan(\Omega / \alpha_g N \ln 2)}{2\alpha_g N \ln 2 \Omega} = 4\alpha_g \pi^2 G/2, \quad (\text{A21})$$

where  $G = 0.916$  is the Catalan constant. Together, Eqs. (A20) and (A21) yield the result (10) from the main text.

### b. The limit $|\mu| \gg T$

In the case  $|\mu| \gg T$  we find that the rate  $\Gamma$  is determined by small energy and momentum transfer,  $|\Omega|, Q < \alpha_g N |\tilde{\mu}|/2$ .

$$\Gamma = 2T \int_0^{\alpha_g N |\tilde{\mu}|/2} d\Omega \left( \frac{\alpha_g N |\tilde{\mu}|}{2} \right)^{-2} \int_{|\Omega|}^{\alpha_g N |\tilde{\mu}|/2} \frac{dQ}{Q} = 8\alpha_g \pi^2 |\mu|. \quad (\text{A22})$$

## APPENDIX B: THE POLARIZATION OPERATOR IN GRAPHENE

We use the dimensionless variables introduced in the preceding sections. Starting from the definition of the polarization operator in the Keldysh technique [18],

$$\Pi^R = \frac{i}{2} \int (d\varepsilon) \text{Tr} [\hat{G}^R(\varepsilon) \hat{G}^K(\varepsilon + \omega) + \hat{G}^K(\varepsilon) \hat{G}^A(\varepsilon + \omega)], \quad (\text{B1})$$

we obtain the following expressions for  $\Pi^R$  for arbitrary chemical potential and temperature,

$$\begin{aligned} \text{Im} \Pi^R &= \frac{TQ}{8\pi} \left\{ \frac{\Theta(1 - |\beta|)}{\sqrt{1 - \beta^2}} \int_1^\infty d\xi \sum_{s=\pm 1} \sqrt{\xi^2 - 1} \frac{\sinh(\beta Q)}{\cosh(\beta Q) + \cosh(s\xi Q - \tilde{\mu})} \right. \\ &\quad \left. - \frac{\Theta(|\beta| - 1)}{\sqrt{\beta^2 - 1}} \int_{-1}^1 d\eta \sqrt{1 - \eta^2} \frac{\sinh(\beta Q)}{\cosh(\beta Q) + \cosh(\text{sign}(\beta)\eta Q + \tilde{\mu})} \right\}, \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \text{Re} \Pi^R &= -\frac{TQ}{8\pi^2} P \int_{-1}^1 d\eta \int_1^\infty d\xi \sum_{s=\pm 1} \left\{ \frac{1}{\beta - s\eta} \sqrt{\frac{\xi^2 - 1}{1 - \eta^2}} \frac{\sinh(s\eta Q)}{\cosh(s\eta Q) + \cosh(s\xi Q - \tilde{\mu})} \right. \\ &\quad \left. - \frac{1}{\beta - s\xi} \sqrt{\frac{1 - \eta^2}{\xi^2 - 1}} \frac{\sinh(s\xi Q)}{\cosh(\xi Q) + \cosh(s\eta Q + \tilde{\mu})} \right\}. \end{aligned} \quad (\text{B3})$$

Here  $P \int \dots$  denotes the principal value. The asymptotics for  $|\tilde{\mu}| \gg 1$  in all relevant integration regions are given in Table I. For  $|\tilde{\mu}| \ll 1$  they can be found in Ref. [18].

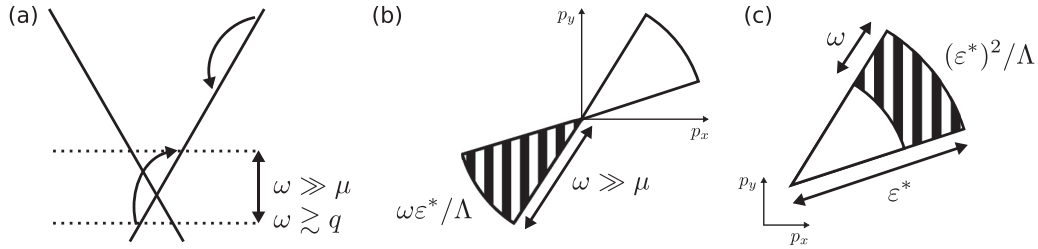


FIG. 8. (a) Auger process and (b) phase space for low energy electrons in the  $p_x$ - $p_y$  plane; (c) phase space for high energy electron.

### APPENDIX C: PHASE SPACE OF TWO-PARTICLE SCATTERING—THE KINETIC KERNEL

Finally we give the asymptotics of the kinetic kernel,

$$\mathcal{K}(\Omega, Q) = 2T^2 e^\Omega \int_{-\infty}^{+\infty} d\xi \frac{\text{Re}[\text{sgn}(1 - |\beta|)(\xi^2 - Q^2)]^{1/2}}{4 \cosh \frac{\xi - \Omega - \tilde{\mu}}{2} \cosh \frac{\xi + \Omega - \tilde{\mu}}{2}}, \quad (\text{C1})$$

for all integration regions in Table II.

### APPENDIX D: ESTIMATE OF THE SCATTERING RATE FROM AUGER PROCESSES

The phase space for Auger processes is controlled by the parameter  $\varepsilon^*/\Lambda$ , which describes the curvature. To estimate the contribution to the JSD from Auger processes, Fig. 8(a), we need the phase space for the high energy electron, which is given by  $\sim \omega(\varepsilon^*)^2/\Lambda$  [Fig. 8(c)], and the phase space for the thermal low energy electrons  $\sim \omega^2 \varepsilon^*/\Lambda$  [Fig. 8(b)]. Their product multiplied by the matrix element of scattering gives the following estimate for the JSD due to Auger processes,

$$P_{\text{Auger}}(\omega) = \frac{\omega^3 (\varepsilon^*)^3}{\Lambda^2} \left[ \frac{|V(\omega, q)|^2}{|\omega^2 - q^2|} \right]_{\omega \gtrsim q} \sim \frac{(\varepsilon^*)^3}{\Lambda^2 \omega}. \quad (\text{D1})$$

Comparing  $P_{\text{Auger}}$  with  $P$  due to intraband transitions we find that for  $\omega \gtrsim T(T^{1/3} \Lambda^{2/3}/\varepsilon^*)^2$ , Auger processes dominate. However, if this threshold lies beyond the particle energy  $\varepsilon^*$  we can neglect them, i.e., for  $\varepsilon^* \lesssim \Lambda(T/\Lambda)^{5/9}$ . This applies irrespective of the relation between  $T$  and  $\mu$ , provided  $\omega \gg |\mu|, T$ .

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