Thermal Hall effect of magnons in magnets with dipolar interaction

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Thermal Hall conductivity of magnons described by a noninteracting boson Hamiltonian is derived by the linear response theory. The thermal Hall conductivity is expressed by the Berry curvature in momentum space, which also has the prevailing form for bosonic systems. This theory covers various spin waves, such as spin waves in antiferromagnets and magnetostatic spin waves. As an example, we calculate the thermal Hall conductivity by the magnetostatic spin wave in yttrium iron garnet and reveal its dependence on a magnetic field and temperature.

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I. INTRODUCTION

A magnon (spin wave) is a low-energy collective excitation in magnetically ordered media [1,2]. Particularly in insulating magnets such as yttrium iron garnet (YIG), magnons can carry spin information for a long distance without dissipation by Joule heating [3]. With these characteristics, magnons attract much attention in the field of spintronics [4,5], and nowadays lead us to a new field of physics called "magnonics" [6–9]. This field aims to control and process information using magnons and a number of magnonic devices have been proposed. For these purposes, a precise manipulation of spin-wave propagation is vital.

Recently, the thermal Hall effect due to a transversal magnon current (magnon Hall effect) has been studied both theoretically [10–12] and experimentally [13,14]. It was theoretically predicted that the transversal current appears by a gradient of magnetic field in a noncoplanar spin structure [10] and by a temperature gradient even in a collinear ferromagnet with a particular lattice structure, such as a kagome lattice [11]. In experiments, the thermal Hall effect is observed in $Lu_2V_2O_7$, a ferromagnetic insulator with pyrochlore structure [13], and in various ferromagnetic insulators [14] with the Dzyaloshinskii-Moriya (DM) interaction, where topological aspects of the Hall effect has been suggested.

Using an analogy between a semiclassical equation of motion for an electron wave packet [15] and that for a magnon wave packet, two of the authors recently identified a correction term to the thermal Hall conductivity, which was missing in the previous theories [11,13]. The term physically results from orbital motions of the magnon wave packet, which can also be derived by the linear response theory with spatial gradients of temperature [16]. With this correction term, they revealed that the thermal Hall current of magnons is indeed generated by the so-called Berry curvature associated with Bloch wave functions for spin-wave bands in the momentum space. All of these preceding theories, however, are applicable only to those magnets where a magnon's current and density operators respect a continuity equation.

Nonzero Berry curvature for spin-wave bands usually results from interactions with "spin-orbit locking," i.e., locks of a relative rotation between spin space and orbital space. This transfers a complex valued character in spin-wave functions (one of the three Pauli matrices) into wave functions in the orbital space, leading to a finite Berry curvature in the momentum space. In magnetic insulators, either the short-ranged DM interaction [11,13,14] or a long-ranged dipole-dipole interaction [12,17,18] plays such a role. Apart from some exceptions [11,13], these spin-orbit-locking interactions break global spin-rotation symmetry completely, so that systems do not have any axis with a continuous spin-rotational symmetry; spin-wave Hamiltonians with spin-orbit lockings usually do not conserve the total number of magnons and a continuity equation for the magnon's density and current no longer holds true. This prevents us from utilizing the previous theories.

In the present paper, we develop a comprehensive theory for the magnon Hall effect in magnets where a magnon number is not necessarily conserved. Following a theory of thermal Hall effect in superconductors [19–21], we begin with a continuity equation for the magnon's *energy* density, to introduce a thermal current associated with magnon transport. Using the linear response theory developed by Smrčka and Středa [16], we derive the thermal transport coefficient. It is shown that the thermal Hall conductivity is directly related to the Berry curvature [17,18] in momentum space (Secs. II and III). Our theory is widely applicable to various types of magnets, including dipolar ferromagnets with magnetostatic spin waves and antiferromagnets with the DM interaction. Armed with this theory, we next calculate the magnetic-field and temperature dependence of the thermal Hall conductivity in ferromagnetic thin films (Sec. IV). We clarify that the thermal Hall conductivity via the magnetostatic forward volume wave [6–9] is mostly independent of the temperature. Throughout this paper, we assume that magnons do not interact with each other.

II. NONINTERACTING BOSON HAMILTONIAN

A spin-wave system considered in this paper is described by a noninteracting boson Hamiltonian. It is given by a quadratic form of a magnon field (creation/annihilation operator):

$$\mathcal{H} \equiv \frac{1}{2}\int d\boldsymbol{r}\Psi^{\dagger}(\boldsymbol{r})H_0\Psi(\boldsymbol{r}), \tag{1}$$

where H_0 is an arbitrary $2N \times 2N$ Hermite matrix, $\Psi^{\dagger}(\boldsymbol{r}) = [\beta_1^{\dagger}(\boldsymbol{r}),\ldots,\beta_N^{\dagger}(\boldsymbol{r}),\beta_1(\boldsymbol{r}),\ldots,\beta_N(\boldsymbol{r})]$, $\beta_i^{\dagger}(\boldsymbol{r})$ and $\beta_i(\boldsymbol{r})$ $(1 \leqslant i \leqslant N)$ are the bosonic (magnon) creation and annihilation

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operators, $[\beta_i(\boldsymbol{r}),\beta_i^\dagger(\boldsymbol{r}')] = \delta_{\boldsymbol{r}\boldsymbol{r}'}\delta_{ij}$, and N is a number of bosons within a unit cell (or a number of spin-wave bands). In the presence of the spin-orbit-locking interactions, the spin-wave Hamiltonian usually contains particle-particle pairing terms, $\beta_i^\dagger\beta_j^\dagger$ or $\beta_i\beta_j$, so that the total number of magnons is not conserved. A fermionic counterpart of this Hamiltonian is known as a Bogoliubov-de Gennes Hamiltonian, which describes superconductors. Under the Fourier transformation,

$$\beta_i(\boldsymbol{r}) = \frac{1}{\sqrt{N_\Lambda}}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\beta_{i,\boldsymbol{k}}, \tag{2}$$

$$\beta_i^\dagger(\boldsymbol{r}) = \frac{1}{\sqrt{N_\Lambda}}\sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\beta_{i,\boldsymbol{k}}^\dagger, \tag{3}$$

the Hamiltonian is written as

$$\mathcal{H} = \frac{1}{2}\sum_{\boldsymbol{k}}(\boldsymbol{\beta}_{\boldsymbol{k}}^\dagger \boldsymbol{\beta}_{-\boldsymbol{k}})H_{\boldsymbol{k}}\begin{pmatrix}\boldsymbol{\beta}_{\boldsymbol{k}}\\ \boldsymbol{\beta}_{-\boldsymbol{k}}^\dagger\end{pmatrix}, \tag{4}$$

where $\boldsymbol{\beta}_{\boldsymbol{k}}^\dagger = (\beta_{1,\boldsymbol{k}}^\dagger,\ldots,\beta_{N,\boldsymbol{k}}^\dagger)$ and $H_{\boldsymbol{k}}$ is a Fourier transformation of H_0. N_Λ is a number of the unit cells. Though a lattice model is adopted for simplicity, it is straightforward to apply our theory to a continuous model. The Hamiltonian is diagonalized by a paraunitary matrix $T_{\boldsymbol{k}}$ [22],

$$\mathcal{H} = \frac{1}{2}\sum_{\boldsymbol{k}}(\boldsymbol{\gamma}_{\boldsymbol{k}}^\dagger\boldsymbol{\gamma}_{-\boldsymbol{k}})\mathcal{E}_{\boldsymbol{k}}\begin{pmatrix}\boldsymbol{\gamma}_{\boldsymbol{k}}\\ \boldsymbol{\gamma}_{-\boldsymbol{k}}^\dagger\end{pmatrix} = \sum_{\boldsymbol{k}}\sum_{n=1}^{N}\varepsilon_{n\boldsymbol{k}}\left(\gamma_{n\boldsymbol{k}}^\dagger\gamma_{n\boldsymbol{k}} + \frac{1}{2}\right), \tag{5}$$

where $\boldsymbol{\gamma}_{\boldsymbol{k}}^\dagger = (\gamma_{1,\boldsymbol{k}}^\dagger,\ldots,\gamma_{N,\boldsymbol{k}}^\dagger)$, $\varepsilon_{n\boldsymbol{k}}$ is the nth band energy, and

$$\begin{pmatrix}\boldsymbol{\gamma}_{\boldsymbol{k}}\\ \boldsymbol{\gamma}_{-\boldsymbol{k}}^\dagger\end{pmatrix} = T_{\boldsymbol{k}}^{-1}\begin{pmatrix}\boldsymbol{\beta}_{\boldsymbol{k}}\\ \boldsymbol{\beta}_{-\boldsymbol{k}}^\dagger\end{pmatrix}, \tag{6}$$

$$\mathcal{E}_{\boldsymbol{k}} = T_{\boldsymbol{k}}^\dagger H_{\boldsymbol{k}} T_{\boldsymbol{k}} = \begin{pmatrix}E_{\boldsymbol{k}} & \\ & E_{-\boldsymbol{k}}\end{pmatrix}, \tag{7}$$

$$E_{\boldsymbol{k}} = \begin{pmatrix}\varepsilon_{1\boldsymbol{k}} & & \\ & \ddots & \\ & & \varepsilon_{N\boldsymbol{k}}\end{pmatrix}. \tag{8}$$

Since the matrix $T_{\boldsymbol{k}}$ diagonalizes the Hamiltonian, it can be regarded as an alignment of the eigenstates. The boson commutation relation for $\boldsymbol{\gamma}_{\boldsymbol{k}}$ requires that $T_{\boldsymbol{k}}$ must satisfy the paraunitary conditions,

$$T_{\boldsymbol{k}}^\dagger\sigma_3 T_{\boldsymbol{k}} = \sigma_3, \tag{9}$$

$$T_{\boldsymbol{k}}\sigma_3 T_{\boldsymbol{k}}^\dagger = \sigma_3, \tag{10}$$

where $\sigma_3 = \begin{pmatrix}1_{N\times N} & 0\\ 0 & -1_{N\times N}\end{pmatrix}$. As seen from Eq. (7), the Hamiltonian consists of two copies of the same eigenstates. In analogy with superconductors for fermions, we refer to the space with indices $n = 1,\ldots,N$ as particle space and that with indices $n = N+1,\ldots,2N$ as hole space.

III. THERMAL TRANSPORT COEFFICIENT

In this section, we calculate thermal transport coefficients and the thermal Hall conductivity. In Sec. III A, we review the linear response theory with a temperature gradient and calculate the thermal current operator from the continuity equation. The operator is separated into two parts: one is independent of an external field and the other is linear in the field. They, respectively, produce thermal transport coefficients which we calculate in Secs. III B 1 and III B 2, and their sum is the total coefficient. Finally, the thermal Hall conductivity is derived from the coefficient in Sec. III B 3. The thermal Hall conductivity is expressed by the Berry curvature in momentum space, whose properties are also discussed in Sec. III B 4.

A. Pseudogravitational potential and thermal current operator

Theoretical treatment of the linear response to the temperature gradient requires some care. In the standard linear response theory to an external field, the field should enter the Hamiltonian as a perturbation. On the other hand, the temperature gradient does not affect the Hamiltonian but affects the Boltzmann factor $e^{-H/(k_BT)}$, where H is a Hamiltonian and k_B is the Boltzmann constant. Luttinger showed that the introduction of a fictitious pseudogravitational potential [16,23] removes this difficulty. A force due to a gradient of the pseudogravitational potential is defined to be proportional to the particle energy, due to the following reason. In the Boltzmann factor $e^{-H/(k_BT)}$, the temperature gradient $T(\mathbf{r}) = T_0[1-\chi(\mathbf{r})]$, where T_0 is a constant and χ is a space-dependent small parameter, can be regarded as a space-dependent prefactor to the Hamiltonian,

$$e^{-H/[k_BT(\mathbf{r})]} \simeq e^{-(1+\chi)H/(k_BT_0)}. \tag{11}$$

Thus, χH is regarded as a perturbation to the Hamiltonian due to the temperature gradient, and its gradient$(\boldsymbol{\nabla}\chi)H$ represents a force which is proportional to the energy. In this way, one can incorporate the temperature gradient into the Hamiltonian as a perturbation by using the pseudogravitational potential. In this sense, the pseudogravitational potential is a dynamical force and the temperature gradient is a statistical force; the former exerts a force to a particle, while the latter does not but affects a particle motion through the distribution function. This is analogous to the situation in which the transport coefficients from the chemical potential in electron systems are derived from the response to the electric field.

Since we are interested in the linear response, we assume the pseudogravitational potential χ to be linear in the position and expand the response in terms of $\boldsymbol{\nabla}\chi$. By using this potential, a perturbing field from the temperature gradient is written as

$$F \equiv \frac{1}{4}\int d\boldsymbol{r}\Psi^\dagger(\boldsymbol{r})(H_0\chi + \chi H_0)\Psi(\boldsymbol{r}), \tag{12}$$

and the total Hamiltonian H_{T} is

$$H_{\mathrm{T}} = \mathcal{H} + F. \tag{13}$$

Thermal transport coefficients are calculated as a linear response to the gradient of the pseudogravitational potential [23],

$$\langle J_\mu^{\mathrm{Q}}\rangle = L_{\mu\nu}\left(T\nabla_\nu\frac{1}{T} - \nabla_\nu\chi\right), \tag{14}$$

where $L_{\mu\nu}$ is the thermal transport coefficient and $\mu,\nu = x,y$. $\langle J_\mu^{\mathrm{Q}}\rangle$ is a macroscopic thermal current [16] where J_μ^{Q} is defined as $J^{\mathrm{Q}}_\mu \equiv \frac{1}{V}\int d\boldsymbol{r}\, j^{\mathrm{Q}}_\mu(\boldsymbol{r})$, with $j^{\mathrm{Q}}_\mu(\boldsymbol{r})$ being a thermal current density operator, V being a volume of the system, and $\langle\cdot\rangle$ denoting a thermal and quantum-mechanical average. The thermal Hall conductivity $\kappa_{\mu\nu}$ is expressed as

$$\kappa_{\mu\nu} = \frac{L_{\mu\nu}}{T}. \tag{15}$$

Hereafter, we take the system volume V and $\hbar$ to be 1 and restore them as necessary.

To calculate the thermal Hall conductivity, let us first calculate the thermal current density (operator) in the presence of the pseudogravitational field $\nabla\chi$. Since χ is small, the total Hamiltonian is rewritten as

$$H_{\mathrm{T}} = \frac{1}{2}\int d\boldsymbol{r}\left(1+\frac{\chi}{2}\right)\Psi^\dagger(\boldsymbol{r})H_0\left(1+\frac{\chi}{2}\right)\Psi(\boldsymbol{r}). \tag{16}$$

From the conservation of the energy density, the continuity equation is

$$\dot{h}_{\mathrm{T}} + \nabla\cdot\boldsymbol{j}^{\mathrm{Q}}(\boldsymbol{r}) = 0, \tag{17}$$

where $h_{\mathrm{T}} = \frac{1}{2}(1+\frac{\chi}{2})\Psi^\dagger(\boldsymbol{r})H_0(1+\frac{\chi}{2})\Psi(\boldsymbol{r})$ is an energy density. From Eq. (17), the thermal current operator up to the linear order in the external field $\nabla\chi$ is derived as the following (see Appendix A for details):

$$j^{\mathrm{Q}}_\mu(\boldsymbol{r}) = j^{\mathrm{Q}}_{0,\mu}(\boldsymbol{r}) + j^{\mathrm{Q}}_{1,\mu}(\boldsymbol{r}), \tag{18}$$

where

$$j^{\mathrm{Q}}_{0,\mu}(\boldsymbol{r}) = \frac{1}{4}\Psi^\dagger(\boldsymbol{r})(V_\mu\sigma_3H_0 + H_0\sigma_3V_\mu)\Psi(\boldsymbol{r}), \tag{19}$$

$$\begin{aligned} j^{\mathrm{Q}}_{1,\mu}(\boldsymbol{r}) = &-\frac{i}{8}\nabla_\nu\chi\,\Psi^\dagger(\boldsymbol{r})(V_\mu\sigma_3V_\nu - V_\nu\sigma_3V_\mu)\Psi(\boldsymbol{r}) \\ &+\frac{1}{8}\nabla_\nu\chi[\Psi^\dagger(\boldsymbol{r})(x_\nu V_\mu\sigma_3 + 3V_\mu\sigma_3x_\nu)H_0\Psi(\boldsymbol{r}) \\ &+\Psi^\dagger(\boldsymbol{r})H_0(3x_\nu\sigma_3V_\mu + \sigma_3V_\mu x_\nu)\Psi(\boldsymbol{r})], \end{aligned} \tag{20}$$

x_μ is a position operator, and $V_\mu = \frac{1}{i\hbar}[x_\mu, H_0]$ is a velocity operator. Thermal current density consists of two parts: $j^{\mathrm{Q}}_{0,\mu}(\boldsymbol{r})$ is independent of $\nabla\chi$ and $j^{\mathrm{Q}}_{1,\mu}(\boldsymbol{r})$ is linear in $\nabla\chi$. They both contribute to the thermal transport coefficients.

B. Calculation of thermal transport coefficients

From Eq. (14), the thermal transport coefficient is obtained from a thermal and quantum-mechanical average of the thermal current operator:

$$\langle J^{\mathrm{Q}}_\mu\rangle = \langle J^{\mathrm{Q}}_{0,\mu}\rangle + \langle J^{\mathrm{Q}}_{1,\mu}\rangle, \tag{21}$$

$$\langle J^{\mathrm{Q}}_{0,\mu}\rangle \equiv -S_{\mu\nu}\nabla_\nu\chi, \tag{22}$$

$$\langle J^{\mathrm{Q}}_{1,\mu}\rangle \equiv -M_{\mu\nu}\nabla_\nu\chi. \tag{23}$$

$J^{Q}_{0/1,\mu}$ is given by the spatial integral of $j^{Q}_{0/1,\mu}$ over an entire system [see Eqs. (B1) and (B2)]. The total thermal transport coefficient is the sum of these two contributions: $L_{\mu\nu} = S_{\mu\nu} + M_{\mu\nu}$. In the following, we derive an expression for the thermal transport coefficients $S_{\mu\nu}$ and $M_{\mu\nu}$ in terms of the spin-wave dispersion $\varepsilon_{n,\boldsymbol{k}}$ and the paraunitary matrix $T_{\boldsymbol{k}}$.

1. Calculation of $S_{\mu\nu}$

The first term $S_{\mu\nu}$ represents the usual Kubo-Greenwood contribution to $L_{\mu\nu}$. Because $J^{Q}_{0,\mu}$ is independent of $\nabla\chi$, the linear response coefficient $S_{\mu\nu}$ is calculated from the deviation of the density matrix out of equilibrium due to $\nabla\chi$. This contribution reads [24,25]

$$S_{\mu\nu} = -\frac{\delta\langle J^{\mathrm{Q}}_{0\mu}\rangle}{\delta\partial_\nu\chi} = -\lim_{\Omega\to 0}\frac{P^R_{\mu\nu}(\Omega) - P^R_{\mu\nu}(0)}{i\Omega}. \tag{24}$$

Now that $\dot{F} = \frac{i}{\hbar}[\mathcal{H}, F] = J^{Q}_{0,\mu}\nabla_\mu\chi$, $P^R_{\mu\nu}(\Omega)$ is a retarded current-current correlation function. It is also given by the imaginary time-ordered correlation function as

$$P^R_{\mu\nu}(\Omega) = P_{\mu\nu}(i\Omega\to\Omega + i0), \tag{25}$$

with

$$P_{\mu\nu}(i\Omega) = -\int_0^\beta d\tau\, e^{i\Omega\tau}\langle T_\tau J^{\mathrm{Q}}_{0,\mu}(\tau)J^{\mathrm{Q}}_{0,\nu}(0)\rangle. \tag{26}$$

Here, $\beta = 1/k_BT$, T_τ is a time-ordering operator, and $J^{\mathrm{Q}}_{0,\mu}(\tau)$ is an interaction representation of the thermal current operator: $J^{\mathrm{Q}}_{0,\mu}(\tau) = e^{\tau\mathcal{H}}J^{\mathrm{Q}}_{0,\mu}e^{-\tau\mathcal{H}}$. From these equations, the Kubo contribution to the thermal transport coefficient is derived (see Appendix C for details) as

$$\begin{aligned} S_{\mu\nu} = -\frac{i}{8}\sum_{n,m=1}^{N}\sum_{\boldsymbol{k}}\Bigg[&\frac{g(\varepsilon_{n\boldsymbol{k}}) - g(\varepsilon_{m\boldsymbol{k}})}{(\varepsilon_{n\boldsymbol{k}} - \varepsilon_{m\boldsymbol{k}})^2}(\varepsilon_{n\boldsymbol{k}} + \varepsilon_{m\boldsymbol{k}})^2(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{nm}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{mn} \\ &-\frac{g(\varepsilon_{n\boldsymbol{k}}) - g(-\varepsilon_{m,-\boldsymbol{k}})}{(\varepsilon_{n\boldsymbol{k}} + \varepsilon_{m,-\boldsymbol{k}})^2}(\varepsilon_{n\boldsymbol{k}} - \varepsilon_{m,-\boldsymbol{k}})^2(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n,m+N}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m+N,n} \\ &-\frac{g(-\varepsilon_{n,-\boldsymbol{k}}) - g(\varepsilon_{m\boldsymbol{k}})}{(\varepsilon_{n,-\boldsymbol{k}} + \varepsilon_{m\boldsymbol{k}})^2}(\varepsilon_{n,-\boldsymbol{k}} - \varepsilon_{m\boldsymbol{k}})^2(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n+N,m}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m,n+N} \\ &+\frac{g(-\varepsilon_{n,-\boldsymbol{k}}) - g(-\varepsilon_{m,-\boldsymbol{k}})}{(\varepsilon_{n,-\boldsymbol{k}} - \varepsilon_{m,-\boldsymbol{k}})^2}(\varepsilon_{n,-\boldsymbol{k}} + \varepsilon_{m,-\boldsymbol{k}})^2(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n+N,m+N}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m+N,n+N}\Bigg], \end{aligned} \tag{27}$$

with $V_{\boldsymbol{k},\mu} \equiv \frac{1}{\hbar}\frac{\partial H_{\boldsymbol{k}}}{\partial k_\mu}$. $g(\varepsilon)$ is the Bose distribution function $g(\varepsilon) = 1/[\exp(\varepsilon/k_BT) - 1]$.

2. Calculation of $M_{\mu\nu}$

The second term $M_{\mu\nu}$ is associated with orbital motions of magnons [12]. Generally speaking, in a system with the time-reversal symmetry breaking, there is a circulation of heat current, leading to an additional contribution to the thermal transport coefficient which is called an energy magnetization term [20,26,27]. $M_{\mu\nu}$ is calculated from the expectation value of Eq. (20) with respect to the unperturbed ($\nabla\chi = 0$) distribution function,

$$\begin{aligned} M_{\mu\nu} = -\frac{\delta\langle J^{\mathrm{Q}}_{1\mu}\rangle}{\delta\partial_\nu\chi} &= \frac{i}{8}\sum_{n,m=1}^{N}\sum_{\boldsymbol{k}}[g(\varepsilon_{n\boldsymbol{k}})(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{nm}(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{mn} - g(\varepsilon_{n\boldsymbol{k}})(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n,m+N}(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m+N,n} \\ &- g(-\varepsilon_{n,-\boldsymbol{k}})(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n+N,m}(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m,n+N} + g(-\varepsilon_{n,-\boldsymbol{k}})(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n+N,m+N}(T_{\boldsymbol{k}}^\dagger V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m+N,n+N}] \\ &- (\mu\leftrightarrow\nu) - \frac{1}{2}\sum_{n=1}^{N}\sum_{\boldsymbol{k}}\{[T_{\boldsymbol{k}}^\dagger(x_\nu V_{\boldsymbol{k},\mu} + V_{\boldsymbol{k},\mu}x_\nu)T_{\boldsymbol{k}}]_{nn}\varepsilon_{n\boldsymbol{k}}g(\varepsilon_{n\boldsymbol{k}}) + [T_{\boldsymbol{k}}^\dagger(x_\nu V_{\boldsymbol{k},\mu} + V_{\boldsymbol{k},\mu}x_\nu)T_{\boldsymbol{k}}]_{n+N,n+N}\varepsilon_{n,-\boldsymbol{k}}g(-\varepsilon_{n,-\boldsymbol{k}})\}. \end{aligned} \tag{28}$$

$M_{\mu\nu}$ is related to M^z_{Q} in Ref. [26] as $M_{xy} = 2M^z_{\mathrm{Q}}$.

3. Expression of thermal transport coefficients in terms of Bloch eigenstates

In the following, we rewrite Eqs. (27) and (28) in terms of the Bloch eigenstates $T_{\boldsymbol{k}}$. In terms of Eq. (28) and Eq. (D13) derived in Appendix D, $M_{\mu\nu}$ can be explicitly calculated as

$$\begin{aligned} M_{xy} &= i\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\,\mathrm{Tr}\left[\delta(\tilde{\eta}-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^\dagger}{\partial k_x}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_y}\right]\cdot\int_0^{\tilde{\eta}}\eta g(\eta)d\eta \\ &\quad -\frac{i}{8}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\,\mathrm{Tr}\left[\delta(\tilde{\eta}-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^\dagger}{\partial k_x}(3\sigma_3\tilde{\eta}^2 - 2\tilde{\eta}H_{\boldsymbol{k}} - H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}})\frac{\partial T_{\boldsymbol{k}}}{\partial k_y}\right]g(\tilde{\eta}) - (x\leftrightarrow y). \end{aligned} \tag{29}$$

Similarly, S_{xy} is rewritten as

$$S_{xy} = -\frac{i}{8}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\eta g(\eta)\mathrm{Tr}\left[\delta(\eta-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^\dagger}{\partial k_x}(\eta + H_{\boldsymbol{k}}\sigma_3)^2\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_y} - (x\leftrightarrow y)\right]. \tag{30}$$

These lead to L_{xy} as

$$L_{xy} = S_{xy} + M_{xy} = -\frac{i}{2}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\,\mathrm{Tr}\left[\delta(\tilde{\eta}-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^\dagger}{\partial k_x}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_y} - (x\leftrightarrow y)\right]\int_0^{\tilde{\eta}}\eta^2\frac{dg(\eta)}{d\eta}d\eta. \tag{31}$$

Finally, the thermal Hall conductivity in a clean limit is expressed as follows (see Appendix D for details):

$$\kappa_{xy} = -\frac{k_B^2 T}{\hbar V}\sum_{\boldsymbol{k}}\sum_{n=1}^{N}\left\{c_2[g(\varepsilon_{n\boldsymbol{k}})] - \frac{\pi^2}{3}\right\}\Omega_{n\boldsymbol{k}}. \tag{32}$$

Here, $c_2(x)$ is defined as

$$c_2(x) \equiv \int_0^x dt\left(\ln\frac{1+t}{t}\right)^2 = (1+x)\left(\ln\frac{1+x}{x}\right)^2 - (\ln x)^2 - 2\mathrm{Li}_2(-x), \tag{33}$$

where $\mathrm{Li}_2(x)$ is a polylogarithm function $\mathrm{Li}_n(x)$ for $n = 2$. $\Omega_{n\boldsymbol{k}}$ is the Berry curvature in momentum space for a noninteracting boson Hamiltonian, which is defined as [17]

$$\Omega_{n\boldsymbol{k}} \equiv i\epsilon_{\mu\nu}\left[\sigma_3\frac{\partial T_{\boldsymbol{k}}^\dagger}{\partial k_\mu}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]_{nn}\quad (n = 1,2,\ldots,2N). \tag{34}$$

Equation (32) is the central result of this paper. The thermal Hall conductivity is given by the sum of the conductivities from each of the Bloch eigenstates. Each contribution is a product of the Berry curvature $\Omega_{n\boldsymbol{k}}$ and $c_2[g(\varepsilon_{n\boldsymbol{k}})] - \frac{\pi^2}{3}$, which is a function of the Bose distribution function $g(\varepsilon_{n\boldsymbol{k}})$. This function $c_2[g(\varepsilon)] - \frac{\pi^2}{3}$ is a monotonically decreasing function of ε. It has maximum value 0 at $\varepsilon = 0$ and minimum value $-\frac{\pi^2}{3}$ at $\varepsilon \to \infty$.

4. Properties of the Berry curvature and thermal Hall conductivity

Equation (32) is written only with the Berry curvature in particle space: $\Omega_{n\boldsymbol{k}}(1 \leqslant n \leqslant N)$. To obtain Eq. (32), we used a formula

$$\Omega_{n\boldsymbol{k}} = -\Omega_{n+N,-\boldsymbol{k}}, \tag{35}$$

which relates the Berry curvature in the hole space ($N+1 \leqslant n \leqslant 2N$) with that of the particle space. It is natural to have such a relation between the particle space and hole space because the hole space is a copy of the particle space. The formula comes from the following particle-hole symmetry associated with the boson commutation relation:

$$H_{\boldsymbol{k}} = \sigma_1 (H_{-\boldsymbol{k}})^{\mathrm{t}} \sigma_1, \tag{36}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1_{N\times N} \\ 1_{N\times N} & 0 \end{pmatrix}. \tag{37}$$

The derivation of Eq. (35) is shown in Appendix F. In the absence of the particle-particle pairing terms, namely the off-block-diagonal terms in Hamiltonian $H_{\boldsymbol{k}}$, one can retrieve from Eq. (32) the previous results [12].

When the momentum integral in Eq. (32) is taken over the first Brillouin zone (a *closed* surface), which is the case in a two-dimensional system with spatial periodicity, $C_n \equiv \frac{2\pi}{V}\sum_{\boldsymbol{k}\in\mathrm{BZ}} \Omega_{n\boldsymbol{k}}$ is quantized to be an integer m, which is called the first Chern integer [17,28,29]. The integer determines a number of chiral edge modes for spin-wave propagations, whose dispersion crosses a band gap for the spin-wave band [17,18,30–32]. Using the quantization, one can further prove that a sum of the Chern integer over the particle bands ($1 \leqslant n \leqslant N$) reduces to zero [17]. In such a case, we can drop $\pi^2/3$ within the curly brackets in the right-hand side of Eq. (32). However, when the momentum integral in Eq. (32) is *not* taken over the closed surface (as we show an example below), the sum is not required to be quantized, and thus one needs to keep $\pi^2/3$ in the right-hand side of Eq. (32).

IV. APPLICATION TO THE MAGNETOSTATIC SPIN WAVES

In this section, we apply the above theory to the magnetostatic spin wave. When a wavelength of the spin-wave excitation gets into the micrometer length scale, the short-range exchange interaction becomes relatively less dominant. Instead, the spin-wave propagation is mainly driven by the long-range dipole-dipole interaction (dipolar regime). Playing the role of the spin-orbit locking, the dipolar interaction brings about a finite Berry curvature and thermal Hall effect in magnets.

In the following, we consider a two-dimensional (2D) ferromagnetic film (e.g., YIG film) in the dipolar regime, where the exchange interaction is negligible. Take the 2D plane to be the xy plane. The saturation magnetization $\boldsymbol{M}_{\mathrm{s}}$ and internal static magnetic field $\boldsymbol{H}_0$ are parallel to the z direction; $\boldsymbol{H}_0 = \boldsymbol{H}_{\mathrm{ex}} - \boldsymbol{M}_{\mathrm{s}}$, where $\boldsymbol{H}_{\mathrm{ex}}$ is an external magnetic field. The spin-wave mode with this geometry is called the magnetostatic forward volume wave (MSFVW) [33]. We assume the spin-wave mode to be a plane wave and write the magnetization in the xy direction as $\boldsymbol{m}(\boldsymbol{r},t) = \binom{m_x(z)}{m_y(z)} \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}_\parallel - \omega t)]$, where $\boldsymbol{r}_\parallel = (x,y)$, $\boldsymbol{k}$ is a wave vector and ω is a frequency of the spin wave. The magnetization obeys the following equation of motion [34] with the SI units:

$$\omega_H \boldsymbol{m}(z) - \omega_M \int_{-L/2}^{L/2} dz' \hat{G}(z,z')\boldsymbol{m}(z') = \omega\sigma_z \boldsymbol{m}(z),$$
$$\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{m}(z) \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} m_x(z) - i m_y(z) \\ m_x(z) + i m_y(z) \end{pmatrix}. \tag{38}$$

L is a thickness of the film, $\omega_H \equiv \gamma H_0$, $\omega_M \equiv \gamma M_{\mathrm{s}}$, and γ is the gyromagnetic ratio. $\hat{G}(z,z')$ is the 2×2 complex-valued matrix of the Green's function, which comes from the magnetic dipole-dipole interaction:

$$\hat{G}(z,z') = -\frac{1}{2} G_P(z,z') \begin{pmatrix} 1 & e^{-2i\varphi} \\ e^{2i\varphi} & 1 \end{pmatrix}, \tag{39}$$

$$G_P(z,z') = \frac{k}{2}\exp(-k|z-z'|). \tag{40}$$

φ specifies a direction of the wave vector $\boldsymbol{k}$ as $\boldsymbol{k} \equiv k(\cos\varphi, \sin\varphi)$. This integral equation includes the Landau-Lifshitz equation $d\boldsymbol{M}/dt = -\gamma(\boldsymbol{M}\times\boldsymbol{H})$ (here we do not take into account the damping term), Maxwell equation in the magnetostatic limit $\boldsymbol{\nabla}\times\boldsymbol{H} = 0$, $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, and the usual boundary conditions for $\boldsymbol{H}$ and $\boldsymbol{B}$. By assuming the form of the magnetic potential of the inside and outside of the thin film in the conventional way [33,35], Eq. (38) gives a band structure $\omega_{n\boldsymbol{k}}$, where n denotes the band index.

Equation (38) is nothing but a generalized eigenvalue problem,

$$\int dz' H^{\boldsymbol{k}}_{z,z'} \boldsymbol{m}_{n\boldsymbol{k}} = \sigma_3 \omega_{n\boldsymbol{k}} \boldsymbol{m}_{n\boldsymbol{k}}, \tag{41}$$

$$H^{\boldsymbol{k}}_{z,z'} \equiv \omega_H \delta(z-z') - \omega_M \hat{G}(z,z'), \tag{42}$$

where $\boldsymbol{m}_{n\boldsymbol{k}}$ is the nth eigensolution with its eigenfrequency being $\omega_{n\boldsymbol{k}}$. In the present case, $H^{\boldsymbol{k}}_{z,z'} = \sigma_1 (H^{-\boldsymbol{k}}_{z,z'})^* \sigma_1$ holds, which is equivalent to Eq. (36). This leads to $\int dz' H^{\boldsymbol{k}}_{z,z'}(\sigma_1 \boldsymbol{m}^*_{n,-\boldsymbol{k}}) = -\sigma_3 \omega_{n,-\boldsymbol{k}} (\sigma_1 \boldsymbol{m}^*_{n,-\boldsymbol{k}})$. Including these counterparts in the hole space, Eq. (41) more generally takes the following form:

$$\int dz' H^{\boldsymbol{k}}_{z,z'} T_{z'} = \sigma_3 T_z \tilde{E}, \tag{43}$$

$$T_z \equiv \left(\boldsymbol{m}_{1,\boldsymbol{k}} \cdots \boldsymbol{m}_{N,\boldsymbol{k}} \quad \sigma_1 \boldsymbol{m}^*_{1,-\boldsymbol{k}} \cdots \sigma_1 \boldsymbol{m}^*_{N,-\boldsymbol{k}} \right), \tag{44}$$

$$\tilde{E} \equiv \begin{pmatrix} E_{\boldsymbol{k}} & \\ & -E_{-\boldsymbol{k}} \end{pmatrix}, \tag{45}$$

$$E_{\boldsymbol{k}} \equiv \begin{pmatrix} \omega_{1,\boldsymbol{k}} & & \\ & \ddots & \\ & & \omega_{N,\boldsymbol{k}} \end{pmatrix}, \tag{46}$$

with a normalization condition

$$\int dz T^\dagger_z \sigma_3 T_z = \sigma_3.$$

Here the number of eigenmodes should be bounded, $N < \infty$, by the exchange interaction length l_{ex}. Namely, for a larger mode index n, the corresponding wave function $m_{n\boldsymbol{k}}$ has many nodes along the z direction, where the short-range exchange interaction becomes more relevant than the dipole-dipole interaction in the Landau-Lifshitz equation. On the other hand, Eq. (38) ignores the short-range interaction from the outset, so that the usage is limited for those eigenmodes, which have nodes along the z direction much less than L/l_{ex}.

Now we calculate the thermal Hall conductivity of the ferromagnetic film. The Berry curvature for the MSFVW mode is given as [12]

$$\Omega_{nk} = \frac{1}{2\omega_H}\frac{1}{k}\frac{\partial\omega_{nk}}{\partial k}\left(1-\frac{\omega_H^2}{\omega_{nk}^2}\right)\cdot(\sigma_3)_{nn}, \quad (47)$$

whose behavior as a function of k is shown in Ref. [12]. With Eqs. (32) and (47), the thermal Hall conductivity κ_{xy} of the ferromagnetic film is derived as follows:

$$\kappa_{xy} = -\frac{\pi k_B^2 T}{(2\pi)^2\hbar\omega_H}\sum_{n=1}^{N}\int_{\omega_H}^{\sqrt{\omega_H(\omega_M+\omega_H)}} d\omega_{nk} \times\left\{c_2[g(\hbar\omega_{nk})]-\frac{\pi^2}{3}\right\}\left(1-\frac{\omega_H^2}{\omega_{nk}^2}\right). \quad (48)$$

Here we note that all energy bands of the MSFVW mode begin from ω_H at $k=0$ and approach $\sqrt{\omega_H(\omega_M+\omega_H)}$ at $k\to\infty$ [33]. To show the results in a universal way, we introduce the following dimensionless parameters: $\tilde{\kappa}_{xy}\equiv\kappa_{xy}/(\frac{k_B\omega_M N}{4\pi})$, $r\equiv H_0/M_{\mathrm{s}}$ denoting a ratio between the internal magnetic field H_0 and saturation magnetization M_{s}, and $u\equiv k_BT/\hbar\omega_M$ denoting a ratio between the temperature and the saturation magnetization. N is an upper bound of n. By using these parameters, Eq. (48) is rewritten as

$$\tilde{\kappa}_{xy} = -\frac{u}{r}\int_r^{\sqrt{r(1+r)}} dx\left[c_2\left(\frac{1}{e^{x/u}-1}\right)-\frac{\pi^2}{3}\right]\left(1-\frac{r^2}{x^2}\right). \quad (49)$$

$\tilde{\kappa}_{xy}$ converges to zero in the zero-temperature limit. However, in most realistic cases, $u\gg r$ holds true (e.g., when $T=300$ K and $H_0=1$ T in YIG film, $u/r=k_BT/\hbar\omega_H=1.5\times10^5$), so that Eq. (49) is approximated to

$$\tilde{\kappa}_{xy}\simeq\frac{1}{2}-\frac{r}{2}\ln\left(1+\frac{1}{r}\right). \quad (50)$$

This shows that the thermal Hall conductivity via the MSFVW seldom depends on the temperature. Figure 1 shows a plot of Eq. (50). It is easily shown that $\tilde{\kappa}_{xy}\to 1/2$ when $r\to 0$ and $\tilde{\kappa}_{xy}\to 0$ when $r\to\infty$.

The magnitude of the thermal Hall conductivity via the MSFVW is determined not only by the ratios among saturation magnetization, internal static field, and temperature, but also by the ratio between the exchange length l_{ex} and thickness of the film L. Namely, when the wavelength in the direction normal to the film becomes shorter than exchange length l_{ex}, spin-wave bands are mainly determined by the short-range exchange interaction, where no finite Berry curvature is expected. Since the nth spin-wave band obtained from Eq. (38) has n nodes along the z direction [34], the upper bound of n should be roughly estimated as $N=L/l_{\mathrm{ex}}$, where $l_{\mathrm{ex}}=1.72\times10^{-8}$ m for YIG film. A typical value for a film much thicker than l_{ex} is $\kappa_{xy}/L=5.9\times10^{-8}$ W/Km, where parameters are set as $\gamma=2.8$ MHz/Oe, $M_{\mathrm{s}}=1750$ G, $T=300$ K, $H_{\mathrm{ex}}=3000$ Oe. It is almost temperature independent above $\hbar\omega_H/k_B\sim 27$ mK.

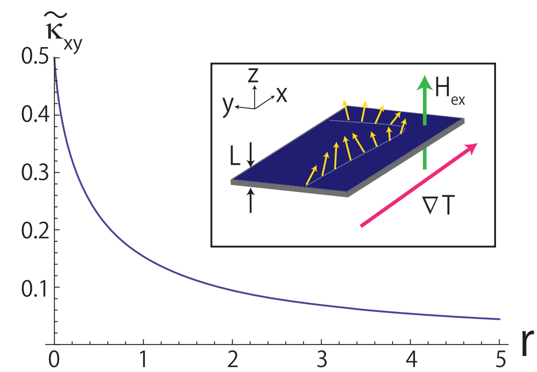

FIG. 1. (Color online) Dependence of the thermal Hall conductivity on a magnetic field. $\tilde{\kappa}_{xy}\equiv\kappa_{xy}/(\frac{k_B\omega_M N}{4\pi})$ and $r\equiv H_0/M_{\mathrm{s}}$ denote a dimensionless thermal Hall conductivity and magnetic field, respectively. $r=0$ ($H_0=0$) corresponds to the field at the saturation field; $H_{\mathrm{ex}}=M_s$. The inset shows a geometry of the thermal Hall effect via the MSFVW mode.

At a glance, one may think it strange that the thermal Hall conductivity is almost independent of the temperature because more and more magnons are excited as the temperature increases. This arise from the energy scale of magnons, as we see below. First, in Eq. (48), we assumed that the spectrum of the spin wave is bounded at the maximum value $\sqrt{\omega_H(\omega_M+\omega_H)}$. This value is in the range of gigahertz, which is much less than 1 K. This means that there are a number of magnons and $g(\varepsilon)\to\infty$ even at $T=1$ K. Thus, even if the temperature increases in an energy scale much higher than the characteristic energy scale of magnons, the resulting increase of the magnon number is much smaller than the total magnon number already excited, and barely affects the thermal Hall conductivity. Furthermore, when we take into account the exchange interaction, magnons with higher energy do not contribute to the thermal Hall conductivity. This is because the exchange coupling does not give rise to a nonzero Berry curvature.

V. CONCLUSIONS

We derived the thermal current operator and thermal Hall conductivity for magnons described by the general noninteracting boson Hamiltonian. κ_{xy} is expressed by the Berry curvature in the momentum space. We applied our theory to the magnetostatic spin wave in YIG and clarified the dependence of the thermal Hall conductivity on temperature and magnetic field. The present theory also can be widely applied to magnons described by the noninteracting boson Hamiltonian. This includes magnons in ferromagnets with DM interactions or dipolar interactions, as well as magnons in ferrimagnets and antiferromagnets with noncollinear or noncoplanar spin structures. It is also applicable to other bosonic systems such as phonons or photons, as long as their Hamiltonian is given by a noninteracting boson Hamiltonian.

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APPENDIX A: DERIVATION OF THERMAL CURRENT OPERATOR j^{Q}

Here we derive the thermal current operator in Eqs. (18)–(20) from the continuity equation

$$\dot{h}_{\mathrm{T}} + \nabla \cdot \boldsymbol{j}^{\mathrm{Q}}(\boldsymbol{r}) = 0. \tag{A1}$$

We consider a lattice model for simplicity. The Hamiltonian is expressed as Eq. (1),

$$\mathcal{H} = \frac{1}{2}\sum_{\boldsymbol{r}} \Psi^\dagger(\boldsymbol{r}) H_0 \Psi(\boldsymbol{r}), \tag{A2}$$

where

$$H_0 = \sum_{\boldsymbol{\delta}} H_{\boldsymbol{\delta}} e^{i\hat{\boldsymbol{p}}\cdot\boldsymbol{\delta}}, \tag{A3}$$

$$H_{\boldsymbol{\delta}} = \begin{pmatrix} h_{\boldsymbol{\delta}} & \Delta_{\boldsymbol{\delta}} \\ \Delta^*_{\boldsymbol{\delta}} & h^t_{-\boldsymbol{\delta}} \end{pmatrix}, \tag{A4}$$

$$\Psi_i(\boldsymbol{r}) = \begin{cases} \beta_i(\boldsymbol{r}) & (i = 1,\dots,N) \\ \beta^\dagger_{i-N}(\boldsymbol{r}) & (i = N+1,\dots,2N), \end{cases} \tag{A5}$$

$$[\beta_i(\boldsymbol{r}),\beta^\dagger_j(\boldsymbol{r}')] = \delta_{ij}\delta_{\boldsymbol{r},\boldsymbol{r}'}, \tag{A6}$$

and N is the number of degrees of freedom within the unit cell (e.g., sublattice and orbital degrees of freedom). Here, $h_{\boldsymbol{\delta}}$ and $\Delta_{\boldsymbol{\delta}}$ represent hopping terms between sites belonging to unit cells apart by $\boldsymbol{\delta}$ with a translation operator:

$$e^{i\hat{\boldsymbol{p}}\cdot\boldsymbol{\delta}}\beta_i(\boldsymbol{r}) = \beta_i(\boldsymbol{r}+\boldsymbol{\delta}). \tag{A7}$$

H_0 is a Hermitian operator so that $H_{\boldsymbol{\delta}}$ satisfies

$$H^\dagger_{\boldsymbol{\delta}} = H_{-\boldsymbol{\delta}}. \tag{A8}$$

Thanks to the bosonic commutation relations, $H_{\boldsymbol{\delta}}$ satisfies

$$\sigma_1 H_{\boldsymbol{\delta}} \sigma_1 = H^t_{-\boldsymbol{\delta}}, \tag{A9}$$

where σ_1 is defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1_{N\times N} \\ 1_{N\times N} & 0 \end{pmatrix}. \tag{A10}$$

Note that Eq. (A5) satisfies the following commutation relations:

$$[\Psi_i(\boldsymbol{r}),\Psi^\dagger_j(\boldsymbol{r}')] = (\sigma_3)_{ij}\delta_{\boldsymbol{r},\boldsymbol{r}'}, \tag{A11}$$

$$[\Psi^\dagger_i(\boldsymbol{r}),\Psi^\dagger_j(\boldsymbol{r}')] = -i(\sigma_2)_{ij}\delta_{\boldsymbol{r},\boldsymbol{r}'}, \tag{A12}$$

$$[\Psi_i(\boldsymbol{r}),\Psi_j(\boldsymbol{r}')] = i(\sigma_2)_{ij}\delta_{\boldsymbol{r},\boldsymbol{r}'}, \tag{A13}$$

with

$$\sigma_2 = \begin{pmatrix} 0 & -i1_{N\times N} \\ i1_{N\times N} & 0 \end{pmatrix}, \tag{A14}$$

$$\sigma_3 = \begin{pmatrix} 1_{N\times N} & 0 \\ 0 & -1_{N\times N} \end{pmatrix}. \tag{A15}$$

Under a pseudogravitational potential χ, the total Hamiltonian is written as

$$H_{\mathrm{T}} = \sum_{\boldsymbol{r}} h_{\mathrm{T}}(\boldsymbol{r}), \tag{A16}$$

where $h_{\mathrm{T}}(\boldsymbol{r})$ is a Hamiltonian density operator

$$h_{\mathrm{T}}(\boldsymbol{r}) \equiv \tfrac{1}{2}\tilde{\Psi}^\dagger(\boldsymbol{r}) H_0 \tilde{\Psi}(\boldsymbol{r}), \tag{A17}$$

with $\tilde{\Psi}(\boldsymbol{r}) \equiv (1+\frac{\chi}{2})\Psi(\boldsymbol{r})$. Now the continuity equation leads to

$$\begin{aligned}
\dot{h}_{\mathrm{T}}(\boldsymbol{r}) &= \frac{i}{\hbar}[H_{\mathrm{T}},h_{\mathrm{T}}(\boldsymbol{r})] = \frac{i}{4\hbar}\{[H_{\mathrm{T}},\tilde{\Psi}^\dagger(\boldsymbol{r})]H_0\tilde{\Psi}(\boldsymbol{r}) + \tilde{\Psi}^\dagger(\boldsymbol{r})[H_{\mathrm{T}},H_0\tilde{\Psi}(\boldsymbol{r})] + \mathrm{H.c.}\}\\
&= \frac{i}{8\hbar}\Bigg\{\sum_{\boldsymbol{\delta},\boldsymbol{\delta}'} \tilde{\Psi}^\dagger(\boldsymbol{r}+\boldsymbol{\delta})\sigma_1 H^t_{\boldsymbol{\delta}}(-i\sigma_2)\left[1+\frac{\chi(\boldsymbol{r})}{2}\right]^2 H_{\boldsymbol{\delta}'}\tilde{\Psi}(\boldsymbol{r}+\boldsymbol{\delta}') + \tilde{\Psi}^\dagger(\boldsymbol{r}-\boldsymbol{\delta})H_{\boldsymbol{\delta}}\sigma_3\left[1+\frac{\chi(\boldsymbol{r})}{2}\right]^2 H_{\boldsymbol{\delta}'}\tilde{\Psi}(\boldsymbol{r}+\boldsymbol{\delta}')\\
&\quad - \tilde{\Psi}^\dagger(\boldsymbol{r})H_{\boldsymbol{\delta}}\sigma_3\left[1+\frac{\chi(\boldsymbol{r}+\boldsymbol{\delta})}{2}\right]^2 H_{\boldsymbol{\delta}'}\tilde{\Psi}(\boldsymbol{r}+\boldsymbol{\delta}+\boldsymbol{\delta}') + \tilde{\Psi}^\dagger(\boldsymbol{r})H_{\boldsymbol{\delta}}(-i\sigma_2)\left[1+\frac{\chi(\boldsymbol{r}+\boldsymbol{\delta})}{2}\right]^2 \left(H^t_{\boldsymbol{\delta}'}\right)\sigma_1\tilde{\Psi}(\boldsymbol{r}+\boldsymbol{\delta}-\boldsymbol{\delta}') + \mathrm{H.c.}\Bigg\}\\
&= \frac{i}{4\hbar}\sum_{\boldsymbol{\delta},\boldsymbol{\delta}'}\sum_{\mu=x,y}\left\{\nabla_\mu\left[\delta_\mu H_{\boldsymbol{\delta}}\tilde{\Psi}(\boldsymbol{r}+\boldsymbol{\delta})\right]^\dagger\sigma_3\left[1+\frac{\chi(\boldsymbol{r})}{2}\right]^2 H_{\boldsymbol{\delta}'}\tilde{\Psi}(\boldsymbol{r}+\boldsymbol{\delta}') + \mathrm{H.c.}\right\}.
\end{aligned} \tag{A18}$$

In Eq. (A18), we have used $\tilde{\Psi}(\boldsymbol{r}) = \sigma_1\tilde{\Psi}^\dagger(\boldsymbol{r})$ and $\tilde{\Psi}^\dagger(\boldsymbol{r}) = \sigma_1\tilde{\Psi}(\boldsymbol{r})$. In terms of a velocity operator V_μ,

$$V_\mu \equiv \frac{1}{i\hbar}[x_\mu,H_0] = \frac{i}{\hbar}\sum_{\boldsymbol{\delta}}\delta_\mu H_{\boldsymbol{\delta}} e^{i\hat{\boldsymbol{p}}\cdot\boldsymbol{\delta}}, \tag{A19}$$

one obtains the thermal current operator $j^{\mathrm{Q}}_\mu(\boldsymbol{r})$ from Eq. (A18),

$$j^{\mathrm{Q}}_\mu(\boldsymbol{r}) = \frac{1}{4}\Psi^\dagger(\boldsymbol{r})\left[1+\frac{\chi(\boldsymbol{r})}{2}\right]\left\{V_\mu\sigma_3\left[1+\frac{\chi(\boldsymbol{r})}{2}\right]^2 H_0 + H_0\left[1+\frac{\chi(\boldsymbol{r})}{2}\right]^2\sigma_3 V_\mu\right\}\left[1+\frac{\chi(\boldsymbol{r})}{2}\right]\Psi(\boldsymbol{r}). \tag{A20}$$

By using a relation $\chi(\boldsymbol{r}) = \boldsymbol{r}\cdot\nabla\chi$ and expanding (A20) in terms of $\nabla\chi$, one obtains Eqs. (19) and (20).

APPENDIX B: FOURIER TRANSFORMATION

The Hamiltonian is defined in Eqs. (A2)–(A6) and the Fourier transformation is introduced as Eqs. (2) and (3). Substituting these equations into Eq. (A2), one obtains the Fourier transformation of the Hamiltonian in Eq. (4) with $H_{\boldsymbol{k}} \equiv \sum_{\boldsymbol{\delta}} H_{\boldsymbol{\delta}} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}}$. Similarly, one can obtain the Fourier transformation of the total thermal current operators from Eqs. (19) and (20),

$$J^{\mathrm{Q}}_{0,\mu} \equiv \int d\boldsymbol{r}\, j^{\mathrm{Q}}_{0,\mu}(\boldsymbol{r}) = \frac{1}{4}\sum_{\boldsymbol{k}} \Psi^{\dagger}_{\boldsymbol{k}}(V_{\boldsymbol{k},\mu}\sigma_3 H_{\boldsymbol{k}} + H_{\boldsymbol{k}}\sigma_3 V_{\boldsymbol{k},\mu})\Psi_{\boldsymbol{k}}, \tag{B1}$$

$$\begin{aligned} J^{\mathrm{Q}}_{1,\mu} \equiv \int d\boldsymbol{r}\, j^{\mathrm{Q}}_{1,\mu}(\boldsymbol{r}) = -\frac{i}{8}\nabla_\nu\chi \sum_{\boldsymbol{k}} \Psi^{\dagger}_{\boldsymbol{k}}(V_{\boldsymbol{k},\mu}\sigma_3 V_{\boldsymbol{k},\nu} - V_{\boldsymbol{k},\nu}\sigma_3 V_{\boldsymbol{k},\mu})\Psi_{\boldsymbol{k}} + \frac{1}{8}\nabla_\nu\chi \sum_{\boldsymbol{k}}[\Psi^{\dagger}_{\boldsymbol{k}}(x_\nu V_{\boldsymbol{k},\mu}\sigma_3 + 3V_{\boldsymbol{k},\mu}\sigma_3 x_\nu)H_{\boldsymbol{k}}\Psi_{\boldsymbol{k}} \\ + \Psi^{\dagger}_{\boldsymbol{k}} H_{\boldsymbol{k}}(3x_\nu\sigma_3 V_{\boldsymbol{k},\mu} + \sigma_3 V_{\boldsymbol{k},\mu}x_\nu)\Psi_{\boldsymbol{k}}], \end{aligned} \tag{B2}$$

where $V_{\boldsymbol{k},\mu} \equiv \sum_{\boldsymbol{\delta}} \frac{i}{\hbar}\delta_\mu H_{\boldsymbol{\delta}} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}} = \frac{1}{\hbar}\frac{\partial H_{\boldsymbol{k}}}{\partial k_\mu}$ and

$$\Psi_{i,\boldsymbol{k}} = \begin{cases} \beta_{i,\boldsymbol{k}} & (i = 1,\ldots,N) \\ \beta^{\dagger}_{i-N,-\boldsymbol{k}} & (i = N+1,\ldots,2N). \end{cases} \tag{B3}$$

By using the basis defined in Eq. (6), one can rewrite the bosonic field operator $\Psi_{\boldsymbol{k}}$ as

$$\Psi_{i,\boldsymbol{k}} = \sum_{n=1}^{N}(T_{\boldsymbol{k}})_{in}\gamma_{n\boldsymbol{k}} + \sum_{n=1}^{N}(T_{\boldsymbol{k}})_{i,n+N}\gamma^{\dagger}_{n,-\boldsymbol{k}}, \tag{B4}$$

$$\Psi^{\dagger}_{i,\boldsymbol{k}} = \sum_{n=1}^{N}(T^{\dagger}_{\boldsymbol{k}})_{ni}\gamma^{\dagger}_{n\boldsymbol{k}} + \sum_{n=1}^{N}(T^{\dagger}_{\boldsymbol{k}})_{n+N,i}\gamma_{n,-\boldsymbol{k}}. \tag{B5}$$

These equations are useful to calculate the thermal transport coefficients $S_{\mu\nu}$ and $M_{\mu\nu}$.

APPENDIX C: CALCULATION OF $S_{\mu\nu}$

In this section, we show how to calculate Eq. (26). It is written as

$$\begin{aligned} P_{\mu\nu}(i\Omega) &= -\frac{1}{16}\int_0^\beta d\tau e^{i\Omega\tau}\sum_{\boldsymbol{k},\boldsymbol{k}'}\langle T_\tau[\Psi^{\dagger}_{\boldsymbol{k}}(\tau'+\tau)X_{\boldsymbol{k},\mu}\Psi_{\boldsymbol{k}}(\tau'+\tau)\Psi^{\dagger}_{\boldsymbol{k}'}(\tau')X_{\boldsymbol{k}',\nu}\Psi_{\boldsymbol{k}'}(\tau')]\rangle \\ &= -\frac{1}{16}\int_0^\beta d\tau e^{i\Omega\tau}\sum_{\boldsymbol{k},\boldsymbol{k}'}(X_{\boldsymbol{k},\mu})_{\alpha,\beta}(X_{\boldsymbol{k}',\nu})_{\gamma,\delta}[\langle T_\tau\Psi^{\dagger}_{\alpha,\boldsymbol{k}}(\tau'+\tau)\Psi_{\delta,\boldsymbol{k}'}(\tau')\rangle\langle T_\tau\Psi_{\beta,\boldsymbol{k}}(\tau'+\tau)\Psi^{\dagger}_{\gamma,\boldsymbol{k}'}(\tau')\rangle \\ &\quad + \langle T_\tau\Psi^{\dagger}_{\alpha,\boldsymbol{k}}(\tau'+\tau)\Psi^{\dagger}_{\gamma,\boldsymbol{k}'}(\tau')\rangle\langle T_\tau\Psi_{\beta,\boldsymbol{k}}(\tau'+\tau)\Psi_{\delta,\boldsymbol{k}'}(\tau')\rangle + \langle T_\tau\Psi^{\dagger}_{\alpha,\boldsymbol{k}}(\tau'+\tau)\Psi_{\beta,\boldsymbol{k}}(\tau'+\tau)\rangle\langle T_\tau\Psi^{\dagger}_{\gamma,\boldsymbol{k}'}(\tau')\Psi_{\delta,\boldsymbol{k}'}(\tau')\rangle], \end{aligned} \tag{C1}$$

where $\Omega = 2\pi n/\beta$, $n \in Z$, T_τ is a time-ordering operator, and $X_{\boldsymbol{k},\mu} \equiv V_{\boldsymbol{k},\mu}\sigma_3 H_{\boldsymbol{k}} + H_{\boldsymbol{k}}\sigma_3 V_{\boldsymbol{k},\mu}$. The last term in the right-hand side of Eq. (C1) does not contribute since it cancels out via integration over τ. The remaining contraction, for example $\langle T_\tau\Psi^{\dagger}_{\alpha,\boldsymbol{k}}(\tau'+\tau)\Psi_{\delta,\boldsymbol{k}'}(\tau')\rangle$, is calculated as follows:

$$\langle T_\tau\Psi^{\dagger}_{\alpha,\boldsymbol{k}}(\tau'+\tau)\Psi_{\delta,\boldsymbol{k}'}(\tau')\rangle = \sum_{n=1}^{N}[(T^{\dagger}_{\boldsymbol{k}})_{n,\alpha}(T_{\boldsymbol{k}})_{\delta,n}e^{\tau\varepsilon_{n\boldsymbol{k}}}g(\varepsilon_{n\boldsymbol{k}}) - (T^{\dagger}_{\boldsymbol{k}})_{n+N,\alpha}(T_{\boldsymbol{k}})_{\delta,n+N}e^{-\tau\varepsilon_{n,-\boldsymbol{k}}}g(-\varepsilon_{n,-\boldsymbol{k}})]. \tag{C2}$$

We have used relations

$$\langle\gamma^{\dagger}_{n,\boldsymbol{k}}\gamma_{m,\boldsymbol{k}}\rangle = \delta_{n,m}g(\varepsilon_{n\boldsymbol{k}}), \quad \langle\gamma_{n,\boldsymbol{k}}\gamma^{\dagger}_{m,\boldsymbol{k}}\rangle = -\delta_{n,m}g(-\varepsilon_{n\boldsymbol{k}}). \tag{C3}$$

By integrating over τ in Eq. (C1), we get Eq. (27).

APPENDIX D: DERIVATION OF THERMAL TRANSPORT COEFFICIENT $L_{\mu\nu}$

Here we calculate the thermal transport coefficient. It consists of two parts:

$$\langle J^{\mathrm{Q}}_{\mu}\rangle = \langle J^{\mathrm{Q}}_{0,\mu}\rangle + \langle J^{\mathrm{Q}}_{1,\mu}\rangle \equiv -(S_{\mu\nu} + M_{\mu\nu})\nabla_\nu\chi, \tag{D1}$$

where $S_{\mu\nu}$ and $M_{\mu\nu}$ are written as Eqs. (27) and (28), respectively. The total thermal transport coefficient is obtained by $L_{\mu\nu} = S_{\mu\nu} + M_{\mu\nu}$. In the following, we first separate the coefficients into two parts, respectively, to avoid complexity: $S_{\mu\nu} = S^{(1)}_{\mu\nu} + S^{(2)}_{\mu\nu}$

and $M_{\mu\nu} = M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu}$. Here, the sum $S^{(2)}_{\mu\nu} + M^{(2)}_{\mu\nu}$ cancels out and thus what we should calculate are the remaining parts. Next we calculate $M^{(1)}_{\mu\nu}$ in the same manner as Smrčka and Středa [16]. Finally, we obtain $L_{\mu\nu}$, which is expressed by the Berry curvature in momentum space $\Omega_{n\boldsymbol{k}}$.

Now we start the calculation. Using the relation $(\varepsilon_{n\boldsymbol{k}} \mp \varepsilon_{m\boldsymbol{k}})^2 = (\varepsilon_{n\boldsymbol{k}} \pm \varepsilon_{m\boldsymbol{k}})^2 \mp 4\varepsilon_{n\boldsymbol{k}}\varepsilon_{m\boldsymbol{k}}$ in Eq. (27), we decompose $S_{\mu\nu}$ as $S_{\mu\nu} = S^{(1)}_{\mu\nu} + S^{(2)}_{\mu\nu}$, which corresponds to $\mp 4\varepsilon_{n\boldsymbol{k}}\varepsilon_{m\boldsymbol{k}}$ and $(\varepsilon_{n\boldsymbol{k}} \pm \varepsilon_{m\boldsymbol{k}})^2$, respectively. $M_{\mu\nu}$ is also decomposed as $M_{\mu\nu} = M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu}$, where $M^{(1)}_{\mu\nu}$ denotes the term containing $T^\dagger_{\boldsymbol{k}}(x_\nu V_{\boldsymbol{k},\mu} + V_{\boldsymbol{k},\mu}x_\nu)T_{\boldsymbol{k}}$ in Eq. (28). Then one finds that $S_{\mu\nu}$ and $M_{\mu\nu}$ cancel out partially, $S^{(2)}_{\mu\nu} + M^{(2)}_{\mu\nu} = 0$ and thus $L_{\mu\nu} = S^{(1)}_{\mu\nu} + M^{(1)}_{\mu\nu}$. The remainder, $S^{(1)}_{\mu\nu}$ and $M^{(1)}_{\mu\nu}$, are written, respectively, as follows:

$$\begin{aligned} S^{(1)}_{\mu\nu} &= -\frac{i}{2}\sum_{n,m=1}^{2N}\sum_{\boldsymbol{k}}\left((T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{nm}\left\{\frac{(\mathcal{E}_{\boldsymbol{k}})_{nn}\mathcal{E}_{\boldsymbol{k}}}{[(\sigma_3\mathcal{E}_{\boldsymbol{k}})_{nn} - \mathcal{E}_{\boldsymbol{k}}\sigma_3]^2}\right\}_{mm}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{mn}[g(\sigma_3\mathcal{E}_{\boldsymbol{k}})]_{nn}\right) - (\mu \leftrightarrow \nu) \\ &= -\frac{i}{2}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}\eta g(\eta)\mathrm{Tr}\left\{\delta(\eta - \sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\left[T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}}\frac{\mathcal{E}_{\boldsymbol{k}}}{(\eta - \sigma_3\mathcal{E}_{\boldsymbol{k}})^2}T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}}\right]\right\}d\eta - (\mu \leftrightarrow \nu) \\ &= -\frac{i}{2}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}\eta g(\eta)\mathrm{Tr}\left[\delta(\eta - \sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T^\dagger_{\boldsymbol{k}}}{\partial k_\mu}H_{\boldsymbol{k}}\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]d\eta - (\mu \leftrightarrow \nu), \end{aligned} \tag{D2}$$

$$M^{(1)}_{\mu\nu} = -\frac{1}{2}\sum_{n=1}^{2N}\sum_{\boldsymbol{k}}[T^\dagger_{\boldsymbol{k}}(x_\nu V_{\boldsymbol{k},\mu} + V_{\boldsymbol{k},\mu}x_\nu)T_{\boldsymbol{k}}\mathcal{E}_{\boldsymbol{k}}g(\sigma_3\mathcal{E}_{\boldsymbol{k}})]_{nn} = -\frac{1}{2}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}\eta g(\eta)\mathrm{Tr}[\sigma_3(x_\nu V_{\boldsymbol{k},\mu} - x_\mu V_{\boldsymbol{k},\nu})\delta(\eta - \sigma_3 H_{\boldsymbol{k}})]d\eta. \tag{D3}$$

In Eq. (D2), we have used the relation $T^\dagger_{\boldsymbol{k}}\sigma_3 T_{\boldsymbol{k}} = \sigma_3$ and $T^{-1}_{\boldsymbol{k}}f(\sigma_3 H_{\boldsymbol{k}})T_{\boldsymbol{k}} = f(\sigma_3\mathcal{E}_{\boldsymbol{k}})$, where $f(x)$ is an arbitrary function. The term $\{\frac{(\mathcal{E}_{\boldsymbol{k}})_{nn}\mathcal{E}_{\boldsymbol{k}}}{[(\sigma_3\mathcal{E}_{\boldsymbol{k}})_{nn} - \mathcal{E}_{\boldsymbol{k}}\sigma_3]^2}\}_{mm}$ means $\frac{\varepsilon_{n\boldsymbol{k}}\varepsilon_{m,-\boldsymbol{k}}}{(\varepsilon_{n\boldsymbol{k}} + \varepsilon_{m,-\boldsymbol{k}})^2}$ for $1 \leqslant n \leqslant N$ and $N+1 \leqslant m \leqslant 2N$, for example. Here we also present the expression for $M^{(2)}_{\mu\nu} = -S^{(2)}_{\mu\nu}$, although it does not affect the following calculation due to the cancellation:

$$\begin{aligned} M^{(2)}_{\mu\nu} = -S^{(2)}_{\mu\nu} = \frac{i}{8}\sum_{n,m=1}^{N}\sum_{\boldsymbol{k}}[&g(\varepsilon_{n\boldsymbol{k}})(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{nm}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{mn} - g(\varepsilon_{n\boldsymbol{k}})(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n,m+N}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m+N,n} \\ &- g(-\varepsilon_{n,-\boldsymbol{k}})(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n+N,m}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m,n+N} + g(-\varepsilon_{n,-\boldsymbol{k}})(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}})_{n+N,m+N}(T^\dagger_{\boldsymbol{k}}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}})_{m+N,n+N}] - (\mu \leftrightarrow \nu) \end{aligned} \tag{D4}$$

$$= -\frac{i}{8}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\,\mathrm{Tr}\left[\delta(\tilde{\eta} - \sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T^\dagger_{\boldsymbol{k}}}{\partial k_\mu}(\sigma_3\tilde{\eta}^2 - 2\tilde{\eta}H_{\boldsymbol{k}} + H_{\boldsymbol{k}}\sigma_3 H_{\boldsymbol{k}})\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]g(\tilde{\eta}) - (\mu \leftrightarrow \nu). \tag{D5}$$

Here we have completed the calculation of $S^{(1)}_{\mu\nu}$. In the following, we further calculate $M^{(1)}_{\mu\nu}$ to express it in terms of the spin-wave dispersion $\varepsilon_{n\boldsymbol{k}}$ and the paraunitary matrix $T_{\boldsymbol{k}}$. We follow Smrčka and Středa [16] to introduce the two functions $A_{\mu\nu}(\eta)$ and $B_{\mu\nu}(\eta)$ as

$$A_{\mu\nu}(\eta) \equiv i\,\mathrm{Tr}\left[\sigma_3 V_{\boldsymbol{k},\mu}\frac{dG^+}{d\eta}\sigma_3 V_{\boldsymbol{k},\nu}\delta(\eta - \sigma_3 H_{\boldsymbol{k}}) - \sigma_3 V_{\boldsymbol{k},\mu}\delta(\eta - \sigma_3 H_{\boldsymbol{k}})\sigma_3 V_{\boldsymbol{k},\nu}\frac{dG^-}{d\eta}\right], \tag{D6}$$

$$B_{\mu\nu}(\eta) \equiv i\,\mathrm{Tr}[\sigma_3 V_{\boldsymbol{k},\mu}G^+\sigma_3 V_{\boldsymbol{k},\nu}\delta(\eta - \sigma_3 H_{\boldsymbol{k}}) - \sigma_3 V_{\boldsymbol{k},\mu}\delta(\eta - \sigma_3 H_{\boldsymbol{k}})\sigma_3 V_{\boldsymbol{k},\nu}G^-], \tag{D7}$$

where $G^\pm$ is defined as $G^\pm \equiv \frac{1}{\eta \pm i0 - \sigma_3 H_{\boldsymbol{k}}}$. They obey the following identity:

$$\begin{aligned} A_{\mu\nu}(\eta) - \frac{1}{2}\frac{dB_{\mu\nu}(\eta)}{d\eta} &= \frac{1}{4\pi}\mathrm{Tr}[\sigma_3 V_{\boldsymbol{k},\mu}(G^+)^2\sigma_3 V_{\boldsymbol{k},\nu}G^+ - \sigma_3 V_{\boldsymbol{k},\mu}(G^-)^2\sigma_3 V_{\boldsymbol{k},\nu}G^-] - (\mu \leftrightarrow \nu) \\ &= \frac{i}{4\pi}\mathrm{Tr}[x_\mu G^+\sigma_3 V_{\boldsymbol{k},\nu}G^+ - x_\mu(G^+)^2\sigma_3 V_{\boldsymbol{k},\nu} - x_\mu G^-\sigma_3 V_{\boldsymbol{k},\nu}G^- + x_\mu(G^-)^2\sigma_3 V_{\boldsymbol{k},\nu}] - (\mu \leftrightarrow \nu) \\ &= \frac{1}{4\pi i}\mathrm{Tr}\{x_\mu[(G^+)^2 - (G^-)^2]\sigma_3 V_{\boldsymbol{k},\nu}\} - (\mu \leftrightarrow \nu) \\ &= -\frac{1}{2}\mathrm{Tr}\left[\sigma_3(x_\nu V_{\boldsymbol{k},\mu} - x_\mu V_{\boldsymbol{k},\nu})\frac{d}{d\eta}\delta(\eta - \sigma_3 H_{\boldsymbol{k}})\right]. \end{aligned} \tag{D8}$$

To see this, we have used the relations $G^+ - G^- = -2\pi i\delta(\eta - \sigma_3 H_{\boldsymbol{k}})$ and $V_{\boldsymbol{k},\mu} = i[x_\mu,\sigma_3(G^\pm)^{-1}]$. We now integrate Eq. (D8) to obtain

$$\mathrm{Tr}[\sigma_3(x_\nu V_{\boldsymbol{k},\mu} - x_\mu V_{\boldsymbol{k},\nu})\delta(\eta - \sigma_3 H_{\boldsymbol{k}})] = 2\int_\eta^\infty d\tilde{\eta}\left[A_{\mu\nu}(\tilde{\eta}) - \frac{1}{2}\frac{dB_{\mu\nu}(\tilde{\eta})}{d\tilde{\eta}}\right] = -2\int_{-\infty}^{\eta}d\tilde{\eta}\left[A_{\mu\nu}(\tilde{\eta}) - \frac{1}{2}\frac{dB_{\mu\nu}(\tilde{\eta})}{d\tilde{\eta}}\right], \tag{D9}$$

where the magnon spectrum is supposed to be bounded. The last equality in Eq. (D9) is based on the following identity:

$$\begin{aligned}
\int_{-\infty}^{\infty} d\tilde{\eta}\left[A_{\mu\nu}(\tilde{\eta})-\frac{1}{2}\frac{dB_{\mu\nu}(\tilde{\eta})}{d\tilde{\eta}}\right] &= i\int_{-\infty}^{\infty}\mathrm{Tr}\left[\sigma_3 V_{\boldsymbol{k},\mu}\frac{dG^{+}}{d\tilde{\eta}}\sigma_3 V_{\boldsymbol{k},\nu}\delta(\tilde{\eta}-\sigma_3 H_{\boldsymbol{k}})-\sigma_3 V_{\boldsymbol{k},\mu}\delta(\tilde{\eta}-\sigma_3 H_{\boldsymbol{k}})\sigma_3 V_{\boldsymbol{k},\nu}\frac{dG^{-}}{d\tilde{\eta}}\right]d\tilde{\eta}\\
&= -i\int_{-\infty}^{\infty}\sum_{n=1}^{2N}(\sigma_3)_{nn}\delta[\tilde{\eta}-(\sigma_3\mathcal{E}_{\boldsymbol{k}})_{nn}]\left\{T_{\boldsymbol{k}}^{\dagger}V_{\boldsymbol{k},\mu}T_{\boldsymbol{k}}\frac{1}{[(\sigma_3\mathcal{E}_{\boldsymbol{k}})_{nn}-\sigma_3\mathcal{E}_{\boldsymbol{k}}]^2}\sigma_3 T_{\boldsymbol{k}}^{\dagger}V_{\boldsymbol{k},\nu}T_{\boldsymbol{k}}\right\}_{nn}d\tilde{\eta}-(\mu\leftrightarrow\nu)\\
&= -i\int_{-\infty}^{\infty}\mathrm{Tr}\left[\delta(\tilde{\eta}-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\mu}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]d\tilde{\eta}-(\mu\leftrightarrow\nu) = -\sum_{n=1}^{2N}\Omega_{n\boldsymbol{k}}=0.
\end{aligned}\tag{D10}$$

Here $\Omega_{n\boldsymbol{k}}$ is a Berry curvature in momentum space,

$$\Omega_{n\boldsymbol{k}} \equiv i\epsilon_{\mu\nu}\left[\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\mu}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]_{nn},\tag{D11}$$

which can be naturally defined in terms of a projection operator [17,36]. In fact, the Berry curvature thus introduced satisfies the following sum rule, which was used in Eq. (D10):

$$\begin{aligned}
\sum_{n=1}^{2N}\Omega_{n\boldsymbol{k}} &= i\mathrm{Tr}\left[\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\mu}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}-(\mu\leftrightarrow\nu)\right] = i\mathrm{Tr}\left[\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\mu}\sigma_3 T_{\boldsymbol{k}}\sigma_3 T_{\boldsymbol{k}}^{\dagger}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}-(\mu\leftrightarrow\nu)\right]\\
&= -i\mathrm{Tr}\left[\sigma_3 T_{\boldsymbol{k}}^{\dagger}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\mu}\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\nu}\sigma_3 T_{\boldsymbol{k}}-(\mu\leftrightarrow\nu)\right] = -\sum_{n=1}^{2N}\Omega_{n\boldsymbol{k}}=0.
\end{aligned}\tag{D12}$$

Now we calculate $M_{\mu\nu}^{(1)}$ in Eq. (D3). By using Eq. (D9),

$$\begin{aligned}
M_{\mu\nu}^{(1)} &= -\sum_{\boldsymbol{k}}\left(\int_0^{\infty}d\eta\int_{\eta}^{\infty}d\tilde{\eta}+\int_{-\infty}^{0}d\eta\int_{\eta}^{-\infty}d\tilde{\eta}\right)\eta g(\eta)\left[A_{\mu\nu}(\tilde{\eta})-\frac{1}{2}\frac{dB_{\mu\nu}(\tilde{\eta})}{d\tilde{\eta}}\right]\\
&= -\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\left[A_{\mu\nu}(\tilde{\eta})-\frac{1}{2}\frac{dB_{\mu\nu}(\tilde{\eta})}{d\tilde{\eta}}\right]\cdot\int_0^{\tilde{\eta}}\eta g(\eta)d\eta = i\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\mathrm{Tr}\left[\delta(\tilde{\eta}-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\mu}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]\cdot\int_0^{\tilde{\eta}}\eta g(\eta)d\eta\\
&\quad -\frac{i}{2}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\mathrm{Tr}\left[\delta(\tilde{\eta}-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\mu}\sigma_3(\tilde{\eta}-\sigma_3 H_{\boldsymbol{k}})\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]\tilde{\eta}g(\tilde{\eta})-(\mu\leftrightarrow\nu).
\end{aligned}\tag{D13}$$

Finally, from Eqs. (D2) and (D13), the total thermal transport coefficient $L_{\mu\nu}$ is calculated as follows:

$$\begin{aligned}
L_{\mu\nu} &= S_{\mu\nu}^{(1)}+M_{\mu\nu}^{(1)} = \frac{i}{2}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}d\tilde{\eta}\mathrm{Tr}\left[\delta(\tilde{\eta}-\sigma_3\mathcal{E}_{\boldsymbol{k}})\sigma_3\frac{\partial T_{\boldsymbol{k}}^{\dagger}}{\partial k_\mu}\sigma_3\frac{\partial T_{\boldsymbol{k}}}{\partial k_\nu}\right]\left[2\int_0^{\tilde{\eta}}\eta g(\eta)d\eta-\tilde{\eta}^2 g(\tilde{\eta})\right]-(\mu\leftrightarrow\nu)\\
&= -\frac{1}{2}\sum_{\boldsymbol{k}}\sum_{n=1}^{2N}\int_0^{(\sigma_3\mathcal{E}_{\boldsymbol{k}})_{nn}}\eta^2\frac{dg(\eta)}{d\eta}d\eta\Omega_{n\boldsymbol{k}} = -\sum_{\boldsymbol{k}}\sum_{n=1}^{N}(k_B T)^2\left\{c_2[g(\varepsilon_{n\boldsymbol{k}})]-\frac{\pi^2}{3}\right\}\Omega_{n\boldsymbol{k}}.
\end{aligned}\tag{D14}$$

Here, $c_2(x)$ is defined in Eq. (33) and we have used $c_2(\infty)=\pi^2/3$ in Eq. (D14). These results are identical to Eq. (32).

APPENDIX E: SUM RULE OF THE CHERN INTEGER OVER ALL PARTICLE BANDS

To show the sum rule for the Chern integer over particle bands, we follow the argument given in Ref. [17] henceforth. We first separate a $2N\times 2N$ bosonic Hamiltonian as follows:

$$H_{\boldsymbol{k}} \equiv \begin{pmatrix} A_{\boldsymbol{k}} & B_{\boldsymbol{k}} \\ B_{-\boldsymbol{k}}^{*} & A_{-\boldsymbol{k}}^{*}\end{pmatrix} = t_{\boldsymbol{k}}1_{2N\times 2N}+C_{\boldsymbol{k}},\tag{E1}$$

where $C_{\boldsymbol{k}}$ is a traceless part of $H_{\boldsymbol{k}}$. We suppose $H_{\boldsymbol{k}}$ to be paraunitarily positive definite for any $\boldsymbol{k}$. This leads to $t_{\boldsymbol{k}}>0$.

We introduce a parameter λ as

$$\begin{aligned}
H_{\boldsymbol{k}}(\lambda) &= t_{\boldsymbol{k}}1_{2N\times 2N}+\lambda C_{\boldsymbol{k}},\\
&= (1-\lambda)t_{\boldsymbol{k}}1_{2N\times 2N}+\lambda H_{\boldsymbol{k}},
\end{aligned}\tag{E2}$$

with $H_{\boldsymbol{k}}(1)=H_{\boldsymbol{k}}$ and $H_{\boldsymbol{k}}(0)=t_{\boldsymbol{k}}1_{2N\times 2N}$. While changing λ from zero to one, $H_{\boldsymbol{k}}(\lambda)$ keeps unitarily positive definite for any $\boldsymbol{k}$; the eigenvalues of $H_{\boldsymbol{k}}(\lambda)$ are the sum of the eigenvalues of $\lambda H_{\boldsymbol{k}}$ and $(1-\lambda)t_{\boldsymbol{k}}$, both of which are positive.

Being unitarily positive definite, $H_{\boldsymbol{k}}(\lambda)$ is also paraunitarily positive definite [22]. Thus, there always exists a band gap between particle bands ($1\leqslant n\leqslant N$) and hole bands ($N+1\leqslant n\leqslant 2N$) during $\lambda=0$ to $\lambda=1$. This guarantees that the sum of the Chern integer for all positive bands is invariant during the interpolation,

$$\sum_{n=1}^{N} C_n(\lambda) = \text{const.} \tag{E3}$$

On the other hand, Eq. (E3) vanishes at $\lambda = 0$, which leads to

$$\sum_{n=1}^{N} C_n(\lambda = 1) = 0. \tag{E4}$$

Namely, provided that $H_{\boldsymbol{k}}$ is paraunitarily positive definite, the sum of the Berry curvature over the Brillouin zone (BZ) and over all particle bands is always zero.

APPENDIX F: RELATION OF THE BERRY CURVATURE BETWEEN PARTICLE SPACE AND HOLE SPACE

In order to derive the relation given by Eq. (35), we first study the relation between $T_{\boldsymbol{k}}$ and $T_{-\boldsymbol{k}}$. From Eq. (4), the particle-hole symmetry leads to

$$H_{\boldsymbol{k}} = \sigma_1 (H_{-\boldsymbol{k}})^{\mathrm{t}} \sigma_1. \tag{F1}$$

By using the paraunitarity $T_{\boldsymbol{k}}^\dagger \sigma_3 T_{\boldsymbol{k}} = \sigma_3$, the eigenvalue problem [Eq. (7)] is written as

$$H_{\boldsymbol{k}} T_{\boldsymbol{k}} = \sigma_3 T_{\boldsymbol{k}} \begin{pmatrix} E_{\boldsymbol{k}} & \\ & -E_{-\boldsymbol{k}} \end{pmatrix}. \tag{F2}$$

By the replacement of $\boldsymbol{k} \to -\boldsymbol{k}$ and utilizing Eq. (F1), Eq. (F2) becomes

$$H_{\boldsymbol{k}} \sigma_1 T_{-\boldsymbol{k}}^* \sigma_1 = \sigma_3 \sigma_1 T_{-\boldsymbol{k}}^* \sigma_1 \begin{pmatrix} E_{\boldsymbol{k}} & \\ & -E_{-\boldsymbol{k}} \end{pmatrix}. \tag{F3}$$

The equation means that $\sigma_1 T_{-\boldsymbol{k}}^* \sigma_1$ also satisfies the same eigenvalue equation as $T_{\boldsymbol{k}}$. Thus, $\sigma_1 T_{-\boldsymbol{k}}^* \sigma_1$ can be expressed as

$$T_{\boldsymbol{k}} = \sigma_1 T_{-\boldsymbol{k}}^* \sigma_1 M_{\boldsymbol{k}}. \tag{F4}$$

Here, $M_{\boldsymbol{k}}$ is, in generic situations, a diagonal matrix with its diagonal elements being a phase factor. Imposing the paraunitarity onto the right-hand side, one finds that $M_{\boldsymbol{k}}$ is an unitary matrix:

$$M_{\boldsymbol{k}}^\dagger M_{\boldsymbol{k}} = 1_{2N\times 2N}, \tag{F5}$$

or, equivalently, $(M_{\boldsymbol{k}})_{ij} = \delta_{ij} \exp[i\theta_{j,\boldsymbol{k}}]$. On the other hand, applying a replacement $\boldsymbol{k} \to -\boldsymbol{k}$ and taking the complex conjugate of Eq. (F4), one obtains $T_{-\boldsymbol{k}}^* = \sigma_1 T_{\boldsymbol{k}} \sigma_1 M_{-\boldsymbol{k}}^*$. By substituting this equation into Eq. (F4) again, one finds

$$\sigma_1 M_{-\boldsymbol{k}}^* \sigma_1 M_{\boldsymbol{k}} = 1_{2N\times 2N}, \tag{F6}$$

which means $\theta_{j,\boldsymbol{k}} = \theta_{j+N,-\boldsymbol{k}}$ for $1 \leqslant j \leqslant N$.

Now we investigate the relation between the Berry curvature of the particle space and that of the hole space. It is convenient to introduce a gauge field $A_{n,\boldsymbol{k}}^\nu$ as

$$A_{n,\boldsymbol{k}}^\nu \equiv i\mathrm{Tr}(\Gamma_n \sigma_3 T_{\boldsymbol{k}}^\dagger \sigma_3 \partial_{k_\nu} T_{\boldsymbol{k}}), \tag{F7}$$

where $(\Gamma_n)_{ij} \equiv \delta_{ij}\delta_{in}$. Then, Eq. (F4) leads to

$$\begin{aligned} A_{n,\boldsymbol{k}}^\nu &= -i\mathrm{Tr}\big[\partial_{k_\nu}\big(M_{\boldsymbol{k}}^t \sigma_1 T_{-\boldsymbol{k}}^\dagger\big)\sigma_3 T_{-\boldsymbol{k}} \sigma_1 M_{\boldsymbol{k}}^* \sigma_3 \Gamma_n\big] \\ &= i\mathrm{Tr}\big[\Gamma_n\big(\partial_{k_\nu} M_{\boldsymbol{k}}^t\big) M_{\boldsymbol{k}}^*\big] - i\mathrm{Tr}[\Gamma_{n+N} T_{-\boldsymbol{k}}^\dagger \sigma_3 \partial_{k_\nu} T_{-\boldsymbol{k}} \sigma_3] \\ &= -\partial_{k_\nu}\theta_{n,\boldsymbol{k}} + A_{n+N,-\boldsymbol{k}}^\nu, \end{aligned} \tag{F8}$$

where we used a relation $(\partial_{k_\nu} T_{\boldsymbol{k}}^\dagger)\sigma_3 T_{\boldsymbol{k}} + T_{\boldsymbol{k}}^\dagger \sigma_3 (\partial_{k_\nu} T_{\boldsymbol{k}}) = 0$. Because the gauge field generates the Berry curvature as $\Omega_{n\boldsymbol{k}} = \partial_{k_x} A_{n,\boldsymbol{k}}^y - \partial_{k_y} A_{n,\boldsymbol{k}}^x$, the Berry curvature of the hole space $\Omega_{n+N,\boldsymbol{k}}$ is related to that of the particle space as

$$\Omega_{n,\boldsymbol{k}} = -\Omega_{n+N,-\boldsymbol{k}}. \tag{F9}$$

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