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Hydrodynamics of Euler incompressible fluid and the fractional quantum Hall effect

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We show that the fractional quantum Hall effect can be phenomenologically described as a special flow of a quantum incompressible Euler liquid. This flow consists of a large number of vortices of the same chirality. In this approach each vortex is identified with an electron while the fluid is neutral. We show that the Laughlin wave function naturally emerges as a stationary flow of the system of vortices in quantum fluid dynamics. Here we develop the hydrodynamics of the vortex liquid and are able to consistently quantize it. As a demonstration of the efficiency of the hydrodynamics we show how subtle features of the fractional quantum Hall effect such as effects of Lorentz shear stress, the structure function, the Hall current in a nonuniform magnetic field, and Hall conductance in a curved spatial landscape naturally follow from the hydrodynamics approach.

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In the fractional quantum Hall regime (FQH) electrons form a perplexing quantum liquid. Some major characteristics of this liquid are well established theoretically and experimentally: the liquid is incompressible,[1] almost dissipation free,[2,3] the Hall conductance is quantized,[2] excitations are vortices which carry a fraction of the negative electronic charge,[1] and neutral bulk excitations are gapped.[3] More subtle properties in the focus of recent interest are the Lorentz shear force and odd viscosity (also known as anomalous or Hall viscosity).[5–10]

A natural approach to the fractional quantum Hall effect (FQHE) advocated in a seminal paper[4] is quantum hydrodynamics. Quantum hydrodynamics is based on a set of fundamentally restrictive assumptions that long and slow waves are described exclusively by a closed system of conservation laws. Hydrodynamics is often difficult to derive from an *ab initio* microscopic basis, but once developed has a predictive power and could be tested against the known properties.

Here we extend this approach. First, irrespectively of the FQHE, we develop the quantum hydrodynamics of the vortex flow in the two-dimensional incompressible Euler fluid. Then we see how major concepts of the FQHE, such as Laughlin's wave function and fractionalization of the Hall conductance and excitations, emerge in the Euler hydrodynamics. Then we obtain more subtle properties of FQHE, the Lorentz shear force and anomalous viscosity. All naturally follow from the hydrodynamics of the quantum Euler fluid.

This is, of course, not an accidental coincidence, but rather a confirmation of the conceptional viewpoint that the major properties of the FQHE are governed by symmetries and the underlying geometry of the states. The hydrodynamics reveals and clarifies the symmetries.

As a demonstration of the effectiveness of the hydrodynamic approach, we compute the spectral function and linear response of the electronic fluid to nonuniform electric and magnetic fields, the density profile at the quasihole, and accumulation of charge on a curved surface.

We consider only Laughlin's cases, where fraction ν is an inverse of an odd integer, say $1/3$. Extensions of the hydrodynamic approach to FQH states possessing external symmetries will be discussed elsewhere.

Historically the quantum hydrodynamics goes back to studies of the superfluid helium by Landau[11] and Feynman.[12] A quest for the hydrodynamics of the FQH liquid was originated in Ref. 4. Earlier approaches to FQHE[13–16] were essentially related to hydrodynamics as explained in Ref. 16. Hydrodynamics of FQH liquid is a focus of renewed interest.[5–10]

Among the vast variety of flows in the incompressible Euler fluid only one special class of flows is relevant to FQHE. This is a *turbulent flow* where vorticity is proportional to the volume. Such flow consists of a dense system of quantized vortices, all oriented in one direction. We will be interested in a regime where vortices themselves constitute a liquid, the *vortex liquid*.

In this Rapid Communication we present the development of the hydrodynamics of such vortex fluid in a close analog of the Feynman theory of rotating superfluid helium;[12] see also Ref. 17, where a similar setting occurs in the regime when the lattice of vortices is melted. The difference, however, is crucial: in contrast to helium, the FQH liquid is *incompressible*.

Then we observe that properties of the vortex liquid are identical to the FQH electronic liquid. In other words, external forces applied to the vortex liquid (not to the liquid itself, but to vortices) generate the same motion as FQH-electronic liquid under electric and magnetic fields.

This observation suggests a phenomenological picture of FQHE: collective electronic states are localized on vortices, the topological configurations, of a neutral incompressible liquid. The liquid itself is a neutral agent which mediates interaction between electrons. The similar picture is known in organic conductors (see, e.g., Ref. 18). There electrons occupy the core of topological configurations (kinks) of ion displacements, the neutral field mediating electronic interaction.

Quantization of incompressible hydrodynamics is a subtle matter due to its nonlinear nature. In this Rapid Communication we present the consistently quantized hydrodynamics. We achieve it through quantization of Kirchhoff equations for vortices dynamics.

We start by a brief discussion of the energy and length scales in the fluid mechanics and FQHE. In hydrodynamics only few basic principles, symmetries, and phenomenological parameters suffice to formulate fundamental equations. The phenomenological parameters of the quantum hydrodynamics is the circulation of each vortex $2\pi\Gamma$. The characteristic of the flow is the mean density of vortices $\bar{\rho}$. We assume that the liquid performs a solid rotation with the frequency $\Omega = \pi\Gamma\bar{\rho}$, such that the net vorticity vanishes. The energy of the solid rotation $\hbar\Omega$ is the only energy scale of the flow.

On the other hand the only energy scale in the FQHE is given by the gap in the excitation spectrum Δ_ν, typically $\Delta_\nu \sim 10$ K.[3] This scale is controlled by the Coulomb interaction. It is customary to introduce a scale of mass associated with the this energy setting $m_\nu \sim \hbar^2/\Delta_\nu \ell^2$, where $\ell = \sqrt{\hbar/eB}$ is magnetic length. The very existence of the FQH state requires that the gap to be less than the cyclotron frequency $\Delta_\nu \ll \hbar\omega_c$, so that all states are confined on the lowest Landau level. This means that m_ν exceeds the band electronic mass m_b.

In the absence of other scales it appears that $\hbar\Omega$ and Δ_ν, and that $\bar\rho$ and ℓ^{-2} are of the same order. Then the scale of the vortex circulation is $\Gamma \sim \hbar/m_\nu$ and m_ν is the inertia of the fluid.

The states on the lowest Landau level are holomorphic. We will see that this property means that the electronic liquid is incompressible. Velocity is divergence free.

The existence of the energy scale within the Landau level is the physical input justifying the hydrodynamics of the FQHE. For that reason the hydrodynamics description does not extend to the integer case, where interaction is weak and the cyclotron energy is the only scale. The role of interaction could be seen within the hydrodynamics itself. An incompressible liquid does not possess linear waves except on the edge.[19] All flows are nonlinear.

Fractional and integer Hall effects can be treated in parallel and within the hydrodynamic approach only in the *topological sector* singled out by the limit $\Delta_\nu \to \infty$. Flows in this sector are steady, such as the Hall current. After these comments we turn to the Euler hydrodynamics. We start from the classical case.

Incompressible $\nabla \cdot u = 0$ flows in two dimensions are fully characterized by its vorticity $\omega = \nabla \times u$, where u is the fluid velocity. Vorticity obeys a single equation, which in the case of inviscid fluid has a simple geometrical meaning: the material derivative of the vorticity vanishes

$$D_t\omega \equiv \left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\omega = 0, \quad \nabla \cdot u = 0. \tag{1}$$

Vorticity is transported along divergence-free velocity.

In the class of Helmholtz solutions the complex velocity is a meromorphic function. In the rotating frame

$$u(z,t) = -i\Omega\bar z + i\sum_{j=1}^{N}\frac{\Gamma_j}{z - z_j(t)}. \tag{2}$$

Here Γ_j and $z_j(t)$ are circulations and positions of vortices. The Kelvin theorem insures that the number of vortices N and their circulations Γ_i do not evolve.

A substitution of the "pole Ansatz" into the Helmholtz equation (1) expresses the velocity of vortices as a sum the Magnus forces exerted by other vortices

$$v_i \equiv \dot{\bar z}_i = i\Omega\bar z_i + i\sum_{i\neq j}^{N}\frac{\Gamma_j}{z_i(t) - z_j(t)}. \tag{3}$$

This dynamical system is called the Kirchhoff equations.[20] It replaces the nonlinear PDE (1). The equations describe chaotic motions if $N > 3$. In a proper limit of large N and small Γ solutions approximate virtually any flow.

We will be interested in the system of a large number of vortices $N \to \infty$, the turbulent flow, and specifically in the *chiral flow*, where all vortices have the same (minimal) circulation $\Gamma_i = \Gamma$. In this limit the vortex system must be treated as a liquid itself.

In the turbulent flow we distinguish two types of motion: a fast motion of the fluid around vortex cores, and a slow motion of vortices fluid. In particular, in the ground state of the vortex liquid, the vortices do not move, but the fluid does. In the stationary flow vortices are distributed uniformly with the mean density $\bar\rho = \Omega/(\pi\Gamma)$.

Kirchhoff equations are scale invariant. They do not change under a dilatation $z_i \to \lambda z_i, t \to \lambda^2 t, \bar\rho \to \lambda^{-2}\bar\rho$ and for that reason do not consist of any energy scale. In order to write the Hamiltonian one needs to introduce an *ad hoc* scale of energy. Bearing in mind application to FQHE, we set it to be Δ_ν. Then the Hamiltonian

$$\mathcal{H} = \Delta_\nu \sum_i\left(\pi\bar\rho|z_i|^2 - \sum_{j\neq i}\log|z_i - z_j|^2\right), \tag{4}$$

and the Poisson brackets $\{\bar z_i, z_j\}_{\rm PB} = \frac{\Gamma}{i\Delta_\nu}\delta_{ij}$ reveal the Kirchhoff equation (3). The scale Δ_ν disappears from the equations.

Now we proceed with the quantization. The first step is to replace the Poisson brackets by the commutators

$$\{\bar z_i, z_j\}_{\rm PB} \to [\bar z_i, z_j] = 2\ell^2\delta_{ij}, \tag{5}$$

where we denote $2\ell^2 = \hbar\Gamma/\Delta_\nu$. It has a dimension of area. The ratio between this scale and the area per particle $\nu = 2\pi\bar\rho\ell^2$ is the dimensionless semiclassical parameter. We will see in a moment that ν appears to be the filling fraction, and ℓ to be the magnetic length.

At the next step we must specify the space of states. We assume that states are holomorphic polynomials of z_i. Then operators $\bar z_i$ are canonical momenta

$$\bar z_i = 2\ell^2\partial_{z_i}. \tag{6}$$

The last step is to specify the inner product. We impose the *chiral condition*: operators $\bar z_i$ and z_i are assumed to be Hermitian conjugated

$$\text{chiral condition}: \quad \bar z_i^\dagger = z_i. \tag{7}$$

The conditions (6) and (7) identify the set of states with the Bargmann space:[4,21] the Hilbert space of analytic polynomials $\psi(z_1,\ldots,z_N)$ with the inner product

$$\langle\psi'|\psi\rangle = \int e^{-\sum_i\frac{|z_i|^2}{2\ell^2}}\,\overline{\psi'}\psi\, d^2z_1\ldots d^2z_N. \tag{8}$$

Equations (3) and (6) help to write quantum velocity operators as

$$p_i = -i\hbar\left(\partial_{z_i} - \sum_{j\neq i}\frac{\beta}{z_i - z_j}\right), \quad \beta = \nu^{-1}, \tag{9}$$

where we set $p_i = m_\nu v_i$ and the effective mass $m_\nu = \hbar/(\nu\Gamma)$. Operators p_i are the many-body version of the guidance center coordinates or coordinates of magnetic translations.

At this stage the Kirchhoff equations are readily identified with the FQHE in a disk geometry. There the electronic droplet occupies a volume confined by a weak potential.

We recall that the Bargmann space is just another way to say that all states belong to the lowest Landau level. In that representation the wave functions are written in the radial gauge with respect to a marked point (the origin) inside the droplet (see, e.g., Ref. 4 for details). Apart from the factor $\exp(-\sum_i |z_i|^2/2\ell^2)$, treated as a measure, the states are holomorphic polynomials.

Let us determine the ground state of the vortex liquid. There all velocities vanish, $p_i\psi_0 = 0$. The common solution of the set of first-order PDEs is the Laughlin function

$$p_i\psi_0 = 0, \quad \psi_0 = \prod_{i>j}(z_i - z_j)^\beta, \quad \nu = \beta^{-1}. \tag{10}$$

The wave function is single valued if β is an integer. Depending on whether β is chosen to be an odd or even integer the vortices are fermions or bosons. In particular, $\beta = 2$ is believed to describe the rotating Bose condensate of trapped atoms. At $\beta = 3$ we obtain the Laughlin $\nu = 1/3$ state.

We observe that the Laughlin states, fermionic or bosonic alike, naturally emerge from the quantum hydrodynamics of the vortex fluid. In this approach, the fraction appears as a parameter of the quantization.

In the hydrodynamics interpretations "particles" entered into the Laughlin function are vortices of the incompressible fluid. In the FQHE particles are electrons, with electric charge. To complete the hydrodynamics description we must identify electric and magnetic field as field acting on vortex cores. To this end we add a potential $\sum_i U(r_i)$ to the energy (4), where r_i are coordinates of vortices. It exerts the force $-i[U,\bar z_i] = i2\ell^2\partial_{z_i}U$ added to the Kirchhoff equations

$$v_i = -i\Omega\bar z_i + i\sum_{j\neq i}\frac{\Gamma}{z_i - z_j} + i\ell^2 eE, \tag{11}$$

where $eE = -\nabla U$ plays the role of the electric field. The electric field acts normal to velocity. It does not accelerate the flow since vortices have no "mass." It must not be confused with the $m_\nu = \hbar/(\nu\Gamma)$, the inertia of the fluid. Thus we identify the angular velocity with the cyclotron frequency of vortices $\Omega = eB/m_\nu = (m_b/m_\nu)\omega_c \ll \omega_c$,$\ell$ with the magnetic length $\ell = \sqrt{\hbar/eB}$, and $\nu = \hbar/(m_\nu\Gamma)$ the filling fraction.

To illustrate the assignment of electric charges to vortices we invoke a similar phenomena known in organic conductors.[18] There electronic states are localized on cores of kinks of ion displacements and move together if the motion is adiabatic. The kinks are the topological configurations of the 1D phonon field. Here, in a very similar manner electronic states are trapped by vortices, the topological configurations in 2D. This is only the illustration. It does not explain a microscopical mechanism of attachment of the vortex circulation to the electron, but rather provides a hydrodynamics interpretation to the commonly used concept of the "flux attachment."

Quantization of the Hall conductance elegantly follows from the Kirchhoff equations (11). Let us assume that the electric field is uniform and sum up all the equations. We obtain the relation between the e.m. current and the electric field $N^{-1}e\sum_i(v_i + i\Omega\bar z_i) = i\ell^2e^2E$ with the fractionally quantized conductance $\sigma_{xy} = \nu(e^2/h)$. If the electric field is not uniform, the Hall conductance possesses universal corrections described below.

The fractionalization of quasiholes is another easy consequence of the Kirchhoff equations. The quasihole[1] is a state with the wave function $\psi_h = \prod_i(z - z_i)\psi_0$. The operator (9) acting on this state is

$$p_i\psi_h = -i\hbar\left(\partial_{z_i} + \frac{1}{z_i - z} - \sum_{j\neq i}\frac{\beta}{z_i - z_j}\right)\psi_0.$$

It shows that the Magnus force exerted by vortices to the quasihole is the the fraction ν of the forces between vortices and acts in the opposite direction. Thus in the hydrodynamic interpretation the quasihole appears is a vortex with a fractional negative circulation $-\nu$, an antivortex, or a hole in the uniform "Fermi sea" of vortices.

These arguments seem to justify Eqs. (4), (5), (8), and (11) as a complete minimal set of FQHE dynamics.

Our next goal is to obtain the hydrodynamic description of the vortex fluid. From the hydrodynamics standpoint, the coordinates of vortices are treated as Lagrangian specification of fluid parcels. To pass to the Eulerian specification we must consider the macroscopic conserved fields, the vortex density and the vortex flux

$$\rho(r) = \sum_i\delta(r - r_i) = \bar\rho + \frac{1}{2\pi\Gamma}(\nabla\times u), \tag{12}$$

$$\mathcal{J}(r) = \sum_i\delta(r - r_i)v_i, \tag{13}$$

compute them, and determine velocity through the relation

$$\mathcal{J}(r) = \rho(r)v(r).$$

By construction the flux annihilates the ground state

$$\mathcal{J}|0\rangle = \langle 0|\mathcal{J}^\dagger = 0.$$

To the best of our knowledge this program has never been set up even for the classical fluids. Below we outline the major step. To simplify the formulas we compute the flux classically. The quantum result is the same, providing the ordering of operators is kept.

We write the vortex flux

$$\mathcal{J} = \sum_i\delta(r - r_i)\left[-i\Omega\bar z_i + \sum_{j\neq i}\frac{\Gamma}{z_i - z_j}\right] \tag{14}$$

and use the $\bar\partial$ formula $\pi\delta = \bar\partial(\frac{1}{z})$ and the identity

$$2\sum_{i\neq j}\frac{1}{z - z_i}\frac{1}{z_i - z_j} = \left(\sum_i\frac{1}{z - z_i}\right)^2 - \sum_i\left(\frac{1}{z - z_i}\right)^2.$$

A simple computation yields the important relation between the vortex flux and the vorticity flux

$$\begin{aligned}\mathcal{J} &= -i\rho\Omega\bar z + i\frac{\Gamma}{2}\bar\partial\left[\left(\sum_i\frac{1}{z - z_i}\right)^2 - \sum_i\frac{1}{(z - z_i)^2}\right]\\ &= \rho\left[-i\Omega\bar z + i\sum_j\frac{\Gamma}{z - z_j}\right] + i\frac{\Gamma}{2}\partial\rho\\ &= \rho u + i\frac{\Gamma}{2}\partial\rho.\end{aligned} \tag{15}$$

The first term on the right-hand side is the vorticity flux ρu. The second is the anomalous term. It appears because the velocity of the fluid u diverges at a core of an isolated vortex [as in Eq. (2)]. However, velocities of vortices are finite. The anomalous term removes that singularity.

In Cartesian coordinates the relation between velocity of the fluid and velocity of the vortex fluid reads [we denote $(\nabla\times)_a = \epsilon_{ab}\nabla_b$][22]

$$\mathcal{J} \equiv \rho v = \rho u + \frac{\Gamma}{4}\nabla\times\rho = \rho u - \frac{1}{4\pi}\Delta u. \tag{16}$$

The meaning of the anomalous term is seen from the geometric phase of the FQH states. That is the phase acquired by the state when a chosen particle moved around a closed path encompassed all other particles. In units of 2π it equals the number of zeros of the wave function with respect to a chosen particle N_ϕ and equals the number of fluxes of magnetic field $N_\phi = (N-1)\beta$ in the disk geometry. The "shift," i.e., the difference between N and νN_ϕ, is the contribution of the anomalous term. It can be seen as a result of integration of the shift relation (16) over a contour encompassing the droplet. The condition (16) is the local version of the "shift," the global relation between the magnetic flux and the number of particles (see, e.g., Ref. 23).

We see that the vortex flow is incompressible like the fluid itself and that the Helmholtz equation (1) emerges as the continuity equation for the vortex liquid

$$\mathcal{D}_t\rho = 0, \quad \mathcal{D}_t = \partial_t + v\cdot\nabla, \quad \nabla\cdot v = 0. \tag{17}$$

The relation (16) has far reaching consequences. One of them is the Lorentz shear stress.

The rotating fluid parcel experiences the Coriolis force $\rho F = -m_\nu\Omega\times(\rho u)$. This force also acts on vortices. To find its action we express it through the velocity of the vortex fluid. With the help of the shift formula (16) neglecting higher orders in gradients we obtain

$$\rho F \approx eB\times(\rho v) - \frac{\hbar}{4\nu}\bar{\rho}\nabla(\nabla\times v). \tag{18}$$

The first term here is the familiar Lorentz force; the second is the Lorentz shear force. The universal coefficient translated to this formula from (16) is the anomalous viscosity.

The anomalous force could be written as a divergence of the symmetric Lorentz shear stress tensor $F_a = \nabla_b\sigma'_{ab}$, which is best written in terms of the stream function

$$\sigma'_{ab} = \frac{\hbar}{2\nu}\left(\nabla_a\nabla_b - \frac{1}{2}\delta_{ab}\Delta\right)\Psi, \quad v = -\nabla\times\Psi. \tag{19}$$

The anomalous stress is conservative and traceless. To compare, the dissipative shear viscous tensor is given by the same formula where the stream function is replaced by the hydrodynamic potential.

Initially introduced for the integer QHE in Ref. 5 it has been extended to the FQHE in Refs. 6 and 7. In fact, the Lorentz shear force is the hydrodynamic and also classical phenomena reflecting the discreteness of vortices.

The anomalous force could be visualized as a strain of orbits of the fluid around the vortex cores by the shear flow. The flow elongate them normal to the shear squeezing together flow lines with different velocity exerting additional force toward the boundary.

We see that the Lorentz shear force naturally emerges in the hydrodynamics of the vortex flow. To obtain further applications, we need the hydrodynamic form *chiral condition* (7). From now on we set $m_\nu = 1$, or $\Gamma = \hbar/\nu$.

In classical incompressible fluids positions of vortices determine their velocities, as seen from the classical Kirchhoff equation. The chiral condition (7) insures that the same is true in the quantum case. In hydrodynamics terms this means that the vortex flux $\mathcal{J}$ and the velocity are determined by the density of vortices ρ. This is the chiral consistency condition we want to obtain. It reflects the holomorphic nature of states, or equivalently the incompressibility of the fluid, or that all states belong to the first Landau level.

The chiral consistency relation is obtained when we apply "normal ordering" to the shift equation (15). This means to place the holomorphic operator of velocity u to the left next to the "bra" antiholomorphic state. Then u possesses no differential operators and acts classically as a solution of (12)

$$\langle\ldots|u = \langle\ldots|(-ih/\nu)\partial\varphi, \quad \Delta\varphi = -4\pi(\rho-\bar{\rho}).$$

The normal ordering is achieved with the help of canonical equal point commutation relation $[u(r),\rho(r)] = i\hbar\partial\rho(r)$. It essentially changes the coefficient in the shift equation (16)

$$\mathcal{J} = u\rho - i\hbar\partial\rho = i\frac{h}{\nu}\rho\left(\partial\varphi + \left[\frac{1}{2}-\nu\right]\partial\rho\right). \tag{20}$$

This is the *chiral constituency condition*.[24] It expresses the flux in terms of one- and two-point density functions. The consistency condition is especially efficient in the topological sector, where physics is bound to the leading gradients. In this regime we may treat the relation (20) classically as we assume below. In the remaining part of the Rapid Communication we list an incomplete set of applications emphasizing the role of the anomalous term.

(a) *Flux attachment and the profile of the quasihole*. Let us divide Eq. (18) and (20) by ρ and take a curl of (20)

$$\nabla\times v = \frac{h}{\nu}\left[\rho - \bar{\rho} + \frac{1}{4\pi}\left(\frac{1}{2}-\nu\right)\Delta\log\rho\right], \tag{21}$$

$$F \approx eB\times v + \frac{\hbar}{2\nu}\left(\frac{1}{2}-\nu\right)\nabla(\nabla\times v). \tag{22}$$

If the last term in (21) is ignored the vorticity of the flow follows the density of particles times the filling fraction. This condition has been suggested in Ref. 16 as a basis for the hydrodynamics of FQHE and reflects a popular picture that FQH states are electronic states with attached additional magnetic flux. The *anomalous term* corrects this concept.

In the linear approximation a modulation of the density $\rho_k = \sum_i e^{ik\cdot r_i}$ causes velocity

$$v_k \approx \frac{h}{\nu}\frac{k}{k^2}\left[1 - \frac{1}{2\nu}\left(\frac{1}{2}-\nu\right)(k\ell)^2\right]\rho_k. \tag{23}$$

Equation (21) can be used to find density profiles for various coherent states. For example a quasihole is a source $\nabla\times v = -h\delta(r-r_0)$ in the equation (21). Outside the core and in the leading gradients the quasiholes cause a modulation:[25]

$$\rho_k^{(h)} \approx (\bar{\rho}-\nu)\delta_{k,0} - \left[\nu - \tfrac{1}{2}\left(\tfrac{1}{2}-\nu\right)(k\ell)^2\right].$$

(b) *Structure function*. The structure function

$$s_\nu(k) = N^{-1}\langle 0|\rho_k\rho_{-k}|0\rangle$$

is the correlation of density modes. To compute it we use the hydrodynamics commutation relation

$$[v(r),\rho(r')] = -i\hbar\partial\delta_{rr'}$$

followed from (13) and (9). We recall that the holomorphic velocity annihilates the "ket" vacuum. Therefore

$$\langle 0|v_k,\rho_{-k'}|0\rangle = \frac{1}{2}\hbar k\delta_{k,k'}.$$

Substitute (23) there and obtain the celebrated result of Ref. 4 (see Ref. 26),

$$s_\nu(k) \approx \frac{1}{2}(k\ell)^2\left[1+\frac{1}{2\nu}\left(\frac{1}{2}-\nu\right)(k\ell)^2\right]. \quad (24)$$

(c) *Nonuniform electric field*. At the steady state the electric field balances Lorentz force plus the Lorentz shear force (22) balanced $F = eE$. Solution of this equation gives the Hall current $e\bar\rho v_k = \sigma_{xy}(k)E_k$. The Hall conductance acquires the universal correction[8]

$$\sigma_{xy}(k) = \frac{\nu e^2}{h}\left[1+\frac{1}{2\nu}\left(\frac{1}{2}-\nu\right)(k\ell)^2\right]. \quad (25)$$

Equations (24) and (25) reflect a general relation between the structure function and the Hall conductance. That is $es_\nu(k) = (B/4\pi\nu)k^2\sigma_{xy}(k)$.

(d) *Nonuniform magnetic field*. A similar relation occurs between the density and a nonuniform magnetic field. A nonuniform magnetic field enters into the relation (21) through the mean density $\bar\rho = \frac{\nu}{h}eB$. At the ground state where velocity vanishes Eq. (21) becomes the Liouville-like equation for the density. In the leading gradients this equation yields the generalized Streda formula, that is the relation between the density and a weakly nonuniform magnetic field $e\langle 0|\rho_k|0\rangle = \sigma_{xy}(k)B_k$. The Hall conductance $\sigma_{xy}(k)$ in this formula is given by (25). It has been computed for the case of a uniform magnetic field and a nonuniform electric field.

(e) *Accumulation of charges in curved space*. Anomalous properties of FQHE are seen in a curved space. Here we mention just one. In a curved space the density (the number of particles per unit area $\rho\sqrt{g}dzd\bar z$) is not uniform but rather depends on the curvature

$$\rho = \bar\rho + \frac{1}{4\pi}R + O(\ell^2\Delta R). \quad (26)$$

The first term of the gradient expansion in the curvature follows from the shift formulas (16) In the curved space the density transformed as $\rho \to \rho\sqrt{g}$. Under this transformation the anomalous term in (16) acquires an addition $\frac{\hbar}{4\nu}\nabla\times\sqrt{g}$ which yields the term $-\frac{1}{2\pi}\frac{1}{\sqrt{g}}\Delta\log\sqrt{g}$ on the right-hand side of (21) and subsequently (26). Recall that $R = -\frac{2}{\sqrt{g}}\Delta\log\sqrt{g}$ is the Gaussian curvature. The next term in the expansion (26) is also universal, but requires a more involved analysis.

Particles/vortices accumulate at curved parts being pushed there by the Lorentz shear force. For example, a cone with the deficit angle α possesses extra $\alpha/4\pi$ particles located right at the vertex.

Equation (26) can be checked against the known formula for the number of particles at the Laughlin state on a Riemannian manifold. Integrating (26) and using the Gauss-Bonnet theorem we obtain

$$N = \nu N_\phi + \tfrac{1}{2}\chi,$$

where χ is Euler characteristic. This formula remains valid for surfaces with boundaries, cones, and parabolic singularities.

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[26]The authors of Ref. 4 computed the projected structure function $\bar{s}_\nu = s_\nu - s_1 = \frac{1}{8}(\frac{1}{\nu}-1)(k\ell)^4$.