Induced tunneling and localization for a quantum particle in tilted two-dimensional lattices

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We consider a quantum particle in tilted two-dimensional lattices in the tight-binding approximations. We show that for certain lattice geometries the particle can freely move across the lattice in the direction perpendicular to the vector of the static force. This effect is argued to be analog of the photon-induced tunneling in driven one-dimensional lattices. We calculate the particle dispersion relation by using a method based on the Bogoliubov-Mitropolskii averaging technique from the theory of dynamical systems. This dispersion relation draws the analogy with driven one-dimensional lattices further by eventually showing band collapses when a control parameter is varied.

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I. INTRODUCTION

Band collapse or dynamic localization is a destructive interference effect that occurs in periodically driven one-dimensional (1D) lattices. This effect has been of large theoretical interest since the seminal work by Dunlap and Kenkre [1] and was experimentally observed for cold atoms in optical lattices [2–4] and the light in coupled waveguide arrays [5]. It takes place for certain amplitudes of the driving force F_ω that are determined by roots of the equation $\mathcal{J}_0(aF_\omega/\hbar\omega)=0$, where $\mathcal{J}_0(z)$ is the zero-order Bessel function, ω the driving frequency, and a the lattice period. One meets the same effect in tilted 1D lattices provided the Bloch frequency $\omega_B = aF/\hbar$ is a multiple of the driving frequency, i.e., $\omega_B = q\omega$ [6–8]. In this case the original dispersion relation $E(\kappa)=2t\cos(\kappa a)$ for the quantum particle in the 1D lattice is substituted by

$$E(\kappa) = t\mathcal{J}_q\left(\frac{aF_\omega}{\hbar\omega_B}\right)\cos(\kappa a) \tag{1}$$

(here t is the hopping matrix element and κ the particle quasimomentum). Then the localization condition is given by roots of the qth order Bessel function [9]. Notice that because $\mathcal{J}_q(z)\sim z^q$ for $z\to 0$ the particle can be delocalized in the tilted lattice only in the presence of driving—the phenomenon known as photon-induced or photon-assisted tunneling. Thus the resonant driving first induces the tunneling and then suppresses it for higher amplitude of the driving force.

In the present paper we discuss emergence and suppression of tunneling for a quantum particle in tilted two-dimensional (2D) lattices. Similar to tilted 1D lattices, the necessary condition to observe these effects is the resonance condition, now on two Bloch frequencies associated with two components of the vector $\mathbf{F}$ of a static force. This condition, however, is not sufficient, and the effects are absent in simple lattices like the square or triangle lattices. The other requirement is that the 2D lattice should have nontrivial geometry and consist of at least two sublattices. In this case the particle can freely propagate in the direction orthogonal to the vector $\mathbf{F}$ [10]. As shown in Refs. [11] and [12], the physics behind this effect is the Landau-Zener tunneling between Bloch subbands of the unbiased lattice. Here we focus on the mathematical aspect of the problem. Specifically we address the geometry shown in Fig. 1— a square lattice with three different hopping matrix elements. This model covers three important cases: (i) if $t_3=t_2$ we have a simple square lattice with no Bloch subbands; (ii) for $t_3=0$ it is topologically equivalent to the honeycomb lattice with two subbands touching at the Dirac points; (iii) finally, the case $t_3=-t_2$ corresponds to a staggered magnetic field with π flux through the elementary cell. In what follows we assume t_1 to be the maximal hopping matrix element and shall measure the other two elements in units of t_1. We mention that a square lattice with three different, independently variable hopping matrix elements have been recently realized with cold atoms [13,14], where $t_j/\hbar$ are of the order of 1 kHz.

II. DISPERSION RELATION

For the considered square lattice the resonance condition reads $F_x/F_y = r/q$, where r and q are co-prime numbers and we refer to the coordinate system determined by the lattice primary axes. Our aim is to obtain an analog of Eq. (1) where control parameters are magnitude of the static force $F=|\mathbf{F}|$ and two prime numbers r and q. The initial step of the analysis closely follows our recent work [15] devoted to the Wannier-Stark states in the honeycomb lattice. First we introduce the coordinate system (η,ξ) which is rotated by the angle $\theta=\arctan(F_x/F_y)=\arctan(r/q)$ with respect to the lattice primary axes. In this coordinate system the Stark energy $\mathbf{Fr}$ depends only on ξ and, hence, we can use the ansatz $\Psi(\eta,\nu)=\exp(i\kappa\eta)\psi(\xi)$ to separate the variables [16]. Notice that in new variables the coordinates $\mathbf{r}_i$ of the lattice sites are a multiple of the period

$$d = \frac{\sqrt{2}a}{\sqrt{r^2+q^2}}, \tag{2}$$

where a is the period of the square lattice with no sublattices ($t_3=t_2$). Using the above ansatz the stationary Schrödingier equation reduces to the following system of coupled algebraic equations

$$\begin{aligned} E\psi_j^A &= -t_1e^{-ir\kappa d}\psi_{j-q}^B - t_1e^{iq\kappa d}\psi_{j-r}^B - t_2e^{i(q-r)\kappa d}\psi_{j-q-r}^B \\ &\quad - t_3\psi_j^B + Fdj\psi_j^A, \\ E\psi_j^B &= -t_1e^{ir\kappa d}\psi_{j+q}^A - t_1e^{-iq\kappa d}\psi_{j+r}^A - t_2e^{i(r-q)\kappa d}\psi_{j+r+q}^A \\ &\quad - t_3\psi_j^A + (Fdj+E_0)\psi_j^B, \end{aligned} \tag{3}$$

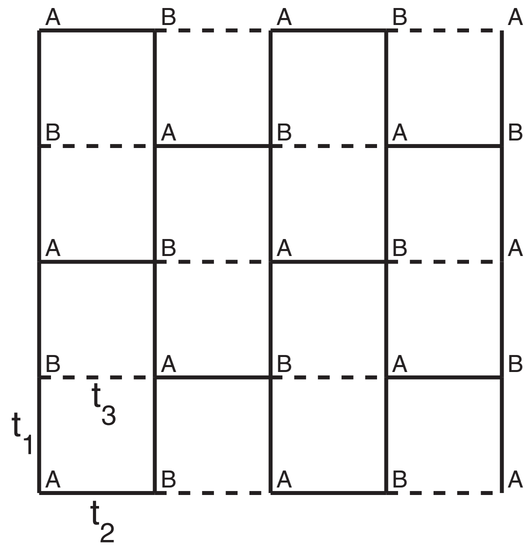

FIG. 1. The tight-binding model with three different hopping amplitudes. For $t_3 = t_2$ the model reduces to the simple square lattice while for $t_3 = 0$ it is topologically equivalent to the honeycomb lattice. Notice that primary axes of the square and honeycomblike lattices are rotated by $\pi/4$ relative to each other.

where $E_0 = Fd(r+q)/2$ and A and B are the sublattice indexes. A similar equation was obtained in Ref. [15] where we solved it numerically. It follows from the general structure of Eq. (3) that the energy spectrum consists of a replica of the (yet unknown) energy band $E = E(\kappa)$ and its symmetric counterpart $E = -E(\kappa)$ arranged into two Wannier-Stark ladders (see inset in Fig. 2). In this sense the problem is effectively one dimensional and, hence, the question about the analog of Eq. (1) is well posed.

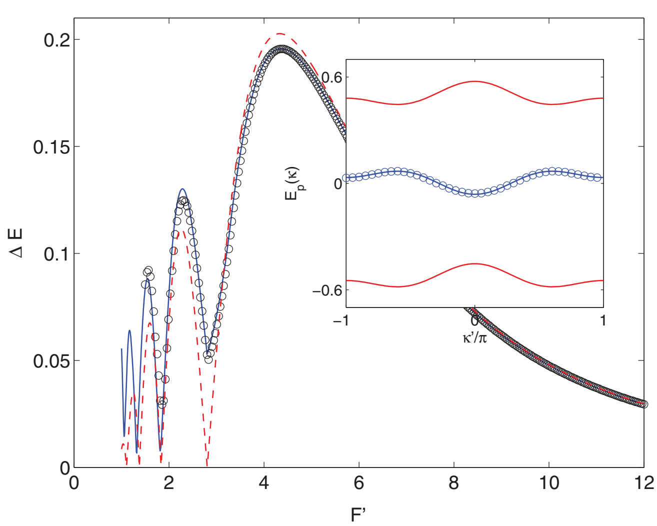

FIG. 2. (Color online) Width of the energy bands in units of t_1 as the function of dimensionless static force $F' = \sqrt{2}aF/t_1$. The system parameters are $F_x/F_y = r/q = 2/1$ and $(t_1,t_2,t_3) = (1,0.5,0.25)$. The dashed and solid lines are plotted by keeping in Eq. (13) the first two and four terms, respectively, and symbols are the exact numerical result. The inset shows a fragment of the energy spectrum for $F' = 2.3$ where open circles are the analytical result according to Eq. (13). The bands are shown as the functions of dimensionless quasimomentum $\kappa' = (r^2+q^2)d\kappa = \sqrt{2(r^2+q^2)}a\kappa$.

To find the above dispersion relation, i.e., to approach Eq. (3) analytically, we Fourier transform it. This substitutes algebraic equations by ordinary differential equations. Denoting by $\tilde{\psi}^{A,B}(\chi)$ the Fourier images of the functions $\psi^{A,B}(\xi) = \sum_j \psi_j^{A,B}\delta(\xi - dj)$ we have

$$iF\frac{\mathrm{d}\tilde{\psi}^A}{\mathrm{d}\chi} = E\tilde{\psi}^A + \tilde{G}\tilde{\psi}^B,$$
$$iF\frac{\mathrm{d}\tilde{\psi}^B}{\mathrm{d}\chi} = (E-E_0)\tilde{\psi}^B + \tilde{G}^*\tilde{\psi}^A, \tag{4}$$

where the coefficient $\tilde{G}$ is a function of χ and κ, $\tilde{G}(\chi,\kappa) = t_1\exp[-id(\kappa r + \chi q)] + t_1\exp[-id(\chi r - \kappa q)] + t_2\exp[-id\chi(r+q) + id\kappa(q-r)] + t_3$. It is convenient to eliminate the diagonal part in the right hand side of Eq. (4) and rewrite the function $\tilde{G}(\chi,\kappa)$ in the rotated variables $\alpha = (\chi\cos\theta - \kappa\sin\theta)/\sqrt{2}a$ and $\beta = (\chi\sin\theta + \kappa\cos\theta)/\sqrt{2}a$. We have

$$i\frac{\mathrm{d}c_1}{\mathrm{d}\chi} = \frac{1}{F}Gc_2, \quad i\frac{\mathrm{d}c_2}{\mathrm{d}\chi} = \frac{1}{F}G^*c_1, \tag{5}$$

where $c_1 = \tilde{\psi}^A\exp(iE\chi/F)$, $c_2 = \tilde{\psi}^B\exp(iE\chi/F)\exp[-i(\alpha+\beta)/2]$, and

$$G = 2t_1\cos\left(\frac{\alpha-\beta}{2}\right) + 2t_2\cos\left(\frac{\alpha+\beta}{2}\right) + (t_3-t_2)\exp\left(i\frac{\alpha+\beta}{2}\right). \tag{6}$$

Equation (5) has the structure of a classical dynamical system with the variable χ playing the role of time [17]. Moreover, since the coefficient (6) is a periodic function of χ, we can use the Bogoliubov-Metropolskii technique [18] to analyze it. This method involves averaging the function G and gives the solution of Eq. (5) in the form of a series over the parameter $\epsilon = 1/F$.

Before proceeding with the above mentioned method we show that for $t_3 = t_2$ Eq. (5) has a trivial solution that corresponds to flat bands of the simple square lattice,

$$E_p(\kappa) = F\frac{a}{\sqrt{r'^2+q'^2}}p, \quad \frac{F'_x}{F'_y} = \frac{r'}{q'}, \quad p = 0,\pm1,\ldots \tag{7}$$

(here F'_x and F'_y are components of $\mathbf{F}$ in the coordinate system determined by the primary axes of the square lattice [19]). Let us denote by G_R and G_I the real and imaginary parts of the function (6). After the substitution $u = (c_1+c_2)\exp(i\epsilon\int G_R\mathrm{d}\chi)$ and $v = (c_1 - c_2)\exp(-i\epsilon\int G_R\mathrm{d}\chi)$ Eq. (5) takes the form

$$i\frac{\mathrm{d}u}{\mathrm{d}\chi} = \epsilon Xv, \quad i\frac{\mathrm{d}v}{\mathrm{d}\chi} = \epsilon X^*u, \tag{8}$$

where $X = -iG_I\exp(2i\epsilon\int G_R\mathrm{d}\chi)$. Since for $t_3 = t_2$ the imaginary part of the function (6) vanishes, the solution of (8) is the constant function $(u,v)^T = (u_0,v_0)^T$. Then the energy spectrum (7) follows from the requirement (quantization condition) that the functions $\tilde{\psi}^{A,B} = \tilde{\psi}^{A,B}(\chi;\kappa)$ are periodic functions of χ with the period $2\pi/d$.

We proceed with the general case $t_3 \neq t_2$. Restricting ourselves by the second order over the parameter $\epsilon = 1/F$, the Bogoliubov equation for the column function $(u,v)^T$ reads

$$i\frac{d}{d\chi}\begin{pmatrix} u \\ v \end{pmatrix} = \epsilon\begin{pmatrix} 0 & \langle X\rangle \\ \langle X^*\rangle & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \epsilon^2\begin{pmatrix} \langle -iX'^*X\rangle & 0 \\ 0 & \langle -iX'X^*\rangle \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \quad (9)$$

where angular brackets denote the average over the period $2\pi/d$ and the prime sign is a shortcut for integral from the oscillating part of X, i.e., $X'(\chi) = \sum_{\nu\neq 0}\frac{\exp(i\nu\chi)}{i\nu}X_\nu$. The solution of Eq. (9) is $(u,v)^T = \exp(-i\epsilon\lambda\chi)(u_0,v_0)^T$ where

$$\lambda = \pm\sqrt{|\langle X\rangle|^2 + \epsilon^2\langle -iX'X^*\rangle^2} \quad (10)$$

(notice that $\langle -iX'X^*\rangle$ is real). The meaning of the quantity $\lambda = \lambda(\kappa)$ is the correction to the energy spectrum (7). Using the explicit form of X this correction can be expressed in terms of the Bessel functions. For the mean value of X we have

$$\langle X\rangle = \begin{cases} -i(t_3 - t_2)\sum_{n,m}\mathcal{J}_m(z_1)\mathcal{J}_n(z_2)\sin\left[\kappa d\frac{r^2+q^2}{r-q}(1+n)\right], & (n+m) \text{ is even}, \\ -(t_3 - t_2)\sum_{n,m}\mathcal{J}_m(z_1)\mathcal{J}_n(z_2)\cos\left[\kappa d\frac{r^2+q^2}{r-q}(1+n)\right], & (n+m) \text{ is odd}, \end{cases} \quad (11)$$

where integer numbers n and m satisfy the equation $(r-q)m = -(r+q)(1+n)$ and arguments of the Bessel functions are

$$z_1 = \frac{8t_1}{Fd(r-q)}, \quad z_2 = \frac{4(t_2+t_3)}{Fd(r+q)}. \quad (12)$$

Analogously, the mean $\langle -iX'X^*\rangle$ is given by the product of four Bessel functions (see Appendix). Equations (10)–(12) constitute the main result of the paper. As shown below, these equations provide an estimate for the maximal width of the bands and describe asymptotic behavior of the energy bands in the limit $F \to \infty$.

III. NUMERICAL EXAMPLES

Let us consider as a generic example the case $(r,q) = (2,1)$. For this direction of the static force Eq. (11) takes the form

$$\langle X\rangle = (t_3 - t_2)[\mathcal{J}_0(z_1)\mathcal{J}_1(z_2) + \mathcal{J}_3(z_1)\mathcal{J}_0(z_2)\cos(5\kappa d) - \mathcal{J}_3(z_1)\mathcal{J}_2(z_2)\cos(5\kappa d) - \mathcal{J}_6(z_1)\mathcal{J}_1(z_2)\cos(10\kappa d) + \ldots]. \quad (13)$$

Notice that arguments of the Bessel functions are proportional to $1/F$. Thus different terms in the square brackets have different asymptotic if $F \to \infty$. In Eq. (13) we keep all terms up to the seventh power of $1/F$, and we checked that the next order Bogoliubov correction does not contain terms larger than $(1/F)^8$. The inset in Fig. 2 compares the dispersion relation calculated by using Eq. (13), open circles, with the exact numerical results, solid lines [20]. Nice correspondence is noticed. An important characteristic of the depicted dispersion relation is the total band width ΔE which we focus on from now on. The band width ΔE as the function of F is shown in the main panel in Fig. 2. It follows from Eqs. (10) and (13) that for $F \to \infty$ the width decreases as $1/F^3$. In the opposite limit the width takes its maximal value at $F \approx 4.5$. Remarkably, already the first term with κ dependence in the series (13) provides an accurate estimate for this maximal value (compare the solid and dashed lines in Fig. 2).

Next we discuss an interesting case $t_3 = -t_2$, which can be viewed as a charged particle in a square lattice in the presence of a staggered magnetic field with π flux through the elementary cell. The system of this kind can be realized with cold atoms in a square optical lattice by properly driving the atoms by additional laser beams [21]. As follows from Eq. (12), for $t_3 = -t_2$ the argument z_2 equals to zero, that essentially simplifies all equations. For example, in the above considered case $(r,q) = (2,1)$ the first Bogoliubov approximation contains only one term and the dispersion relation reads

$$E(\kappa) = -2t_3\mathcal{J}_3\left(\frac{8t_1}{Fd}\right)\cos(5\kappa d). \quad (14)$$

Figure 3 compares Eq. (14) with numerical results for two sets of hopping amplitudes: $(t_1,t_2,t_3) = (1,0.25,-0.25)$ and $(t_1,t_2,t_3) = (1,0.5,-0.5)$. Almost complete band collapses are clearly seen in the figure. Comparing the upper and lower panels we also conclude that the actual parameter of the perturbation theory is $|t_2 - t_3|/F$ but not just $1/F$. In general, the smaller $|t_2 - t_3|$ the further we can go in the region of small F.

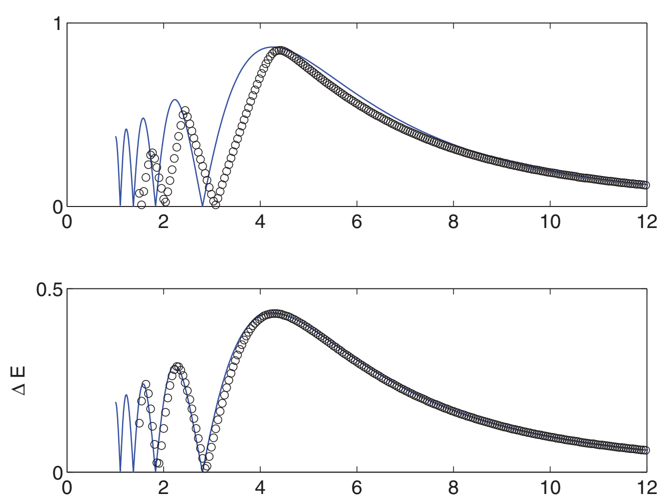

FIG. 3. (Color online) Width of the energy bands as the function of F' for $(r,q) = (2,1)$ and $(t_1,t_2,t_3) = (1,0.25,-0.25)$, lower panel, and $(t_1,t_2,t_3) = (1,0.5,-0.5)$, upper panel.

IV. CONCLUSIONS

In conclusion, we analyzed quantum particle in tilted 2D lattices with square symmetry for orientations of the static force **F** given by the rationality condition $F_x/F_y = r/q$. It is shown that for these orientations the system has common features with the other fundamental problem—the particle in resonantly driven 1D lattices. Namely, the energy bands of both systems show nonmonotonic behavior when a control parameter (for example, the static force F) is varied, with partial or complete band collapse. We developed analytical method which provides explicit expression for the particle dispersion relation. The reported results can be verified in present-day laboratory experiments with cold atoms in 2D optical lattices by studying the ballistic spreading of atoms. Finally, we remark that the presented analysis can be also viewed as a theory of the 2D Wannier-Stark states that remain an intriguing problem of the single-particle quantum mechanics.

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APPENDIX A: THE CASE $r = q$

If $t_3 = t_2$ and the static force is aligned with vertical or horizontal bonds (i.e., r' or q' equals to zero), Eq. (7) takes the form

$$E_p(\kappa) = aFp - 2t_{1,2}\cos(\kappa a). \tag{A1}$$

Notice that the energy bands have a finite width which is independent of F. If $t_3 \neq t_2$ we meet a similar situation for $(r,q) = (1,1)$ where the particle tunnels along the vertical bonds. Including the first-order correction the dispersion relation reads

$$E(\kappa) = \pm\sqrt{(t_2 - t_3)^2\mathcal{J}_1^2(z) + 4t_1^2\cos^2(\kappa d)},\quad z = 2(t_1 + t_3)/Fd, \tag{A2}$$

and is seen to coincide with Eq. (A1) in the limit of large F.

The case $(r,q) = (1,-1)$ is more involved. In this case Eq. (A2) is substituted by

$$E(\kappa) = \pm\sqrt{(t_2 - t_3)^2\mathcal{J}_0^2(z)\sin^2(\kappa d) + (t_2 + t_2)^2\cos^2(\kappa d)},\quad z = 4t_1/Fd. \tag{A3}$$

It follows from Eq. (A3) that $E(\kappa)$ may have different asymptotic depending on the hopping amplitude t_3. Namely, if $t_3 = t_2$ (simple square lattice) we recover the dispersion relation (A1), while for $t_3 = 0$ (honeycomblike lattice) we have

$$E_p(\kappa) \approx t_2\left(1 - \frac{4t_1^2}{F^2}d^2\sin^2(\kappa d)\right), \tag{A4}$$

where the band width decreases as $1/F^2$.

APPENDIX B: SECOND ORDER CORRECTIONS

We present the explicit form of the second-order correction given by the term $\langle -iX'X^*\rangle$. As was mentioned in the main text, this correction is given by the product of four Bessel functions with the indexes n, m, n', and m', respectively. We have two contributions. The first contribution is given by

$$A = \sum_{\nu_+(n,m)=\nu_-(n',m')\neq 0} -\frac{(t_3 - t_2)^2}{d\nu_+(n,m)}\mathcal{J}_m(z_1)\mathcal{J}_n(z_2)\mathcal{J}_{m'}(z_1)\mathcal{J}_{n'}(z_2) \times \cos[(\mu_+(n,m) - \mu_-(n',m'))\kappa d/2], \tag{B1}$$

where

$$\nu_\pm(m,n) = m(r - q) + (n \pm 1)(r + q),\quad \mu_\pm(m,n) = (n \pm 1)(r - q) - m(r + q). \tag{B2}$$

The second contribution refers only to the case where $n + n' + m + m'$ is an odd number and reads

$$A = \sum_{\nu_+(n,m)=\nu_+(n',m')\neq 0} \frac{(t_3 - t_2)^2}{d\nu_+(n,m)}\mathcal{J}_m(z_1)\mathcal{J}_n(z_2)\mathcal{J}_{m'}(z_1)\mathcal{J}_{n'}(z_2) \times \cos[(\mu_+(n,m) - \mu_+(n',m'))\kappa d/2]. \tag{B3}$$

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[20] The bands of two Wannier-Stark ladders are related to each other by a reflection symmetry. Namely, given $E(\kappa)$ the dispersion relation for energy bands of the first ladder, the dispersion relation for energy bands of the second ladder is $-E(\kappa)$ if $(r+q)/2$ is an integer and $-E(\kappa) + Fd/2$ if $(r+q)/2$ is a half integer.

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