Interplay of classical and quantum capacitance in a one-dimensional array of Josephson junctions

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Even in the absence of Coulomb interactions, phase fluctuations induced by quantum size effects become increasingly important in superconducting nanostructures as the mean level spacing becomes comparable with the bulk superconducting gap. Here we study the role of these fluctuations, termed "quantum capacitance," in the phase diagram of a one-dimensional ring of ultrasmall Josephson junctions at zero temperature by using path-integral techniques. Our analysis also includes dissipation due to quasiparticle tunneling and Coulomb interactions through a finite mutual and self-capacitance. The resulting phase diagram has several interesting features: A finite quantum capacitance can stabilize superconductivity even in the limit of only a finite mutual-capacitance energy, which classically leads to breaking of phase coherence. In the case of vanishing charging effects, relevant in cold-atom settings where Coulomb interactions are absent, we show analytically that superfluidity is robust to small quantum finite-size fluctuations and identify the minimum grain size for phase coherence to exist in the array. We have also found that the renormalization group results are in some cases very sensitive to relatively small changes of the instanton fugacity. For instance, a certain combination of capacitances could lead to a nonmonotonic dependence of the superconductor-insulator transition on the Josephson coupling.

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The Josephson's effect [1,2] reveals the central role played by the phase of the order parameter in superconductivity. It has been exploited in a broad spectrum of research problems and applications: from the study of the pseudogap phase in high-T_c materials [3], fluctuations above T_c [4], and cold-atom physics [5] to spintronics [6] and quantum computing [7]. Of special interest is the study of an array of superconducting grains separated by thin tunnel junctions, usually referred to as Josephson junctions (JJs). The physical properties of JJ arrays are very sensitive to the grain dimensionality, the presence of Coulomb interactions, and dissipation [8,9–12] (see also the review [13]). Usually it is assumed that each single grain is sufficiently large so that the amplitude of the order parameter, the superconducting gap, is well described by the bulk Bardeen-Cooper-Schriffer (BCS) theory. Moreover, it is also commonly assumed that a simple capacitance model is sufficient to account for Coulomb interactions. The phase of each grain is therefore the only effective degree of freedom of the JJ array.

Within this general theoretical framework a broad consensus has emerged on the main features of JJ arrays: For long one-dimensional (1D) arrays at zero temperature with negligible dissipation, the existence of long-range order depends on the nature of the capacitance interactions. For situations in which only self-capacitance is important, superconductivity persists for sufficiently small charging effects [8] provided that the Josephson coupling is strong enough. Despite spatial global long-range order a state of zero resistance will strictly occur only in the case in which the supercurrent is induced by threading a flux in a ring-shaped JJ array [14,15]. A current in a long but finite linear JJ array will eventually induce a resistance though for sufficiently strong Josephson coupling it is hard to measure it as its typical time scale can be much longer that the experimental observation time. At any finite temperature the resistivity is always finite as a consequence of the unbinding of phase antiphase slips.

In the opposite limit in which only mutual-capacitance is considered, even small charging effects induce a superconductor-insulator transition. The combined effect of the two types of charging effects, considered in Ref. [16], can also lead to global long-range order. On a single junction, dissipation by quasiparticle tunneling only renormalizes [17] the value of the capacitance. However, dissipation caused by a Ohmic resistance [18] induces long-range correlations between phase slips and antiphase slips that restore superconductivity provided that the normal resistance is smaller than the quantum one. In order to illustrate the profound impact of dissipation, it is worth noting that a state of zero resistance in a 1D JJ array can in some cases coexist [9] with an order parameter whose spatial correlation functions are short ranged.

The closely related problem of a quantum nanowire was addressed in Refs. [14,19] by employing instanton techniques to model phase tunneling and then mapping the resulting effective model onto a $1+1$D Coulomb gas where one of the dimensions is imaginary time. For an infinite wire in the zero-temperature limit a superconductor-insulator Berezinsky-Kosterlitz-Thouless (BKT) transition occurs as a function of the system parameters. The role of vortices in $1+1$D is played by phase slips which correspond to configurations for which the amplitude of the order parameter vanishes and the phase receives a 2π boost. By contrast at finite temperature—a similar argument holds for finite length—the time dimension is compactified so, in the absence of dissipation, the Coulomb gas analogy breaks down since, for long separations, phase and antiphase slips become uncorrelated. As a consequence, phase coherence is lost and the resistance is always finite [14,16,20].

As was mentioned previously, all these results assume that the amplitude of the order parameter of each grain, which enters in the definition of the Josephson coupling energy, is not affected by any deviations from the bulk limit and that the phase dynamics is induced only by classical charging effects. Although these assumptions are in many cases sound, there are situations in which corrections are expected.

In sufficiently small grains close to the critical temperature it is well documented that homogeneous path-integral configurations different from the mean-field prediction, the so-called static paths, contribute significantly to the specific heat and other thermodynamical observables [21]. For single nanograins at intermediate temperatures it has been shown recently [22] that, even in the limit of vanishing Coulomb interactions, deviations from mean-field predictions occur due to the nontrivial interplay of thermal and quantum fluctuations induced by finite-size effects. Experimentally, it is also well established [23,24] that substantial deviations from mean-field predictions occur in isolated nanograins. Indeed, it has recently been reported [23,25] that quantum size effects enhance the superconducting gap of single isolated Sn nanograins with respect to the bulk limit.

It is therefore of interest to understand in more detail the role of these finite-size effects in arrays of ultrasmall JJ where the mean level energy spacing of single grains is smaller, but comparable, to the superconducting gap. This paper is a step in this direction. We study the stability of phase coherence in arrays of 1D JJ at zero temperature. Our formalism includes the above quantum fluctuations induced by size effects, charging effects, and dissipation by quasiparticle tunneling. Starting from a microscopic Hamiltonian for a 1D JJ ring-shaped array of nanograins at zero temperature, we map the problem onto a sine-Gordon Hamiltonian where we identify the region of parameters in which long-range order persists in the presence of phase fluctuations. In the limit of vanishing charging energy, relevant for cold-atom experiments, we find the minimum size for which the JJ array can be superfluid as a function of the wire resistance in the normal state. We also show that quantum fluctuations induced by finite-size effects can, in principle, stabilize superconductivity in the limit of a negligible self-capacitance energy but a finite mutual-capacitance energy. We have also identified a region parameters in which a nonmonotonic dependence of the superconductor-insulator transition on the Josephson coupling is observed.

I. THE MODEL

We consider the system sketched in Fig. 1, consisting of an array of L superconducting grains with periodic boundary conditions and a total magnetic flux Φ passing through it, which can be modeled by the Hamiltonian

$$H = \sum_{r=1}^{L} H_r^{\mathrm{BCS}} + H_r^{\mathrm{S}} + H_{r,r+1}^{\mathrm{M}} + H_{r,r+1}^{\mathrm{T}}. \qquad (1)$$

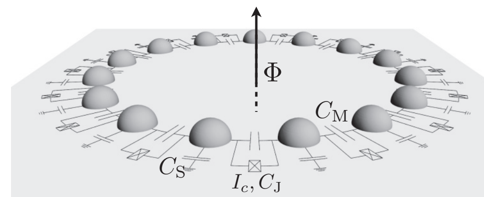

FIG. 1. Sketch a closed ring of JJs pierced by a total flux Φ.

Each isolated superconducting grain is described by the BCS term,

$$H_r^{\mathrm{BCS}} = \sum_{\alpha,\sigma} \epsilon_{\alpha,r}\, c^{\dagger}_{\alpha,\sigma,r} c_{\alpha,\sigma,r} - g_r \delta_r \left(\sum_{\alpha} c^{\dagger}_{\alpha,1,r} c^{\dagger}_{-\alpha,-1,r} \right) \left(\sum_{\alpha'} c_{-\alpha',-1,r} c_{\alpha',1,r} \right), \qquad (2)$$

accounting for the effective attractive electron-electron interactions in the region where the grain size is much smaller then the bulk superconducting coherence length. α, $-\alpha$ label single-particle states related by time-reversal symmetry with energies $\epsilon_\alpha = \epsilon_{-\alpha}$, $\sigma = \pm 1$ is the spin label and δ_r and g_r are, respectively, the mean level spacing (inversely proportional to the grain volume) and the dimensionless coupling constant of grain r. We further assume the presence of self- and mutual capacitive terms of the form

$$H_r^{\mathrm{S}} = \frac{1}{2C_r^{\mathrm{S}}} \left(\hat{N}_r - N_r^{\mathrm{S}} \right)^2, \qquad (3)$$

$$H_{r,r+1}^{\mathrm{M}} = \frac{1}{2C_r^{M}} \left(\hat{N}_r - \hat{N}_{r+1} - N_r^{\mathrm{M}} \right)^2, \qquad (4)$$

accounting for the repulsive Coulomb interaction within each grain and between electrons in neighboring grains. $\hat{N}_r = \sum_{\alpha,\sigma} c^{\dagger}_{\alpha,\sigma,r} c_{\alpha,\sigma,r}$ is the total number of electrons, C_r^{S} is the self-capacitance the of grain r, and C_r^{M} is the the mutual capacitance between nearest-neighbor grains r and $r+1$. The constants N_r^{S} and N_r^{M} can be adjusted by applying suitable gate voltages. Finally, the hopping of electrons between grains is captured by the term

$$H_{r,r+1}^{\mathrm{T}} = \sum_{\alpha\alpha'\sigma} T_{r,r+1}^{\alpha,\alpha'} c^{\dagger}_{\alpha,\sigma,t} c_{\alpha',\sigma,r+1} + \mathrm{H.c.}, \qquad (5)$$

where the hybridization matrix $T_{r,r+1}^{\alpha,\alpha'} \propto \int \psi_{\alpha,\sigma,r}(\boldsymbol{x}) \bar{\psi}_{\alpha',\sigma,r+1}(\boldsymbol{x}) d\boldsymbol{x}$ is proportional to the overlap of the single-particle wave functions of two neighboring grains. In the regime of interest here—small grain sizes with respect to the bulk coherence length—the simplifying assumption that the hybridization is energy independent $T_{r,r+1}^{\alpha,\alpha'} = t_{r,r+1}$ can safely be used, and thus $H_{r,r+1}^{\mathrm{T}}$ simplifies to

$$H_{r,r+1}^{\mathrm{T}} = t_{r,r+1} \sum_{\sigma} \left(\sum_{\alpha} c^{\dagger}_{\alpha,\sigma,r} \right) \left(\sum_{\alpha} c_{\alpha,\sigma,r+1} \right) + \mathrm{H.c.}, \qquad (6)$$

with $\Phi = \sum_i \arg t_{r,r+1}$ the total flux passing through the ring.

II. FINITE-SIZE CORRECTIONS TO THE ACTION OF A JOSEPHSON JUNCTION'S ARRAY

A. Partition function in the path-integral formalism

In this section we write the partition function $Z = \mathrm{Tr}[e^{-\beta H}]$ in the path-integral form and identify the finite-size corrections to the action. This is done by inserting L complex-valued Hubbard-Stratonovich fields (HSFs) Δ_r to decouple the BCS term in the superconducting channel, L real-valued HSF V_r^{S}, conjugate to the number of particles on each grain, and L real-valued HSF V_r^{M}, conjugate to the difference of the number of particles in neighboring grains. Using the notation $\Psi = (c_{\alpha,1,1}, c^\dagger_{\alpha,-1,1}, c_{\alpha,1,2}, c^\dagger_{\alpha,-1,2}, \dots)^T$, the partition function reads $Z = \int Dc\, D\Delta\, DV\, e^{-S}$, with the action

$$S = -\Psi^\dagger G^{-1}\Psi + \int_0^\beta d\tau \sum_r \left[\frac{1}{g_r\delta_r}\Delta_r^\dagger\Delta_r + \frac{C_r^{\mathrm{S}}}{2}\left(V_r^{\mathrm{S}}\right)^2 + iN_r^{\mathrm{S}}V_r^{\mathrm{S}} + \frac{C_r^{\mathrm{M}}}{2}\left(V_r^{\mathrm{M}}\right)^2 + iN_r^{\mathrm{M}}V_r^{\mathrm{M}}\right], \tag{7}$$

where the full Green's function is given by

$$G^{-1} = \begin{pmatrix} G_1^{-1} & T_{21} & \\ T_{21}^\dagger & G_2^{-1} & \ddots \\ & \ddots & \ddots \end{pmatrix}, \tag{8}$$

and

$$G_r^{-1} = \begin{pmatrix} -\partial_\tau - \tilde{\varepsilon}_{\alpha,r}(\tau) & \Delta_r(\tau) \\ \Delta_r^\dagger(\tau) & -\partial_\tau + \tilde{\varepsilon}_{\alpha,r}(\tau) \end{pmatrix}, \tag{9}$$

is the inverse of the electronic propagators restricted to grain r. Here we defined $\tilde{\varepsilon}_{\alpha,r}(\tau) = \varepsilon_{\alpha,r} - iV_r^{\mathrm{S}}(\tau) - iV_r^{\mathrm{M}}(\tau) + iV_{r-1}^{\mathrm{M}}(\tau)$ and the hybridization matrix $T_{r+1,r} = \begin{pmatrix} t_{r+1,r} & 0 \\ 0 & -\bar{t}_{r+1,r}\end{pmatrix}$.

Integrating out Ψ yields the action

$$S = -\mathrm{Tr}\ln[-G^{-1}] + \int_0^\beta d\tau \sum_r \left[\frac{1}{g_i\delta_r}\Delta_r^\dagger\Delta_r + \frac{C_r^{\mathrm{S}}}{2}\left(V_r^{\mathrm{S}}\right)^2 + iN_r^{\mathrm{S}}V_r^{\mathrm{S}} + \frac{C_r^{\mathrm{M}}}{2}\left(V_r^{\mathrm{M}}\right)^2 + iN_r^{\mathrm{M}}V_r^{\mathrm{M}}\right] \tag{10}$$

solely in terms of the HSF.

We apply the unitary transformation

$$U = \mathrm{diag}\{e^{i\frac{1}{2}\phi_1(\tau)}, e^{-i\frac{1}{2}\phi_1(\tau)}, e^{i\frac{1}{2}\phi_2(\tau)}, e^{-i\frac{1}{2}\phi_2(\tau)}, \dots\},$$

with $\phi_r(\tau) = \phi_r(\tau+\beta) + 2\pi n_{\phi_r} (n_{\phi_r} \in \mathbb{Z})$ to the electronic propagator G^{-1} in order to render real its off-diagonal anomalous elements $\Delta_r(\tau) = s_r(\tau)e^{i\phi_r(\tau)}$, where $s_r(\tau), \phi_r(\tau) \in \mathbb{R}$. Note that for odd n_{ϕ_i} one has that $\mathrm{Tr}_f[G^{-1}] = \mathrm{Tr}_b[U^\dagger G^{-1} U]$, where Tr_f denotes the trace over antiperiodic functions (fermionic) and Tr_b the trace over periodic functions (bosonic). For a generic n_{ϕ_r} we denote $\mathrm{Tr}_{n_{\phi_r}} = \mathrm{Tr}_f$ for n_{ϕ_r} even and $\mathrm{Tr}_{n_{\phi_r}} = \mathrm{Tr}_b$ for n_{ϕ_r} odd. Whenever we have two such indices we use $\mathrm{Tr}_{n_{\phi_1}n_{\phi_2}}$ for the time periodicity in indices 1 and 2. Note, however, that this complication is only formal as we are interested in the low-temperature properties of this action where the distinction between even and odd n_ϕ's can be safely ignored [26]. After this transformation we get

$$\tilde{G}^{-1} = U^\dagger G^{-1} U = \begin{pmatrix} \tilde{G}_1^{-1} & \tilde{T}_{21} & \\ \tilde{T}_{21}^\dagger & \tilde{G}_2^{-1} & \ddots \\ & \ddots & \ddots \end{pmatrix}, \tag{11}$$

with

$$\tilde{G}_r^{-1} = -1 \times \begin{pmatrix} \partial_\tau + \tilde{\varepsilon}_{\alpha,r}(\tau) + i\frac{1}{2}\partial_\tau\phi_r(\tau) & -s_r(\tau) \\ -s_r(\tau) & \partial_\tau - \tilde{\varepsilon}_{\alpha,r}(\tau) - i\frac{1}{2}\partial_\tau\phi_r(\tau) \end{pmatrix}, \tag{12}$$

and

$$\tilde{T}_{r+1,r} = \begin{pmatrix} t_{r+1,r}e^{i\frac{1}{2}[\phi_{r+1}(\tau)-\phi_r(\tau)]} & 0 \\ 0 & -\bar{t}_{r+1,r}e^{-i\frac{1}{2}[\phi_{r+1}(\tau)-\phi_r(\tau)]} \end{pmatrix}. \tag{13}$$

Moreover, assuming the hopping amplitude to be small, we may develop the $\mathrm{Tr}\ln[-G^{-1}]$ term to second order in $|t_{r+1,r}|$ and obtain the action

$$S[s,\phi,V] = \sum_r \left\{ \int_0^\beta d\tau \left[\frac{1}{g_i\delta_r}s_r^\dagger s_r + \frac{C_r^{\mathrm{S}}}{2}\left(V_r^{\mathrm{S}}\right)^2 + iN_r^{\mathrm{S}}V_r^{\mathrm{S}} + \frac{C_r^{\mathrm{M}}}{2}\left(V_r^{\mathrm{M}}\right)^2 + iN_r^{\mathrm{M}}V_r^{\mathrm{M}}\right] - \mathrm{Tr}_{n_{\phi_r}}\ln\left[-\tilde{G}_r^{-1}\right] + \mathrm{Tr}_{n_{\phi_r}n_{\phi_{r+1}}}[\tilde{G}_{r+1}\tilde{T}_{r+1,i}\tilde{G}_r\tilde{T}^\dagger_{r,r+1}]\right\}. \tag{14}$$

B. Leading behavior in δ

The action given by Eq. (14) is suitable for a saddle-point expansion in both s and V fields since the action for each grain is an extensive quantity in the number of electrons within that grain $\langle N_r\rangle \simeq E_D/\delta_r$. Notice, however, that the saddle-point equations cannot be explicitly evaluated as $\tilde{G}^{-1}$ depends on $\phi_r(\tau)$. We proceed by noting that $\partial_\tau\phi_r(\tau)$ is small, as the phase varies smoothly as a function of τ for sufficiently low temperatures. Formally, we set $V_r^{\mathrm{S}}(\tau) = V_{r,0}^{\mathrm{S}} + \delta V_r^{\mathrm{S}}(\tau)$, $V_r^M(\tau) = V_{r,0}^M + \delta V_r^M(\tau)$, and $s_r(\tau) = s_{r,0} + \delta s_r(\tau)$, where the subscript 0 denotes the static component (constant in τ) of the different quantities and the fluctuation around the static value, to be considered at quadratic order, are denoted by $\delta V_r^{\rm S}$, $\delta V_r^{\rm M}$, and δs_r. Physically, $s_{r,0}$ is the amplitude of the condensate on grain i and the terms $iV_{r,0}^{\rm S}, iV_{r,0}^{\rm M} \in \mathbb{R}$ lead to a renormalization of the chemical potential: $\tilde{\varepsilon}_{\alpha,r} = \varepsilon_{\alpha,r} - iV_{r,0}^{\rm S} - iV_{r,0}^{\rm M} + iV_{r-1,0}^{\rm M}$.

For equally spaced levels and a particle-hole-symmetric single-particle density of states, the tunneling term can be simplified at low temperatures [27],

$$\begin{aligned}
&\mathrm{Tr}[\tilde{G}_{r+1}\tilde{T}_{r+1,r}\tilde{G}_r\tilde{T}_{r,r+1}^\dagger] \\
&\simeq \frac{C_i^{\rm J}}{8}\int d\tau \{\partial_\tau[\phi_{r+1}(\tau)-\phi_r(\tau)]\}^2 \\
&\quad - \frac{I_i^{\rm c}}{2}\int d\tau \cos[\phi_{r+1}(\tau)-\phi_r(\tau)+\phi_r^t],
\end{aligned}$$

where ϕ_r^t is the phase of the hopping term $t_{r+1,r} = |t_{r+1,r}|e^{i\phi_r^t}$, $C_r^{\rm J}$ is the quasiparticle-induced capacitance, and $I_r^{\rm c}$ is the junction's critical current between grains r and $r+1$, given, respectively, by [28]

$$\begin{aligned}
C_i^{\rm J} = 2\frac{4|t_{r+1,r}|^2}{\delta_r\delta_{r+1}}\int_{s_{r,0}}^\infty d\nu_1 \int_{s_{r+1,0}}^\infty d\nu_2 \\
\times \frac{\nu_1\nu_2}{(\nu_2+\nu_1)^3\sqrt{(\nu_1^2-s_{r,0}^2)(\nu_1^2-s_{r+1,0}^2)}}
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
I_r^{\rm c} = \frac{8|t_{r+1,r}|^2}{\delta_r\delta_{r+1}}\int_{s_{r,0}}^\infty d\nu_1 \int_{s_{r+1,0}}^\infty d\nu_2 \\
\times \frac{s_{r,0}s_{r+1,0}}{(\nu_2+\nu_1)\sqrt{(\nu_1^2-s_{r,0}^2)(\nu_1^2-s_{r+1,0}^2)}}.
\end{aligned} \tag{16}$$

Note that for $s_{r,0} = s_{r+1,0} = s_0$ these expressions simplify to $C_r^{\rm J} = C_{\rm J} = \frac{3\pi}{32}\frac{1}{s_0 R_N}$ and $I_r^{\rm c} = I_{\rm c} = \frac{\pi}{2}\frac{s_0}{R_N}$ with $R_N = (\frac{4|t|^2\pi}{\delta^2})^{-1}$ the normal state resistance of the junction.

With these approximations the action reads

$$\begin{aligned}
S[s,\phi,V] = S_0 + \int d\tau \sum_r \Big\{ \Omega_r \delta s_r^2(\tau) \\
+ \frac{C_r^{\rm S}}{2}\delta V_r^{\rm S}(\tau)^2 + \frac{C_r^{\rm M}}{2}\delta V_r^{\rm M}(\tau)^2 + \frac{1}{2}C_{\delta,r}\varphi_r^2(\tau) \\
- i\langle N_{0,r}\rangle \partial_\tau\phi_r(\tau) + \frac{C_r^{\rm J}}{8}\{\partial_\tau[\phi_{r+1}(\tau)-\phi_r(\tau)]\}^2 \\
- \frac{I_r^{\rm c}}{2}\cos\left[\phi_{r+1}(\tau)-\phi_r(\tau)+\phi_r^t\right]\Big\},
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
S_0 = \sum_i \Big\{ \mathrm{Tr}\ln\left[-\tilde{G}_{r,0}^{-1}\right] + \beta\Big[\frac{1}{g_r\delta_r}s_{r,0}^2 + \frac{C_r^{\rm S}}{2}\left(V_{r,0}^{\rm S}\right)^2 \\
+ iN_r^{\rm S}V_{r,0}^{\rm S} + \frac{C_r^{\rm M}}{2}\left(V_{r,0}^{\rm M}\right)^2 + iN_r^{\rm M}V_{r,0}^{\rm M}\Big]\Big\}
\end{aligned} \tag{18}$$

only depend on the static saddle-point values, $\Omega_r = \frac{1}{g_r\delta_r} - \frac{1}{2}(\sum_\alpha \frac{\tilde{\varepsilon}_{\alpha,r}^2}{\xi_{\alpha,r}^3})$, $\xi_{\alpha,r} = \sqrt{\tilde{\varepsilon}_{\alpha,r}^2 + s_{0,r}^2}$, $\langle N_{0,r}\rangle = \frac{1}{2}\sum_\alpha(1-\frac{\tilde{\varepsilon}_{\alpha,r}}{\xi_{\alpha,r}})$,

$$\tilde{G}_{r,0}^{-1}(i\omega_n) = \begin{pmatrix} i\omega_n - \tilde{\varepsilon}_{\alpha,r} & s_{r,0} \\ s_{r,0} & i\omega_n + \tilde{\varepsilon}_{\alpha,r} \end{pmatrix}, \tag{19}$$

and where we also define

$$\varphi_r(\tau) = \delta V_r^{\rm S}(\tau) + \delta V_r^{\rm M}(\tau) - \delta V_{r-1}^{\rm M}(\tau) - \tfrac{1}{2}\partial_\tau\phi_r(\tau) \tag{20}$$

and the finite-size induced self-capacitance $C_{\delta,r} = \frac{2}{\delta_r}$.

Equation (17) is now suitable to a static-path treatment [22] once the fluctuations are integrated out. Here, as we are only interested in the phase dynamics at low temperatures, we set the static components to their mean-field values and integrate out the gapped fluctuations in both the s and the V fields. In the limit $\beta s_0 \gg 1$ the final action in terms of the phase degrees of freedom and assuming translational invariance in the couplings $C_r^{\rm S} = C_{\rm S}$, $C_r^{\rm M} = C_{\rm M}$, $C_{\delta,r} = C_\delta$, is given by

$$\begin{aligned}
S = \frac{1}{8}\int d\tau \sum_{r,r'} \partial_\tau\phi_r(\tau)[C_R]_{r,r'}\partial_\tau\phi_{r'}(\tau) \\
+ i\frac{1}{2}\sum_r \langle N_{0,r}\rangle \int d\tau\, \partial_\tau\phi_r(\tau) \\
- \frac{I^{\rm c}}{2}\sum_r \int d\tau \cos\left[\phi_{r+1}(\tau)-\phi_r(\tau)+\phi_r^t\right],
\end{aligned} \tag{21}$$

where $C_R = \frac{1}{\tilde{C}_{\rm S}^{-1} - C_{\rm M}^{-1}\Delta_1^2} - C_{\rm J}\Delta_1^2$ is the capacitance matrix, with Δ_1 the discrete derivative: $(\Delta_1\phi)_r = \phi_r - \phi_{r-1}$, $\phi_r^t = \arg t_{r,r+1}$ is the phase of the hopping term, $\langle N_{0,r}\rangle = \frac{1}{2}\sum_\alpha(1 - \frac{\tilde{\varepsilon}_{\alpha,r}}{\xi_{\alpha,r}})$ is the average number of electrons in grain r, and

$$\tilde{C}_{\rm S} = \left(\frac{1}{C_{\rm S}} + \frac{\delta}{2}\right)^{-1} \tag{22}$$

is the grain self-capacitance renormalized by quantum finite-size effects. Note that on the lattice $\sum_r(\Delta_1\phi)_r(\Delta_1\phi')_r = -\sum_r \phi_r(\bar{\Delta}_1\Delta_1\phi')_j$, with $(\bar{\Delta}_1\phi)_r = \phi_{r+1} - \phi_r$; for the sake of simplicity we use the notation Δ_1^2 to denote the lattice Laplacian $\bar{\Delta}_1\Delta_1$.

Equation (21) is the central result of this section, it contains the effective low-energy theory for a junction at $T \ll T_c$, including charging effects, quasiparticle dissipation, and for the first time quantum fluctuations induced by finite-size effects C_δ. The Berry phase term—second term of Eq. (21)—ensures that, in the ground state (i.e., for $T = 0$), the average number of electrons on each grain is even [26]. In the following we assume that this condition is fulfilled and drop this term.

Note that for a set of isolated finite-size grains with $I^{\rm c} = C_{\rm J} = C_{\rm M}^{-1} = 0$ no superconducting phase ensues as the action in Eq. (21) reduces to $\frac{\varrho}{2}\int d\tau[\partial_\tau\phi_r(\tau)]^2$ with the phase stiffness $\varrho = \frac{\tilde{C}_{\rm S}}{4}$ controlling the exponential time decay of the order parameter correlation function $\hat{\Psi}_r(\tau) = g\delta\sum_\alpha c_{r,\alpha}(\tau)c_{r,\bar{\alpha}}(\tau)$: $\langle\Psi_r(\tau)\Psi_{r'}^\dagger(\tau')\rangle \propto \delta_{r,r'}\, s_0^2 e^{-\frac{|\tau-\tau'|}{2\varrho}}$.

III. SUPERCONDUCTING TRANSITION

A. Hamiltonian formulation

In this section we analyze the action given by Eq. (21), without the Berry phase term $\int d\tau \partial_\tau \phi_r(\tau)$ as we assume an even number of electrons in each grain. The calculation is carried out by first mapping this equation onto an equivalent Coulomb gas model. The Coulomb gas is subsequently transformed into a sine-Gordon action for which a perturbative Renormalization Group (RG) treatment can be effectively performed .

First we provide a description of the model in terms of the effective low-energy Hamiltonian for the phase degrees of freedom in order to make contact with previous works where this effective description is taken as the starting point of the calculation. The initial step is the discretization of the imaginary time in Eq. (21): $\tau = \Delta\tau\,\tilde{\tau}$ (with $\tilde{\tau} = 1,\ldots,N$ and $N\Delta\tau = \beta$). Using the identity

$$\lim_{\Delta\tau\to 0} \sum_{\boldsymbol{n}=n_1,\ldots,n_N} e^{-\frac{\Delta\tau}{2}\boldsymbol{n}\cdot\boldsymbol{A}^{-1}\cdot\boldsymbol{n}+\Delta\tau i\boldsymbol{b}\boldsymbol{n}} = \left(\sqrt{\frac{2\pi}{\det(\boldsymbol{A})\Delta\tau}}\right)^N e^{-\frac{\Delta\tau}{2}\boldsymbol{b}\cdot\boldsymbol{A}\cdot\boldsymbol{b}}, \tag{23}$$

the partition function can be rewritten as $Z = \int D\phi \sum_n e^{-iS[\phi,n]}$, with

$$\begin{aligned} S[\phi,n] = &\sum_{\tilde{\tau},r,r'} 2\Delta\tau\, n(\tilde{\tau},r)\left[C_R^{-1}\right]_{rr'} n(\tilde{\tau},r') \\ &- \sum_{\tilde{\tau},r} in(\tilde{\tau},r)[\phi(\tilde{\tau}+1,r)-\phi(\tilde{\tau},r)] \\ &- \frac{I_c}{2}\sum_{\tilde{\tau},r}\Delta\tau\cos\left[\phi(\tilde{\tau},r+1)-\phi(\tilde{\tau},r)+\phi_r^t\right]. \end{aligned} \tag{24}$$

In this form, Eq. (24) can readily be interpreted as the Trotter-sliced action coming from the Hamiltonian

$$H = \sum_{rr'} 2\hat{n}_r\left[C_R^{-1}\right]_{rr'}\hat{n}_{r'} - \frac{I_c}{2}\sum_r \cos\left[\hat{\phi}_{r+1}-\hat{\phi}_r+\phi_r^t\right], \tag{25}$$

where $\hat{n}_r = (-i\partial_{\phi_r})$, the variable conjugated to $\hat{\phi}_r$, is the number of Cooper pairs in grain r.

B. Partition function of the Coulomb gas

We follow the procedure of [29] to rewrite the action of a JJ array in terms of the partition function of a classical Coulomb gas. Using the Villain decomposition of the cosine term,

$$e^{z\cos(\theta)} \simeq I_0(z)\sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}\mu(z)m^2} e^{im\theta}, \tag{26}$$

with $I_0(z)$ a modified Bessel function of the first kind, valid for both large and small z, respectively, with

$$\mu(z) = \begin{cases} -2\ln(z/2) & \text{for } z \ll 1, \\ z^{-1} & \text{for } z \gg 1. \end{cases} \tag{27}$$

Equation (24) can be written as

$$\begin{aligned} S[\phi,n] = &\sum_{\tilde{\tau},r,r'} 2\Delta\tau n_0(\tilde{\tau},r)\left[C_R^{-1}\right]_{rr'} n_0(\tilde{\tau},r') \\ &+ \sum_{\tilde{\tau},r}\left\{\frac{1}{2}\mu\left(\frac{I_c\Delta\tau}{2}\right)n_1^2(\tilde{\tau},r) - i\phi_r^t n_1(\tilde{\tau},r) - i\phi(\tilde{\tau},r)\right. \\ &\left.\times[n_1(\tilde{\tau},r-1)-n_1(\tilde{\tau},r)+n_0(\tilde{\tau}-1,r)-n_0(\tilde{\tau},r)]\right\}, \end{aligned} \tag{28}$$

where we relabel $n \to n_0$ in Eq. (24) and $m \to n_1$ in Eq. (26) in order to interpret $n_\mu(\tilde{\tau},r)$ as an integer field living on links of a square lattice—an integer-valued 1 form on the square lattice—with n_0 corresponding to timelike links and n_1 to spacelike links.

Integrating out the ϕ field yields the divergence-free constraint

$$\partial n \equiv \Delta_1 n_1 + \Delta_0 n_0 = 0, \tag{29}$$

where $\Delta_0 f(\tilde{\tau},r) = f(\tilde{\tau},r) - f(\tilde{\tau}-1,r)$ is the discrete derivative along the time direction. Locally, such constraints can be satisfied by writing n as the rotational of an integer-valued field living on the centers of plaquettes—an integer-valued lattice 2 form—$n = \partial a$ or in components, $n_0 = -\Delta_1 a_{01}$, $n_1 = \Delta_0 a_{01}$, where the subscript of a denotes that this field lives on spatiotemporal plaquettes. The operator ∂ can be seen as the lattice exterior coderivative. Globally, the most generic solution of the constraint in Eq. (29) includes a nontrivial divergence-free field that cannot be written as a rotational. On a torus, such general solution can be decomposed as $n = \partial a + \sum_\alpha c_\alpha b^\alpha$. More explicitly,

$$n_0(\tilde{\tau},r) = -\Delta_1 a_{01}(\tilde{\tau},r) + \sum_{\alpha=0,1} c_\alpha b_0^\alpha(\tilde{\tau},r), \tag{30}$$

$$n_1(\tilde{\tau},r) = \Delta_0 a_{01}(\tilde{\tau},r) + \sum_{\alpha=0,1} c_\alpha b_1^\alpha(\tilde{\tau},r), \tag{31}$$

where b^0 and b^1 (with $\partial b^\alpha = 0$) are integer-valued 1 forms on the lattice that cannot be written as a rotational. They are chosen, see Fig. 2, to have a minimum flux along time and space directions, respectively: $\sum_{\tilde{\tau},r} b_\mu^0(\tilde{\tau},r) = N\delta_{\mu 0}$, $\sum_{\tilde{\tau},r} b_\mu^1(\tilde{\tau},r) = L\delta_{\mu 1}$. $c_{\alpha=0,1}$ are integer-valued coefficients labeling different topological sectors. Note that in the infinite-volume limit, i.e., zero temperature and $L \to \infty$, the b terms can be dropped in the solution as the space becomes

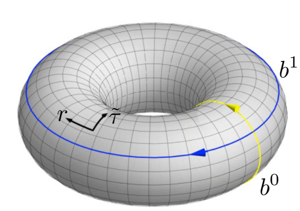

FIG. 2. (Color online) Sketch of two integer-valued 1 forms that cannot be written as $\partial a = \{-\Delta_1 a_{01}, \Delta_0 a_{01}\}$ with a a 2 form. $b_0^0(\tilde{\tau},r) = 1$ and $b_1^1(\tilde{\tau},r) = 1$ for $(\tilde{\tau},r)$ in the yellow and blue lines, respectively, otherwise $b_0^0(\tilde{\tau},r) = b_1^1(\tilde{\tau},r) = b_0^1(\tilde{\tau},r) = b_1^0(\tilde{\tau},r) = 0$.

topologically trivial. Later we drop the b^0 contribution as we are interested in the zero-temperature limit.

In terms of the a field and the integers c_0 and c_1, the partition function is given by the unconstrained sum $Z = \sum_{a,c} e^{-S[a,c]}$, with

$$S[a,c] = \sum_{\tilde{\tau},r,r'} 2\Delta\tau \left[\Delta_1 a(\tilde{\tau},r) - \sum_\alpha c_\alpha b_0^\alpha(\tilde{\tau},r)\right] \times [C_R^{-1}]_{rr'} \left[\Delta_1 a(\tilde{\tau},r') - \sum_\alpha c_\alpha b_0^\alpha(\tilde{\tau},r')\right] - i c_1 \Phi + \frac{1}{2}\sum_{\tilde{\tau},r} \mu\left(\frac{I_c \Delta\tau}{2}\right)\left[\Delta_0 a(\tilde{\tau},r) + c_\alpha b_1^\alpha(\tilde{\tau},r)\right]^2, \tag{32}$$

where the total flux $\Phi = \sum_r \phi_r^t$.

Using the Poisson summation formula $\sum_a f(a) = \sum_m \int d\psi\, f(\psi) e^{2\pi i m \psi}$ to improve the convergence of the sum over Eq. (32) [29] and integrating over ψ yields

$$Z = \sum_{m,c} \delta_{\sum m=0} e^{-S[c,m]} e^{i c_1 \Phi}, \tag{33}$$

where the sum over m is restricted such that the so-called neutrality condition $\sum_{r\tilde{\tau}} m(\tilde{\tau},r) = 0$ is fulfilled [29] and

$$S[c,m] = \frac{(2\pi)^2}{2} \sum_{\tilde{\tau}\tilde{\tau}'rr'} m(\tilde{\tau},r) G(\tilde{\tau}-\tilde{\tau}',r-r') m(\tilde{\tau}',r') - 2\pi i \sum_\alpha c_\alpha \sum_{\tilde{\tau},r} m(\tilde{\tau},r)(\partial^{-1} b^\alpha)(\tilde{\tau},r), \tag{34}$$

with $(\partial^{-1} b^\alpha)_{01} = (\Delta_0^2 + \Delta_1^2)^{-1}(\Delta_1 b_0^\alpha - \Delta_0 b_1^\alpha)$ the inverse of the ∂ operator defined in Eq. (29). The last term in Eq. (34) for b^1 can be simplified to

$$\sum_{\tilde{\tau},r} m(\tilde{\tau},r)(\partial^{-1} b^1)(\tilde{\tau},r) = \sum_j \left[(\Delta_0^2 + \Delta_1^2)^{-1} \bar{\Delta}_0 m\right](0,r).$$

The Green's function is given by

$$G^{-1} = -4\Delta\tau \tilde{C}_S^{-1}\Delta_1^2 \left[\frac{1}{\{1 - C_M^{-1}\tilde{C}_S\Delta_1^2\}} - \tilde{C}_S^{-1} C_J \Delta_1^2\right]^{-1} - \mu\left(\frac{I_c\Delta\tau}{2}\right)\Delta_0^2. \tag{35}$$

In summary, after integrating over the ψ field that represents small phase fluctuations, the action in Eq. (34) is given solely in terms of topological excitations, m, that can be interpreted as an instanton field representing a phase slip. The corrections due to nonvanishing values of $C_S^{-1}C_J$ and $C_M^{-1}\tilde{C}_S$ do not change the nature of the long-range interaction between the phase slips, as they multiply higher powers of the discrete Laplacian. Nonetheless, they appear in Eq. (34) in inequivalent ways; further we see this translates to different contribution to the monopole energy to create monopole pairs.

C. Flux quantization

To understand how the flux piercing the ring gets quantized in the superconducting phase, where the density of instantons (phase slips) vanishes, let us examine the partition function given in Eq. (33). For simplicity, let us first take the zero-temperature limit in order to ignore the b^0 field. The flux Φ is imposed to the system assuming that the magnetic field far from the ring is constant and perpendicular to the z axes in Fig. 1. A complete description of the system array + field should include the dynamics of the electromagnetic field as well. However, this is too involved and not really needed here; the only thing that is required is to remember that the spacial distribution of the electromagnetic field (and thus the flux piercing the ring) is itself determined by an action containing the electromagnetic contribution plus the coupling of the electromagnetic field to the instanton configurations given by the last term of Eq. (34).

Performing the summation over c_1 in Eq. (33) one observes that the partition function of a system with flux Φ can be written as

$$Z = \sum_{m,c_1} \delta_{\sum m=0}\, \delta_{2\pi}\left(\Phi - \Phi_m^1\right) \times e^{-\frac{(2\pi)^2}{2}\sum_{\tilde{\tau}r,\tilde{\tau}'r'} m(\tilde{\tau},r)\cdot G(\tilde{\tau}-\tilde{\tau}',r-r')\cdot m(\tilde{\tau}',r')}, \tag{36}$$

where $\delta_{2\pi}$ is the 2π-periodic δ function and $\Phi_m^1 = 2\pi \sum_r [\frac{\bar{\Delta}_0}{\Delta_0^2+\Delta_1^2} m](0,r) \in \mathbb{R}$. To the action of the free electromagnetic action, one should thus add the monopole contribution $F[\Phi] = -\ln Z$. Directly from Eq. (36) one can observe that if the density of phase slips vanishes (i.e., $\langle \frac{1}{NL}\sqrt{\sum_{\tilde{\tau},r} m^2(\tilde{\tau},r)}\rangle = 0$), then $\Phi_m^1 = 0$ and thus Φ has to be quantized in multiples of 2π. When phase slips proliferate, Φ_m^1 is a fraction of 2π, for a generic configuration of instantons m, the summation over all m configurations allows for a continuum value of Φ.

D. Superconducting-insulating transition

Having understood how the flux gets quantized once instantons are suppressed, let us neglect the topological terms [i.e., set $c_{0,1} = 0$ in Eq. (33)] in order to study the superconducting-insulating transition. A simple way of addressing this question is to map the problem to the sine-Gordon model. The main result we report in this section is that the superconducting-insulating phase transition is Kosterlitz-Thouless-like, even in the presence of a finite C_M and C_J. This extends the results of Ref. [16], where the case $C_M \neq 0$, $C_J = 0$ is considered. Nonetheless, C_M and C_J renormalize the instanton-core energy in rather different ways. By studying how this energy gets renormalized we obtain the behavior of the superconducting-insulating transition line as a function of I_c, $\tilde{C}_S$, C_M, and C_J. We note that $\tilde{C}_S$ also includes a term $\propto 1/\delta$ coming from quantum fluctuations induced by finite-size effects that so far had not investigated in the literature.

The first step to get the sine-Gordon action is to regularize the instanton interaction kernel at the origin $G(\tilde{\tau},j) \to G(\tilde{\tau},j) - G(0,0)$ in Eq. (34) by making use of the neutrality condition. After this procedure the asymptotic $\tilde{\tau},j \to \infty$ form of the instanton (anti-) instanton interaction is given by

$$(2\pi)^2[G(\tilde{\tau},r) - G(0,0)] \simeq \tilde{G}(\tilde{\tau}-\tilde{\tau}',r-r') - \nu, \tag{37}$$

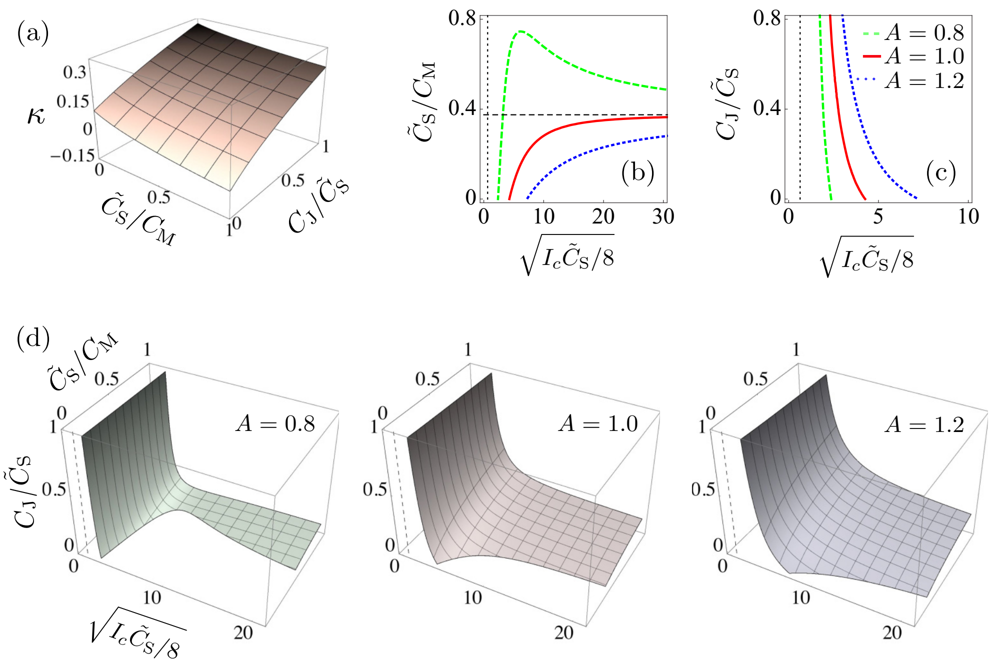

FIG. 3. (Color online) (a) Plot of the function κ as a function of $\tilde{C}_S/C_M$ and $C_J/\tilde{C}_S$ computed numerically from the asymptotic form of $G(\tilde{\tau},r) - G(0,0)$ for $\tau,r \to \infty$ at $\lambda = 1$. (b) Phase diagram in the $\{\sqrt{I_c\tilde{C}_S/8},\tilde{C}_S/C_M\}$ plane for $C_J = 0$ for different values of the nonuniversal constant A. Below the phase transition line the system is a superconductor and above it is an insulator. The horizontal dashed line corresponds to the critical ratio $\tilde{C}_S/C_M \approx 0.375$ above which the system is always in the insulating phase in the $\sqrt{I_c\tilde{C}_S/8} \to \infty$ limit. The vertical dotted line at $\tilde{C}_S/C_M = 2/\pi$ marks the lower bound obtained for $A = 0$. (c) Phase diagram in the $\{\sqrt{I_c\tilde{C}_S/8},C_J/\tilde{C}_S\}$ plane for $C_M \to \infty$. (d) Complete phase diagram in the $\{\sqrt{I_c\tilde{C}_S/8},\tilde{C}_S/C_M,C_J/\tilde{C}_S\}$ space for different values of A.

where

$$\tilde{G}(\tilde{\tau},r) = -2\pi\sqrt{I_c\tilde{C}_S/8}\,\ln(\sqrt{\tilde{\tau}^2/\lambda^2 + r^2}) \tag{38}$$

is the long-range instanton interaction potential and

$$\nu = \sqrt{I_c\tilde{C}_S/8}\,\kappa(\lambda,\tilde{C}_S/C_M,C_J/\tilde{C}_S) \tag{39}$$

is the instanton-core energy. Choosing the regulator $\Delta\tau \approx \sqrt{\tilde{C}_S/2I_c}$ such that $I_c\Delta\tau/2 \gg 1$ and $\Delta\tau \ll 1$ [15,30], we observe by Eq. (27) that $\mu(\frac{I_c\Delta\tau}{2}) \simeq \frac{2}{I_c\Delta\tau}$. The anisotropy between time and space directions $\lambda = \sqrt{\tilde{C}_S/2I_c(\Delta\tau)^2}$ is thus of order 1. κ depends on all ratios $\lambda,\tilde{C}_S/C_M$ and $C_J/\tilde{C}_S$; however, it is mildly varying as a function of λ around $\lambda = 1$. In the following we take the $\lambda = 1$ prescription [15,30] for our numerical analysis.

The function κ can be computed numerically by subtracting the asymptotic behavior $\tilde{G}(\tilde{\tau},r)$ to the right-hand site of Eq. (37) and numerically integrating the resulting expression. After a careful analysis of the numerical data to ensure that the asymptotic values are well reproduced, we obtained the results of Fig. 3(a).

Using the neutrality condition once more, the action acquires the Coulomb (lattice) gas form,

$$S[m] \simeq \sum_{r\tilde{\tau}\neq r'\tilde{\tau}'} m(\tilde{\tau},r)\tilde{G}(\tilde{\tau}-\tilde{\tau}',r-r')m(\tilde{\tau}',r') + \nu\sum_{r\tilde{\tau}}[m(\tilde{\tau},r)]^2. \tag{40}$$

The (lattice) sine-Gordon model can be obtained by inserting a HSF and using the identity given in Eq. (26):

$$Z = \sum_m \delta_{\sum m=0} e^{-S[m]} \propto \int D\psi\, e^{-\frac{1}{2}\psi\tilde{G}^{-1}\psi + u\sum_x \cos(\psi_x)}, \tag{41}$$

with $\mu(u) = \nu$ given by Eq. (27). Note that in this mapping the neutrality condition is assured by the fact that $\tilde{G}^{-1}(\omega = 0,k = 0) = 0$. The usual (continuum) sine-Gordon action, that maintains the universal properties of the lattice model, is obtained taking the continuum limit by formally introducing a regularizing lattice constant a and taking the limit $a \to 0$. In the continuous form the inverse of the kernel $\tilde{G}$ can be straightforwardly identified: $\frac{1}{2\pi}(\frac{1}{\lambda}\partial_{x_1}^2 + \lambda\partial_{x_0}^2)\ln(\sqrt{\frac{x_0^2}{\lambda^2} + x_1^2}) = \delta(x_0)\delta(x_1)$. After a rescaling of the axes in the x_0 direction, one obtains the continuum sine-Gordon action

$$S = -\frac{1}{2}\int d^2x[g(\nabla\psi)^2 - \lambda a^{-2}u\cos(\psi)], \tag{42}$$

with $g = \frac{1}{(2\pi)^2}\sqrt{\frac{8}{I_c\tilde{C}_S}}$. This model has a phase transition for $g = g_c$, which can be estimated by a perturbative renormalization group procedure to first order in u [31],

$$g_c = \frac{1}{8\pi} - y_1\lambda u + O(u^2), \tag{43}$$

where $y_1 \simeq 1/8$ [32] and $\mu(u) \simeq -2\ln(u/2)$.

Substituting these values in Eq. (43), one obtains the phase transition condition

$$\sqrt{\frac{8}{I_c\tilde{C}_S}} = \frac{\pi}{2}\left[1 - Ae^{-\frac{1}{2}\sqrt{\frac{I_c\tilde{C}_S}{8}}\kappa(\lambda,\frac{\tilde{C}_S}{C_M},\frac{C_J}{\tilde{C}_S})}\right], \tag{44}$$

where $A = 16\pi y_1\lambda$.

Equation (44) predicts the form of the Kosterlitz-Thouless transition line as a function of $\tilde{C}_S/C_M$, $C_J/\tilde{C}_S$, and the nonuniversal constant A. We have now all the ingredients to discuss the phase diagram of the 1D JJ array.

IV. DISCUSSION

The phase diagram as a function of $\tilde{C}_S/C_M$ for $C_J = 0$ is depicted in Fig. 3(b). As was expected, the stability of the superconducting phase is reduced upon increasing the ratio $\tilde{C}_S/C_M$, in agreement with Ref. [30] where a perturbative analysis around $\tilde{C}_S/C_M = 0$ was performed. For $\tilde{C}_S/C_M \to \infty$ it is well known that [8] the system is always in the insulating phase independently of the value of $\sqrt{I_c\tilde{C}_S/8}$. The expression Eq. (44) interpolates between these two regimes. It predicts a critical value $\tilde{C}_S/C_M \approx 0.375$, above which the system is always in the insulating phase in the $\sqrt{I_c\tilde{C}_S/8} \to \infty$ limit. For this critical ratio, κ vanishes and becomes negative ($\kappa < 0$) for larger values of $\tilde{C}_S/C_M$ which, for sufficiently large $\sqrt{I_c\tilde{C}_S/8}$, renders the system insulating due to the proliferation of phase slips. The nonlinearity of the relation Eq. (44) induces a striking feature in the transition line for A smaller than unity: Superconductivity is predicted to have a reentrant behavior. Here, upon increasing $\sqrt{I_c\tilde{C}_S/8}$, the system passes from insulator to superconductor and again to insulator. This is a rather contraintuitive behavior, as one would naively expect that an increase of the Josephson energy (proportional to I_c) always enhances superconductivity. It would be very interesting to search for experimental signatures of this phenomena. However, we must also note that A is a nonuniversal constant that depends on various factors including the accuracy to which the instanton fugacity is computed, the exact choice of $\Delta\tau$, and the system parameters. At present we cannot rule out that in the range of plausible parameters for realistic materials $A \geqslant 1$ and this nonmonotonicity is not observed. Another potential limitation of our results is that, since Eq. (43) is only valid for small values of u, the obtained transition lines are only qualitatively correct.

As is observed in Fig. 3(c), the presence of a finite C_J, in the limit $C_M \to \infty$, increases the stability of the superconducting phase. Even away from this limit, a finite C_J makes more robust the superconducting phase. In Fig. 3(d) the full phase diagram is depicted as a function of $\sqrt{I_c\tilde{C}_S/8}$, $\tilde{C}_S/C_M$, and $C_J/\tilde{C}_S$ for different values of the nonuniversal constant A. Another striking feature of the phase diagram, besides the reentrant behavior mentioned previously, is the fact that, even for a relatively large ratio $\tilde{C}_S/C_M$, which brings the system deep into the insulating phase, a fairly small value of $C_J/\tilde{C}_S$ can restore superconductivity.

There are also intriguing features related to the interplay between quantum capacitance and charging effects. For instance, in the limit in which the charging energy is only due to a finite mutual capacitance there is no global superconductivity [8] as phase fluctuations in each grain are independent. However, the inclusion of "quantum" capacitance C_δ, induced by quantum size effects not related to Coulomb interactions, changes this picture qualitatively. From Eq. (44) it is clear that a finite C_δ might stabilize superconductivity in a certain range of parameters even if the self-capacitance energy is zero. Therefore, a finite quantum capacitance, which occurs in all systems no matter the nature of the interactions, can help restore long-range order in some cases.

V. APPLICATION TO COLD-ATOM PHYSICS

In this section we investigate the fate of superconductivity in an array in which Coulomb interactions are absent in the limit in which the grain mean level spacing δ becomes comparable to the bulk gap. For that purpose we study the interplay among the Josephson coupling, the quantum capacitance $C_\delta \sim 2/\delta$, and the quasiparticle dissipation C_J. This question can be easily addressed by solving Eq. (44) in the limit of negligible charging energies. This is not of academic interest as it is possible to study experimentally 1D JJ arrays in a cold-atom setting [5] with no Coulomb interactions at all. Moreover, in cold-atom physics many parameters such the tunneling rate, directly related to R_N, and the gap Δ_0 can be controlled with great precision, so an experimental verification seems feasible.

For sufficiently small grains it is broadly expected that superconductivity will not survive unless the grains are strongly coupled so that the effective granularity of the array is heavily suppressed. Likewise, we expect to have global superconductivity for large grains where quantum fluctuations are negligible. Therefore, for a given value of the normal resistance R_N, there must exist a minimum grain size for which phase coherence can occur despite a finite quantum capacitance C_δ. According to Eq. (44), the best-case scenario for the array to stay superconducting corresponds to the limit of infinite fugacity (or $A = 0$), which sets the following lower bound on the grain mean level spacing $\delta_c \approx \frac{\pi I_c}{32} = \frac{\pi^2\Delta_0 R_q}{32 R_N}$ from which it is possible to estimate the minimum grain size. For metallic superconductors the above estimation results in a minimum grain size of order $L \sim 5$ nm though important variations are expected depending on the material. A finite fugacity is expected to weaken the superfluid state and therefore to decrease δ_c. The evolution of δ_c as a function of R_q/R_N for different values of C_J and the nonuniversal parameter A, depicted in Fig. 4, agrees with this prediction.

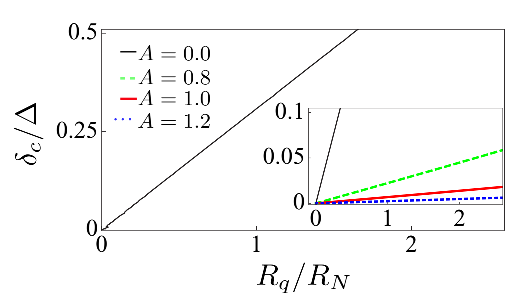

FIG. 4. (Color online) Phase diagram as a function of δ_c/Δ and R_q/R_N plotted for different values of the nonuniversal constant A. Below the curves the system is superconducting and above them it behaves as an insulator.

Note that no reentrant behavior is observed as there is no charging energy related to a mutual capacitance. Finally, we note that our calculation is only valid for $\delta/\Delta_0 \ll 1$ so, from the above expression for δ_c, it is clear that phase coherence is attainable even in the region $R_N \sim R_q$, where the contact among grains is weak and only induces a small smoothing of the spectral density.

VI. SIZE DEPENDENCE OF CLASSICAL AND "QUANTUM" CAPACITANCE

As the grain size decreases, both classical and quantum capacitance play a more important role in the description of the array. Naively, one might think that for sufficiently small grains charging effects are, in general, less important than quantum capacitance effects since the former $E_c \propto 1/L^2$ but the latter is proportional to $\delta \propto 1/L^3$. However, we note the capacitance and the mean level spacing depends on completely different parameters, the former on the dielectric constant of the material and the details of the substrate while the latter on the Fermi energy and the effective electronic mass. As a result, it is plausible that, even if the area scaling holds, both contributions might still be similar for grain sizes $L \sim 10$ nm. This is consistent with the experimental results of [33] for Pb superconducting islands, where it was possible to reproduce the expected classical scaling of the capacitance with the area only for relatively large grains. Indeed, in a Si(111) substrate the charging energy and the mean level spacing of a $L \sim 7$-nm grain with $C \approx 40$ aF can be comparable. Therefore, quantum fluctuations, not related to charging effects, must be taken into account in any quantitative theoretical model of superconducting nanograins.

VII. CONCLUSIONS

We have investigated the robustness of superconductivity in a 1D JJ array of nanograins at zero temperature. We go beyond the standard theoretical treatment of this problem by including quantum fluctuations, not related to Coulomb interactions, induced by finite-size effects, referred to as "quantum capacitance." By using path-integral techniques we have studied the phase diagram of this system including also charging effects and quasiparticle dissipation. We have treated the model analytically by mapping it onto a $1+1$D Coulomb gas and then to a sine-Gordon model which is known to undergo a Kosterlitz-Thouless transition. For sufficiently large grains, long-range order is always robust to small self-capacitance charging effects. However, the combined effect of a vanishing self-capacitance energy and a finite mutual-capacitance energy leads to breaking of phase coherence. We have shown that, even in this limit, superconductivity is stabilized by a quantum capacitance. In systems with vanishing charging effects, relevant in cold-atom experiments, we have shown that long-range order persists up to normal resistances comparable to the quantum one. We have also identified the minimum grain size for global superconductivity to occur in this limit. We have found that the phase diagram resulting from the renormalization group analysis is to some extent sensitive to specific details of the model embodied in a nonuniversal prefactor of the fugacity. As an example, for certain capacitance configurations, small changes in the prefactor of the fugacity can lead to rather contraintuitive results such as a transition from superconductor to insulator by increasing the Josephson coupling.

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