PHYSICAL REVIEW B **88**, 241107(R) (2013)

Bilayer graphene with parallel magnetic field and twisting: Phases and phase transitions in a highly tunable Dirac system

Bitan Roy[1,2] and Kun Yang[3]

[1]*National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32306, USA*
[2]*Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742, USA*
[3]*National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306, USA*

(Received 6 August 2013; published 16 December 2013)

The effective theory for bilayer graphene (BLG), subject to parallel/in-plane magnetic fields, is derived. With a sizable magnetic field the trigonal warping becomes irrelevant, and one ends up with two Dirac points in the vicinity of each valley in the low-energy limit, similar to the twisted BLG. Combining twisting and parallel field thus gives rise to a Dirac system with tunable Fermi velocity and cutoff. If the interactions are sufficiently strong, several fully gapped states can be realized in these systems, in addition to the ones in a pristine setup. Transformations of the order parameters under various symmetry operations are analyzed. The quantum critical behavior of various phase transitions driven by the twisting and the magnetic field is reported. The effects of an additional perpendicular field and possible ways to realize the new massive phases are highlighted.

DOI: 10.1103/PhysRevB.88.241107 PACS number(s): 73.22.Pr, 05.30.Rt, 73.43.Nq

Carbon-based layered materials opened a new frontier in condensed matter physics, where the underlying honeycomb lattice stands responsible for Dirac or Dirac-like fermionic excitations.[1] Two well-studied members of this new class of materials are single-layer and bilayer graphene. Despite the fascinating potential of exhibiting various ordered phases[2,3] and related quantum critical phenomena,[4,5] the Dirac points of monolayer graphene are remarkably stable due to a large quasiparticle Fermi velocity ($v_F \sim 10^6$ m/s); thus far, ordered phases have only been realized in the presence of (perpendicular) magnetic fields.[6] In this regard, BLG appears to be propitious, and has already exhibited phenomena strongly suggestive of spontaneous symmetry breaking,[7–11] possibly realizing a subset of all the possible ordered states available for the fermion to condense into.[12] But the role of the mesoscopic environment, such as gate configuration, substrate, etc., on the nature of the ordered states still lacks a clear understanding.[13] As a result, realization of several interesting ordered states and tuning this system across (quantum) phase transitions are still among future prospects. We here propose that BLG, when immersed in parallel magnetic fields and twisted,[14] yields a unique opportunity to explore some of these interesting possibilities.

Pristine BLG is well described by a two-band model, with quadratic touching of the valence and the conduction bands. Subject to in-plane magnetic fields, each parabolic band touching (PBT) in BLG splits into *two* Dirac cones.[15] A similar scenario also arises when BLG is twisted, if the twisting is commensurate.[16,17] However, such twofold splitting competes with the trigonal warping (TW),[18] which, on the other hand, breaks each PBT into four Dirac cones.[19–21] We here show that when a sufficiently strong in-plane field is applied, one ends up with only two Dirac cones; this happens within accessible magnetic field strength when the field is applied along certain optimal directions (see Fig. 1). More importantly, the field/twisting controls the Fermi velocity of the resultant Dirac points, and thus the (effective) interaction strength; this allows us to tune the system across various transitions between the weak-coupling phase (where interactions are irrelevant[2]) to various ordered phases. Moreover, these setups admit additional fully gapped phases that do not have any analogy in either single-layer graphene or pristine BLG. When a perpendicular magnetic field is present,[22,23] even a richer set of new ordered phases may be realized.

Recently there has been a surge of theoretical[16,17,23–27] and experimental[22,28–31] activities in twisted BLG (mostly focusing on single-particle physics thus far), while BLG with in-plane field has attracted little attention thus far.[15] One of the motivations of the present work is to point out their similarity, and more importantly the fact that they are complementary to each other: Twisting gives rise to a larger effects and does not couple to electron spin, but is discrete. In-plane field has a weaker effect and couples to electron spin as well, but can be tuned continuously, a virtue important for exploring critical phenomena. We show that by combining these two we have a highly tunable Dirac system ideal for exploring various phases and phase transitions.

In terms of the low-energy degrees of freedom of AB-stacked BLG, we define a 4-component spinor $\Psi^\top(\vec{k}) = [v_1(\vec{K}+\vec{k}), v_2(\vec{K}+\vec{k}), v_1(-\vec{K}+\vec{k}), v_2(-\vec{K}+\vec{k})]$. For now, we suppress the fermion's spin degrees of freedom and $v_1(v_2)$ is the fermionic annihilation operator on the B sublattice in layer 1 (2).[32] In this basis the noninteracting Hamiltonian, in the vicinity of two valleys at $\pm\vec{K}$, reads as[33]

$$H^0_{\rm BL} = \gamma_2 \frac{k_x^2 - k_y^2}{2m} - \gamma_1 \frac{2k_x k_y}{2m} + v_2 i \gamma_0(\gamma_1 k_x + \gamma_2 k_y), \qquad (1)$$

where $m = t_\perp/(2v_F^2)$, and $v_F = 3t/(2a)$. t and a are respectively intralayer hopping amplitude and lattice spacing. The interlayer nearest-neighbor hopping is $t_\perp \approx t/10$, and $v_2 \sim v_F/30$ describes the TW.[34] Five mutually anticommuting γ matrices are $\gamma_0 = \sigma_0 \otimes \sigma_3$, $\gamma_1 = \sigma_3 \otimes \sigma_2$, $\gamma_2 = \sigma_0 \otimes \sigma_1$, $\gamma_3 = \sigma_1 \otimes \sigma_2$, and $\gamma_5 = \sigma_2 \otimes \sigma_2$, where $\vec{\sigma}$ are the standard two-dimensional Pauli matrices and σ_0 is the unity matrix. In the absence of the TW ($v_2 = 0$), $H^0_{\rm BL}$ describes PBTs at $\pm\vec{K}$.[35] The TW splits each of the PBTs into four Dirac cones, out of which one is isotropic, while the remaining three are

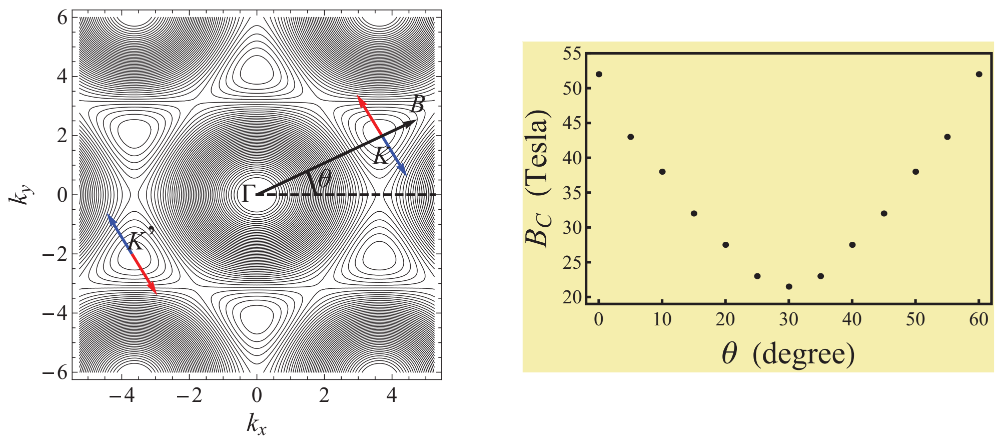

FIG. 1. (Color online) Left: Splitting of the Dirac points (red and blue arrows) due to in-plane field (B) (here $K' = -K$). k_x, k_y are measured in unit $\hbar = a = 1$. Right: A schematic variation of B_C with θ (see left), with $v_F = 10^6$ m/s, $t_\perp/t = 10$, $v_F/v_2 = 30$, $d = 3.5$ Å (Refs. 34,37).

anisotropic, connected by 120° rotations about the isotropic one.[20,21]

Subject to an in-plane magnetic field $\vec{B} = B(\cos\theta, \sin\theta, 0)$, the above Hamiltonian conforms to[36]

$$H[B] = H^0_{\rm BL} + \gamma_2 \left(\frac{k^2_{Bx} - k^2_{By}}{2m}\right) - \gamma_1 \left(\frac{2k_{Bx}k_{By}}{2m}\right), \qquad (2)$$

where $\vec{k}_B = (d/2)(\hat{z} \times \vec{B})$, and $d \sim 3.5$ Å is the interlayer separation.[37] Here θ is measured with respect to the momentum axis k_x, shown in Fig. 1 (left). If $v_2 = 0$, $H[B]$ describes isotropic massless Dirac fermionic excitations in the vicinity of four points $\pm\vec{K} \pm (k_{Bx}\hat{k}_x + k_{By}\hat{k}_y)$. The effect of the in-plane magnetic field (or twisting) is, therefore, qualitatively similar to the nematic order, which split the two PBTs into four Dirac cones by spontaneously breaking lattice rotation symmetry.[38] The in-plane field and twisting, which explicitly breaks lattice rotation symmetry, can thus be viewed as a field coupled to the nematic order parameter (and therefore forces a nonzero value on it). When v_2 is finite, one of the anisotropic Dirac cones and the isotropic one get pushed towards each other by the in-plane field, while the remaining two move apart from each other. If the in-plane magnetic field is applied along the line connecting the isotropic and one of the anisotropic Dirac points, i.e., $\theta = (\pi, 5\pi, 9\pi)/6$, the TW becomes irrelevant for $B > B_C \sim 25$ T, with the currently estimated strength of various band parameters,[34,37] shown in Fig. 1 (right). For the rest of the discussion, we set $v_2 = 0$.

To construct the low-energy theory, we first neglect electron spin for simplicity, and define an 8-component spinor $\Psi^\top_B(\vec{k}) = (\Psi_+, \Psi_-)(\vec{k})$, with $\Psi^\top_\pm(\vec{k}) = [v_1(\vec{K} \pm \vec{k}_B + \vec{k}),\ v_2(\vec{K} \pm \vec{k}_B + \vec{k}),\ v_1(-\vec{K} \pm \vec{k}_B + \vec{k}),\ v_2(-\vec{K} \pm \vec{k}_B + \vec{k})]$, which account for the layer, valley, and $\pm\vec{k}_B$ (hereafter referred as *flavor*) degrees of freedom. In this basis $H[B]$ takes the relativistic invariant form $H_D = -v_B i\Gamma_0(\Gamma_1\, k_x + \Gamma_2\, k_y)$, where $i\Gamma_0\Gamma_1 = 332$, $i\Gamma_0\Gamma_2 = 301$, and $\Gamma_0 = 003$.[39] An identical Hamiltonian also describes the low-energy theory in twisted BLG, when the twisting is commensurate, with $v_B \to v_T$. The effective Fermi velocities near the new Dirac points

$$v_B = \frac{v_F^2 B d}{t_\perp} \sim 1.2 \times 10^3 B(\mathrm{T})\frac{m}{s}, \quad v_T = |\vec{K}| \sin\phi \frac{v_F^2}{\tilde{t}_\perp}, \qquad (3)$$

can be tuned by the in-plane magnetic field and twisting, respectively. Here $\tilde{t}_\perp \approx 0.4 t_\perp$,[16,28] and ϕ measures deviation from the AB stacking. Respectively, in these two systems the dispersion is linear over the energy $v_B\Lambda_B \approx 10^{-5} B^2(\mathrm{T})$ K, and $v_T\Lambda_T \approx 1.76 \times 10^4 \sin^2\phi$ K, where $\Lambda_B \approx 5.6 \times 10^{-29} B(\mathrm{T})$ kg m/s and $\Lambda_T \approx \hbar \sin\phi/a$ are the associated cutoffs for momentum. Therefore, the effect of twisting is much stronger than in-plane field. However, when the in-plane field and twisting are present together, one can tune Dirac dispersion continuously and we focus on this setup, with Fermi velocity v_F and cutoff Λ coming from the combined effects of twisting (we consider only commensurate ones) and in-plane field and continuously tunable, unless specifically noted otherwise. The imaginary-time Lagrangian is $L_0 = \Psi^\dagger_B(\tau,\vec{k})\,(\partial_\tau + H_D)\,\Psi_B(\tau,\vec{k})$. The Dirac Hamiltonian H_D commutes with the generator of translation $I_{\rm Tr} = 330$.[39] It also commutes with $I_L = 001$, and $I_K = 010$, when accompanied by the momentum axis inversion $k_x \to -k_x$. Respectively, these two operators exchange two layers and valleys. Moreover, H_D is invariant under the exchange of the flavors, generated by $I_f = 100$, after taking $\vec{k} \to -\vec{k}$. Additionally, H_D is also invariant under emergent chiral $U_c(4)$ symmetry, generated by $\{x12, x22, x33,\ x03, y11, y21, y30, y00\}$, where $x = 1,2$ and $y = 0,3$.

If the repulsive interactions among the fermions are sufficiently strong, various ordered phases can be realized in this system. We assume there exist metallic gates nearby that render the Coulomb interaction to be short-ranged due to screening. In the long-wavelength limit the interacting Lagrangian for spinless fermions contains 27 quartic terms of the form $\tilde{g}\,(\Psi^\dagger_B M \Psi_B)^2$, out of which 18 are independent.[36] Next we wish to capture the leading instabilities in this system in the large-N scheme, in which all the couplings are independent. After integrating out the fast Fourier modes with the Matsubara frequencies $-\infty < \omega < \infty$ and $\Lambda/b < |\vec{k}| < \Lambda$, where $b > 1$, we obtain the flow equation of dimensionless couplings $g = 4\tilde{g}\Lambda/(v_F\pi)$:

$$\beta_g = \frac{dg}{d\,\log b} = -g - C_M\, g^2 + O\left(\frac{1}{N}\right). \qquad (4)$$

The coefficient $C_M = (1)\ 0$, if M (anti)commutes with H_D, whereas $C_M = 1/2$ if M commutes with either $i\Gamma_0\Gamma_1$ or $i\Gamma_0\Gamma_2$. Therefore, the leading instabilities in this system take place towards the formation of the fully gapped or massive[40] phases at $T = 0$, which minimizes the energy of the filled Dirac-Fermi sea.[2] The linear term in the β functions indicates that interactions need to be sizable to place the system in any ordered phase. Depending on whether $g >$ or < 1 at the scale Λ, which is determined by the marginally relevant flow of pristine BLG at momentum scale beyond Λ,[38,41] the Dirac fermions find themselves in ordered or semimetallic phase. Since Λ is a *tunable* parameter here, BLG subject to twisting and in-plane fields yields unique opportunities to observe various relativistic quantum critical phenomena, described by the Gross-Neveu-Yukawa (GNY) theory, about which in a moment. We note that our estimation from the large-N analysis is expected to hold even for the realistic situation when N (number of 8-component Dirac fermions) $= 1$, since the actual expansion parameter in our theory is $1/8N$.[42]

The spinless flavored Dirac fermions can condense into a plethora of fully gapped phases. Altogether, there are 16 different ways to spontaneously develop mass gaps for the spinless Dirac fermions.[40] They can be arranged into 4 categories, and each of them contain 4 masses. They are $\vec{A} = \{303,200,100,003\}$, $\vec{B} = \{333,230,130,033\}$, $\vec{C} = \{312,211,111,012\}$, $\vec{D} = \{322,221,121,022\}$.[39] The transformations of these masses under various discrete symmetries are shown in Table I. $\vec{A}$, $\vec{B}$ do not mix two valleys, but are respectively even and odd under $\vec{K} \leftrightarrow -\vec{K}$, whereas $\vec{C}$ and $\vec{D}$ represent symmetric and antisymmetric mixing of two valleys, respectively. All four masses of the pristine BLG[35] are flavor insensitive: The layer-polarized (LP) state, corresponding to an imbalance of carriers densities between two layers, is represented by A_4; B_4 represents the quantum anomalous Hall insulators, which in lattice is realized as intralayer circulating currents, orienting in the same direction in two layers. Two translational and time-reversal symmetry breaking Kekulé currents are represented by C_4 and D_4. Eight out of the above sixteen masses, X_js, where $X = A,B,C,D$; $j = 2,3$, mix the flavors, and correspond to periodic orders with periodicity 2Λ. The remaining four masses X_1s where $X = A,B,C,D$, although they do not mix the flavors, are odd under its exchange.

We now restore the spin degrees of freedom. Each of the masses can now be realized in both spin-singlet and spin-triplet channels, yielding all together $2 \times 16 = 32$ possible insulating orders that can fully gap out all the Dirac points. The spin-triplet insulators, besides the discrete symmetries, also break the spin SU(2) rotational symmetry spontaneously (if there is no Zeeman splitting say in twisted BLG with no in-plane field), and the ordered phases are accompanied by 2 massless Goldstone modes. Let us construct a 16-component spinor $\Psi_s = (\Psi_\uparrow, \Psi_\downarrow)^\top$, where $\uparrow,\downarrow$ are the electron's spin projections. This representation is invariant under the electron's spin rotations, which are generated by $\vec{S} = \vec{s} \otimes (000)$.[39] Both $\Psi_\uparrow$ and $\Psi_\downarrow$ take the form $\Psi_B(\vec{k})$. The Dirac Hamiltonian for spinful fermions is $H = s_0 \otimes H_D$, where s_0 is a two-dimensional identity matrix operating on the spin index, and $[H,\vec{S}] = 0$. The Zeeman coupling, when present, reads as $H_Z = \Delta_Z\ (s_3 \otimes I_8)$, where $\Delta_Z = gB(\vec{x}) \sim B$ (T) K, with $g \approx 2$ for electrons in BLG. We note $\Delta_Z \gg v_B \Lambda_B$ for any accessible magnetic field, but $\Delta_Z \ll v_T \Lambda_T$ even for moderate twisting, say $\phi \sim 3^0$–4^0.

While small compared to twisting, Zeeman coupling gives rise to particle- and hole-like Fermi surfaces for opposite spin projections near each Dirac point, and consequently a BCS instability in the particle-hole spin-triplet channel that takes place even at weak interactions.[43] However the critical temperature of such ordering can be extremely small.[44] Perhaps more importantly, the Zeeman coupling restricts the spin degrees of freedom of any triplet order parameter (OP), $\mathcal{M} = \vec{\Delta} \cdot \vec{s} \otimes X$, where $\{X,H_D\} = 0$, within the easy plane, perpendicular to the applied magnetic field. The effective single-particle Hamiltonian then reads as $H_{\rm SP} = H + \mathcal{M} + H_Z$, and its energy eigenvalues are $\pm E_\sigma$, where for $\sigma = \pm$

$$E_\sigma = \left\{\left[\sqrt{v_F^2 k^2 + \Delta_3^2} + \sigma\ \Delta_Z\right]^2 + \Delta_1^2 + \Delta_2^2\right\}^{1/2}. \quad (5)$$

Therefore, the spectrum is maximally gapped with $\Delta_3 = 0$, or when the triplet OP is restricted within the easy plane. Consequently, one of the Goldstone modes becomes massive, and its mass is $\sim\Delta_Z$. Hence, the Zeeman coupling reduces the symmetry of any triplet ordering to $O(2)$, restoring the possibility of Kosterlitz-Thouless transitions at finite temperatures.[45] Recent experiments[10,11] suggest that a layer antiferromagnet (LAF) state can be found in BLG. The LAF OP is $\langle \Psi_s^\dagger (\vec{s} \otimes A_4) \Psi_s \rangle$. Therefore, subject to an in-plane magnetic field, spin of the LAF order gets projected onto the easy plane, known as the *canted antiferromagnet*, which has also been considered in the quantum Hall regime of insulating BLG[46] and single-layer graphene.[47] In twisted BLG, beyond the superlattice AB stacking goes through an AA one and evolves into BA stacking.[17] Therefore, the layer magnetization changes its sign beyond the superlattice; however, the LAF order remains unchanged. A similar situation also arises when the system develops a LP state.

The quantum phase transition towards the formation of the LAF state, driven solely by the twisting, is described by an $O(3)$ GNY theory. A similar theory can also describe the quantum criticality near the antiferromagnet ordering in monolayer graphene, where the number of 4-component Dirac fermions is two, which in our system is four.[4] The Fermi velocity across the transition remains noncritical since $z = 1$ in our problem.[2] The effective action reads as $S = \int d^d x L$, where $L = L_f + L_b + L_{bf}$, with $L_f = \bar{\Psi}_s s_0 \otimes \Gamma_\mu \partial_\mu \Psi_s$, with

TABLE I. Transformation of various masses under the discrete symmetries, which leave the free Hamiltonian invariant. +, − respectively stand for even and odd. Here, $I_T = (110)$ K is the time-reversal operator, where K is the complex conjugation.

$\vec{M}$	I_L	I_K	$I_{\rm Tr}$	I_f	I_T
$\vec{A}$	(−,+,+,−)	(+,+,+,+)	(+,−,−,+)	(−,−,+,+)	(−,+,+,+)
$\vec{B}$	(−,+,+,−)	(−,−,−,−)	(+,−,−,+)	(−,−,+,+)	(+,−,−,−)
$\vec{C}$	(−,+,+,−)	(+,+,+,+)	(−,+,+,−)	(−,−,+,+)	(+,+,+,−)
$\vec{D}$	(−,+,+,−)	(−,−,−,−)	(−,+,+,−)	(−,−,+,+)	(+,+,+,−)

$\bar{\Psi}_s = \Psi_s^\dagger s_0 \otimes \Gamma_0$ and

$$L_b = |\partial_\mu \vec{\Phi}|^2 + m_t^2 |\vec{\Phi}|^2 + \frac{\lambda_t}{2}|\vec{\Phi}|^4, \quad L_{bf} = g_t \vec{\Phi} \cdot \bar{\Psi}_s \vec{s} \otimes I_8 \Psi_s. \tag{6}$$

$\vec{\Phi}$ is a three-component scalar field, and g_t is the Yukawa coupling. The coupling constants λ_t, g_t are dimensionless in $d = 3+1$, and m^2 is the $T=0$ tuning parameter. Upon promoting the theory to the upper-critical dimension ($d=4$), we can perform a controlled $\epsilon(=d-4)$ expansion. In the absence of the Yukawa coupling the transition is described by the Wilson-Fisher fixed point $(\lambda_t, g_t^2) = (6/11, 0)\,\epsilon$.[36] However, this fixed point is unstable against the Yukawa coupling,[4,5] and the critical behavior of the GNY theory is described by a new fixed point $(\lambda_t, g_t^2) = (0.61942, 0.11)\,\epsilon$. Near the $O(3)$ LAF transition the correlation length exponent is $\nu = \frac{1}{2} + 0.531268\epsilon$. The same theory can describe the critical behavior near any triplet ordering in twisted BLG. Due to the nontrivial Yukawa coupling at the critical point, both the bosonic and the fermionic fields acquire nontrivial anomalous dimensions, respectively read as $\eta_b = \frac{8}{9}\epsilon$, $\eta_f = \frac{\epsilon}{6}$.[36] The bicritical fixed point in the GNY theory lies in the unphysical regime $\lambda_t < 0$. As one approaches the critical point from the semimetallic side, the residue of the quasiparticle pole vanishes as $m_t^{z\nu\eta_f}$.

We now consider driving the Dirac semimetal across the LAF transition by tuning a parallel magnetic field, in the presence of twisting. In that situation the universal behavior in the vicinity of the LAF transition will be governed by an $O(2)$ GNY theory, due to the Zeeman coupling. However, such quantum critical behavior can only be probed at a temperature $T > \Delta_Z$. The effective theory is similar to the one in Eq. (6), with $\vec{\Phi}$ as a two-component bosonic field, and $\vec{s} \to \vec{s}_\perp = (s_1, s_2)$. A similar $O(2)$ GNY theory also describes the quantum superconducting transition of the Dirac fermions in graphene and on the surface of topological insulators,[5] and in-plane field-driven transition to any other triplet ordering in twisted BLG. Various critical exponents near this transition are different from the previous ones and read as $\nu = \frac{1}{2} + \frac{3}{10}\epsilon$, $\eta_b = \frac{4}{5}\epsilon$, and $\eta_f = \frac{\epsilon}{10}$.[5]

As a consequence of the restoration of relativistic invariance at the GNY critical point, i.e., $z=1$, the bosonic velocity approaches the fermionic velocity (v_F) (nonuniversal), and consequently the ratio of the specific heats inside the semimetallic and insulating side within the critical region also approaches universal value

$$\frac{C_{\mathrm{SM}}}{C_{\mathrm{Ins}}} = \frac{N_f}{N_G}\,(1 - 2^{-d}) \tag{7}$$

where $N_f = 16$ is the number of gapless fermionic modes and N_G is the number of massless Goldstone modes in the broken-symmetry phase, which is therefore respectively 2 and 1 in the twisting and the Zeeman-driven LAF phase.

In the same framework, we can also address the quantum critical behavior of a Z_2 symmetry breaking transition towards the LP ordering in twisted BLG with[36]

$$L_b = |\partial_\mu \Phi|^2 + m_s^2|\Phi|^2 + \frac{\lambda_s}{2}|\Phi|^4, \quad L_{b-f} = g_s \Phi \bar{\Psi}_s \Psi_s. \tag{8}$$

The transition to a Z_2 symmetry breaking ordering is governed by the critical point $(\lambda_s, g_s^2) = (0.5914, 0.091)\epsilon$. The critical exponents near this critical point are $\nu = \frac{1}{2} + 0.25574\epsilon$, $\eta_b = \frac{8}{11}\epsilon$, and $\eta_f = \frac{\epsilon}{22}$.[36] At $T=0$, however, there is a first-order phase transition from the LP to the $O(2)$ LAF state, which also carries a finite ferromagnetic moment, if one applies a weak in-plane field.

Finally we propose a possible way to realize some of the new masses, which lack any analogy in the pristine BLG. In the vicinity of the neutrality point, we believe that the leading instabilities will likely be similar to those in regular BLG, when it is slightly twisted and placed in weak parallel fields. On the other hand, upon tilting the magnetic field out of the BLG plane, one can develop a set of Landau levels (LLs) at energies $\pm v_F\sqrt{2nB_\perp}$, where $n = 0, 1, \ldots$, and $B_\perp$ is the field's perpendicular component. Relativistic LLs can also be developed by placing the twisted BLG in perpendicular magnetic fields. The twofold *orbital degeneracy* of the zeroth LL in pristine BLG[19] translates into the *flavor degeneracy* when BLG is twisted[16] or when twisting and tilted fields are present simultaneously, which however cannot be lifted by any of the masses in pristine BLG, e.g., A_4, C_1, D_1,[12] since they are flavor independent. If we consider spinless fermions for simplicity, and also neglect flavor and valley mixing, the flavor degeneracy of the zeroth LL can be lifted by A_1 or B_1 masses. Therefore, by placing the chemical potential close to the first excited state within the zeroth LL, additional incompressible Hall states at fillings $f = \pm 1$ can be developed by these new masses that we propose here.[48] The single-particle gap of the $f = 0, \pm 1$ Hall states should scale linearly with the field, if the interaction is sufficiently weak. Scaling then reverts to a sublinear one for moderate interaction strength, and a perfect $\sqrt{B}$ scaling emerges at zero-field criticality.[49] Interestingly, different scaling regimes can be accessed in this system by tuning the twisting and/or in-plane field, which in turn controls the effective interaction strength. A detailed analysis of the quantum Hall physics for spinful fermions is quite rich, but left for future investigation.

B.R. was supported at National High Magnetic Field Laboratory by NSF Cooperative Agreement No. DMR-0654118, the State of Florida, and the U.S. Department of Energy. K.Y. is supported by NSF Grant No. DMR-1004545. We thank Igor F. Herbut and Oskar Vafek for valuable discussions.

[1] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009); S. Das Sarma, S. Adam, E. H. Hwang, and E. Rossi, *ibid.* **83**, 407 (2011).

[2] I. F. Herbut, Phys. Rev. Lett. **97**, 146401 (2006); I. F. Herbut, V. Juričić, and B. Roy, Phys. Rev. B **79**, 085116 (2009); B. Roy and I. F. Herbut, *ibid.* **82**, 035429 (2010).

[3]D. Mesterházy, J. Berges, and L. von Smekal, Phys. Rev. B **86**, 245431 (2012).
[4]I. F. Herbut, V. Juričić, and O. Vafek, Phys. Rev. B **80**, 075432 (2009); B. Roy, *ibid.* **84**, 113404 (2011); F. F. Assaad and I. F. Herbut, Phys. Rev. X **3**, 031010 (2013).
[5]B. Roy, V. Juričić, and I. F. Herbut, Phys. Rev. B **87**, 041401(R) (2013).
[6]K. Yang, Solid State Commun. **143**, 27 (2007); M. O. Goerbig, Rev. Mod. Phys. **83**, 1193 (2011); Y. Barlas, K. Yang, and A. H. MacDonald, Nanotechnology **23**, 052001 (2012), and references therein.
[7]R. T. Weitz, M. T. Allen, B. E. Feldman, J. Martin, and A. Jacoby, Science **330**, 812 (2010).
[8]A. S. Mayorov, D. C. Elias, M. Mucha-Kruczynski, R. V. Gorbachev, T. Tudorovskiy, A. Zhukov, S. V. Morozov, M. I. Katsnelson, V. I. Fal'ko, A. K. Geim, and K. S. Novoselov, Science **333**, 860 (2011).
[9]F. Freitag, J. Trbovic, M. Weiss, and C. Schönenberger, Phys. Rev. Lett. **108**, 076602 (2012).
[10]J. Velasco, Jr., L. Jing, W. Bao, Y. Lee, P. Kratz, V. Aji, M. Bockrath, C. N. Lau, C. Varma, R. Stillwell, D. Smirnov, Fan Zhang, J. Jung, and A. H. MacDonald, Nat. Nanotechnol. **7**, 156 (2012).
[11]F. Freitag, M. Weiss, R. Maurand, J. Trbovic, and C. Schönenberger, Phys. Rev. B **87**, 161402 (2013).
[12]B. Roy, Phys. Rev. B **88**, 075415 (2013).
[13]R. E. Throckmorton and O. Vafek, Phys. Rev. B **86**, 115447 (2012).
[14]When two layers of BLG are neither perfectly Bernel (AB) nor AA stacked is referred to here as twisted BLG.
[15]S. S. Pershoguba and V. M. Yakovenko, Phys. Rev. B **82**, 205408 (2010).
[16]J. M. B. Lopes dos Santos, N. M. R. Peres, and A. H. Castro Neto, Phys. Rev. Lett. **99**, 256802 (2007); R. de Gail, M. O. Goerbig, F. Guinea, G. Montambaux, and A. H. Castro Neto, Phys. Rev. B **84**, 045436 (2011); J. M. B. Lopes dos Santos, N. M. R. Peres, and A. H. Castro Neto, *ibid.* **86**, 155449 (2012).
[17]P. San-Jose, J. Gonzalez, and F. Guinea, Phys. Rev. Lett. **108**, 216802 (2012).
[18]Interlayer next-nearest-neighbor hopping amplitude is commonly referred as *trigonal warping* (TW).
[19]E. McCann and V. I. Falko, Phys. Rev. Lett. **96**, 086805 (2006).
[20]Y. Lemonik, I. L. Aleiner, C. Toke, and V. I. Fal'ko, Phys. Rev. B **82**, 201408 (2010).
[21]J. Nilsson, A. H. Castro Neto, F. Guinea, and N. M. R. Peres, Phys. Rev. B **78**, 045405 (2008).
[22]D. S. Lee, C. Riedl, T. Beringer, A. H. Castro Neto, K. von Klitzing, U. Starke, and J. H. Smet, Phys. Rev. Lett. **107**, 216602 (2011).
[23]M. Kindermann and E. J. Mele, Phys. Rev. B **84**, 161406 (2011).
[24]E. J. Mele, Phys. Rev. B **81**, 161405 (2010); **84**, 235439 (2011).
[25]R. Bistritzer and A. H. MacDonald, Phys. Rev. B **81**, 245412 (2010); Proc. Natl. Acad. Sci. USA **108**, 12233 (2011).
[26]E. Suárez Morell, J. D. Correa, P. Vargas, M. Pacheco, and Z. Barticevic, Phys. Rev. B **82**, 121407(R) (2010); E. Suárez Morell, M. Pacheco, L. Chico, and L. Brey, *ibid.* **87**, 125414 (2013).
[27]J. Gonzalez, Phys. Rev. B **88**, 125434 (2013).
[28]G. H. Li, A. Luican, J. M. B. Lopes dos Santos, A. H. Castro Neto, A. Reina, J. Kong, and E. Y. Andrei, Nat. Phys. **6**, 109 (2010).
[29]A. Luican, G. Li, A. Reina, J. Kong, R. R. Nair, K. S. Novoselov, A. K. Geim, and E. Y. Andrei, Phys. Rev. Lett. **106**, 126802 (2011).
[30]I. Brihuega, P. Mallet, H. González-Herrero, G. Trambly de Laissardire, M. M. Ugeda, L. Magaud, J. M. Gómez-Rodríguez, F. Ynduráin, and J.-Y. Veuillen, arXiv:1209.0991.
[31]W. Yan, W.-Y. He, Z.-D. Chu, M. Liu, L. Meng, R.-F. Dou, Y. Zhang, Z. Liu, J.-C. Nie, and L. He, Nat. Commun. **4**, 2159 (2013); W. Yan, M. Liu, R.-F. Dou, L. Meng, L. Feng, Z.-D. Chu, Y. Zhang, Z. Liu, J.-C. Nie, and L. He, Phys. Rev. Lett. **109**, 126801 (2012).
[32]Sites on the A sublattice are separated by $2t_{\perp} \approx$ 200–300 meV, and are of no dynamical importance at low energy.
[33]V. Cvetkovic, R. E. Throckmorton, and O. Vafek, Phys. Rev. B **86**, 075467 (2012).
[34]L. M. Zhang, Z. Q. Li, D. N. Basov, M. M. Fogler, Z. Hao, and M. C. Martin, Phys. Rev. B **78**, 235408 (2008), and references therein.
[35]O. Vafek, Phys. Rev. B **82**, 205106 (2010).
[36]See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.88.241107 for derivation of effective non-interacting, and interacting Hamiltonians, and one loop calculation of GNY theory.
[37]W. Tao, G. Qing, L. Yan, and S. Kuang, Chin. Phys. B **21**, 067301 (2012).
[38]O. Vafek and K. Yang, Phys. Rev. B **81**, 041401(R) (2010).
[39]Here ijk stands for $\sigma_i \otimes \sigma_j \otimes \sigma_k$, where $i,j,k = 0,1,2,3$.
[40]Any operator that anticommutes with H_D is *mass*.
[41]F. Zhang, H. Min, M. Polini, and A. H. MacDonald, Phys. Rev. B **81**, 041402 (2010).
[42]See A. M. Vasil'ev, S. E. Derkachov, N. A. Kilev, and A. S. Stepanenko, Teor. Mat. Fiz. **92**, 486 (1992); **97**, 364 (1993) for $1/N$ expansion.
[43]L. V. Keldysh and Y. V. Kopaev, Sov. Phys. Solid State **6**, 2219 (1965).
[44]I. L. Aleiner, D. E. Kharzeev, and A. M. Tsvelik, Phys. Rev. B **76**, 195415 (2007).
[45]J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973); J. M. Kosterlitz, *ibid.* **7**, 1046 (1974).
[46]B. Roy, arXiv:1203.6340.
[47]I. F. Herbut, Phys. Rev. B **76**, 085432 (2007).
[48]See also R. Cote, J. Lambert, Y. Barlas, and A. H. MacDonald, Phys. Rev. B **82**, 035445 (2010).
[49]I. F. Herbut and B. Roy, Phys. Rev. B **77**, 245438 (2008); B. Roy and I. F. Herbut, *ibid.* **83**, 195422 (2011); **88**, 045425 (2013).